THE TOPOLOGY OF A TOPOLOGICAL SUM OF ORDERABLE SPACES IS INDUCED BY THE UNION OF TWO ORDER TOPOLOGIES

NOBUYUKI KEMOTO

Abstract. It is known that the subspace $X = (0,1) \cup \{2\}$ in the real line $\mathbb{R}$ is the topological sum of the ordered subspaces $(0,1)$ and $\{2\}$ of $\mathbb{R}$ which is not orderable. In this paper, we prove that the topology of a topological sum of orderable spaces is induced by the union of two order topologies.

In the present paper [4], the orderability of the topological sum of orderable spaces was discussed. Among other results, the orderability of the topological sum of locally connected orderable spaces was characterized, as a corollary, the subspace $(0,1) \cup \{2\}$ in the real line $\mathbb{R}$ is a locally connected suborderable space which is not orderable. The following classical results are known:

- for every subordered space $\langle X, <, \tau \rangle$, there are $\max\{|M|, 2\}$-many orders on X such that the union of the corresponding order topologies induce the original topology τ, where M denotes the set of missing points in it, see [6],
- the topology of the Sorgenfrey line is induced by the union of two order topologies on it, see [5].

In this paper, we prove that if a space is represented as the topological sum of orderable spaces, then there are two orders on it such that the union of these order topologies induce the original topology.

For a collection $\mathcal{S}$ of subsets of a set X, $\tau(\mathcal{S})$ denotes the topology on X generated by $\mathcal{S}$ as a subbase, that is, $\tau(\mathcal{S})$ is the smallest topology containing $\mathcal{S}$. Also $\tau(\mathcal{S})$ is said to be the topology induced by $\mathcal{S}$. Observe that if $\mathbb{S}$ is a collection of collections of subsets of a set X, then $\tau(\bigcup_{\mathcal{S}\in\mathbb{S}} \mathcal{S}) = \tau(\bigcup_{\mathcal{S}\in\mathbb{S}} \tau(\mathcal{S}))$.

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Let $<$ be a linear order on a set X, see [2, page 4]. The pair $\langle X, <\rangle$ is said to be a linearly ordered set or an ordered set, and usually simply denoted by X. So when we say "let X be an ordered set", we will tacitly assume that $X \neq \emptyset$ and a linear order $<$ on X is already given. An ordered set $\langle X, <\rangle$ has the natural topology $\tau(\{(\leftarrow, x)_< : x \in X\} \cup \{(x, \rightarrow)_< : x \in X\})$, which is denoted by $\lambda_<$, where $(\leftarrow, x)_< = \{y \in X : y < x\}$, also $(x, \rightarrow)_<, (\leftarrow, x)_<, (x, y]_<, \cdots$ are similarly defined. The topology $\lambda_<$ is T_2 and called the interval topology or the order topology on $\langle X, <\rangle$. So we can also consider an ordered set as a topological space with the order topology, we say that the triple $\langle X, <, \lambda_<\rangle$ is an ordered space and denote it simply by X. Generally, we will identify the ordered set $\langle X, <\rangle$ with the ordered space $\langle X, <, \lambda_<\rangle$. A subordered space is a triple $\langle X, <, \tau\rangle$ (also denoted by X) such that $<$ is an order on X and τ is a T_1-topology on X having a base, equivalently a subbase, by convex sets in X, where a subset C of X is convex if $(x, y)_< \subset C$ whenever $x, y \in C$ with $x < y$. Notice that the topology τ is stronger than the interval topology $\lambda_<$, that is $\tau \supset \lambda_<$, thus the topological space $\langle X, \tau\rangle$ is T_2, in fact monotonically normal, see [3, 5.21]. Observe that if $\langle X, <, \tau\rangle$ is a subordered space and $Y \subset X$, then $\langle Y, <\restriction Y, \tau \restriction Y\rangle$, where $<\restriction Y$ is the restricted order $<\restriction Y \times Y$on Y of the order $<$ and $\tau \restriction Y$ is the subspace topology $\{U \cap Y : U \in \tau\}$, is automatically a subordered space. In this case, we say "$\langle Y, <\restriction Y, \tau \restriction Y\rangle$ is a subspace of $\langle X, <, \tau\rangle$" or simply "$Y$ is a subspace of the subordered space X". It is known that, in this sense, every subordered space is a subspace of an ordered space. It is easy to see that a connected subordered spaces is ordered, that is, if a subordered space $\langle X, <, \tau\rangle$ is connected as a topological space, then $\tau = \lambda_<$.

A topological space X with a topology τ, which is also simply denoted by X, is said to be orderable if there is an order $<$ on X with $\tau = \lambda_<$. In this case, the order $<$ is said to be a compatible order of τ. Also a topological space X with a topology τ is said to be suborderable if there is an order $<$ on X such that $\langle X, <, \tau\rangle$ is a subordered space. In this case, such an order $<$ is said to be an underlying order of τ. Note that orderable/suborderable spaces generally can have many compatible/underlying orders. In our discussion below, we will distinguish the two notions between (sub)orderable and (sub)ordered.

Let $\mathcal{Y}$ be a pairwise disjoint collection of topological spaces, that is, $Y \cap Y' = \emptyset$ for distinct members $\langle Y, \tau_Y\rangle, \langle Y', \tau_{Y'}\rangle \in \mathcal{Y}$, where τ_Y denotes the topology on Y. A topological space $\langle X, \tau\rangle$ with $X = \bigcup \mathcal{Y}$ is said to be the topological sum of $\mathcal{Y}$, denoted by $X = \bigoplus \mathcal{Y}$, if $\tau = \tau(\bigcup_{Y \in \mathcal{Y}} \tau_Y)$. Obviously if $X = \bigoplus \mathcal{Y}$ and all members of $\mathcal{Y}$ are T_2, then so is X. In particular, $X = \bigoplus \mathcal{Y}$ is written as $X = Y_0 \oplus Y_1$ whenever $\mathcal{Y} = \{Y_0, Y_1\}$ and $X = \bigoplus_{\alpha<\kappa} Y_\alpha$ whenever $\mathcal{Y} = \{Y_\alpha : \alpha < \kappa\}$.

We use a slightly similar notion. Let $\langle X, \tau\rangle$ is a space. Moreover let $\mathcal{Y}$ be a pairwise disjoint collection of ordered spaces with $X = \bigcup \mathcal{Y}$, that is, $Y \cap Y' = \emptyset$ for distinct members $\langle Y, <_Y, \lambda_{<_Y}\rangle, \langle Y', <_{Y'}, \lambda_{<'_Y}\rangle \in \mathcal{Y}$. Then $\mathcal{Y}$ is said to be a ordered decomposition of X if $\tau = \tau(\bigcup_{Y\in\mathcal{Y}} \lambda_{<_Y})$. In this case, the space $\langle X, \tau\rangle$ is T_2. Note that an orderable space can have many ordered decompositions. For example, let $\mathbb{Q} = \langle \mathbb{Q}, <_{\mathbb{Q}}\rangle$ be the usual rational space. Then not only the singleton $\{\langle \mathbb{Q}, <_{\mathbb{Q}}\rangle\}$ but also the following are ordered decompositions of $\langle \mathbb{Q}, \lambda_{<_{\mathbb{Q}}}\rangle$, where $(\leftarrow, \sqrt{2})$ and $(\sqrt{2}, \rightarrow)$ denote the intervals in the real line $\mathbb{R}$ and $<^{-1}$ denotes the reverse order of $<$ defined below,

- $\{\langle \mathbb{Q}, <_{\mathbb{Q}}^{-1}\rangle\}$,
- $\{\langle \mathbb{Q}\cap(\leftarrow, \sqrt{2}), <_{\mathbb{Q}}\restriction \mathbb{Q}\cap(\leftarrow, \sqrt{2})\rangle, \langle \mathbb{Q}\cap(\sqrt{2}, \rightarrow), <_{\mathbb{Q}}\restriction \mathbb{Q}\cap(\sqrt{2}, \rightarrow)\rangle\}$.

We need some definitions and lemmas from [4].

Definition 1. An ordered set X is said to be type 0 if it has neither minimal elements nor maximal elements with respect to the given order. An ordered set X is said to be type 1 if either it has a minimal element but not have maximal elements, or it has a maximal element but not have minimal elements. An ordered set X is said to be type 2 if it has both a minimal element and a maximal element. When X is a singleton, X is considered to be type 2.

Let $\mathcal{X}$ be a pairwise disjoint collection of ordered spaces. For $i \in 3$ $(= \{0, 1, 2\})$, identifying as $X = \langle X, <, \lambda_<\rangle$, let

$$\mathcal{X}^i = \{X \in \mathcal{X} : X \text{ is type } i\ \}.$$

Then $\mathcal{X}$ is decomposed into $\mathcal{X}^0$, $\mathcal{X}^1$ and $\mathcal{X}^2$.

For an order $<$ on X, $<^{-1}$ denotes the reverse order on X, that is, $x <^{-1} y$ iff $y < x$. Note that the reverse order has the same type as the original type and does not change its interval topology, because of $\lambda_{<^{-1}} = \tau((\leftarrow, x)_{<^{-1}} : x \in X\} \cup \{(x, \rightarrow)_{<^{-1}} : x \in X\}) = \tau(\{(x, \rightarrow)_< : x \in X\} \cup \{(\leftarrow, x)_< : x \in X\}) = \lambda_<$.

It is known that a connected (sub)orderable space has exactly two compatible orders, that is, if $<$ is one of its compatible orders, then another one is $<^{-1}$, see [1] or [4].

Definition 2. Let $\mathcal{X}$ be a pairwise disjoint collection of ordered spaces indexed as $\mathcal{X} = \{X_\alpha : \alpha < \kappa\}$ with a cardinal κ with $\kappa \geq 1$ and let $X = \bigcup_{\alpha<\kappa} X_\alpha$. Moreover let $<_\alpha$ be the order on X_α and $\lambda_{<_\alpha}$ its order topology for each $\alpha < \kappa$. For each $x \in X$, let $\alpha(x)$ be the unique $\alpha < \kappa$ with $x \in X_\alpha$.

The symbol $\Sigma_{\alpha<\kappa} <_\alpha$ denotes the order $<$ on X defined by the following rule:

$$x < y \text{ iff } \begin{cases} x <_{\alpha(x)} y & \text{if } \alpha(x) = \alpha(y), \\ \alpha(x) < \alpha(y) & \text{otherwise.} \end{cases}$$

If $\kappa < \omega$, then $\Sigma_{\alpha<\kappa} <_\alpha$ is denoted by $<_0 + <_1 + \cdots + <_{\kappa-1}$. In particular, $<_0 + <_1$ denotes the resulting order on $X_0 \cup X_1$ adding the ordered space X_1 after the ordered space X_0. Similarly if $\kappa = \omega$, then $\Sigma_{\alpha<\kappa} <_\alpha$ is denoted by $<_0 + <_1 + <_2 + \cdots$. Moreover the ordered space $\langle X, \Sigma_{\alpha<\kappa} <_\alpha \rangle$ is also simply denoted by $\Sigma_{\alpha<\kappa} X_\alpha$ if contexts are clear.

In the definition above, letting $<= \Sigma_{\alpha<\kappa} <_\alpha$, $\langle X, <, \tau(\bigcup_{\alpha<\kappa} \lambda_{<_\alpha})\rangle$ is a subordered space, that is, the topological sum $X = \bigoplus_{\alpha<\kappa} X_\alpha$ is suborderable by the underlying order $<$. To see this for $x \in X_\alpha$, note that the intervals $(\leftarrow, x)_{<_\alpha}$ and $(x, \rightarrow)_{<_\alpha}$ in X_α are convex in the ordered set $\langle X, < \rangle$. Then the collection $\bigcup_{\alpha<\kappa}\{(\leftarrow, x)_{<_\alpha} : x \in X_\alpha\} \cup \{(x, \rightarrow)_{<_\alpha} : x \in X_\alpha\}$ of such intervals, which are convex in X, is a subase for the T_2-topology $\tau(\bigcup_{\alpha<\kappa} \lambda_{<_\alpha})$. So we have $\lambda_< \subset \tau(\bigcup_{\alpha<\kappa} \lambda_{<_\alpha})$.

On the other hand, note that $\lambda_{<_0+<_1} = \tau(\lambda_{<_0} \cup \lambda_{<_1})$ is generally not true, for instance, the example $(0, 1) \cup \{2\}$ above is a witness.

We use the following lemma.

Lemma 3. [4, Theorem 10] *Let a space X can be written as the topological sum $X = \bigoplus \mathcal{Y}$ for some pairwise disjoint collection $\mathcal{Y}$ of orderable spaces. If $\mathcal{Y}$ satisfies either* (1) *or* (2) *below, then X is orderable,*

(1) *there are $Y \in \mathcal{Y}$ and an ordered decomposition $\mathcal{Z}_Y$ of Y such that $\mathcal{Z}_Y^1 \neq \emptyset$,*
(2) *there is a sequence $\langle \mathcal{Z}_Y : Y \in \mathcal{Y}\rangle$ of ordered decompositions $\mathcal{Z}_Y$'s of Y's such that $\bigcup_{Y\in\mathcal{Y}} \mathcal{Z}_Y^0 = \emptyset$, or $\bigcup_{Y\in\mathcal{Y}} \mathcal{Z}_Y^2$ is empty or infinite.*

Theorem 4. *If a space X with a topology τ is written as the topological sum $X = \bigoplus \mathcal{Y}$ for some pairwise disjoint collection $\mathcal{Y}$ of orderable spaces, then there are orders $<, <^*$ on X satisfying $\tau = \tau(\lambda_< \cup \lambda_{<^*})$.*

Proof. For each $Y \in \mathcal{Y}$, taking a compatible order $<_Y$ on Y, Y is regarded as the ordered space $\langle Y, <_Y, \lambda_{<_Y}\rangle$.

By the assumption, we have $\tau = \tau(\bigcup_{Y\in\mathcal{Y}} \lambda_{<_Y})$. If X is orderable, then taking a compatible order $<$, we see $\tau = \lambda_< = \tau(\lambda_< \cup \lambda_<)$. Therefore we may assume that $\langle X, \tau \rangle$ is not orderable.

For each $Y \in \mathcal{Y}$, let $\mathcal{Z}_Y = \{Y\}$. Then applying Lemma 3 to the sequence $\langle \mathcal{Z}_Y : Y \in \mathcal{Y} \rangle$, we see that $\mathcal{Y}^1 = \emptyset$, $\mathcal{Y}^0 \neq \emptyset$ and $\mathcal{Y}^2$ is non-empty and finite, otherwise $\langle X, \tau \rangle$ is orderable. So $\mathcal{Y} = \mathcal{Y}^0 \cup \mathcal{Y}^2$. For each $i \in 3$, let $Z_i = \bigcup \mathcal{Y}^i$ and enumerate $\mathcal{Y}^i$ as $\mathcal{Y}^i = \{Y_{i\alpha} : \alpha < \kappa_i\}$ for some cardinal κ_i. Note $\kappa_0 > 0$, $\kappa_1 = 0$, $0 < \kappa_2 < \omega$, $Z_0 \neq \emptyset$, $Z_1 = \emptyset$ and $Z_2 \neq \emptyset$, so we can ignore Z_1. Moreover for $i = 0$ or 2, let $<_{Z_i}$ be the order $\Sigma_{\alpha<\kappa_i} <_{Y_{i\alpha}}$. Since each $Y_{0\alpha}$ is type 0, the order $<_{Z_0}$ on Z_0 is type 0 and $\lambda_{<_{Z_0}} = \tau(\bigcup_{\alpha<\kappa_0} \lambda_{<_{Y_{0\alpha}}})$, see [4, Lemma 6]. Also since each $Y_{2\alpha}$ is type 2 and κ_2 is non-zero and finite, the order $<_{Z_2}$ on Z_2 is type 2 and $\lambda_{<_{Z_2}} = \tau(\bigcup_{\alpha<\kappa_2} \lambda_{<_{Y_{2\alpha}}})$, see [4, Lemma 8 (1)]. So we have

$$\begin{aligned}
\tau &= \tau\Big(\bigcup_{Y\in\mathcal{Y}} \lambda_{<_Y}\Big) \\
&= \tau\Big(\bigcup_{\alpha<\kappa_0} \lambda_{<_{Y_{0\alpha}}} \cup \bigcup_{\alpha<\kappa_2} \lambda_{<_{Y_{2\alpha}}}\Big) \\
&= \tau\Big(\tau\Big(\bigcup_{\alpha<\kappa_0} \lambda_{<_{Y_{0\alpha}}}\Big) \cup \tau\Big(\bigcup_{\alpha<\kappa_2} \lambda_{<_{Y_{2\alpha}}}\Big)\Big) \\
&= \tau(\lambda_{<_{Z_0}} \cup \lambda_{<_{Z_2}}),
\end{aligned}$$

which means $X = Z_0 \bigoplus Z_2$.

Consider the orders $<:=<_{Z_0} + <_{Z_2}$ and $<^*:=<_{Z_0}^{-1} + <_{Z_2}$ on X. Since for every $i = 0, 2$, $\alpha < \kappa_i$ and $x \in Y_{i\alpha}$, the intervals $(\leftarrow, x)_{<_{Y_{i\alpha}}}$ and $(x, \rightarrow)_{<_{Y_{i\alpha}}}$ in $Y_{i\alpha}$ are convex in both order $<$ and $<^*$, we see that both $<$ and $<^*$ are underlying orders for the topology τ, that is, both $\langle X, <, \tau \rangle$ and $\langle X, <^*, \tau \rangle$ are subordered spaces. So we see $\lambda_< \cup \lambda_{<^*} \subset \tau$, therefore $\tau(\lambda_< \cup \lambda_{<^*}) \subset \tau$. Now the following Claim completes the proof.

Claim. $\tau(\lambda_< \cup \lambda_{<^*}) \supset \tau$.

Proof. Since $\tau = \tau(\lambda_{<_{Z_0}} \cup \lambda_{<_{Z_2}})$, it suffices to see both $\lambda_{<_{Z_0}} \subset \tau(\lambda_< \cup \lambda_{<^*})$ and $\lambda_{<_{Z_2}} \subset \tau(\lambda_< \cup \lambda_{<^*})$.

Since $<=<_{Z_0} + <_{Z_2}$ and Z_0 is type 0, for each $y \in Z_0$, we see:

- $(\leftarrow, y)_{<_{Z_0}} = (\leftarrow, y)_< \in \lambda_< \subset \tau(\lambda_< \cup \lambda_{<^*})$,
- $(y, \rightarrow)_{<_{Z_0}} = \bigcup_{x\in(y,\rightarrow)_{<_{Z_0}}} (y, x)_< \in \lambda_< \subset \tau(\lambda_< \cup \lambda_{<^*})$,

so we have $\lambda_{<_{Z_0}} \subset \tau(\lambda_< \cup \lambda_{<^*})$.

Let $y \in Z_2$. It follows from $<=<_{Z_0} + <_{Z_2}$ that $(y, \rightarrow)_{<_{Z_2}} = (y, \rightarrow)_< \in \lambda_< \subset \tau(\lambda_< \cup \lambda_{<^*})$. To see $(\leftarrow, y)_{<_{Z_2}} \in \tau(\lambda_< \cup \lambda_{<^*})$, fix $z_0 \in Z_0$. Note

- $(z_0, \rightarrow)_{<_{Z_0}} \cup (\leftarrow, y)_{<_{Z_2}} = (z_0, y)_< \in \lambda_<$,
- $(z_0, \rightarrow)_{<_{Z_0}^{-1}} \cup (\leftarrow, y)_{<_{Z_2}} = (z_0, y)_{<^*} \in \lambda_{<^*}$,

- $(z_0, \to)_{<_{Z_0}} \cap (z_0, \to)_{<_{Z_0}^{-1}} = (z_0, \to)_{<_{Z_0}} \cap (\leftarrow, z_0)_{<_{Z_0}} = \emptyset$.

Then $Z_0 \cap Z_2 = \emptyset$ shows

$$\begin{aligned}\tau(\lambda_< \cup \lambda_{<^*}) &\ni (z_0, y)_< \cap (z_0, y)_{<^*} \\ &= ((z_0, \to)_{<_{Z_0}} \cup (\leftarrow, y)_{<_{Z_2}}) \cap ((z_0, \to)_{<_{Z_0}^{-1}} \cup (\leftarrow, y)_{<_{Z_2}}) \\ &= ((z_0, \to)_{<_{Z_0}} \cap (z_0, \to)_{<_{Z_0}^{-1}}) \cup (\leftarrow, y)_{<_{Z_2}} \\ &= (\leftarrow, y)_{<_{Z_2}}.\end{aligned}$$

Now we have $\lambda_{<_{Z_2}} \subset \tau(\lambda_< \cup \lambda_{<^*})$. This completes the proof of the claim. □

In the above proof, notice that both $<$ and $<^*$ are underlying orders for the suborderable topology τ with $\lambda_< \neq \lambda_{<^*}$. There is a more simple example of two different underlying orders of a suborderable space:

Example 5. Let τ be the discrete topology on the real line $\mathbb{R}$. Using the usual order $<_{\mathbb{R}}$ on $\mathbb{R}$ and the usual order on $\{-1, 0, 1\}$, we can consider the lexicographic product $\mathbb{R} \times \{-1, 0, 1\}$. Moreover let $<$ be the restricted order on $\mathbb{R} \times \{0\}$ of the lexicographic order. Then the subordered subspace $\mathbb{R} \times \{0\}$ of the lexicographic product is discrete in the topological sense, because of $\{\langle x, 0\rangle\} = (\langle x, -1\rangle, \langle x, 1\rangle) \cap (\mathbb{R} \times \{0\})$ for every $x \in \mathbb{R}$, where $= (\langle x, -1\rangle, \langle x, 1\rangle)$ denotes the interval in the lexicographic product. Identifying $\mathbb{R} = \mathbb{R} \times \{0\}$, we see that $<$ coincides with $<_{\mathbb{R}}$, therefore the order topology $\lambda_<$ on $\mathbb{R}$ is just equal to the Euclidean topology on $\mathbb{R}$. On the other hand, since discrete spaces are orderable, take a compatible order $<^*$ of the discrete topology τ, so $\tau = \lambda_{<^*}$. Thus both $<$ and $<^*$ are underlying orders for the discrete suborderable space $\langle \mathbb{R}, \tau\rangle$ with $\lambda_< \neq \lambda_{<^*} = \tau$.

Corollary 6. *The topology of a locally connected suborderable space is induced by the union of two order topologies.*

Proof. Let $\langle X, \tau\rangle$ be a locally connected suborderable space. Taking its underlying order $<$, we may assume that X is a locally connected subordered space $\langle X, <, \tau\rangle$. Let $\mathcal{Y}$ be the collection of all connected components of X, that is, all maximal connected subsets of X. Since $\langle X, \tau\rangle$ is locally connected, each member Y in $\mathcal{Y}$ is open in the space $\langle X, \tau\rangle$, and convex in the ordered set $\langle X, <\rangle$. Moreover $\langle Y, \tau \restriction Y\rangle$ is connected and the subspace $\langle Y, < \restriction Y, \tau \restriction Y\rangle$ is subordered, so it is ordered, that is, $\tau \restriction Y = \lambda_{<\restriction Y}$. Now apply the theorem above. □

The following classical question may still remain open.

Question 7 ([6])**.** Is every suborderable topology induced by the union of two order topologies?

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Department of Mathematics, Oita University, Oita, 870-1192 Japan
E-mail address: nkemoto@oita-u.ac.jp