COUNTABLE PARACOMPACTNESS OF σ-PRODUCTS

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Abstract. We present an example of a σ-product that is not countably paracompact but all of whose finite subproducts are countably paracompact. This example also shows that countable paracompactness of a σ-product may depend on the choice of base point. We also show that normal non-trivial σ-products are countably paracompact, improving a result of Chiba. Finally we give a new proof that σ-products of ordinals at base point $\mathbf{0}$ are κ-normal and strongly zero-dimensional.

1. Introduction

Throughout the paper, spaces are regular topological spaces. Let X_i be a space for each $i \in \kappa$ and κ a cardinal. $\prod_{i\in\kappa} X_i$ denotes the product space with the usual Tychonoff product topology. For $x \in \prod_{i\in\kappa} X_i$, $x(i)$ denotes the i-th coordinate of x.

A σ-product of X_i's ($i \in \kappa$) with a base point $s \in \prod_{i\in\kappa} X_i$ is the subspace

$$\sigma(\prod_{i\in\kappa} X_i, s) = \{x \in \prod_{i\in\kappa} X_i : |\{i \in \kappa : x(i) \neq s(i)\}| < \omega\}.$$

For x in a σ-product with a base point s, we let $\mathrm{supt}(x)$ denote the set $\{i \in \kappa : x(i) \neq s(i)\}$.

A finite subproduct of $\sigma(\prod_{i\in\kappa} X_i, s)$ means a product $\prod_{i\in B} X_i$ for some finite $B \subset \kappa$.

Similarly a Σ-product is defined by

$$\Sigma(\prod_{i\in\kappa} X_i, s) = \{x \in \prod_{i\in\kappa} X_i : |\{i \in \kappa : x(i) \neq s(i)\}| \leq \omega\}.$$

For a subset $B \subset \kappa$, $p_B : \sigma(\prod_{i\in\kappa} X_i, s) \to \sigma(\prod_{i\in B} X_i, s \restriction B)$ denotes the projection map. For a basic open set U of a product space $\prod_{i\in\kappa} X_i$, $\mathrm{supt}(U)$ denotes the finite set $\{i \in \kappa : p_{\{i\}}(U) \neq X_i\}$.

A general problem about σ-products is whether the full σ-product has a property P assuming that each finite subproduct has P. For example, Kombarov [12] Theorem 3 (Teng [16] Theorem 1) proved that if every finite subproduct of a σ-product is paracompact (metacompact, resp.), then it is also paracompact (metacompact, resp.). It is not difficult to improve the Teng's proof in order to show that if every finite subproduct of a σ-product is λ-metacompact, then it is also λ-metacompact, where λ is an infinite cardinal. So it is natural to ask whether such cardinal restrictions also hold for paracompactness.

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In particular the following are known:

(a) If every finite subproduct of a σ-product is countably metacompact, then it is countably metacompact ([16]).
(b) If every finite subproduct of a normal σ-product is countably paracompact, then it is countably paracompact ([2]).
(c) σ products of ordinals at base point $\mathbf{0}$ are countably paracompact([9]).

Note that the general problem for σ-products also has a negative answer for normality: if X is Dowker, then $\sigma(X \times 2^\omega)$ is not normal. Related to normal σ-products, in addition to the Chiba result cited above, the following is known

(d) if every finite subproduct of a normal σ-product is expandable, then it is expandable ([16]).

In section 2 we show that the assumption that finite subproducts are countably paracompact in (b) is not needed: non-trivial normal σ-products are countably paracompact (in fact, if there are κ many factors, then the space is κ-expandable). In addition, the proof can be modified to give a new proof of Teng's result (d) concerning expandable σ-products.

On the other hand, the authors proved in [9] that countable products of ordinals and Σ-products of ordinals with arbitrary base points are countably paracompact, κ-normal and strongly zero-dimensional. Recall that a space is κ-normal (strongly zero-dimensional) if two disjoint regular closed sets (zero-sets, resp.) are separated by disjoint open sets (clopen sets, resp.). Morever in the same paper, the authors proved the following

(e) Each σ-product of ordinals at base point $\mathbf{0}$ (= the constant function taking value 0) is countably paracompact, κ-normal and strongly zero-dimensional.

In section 2 we present an example of a σ-product of ordinals (at a base point different from $\mathbf{0}$) which is not countably paracompact (nor is it κ-normal). Since finite products of ordinals are countably paracompact, this example shows that the general problem for σ-products fails for countable paracompactness. In addition, it shows that countable paracompactness (as well as κ-normality) of a σ-product may depend on the choice of the base point. While properties of σ-products and Σ-products often do not depend on the choice of the base points, some pathologies do exist: For example, Corson pointed out a family of spaces such that the Σ-product at one base point is not homeomorphic to the Σ-product at a different base point [4]. Also, Chiba gave an example of a family of spaces whose σ-product about one base point is starcompact, and about another base point is not [2]. Van Douwen asked whether normality of a Σ-product may depend on the base point (problem P18 in [14] [15]).

In section 3 we revisit (e): The authors' proofs of (e) involved the use of elementary submodels. We give a new proof, not involving elementary submodels, that σ-products of ordinals at base point $\mathbf{0}$ are κ-normal and strongly zero-dimensional (in fact, it will suffice to assume that each coordinate of the base point has countable cofinality).

2. Countable paracompactness

Example 2.1. *A family of spaces $\{X_i : i \in \omega\}$ with all finite subproducts countably paracompact such that for some base point s, the σ-product of the family at base point s is not countably paracompact.*

For $i \in \omega$, let

$$X_i = \begin{cases} \omega_1, & \text{if } i = 0, \\ \omega_1 + 1, & \text{if } i \geq 1. \end{cases}$$

Now we define a base point $s \in \prod_{i \in \omega} X_i$ by

$$s(i) = \begin{cases} 0, & \text{if } i = 0, \\ \omega_1, & \text{if } i \geq 1. \end{cases}$$

Then finite subproducts of $X = \sigma(\prod_{i \in \omega} X_i, s)$ are countably paracompact. We will show that X is not countably paracompact. To do this, let $D_n = \{x \in X : x \restriction n \text{ is constant }\}$ for each $1 \leq n \in \omega$. Obviously $\{D_n : 1 \leq n \in \omega\}$ is a decreasing sequence of closed sets in X with the empty intersection. Let U_n be an open set containing D_n for each $1 \leq n \in \omega$.

Now fix $1 \leq n \in \omega$ and define $x_{\alpha n} \in X$ for each $\alpha \in \omega_1$ by

$$x_{\alpha n}(i) = \begin{cases} \alpha, & \text{if } i < n, \\ \omega_1, & \text{if } i \geq n. \end{cases}$$

Then for each $\alpha \in \omega_1$, by $x_{\alpha n} \in D_n \subset U_n$, we can find $\beta(\alpha) < \alpha$, $m(\alpha) > n$ and $\gamma(\alpha) \in \omega_1$ such that the basic open set $p^{-1}_{m(\alpha)}((\beta(\alpha), \alpha]^n \times (\gamma(\alpha), \omega_1]^{m(\alpha)-n})$ is contained in U_n, where $p_B : X \to X(B) = \prod_{i \in B} X_i$ denotes the projection map. By the Pressing Down Lemma, we find a stationary set $S_n \subset \omega_1$, $\beta_n \in \omega_1$ and $m_n > n$ such that $\beta(\alpha) = \beta_n$ and $m(\alpha) = m_n$ hold for each $\alpha \in S_n$.

Moving n, take $\delta \in \omega_1$ with $\sup\{\beta_n : 1 \leq n \in \omega\} < \delta$ and define $x \in X$ by

$$x(i) = \begin{cases} \delta, & \text{if } i = 0, \\ \omega_1, & \text{if } i \geq 1. \end{cases}$$

Again fix $1 \leq n \in \omega$ and let W be a basic open neighborhood of x in X. We may assume $W = p^{-1}_m((\beta, \delta] \times (\gamma, \omega_1]^{m-1})$, where $\sup\{\beta_n : 1 \leq n \in \omega\} \leq \beta < \delta$ and $m > m_n$. Take $\alpha \in S_n$ with $\max\{\delta, \gamma\} < \alpha$. Then $p^{-1}_{m_n}((\beta_n, \alpha]^n \times (\gamma(\alpha), \omega_1]^{m_n - n}) \subset U_n$. Define $y \in X$ by

$$y(i) = \begin{cases} \delta, & \text{if } i = 0, \\ \alpha, & \text{if } 0 < i < n, \\ \omega_1, & \text{if } i \geq n. \end{cases}$$

Then we have $y \in W \cap U_n$, thus $x \in \mathrm{cl}_X U_n$. Therefore $x \in \bigcap_{1 \leq n \in \omega} \mathrm{cl}_X U_n$ holds, this shows that X is not countably paracompact. $\square$

Recall that the space $\sigma(\omega_1 \times (\omega_1 + 1)^\omega, \mathbf{0})$ is countably paracompact. Thus we can recognize that the space X above is a delicate example and that countable paracompactness of σ-products can depend on the choice of base point. In addition $\sigma(\omega_1 \times (\omega_1 + 1)^\omega, \mathbf{0})$ is κ-normal and strongly zero-dimensional. Note that the space X above is of cardinality ω_1. So, if CH fails, X must be strongly zero-dimensional. On the other hand, we have the following:

Claim 2.2. *X is not κ-normal.*

Proof. Since $X \times (\omega + 1)$ embeds as a clopen subset of X, it suffices to prove that $X \times (\omega + 1)$ is not κ-normal. To see this, note that the sets D_n in X are regular closed (the subset $I_n = \{x \in X : x \restriction n \text{ is constant with value a successor ordinal}\}$ is open and dense in D_n). Let $H = \bigcup_{n \in \omega}(D_n \times \{2n\})$ and let $K = X \times (\{\omega\} \cup \{2n+1 : n \in \omega\})$. Then both H and K are regular closed in Y and since the D_n's witness the failure of countable paracompactness in X, it follows as in the proof of Dowker's theorem that H and K cannot be separated in Y.

The following proposition, which includes the result [3] above, says that the assumption of the normality of σ-products is quite strong. Also note the result of [16] that if every finite subproduct of a normal σ-product is expandable, then it is expandable. Our approach below is different from the proof of this result and some simple improvements of our approach below give a direct proof of this result.

Proposition 2.3. *Let κ be an infinite cardinal, X_i be a space with $|X_i| \geq 2$ for each $i \in \kappa$ and $s \in \prod_{i\in\kappa} X_i$. If $X = \sigma(\prod_{i\in\kappa} X_i, s)$ is normal, then X is κ-expandable therefore it is countably paracompact and κ-collectionwise normal.*

Proof. Recall that a space is κ-expandable (κ-collectionwise normal) if for every locally finite (discrete, resp.) collection $\mathcal{F}$ of closed sets with $|\mathcal{F}| \leq \kappa$, there is a locally finite (discrete, resp.) collection $\{U(F) : F \in \mathcal{F}\}$ of open sets such that $F \subset U(F)$ for each $F \in \mathcal{F}$. A space is expandable if it is κ-expandable for each infinite cardinal κ. Recall that a space X is normal and κ-expandable iff X is countably paracompact and κ-collectionwise normal iff $X \times A(\kappa)$, where $A(\kappa)$ denotes the one point compactification of the discrete space of size κ, is normal, see [1] and [7].

Let $B \subset \kappa$. We use here the following notation: $X(B) = \sigma(\prod_{i\in B} X_i, s \restriction B)$, $Z(B) = \{x \in X : \mathrm{supt}(x) \subset B\}$, $X^n = \{x \in X : |\mathrm{supt}(x)| \leq n\}$ for each $n \in \omega$ and $p_B : X \to X(B)$ denotes the projection. Note that $p_B \restriction Z(B)$ is a homeomorphism between $Z(B)$ and $X(B)$ and that $X^0 = \{s\}$, each X^n is closed in X and $X = \bigcup_{n\in\omega} X^n$. For $x \in X$, x_B is the element in X defined by

$$x_B(i) = \begin{cases} x(i), & \text{if } i \in B, \\ s(i), & \text{otherwise.} \end{cases}$$

Also note $p_B(x) = p_B(x_B)$.

Claim 1. X contains a copy of $A(\kappa)$.

Proof. For each $i \in \kappa$, fix $t(i) \in X_i$ with $t(i) \neq s(i)$ and define $x_i \in X$ by

$$x_i(j) = \begin{cases} t(j), & \text{if } i = j, \\ s(j), & \text{otherwise.} \end{cases}$$

Then $\{x_i : i \in \kappa\} \cup \{s\}$ is homeomorphic to $A(\kappa)$.

Claim 2. For each finite subset B of κ, $X(B)$ is κ-expandable.

Proof. Since X is homeomorphic to $X(B) \times X(\kappa \setminus B)$, by the argument of Claim 1, $X(\kappa \setminus B)$ contains a copy of $A(\kappa)$. It follows from the normality of X that $X(B) \times A(\kappa)$ is normal, therefore $X(B)$ is κ-expandable.

Let $\mathcal{F}$ be a locally finite collection of closed sets of X with $|\mathcal{F}| \leq \kappa$. First set $U_{-1} = W_{-1} = \emptyset$ and $V_{-1}(F) = G_{-1}(F) = \emptyset$ for each $F \in \mathcal{F}$. We will define open sets U_n, W_n, $V_n(F)$ and $G_n(F)$ for each $n \in \omega$ and $F \in \mathcal{F}$ such that

(a) $X^n \cup \mathrm{cl}W_{n-1} \subset W_n \subset \mathrm{cl}W_n \subset U_n \subset \mathrm{cl}U_n \subset X \setminus \bigcup_{F\in\mathcal{F}}(F \setminus G_n(F))$,
(b) $F \cap U_n \subset G_n(F) = G_{n-1}(F) \cup V_n(F)$ and $V_n(F) \cap W_{n-1} = \emptyset$ for each $F \in \mathcal{F}$,
(c) $\mathcal{V}_n = \{V_n(F) : F \in \mathcal{F}\}$ and $\mathcal{G}_n = \{G_n(F) : F \in \mathcal{F}\}$ are locally finite.

Assume that U_k, W_k, $V_k(F)$ and $G_k(F)$ are defined for $k \le n$ and $F \in \mathcal{F}$. For each $B \in [\kappa]^{n+1}$, since $\mathcal{H}(B) = \{(F \setminus U_n) \cap Z(B) : F \in \mathcal{F}\}$ is locally finite in the closed subspace $Z(B)$ that is homeomorphic (by $p_B \restriction Z(B)$) to the κ-expandable space $X(B)$, we can find a locally finite collection $\mathcal{V}(B) = \{V(F,B) : F \in \mathcal{F}\}$ of open sets in $X(B)$ such that for each $F \in \mathcal{F}$,

(1) $V(F,B) \supset p_B((F \setminus U_n) \cap Z(B))$,
(2) $V(F,B) \cap p_B(\mathrm{cl}W_n \cap Z(B)) = \emptyset$.

Let $V_{n+1}(F) = \bigcup\{p_B^{-1}(V(F,B)) : B \in [\kappa]^{n+1}\} \setminus \mathrm{cl}W_n$ for each $F \in \mathcal{F}$. Then obviously $V_{n+1}(F) \cap W_n = \emptyset$.

Claim 3. $\mathcal{V}_{n+1} = \{V_{n+1}(F) : F \in \mathcal{F}\}$ is locally finite in X.

Proof. Let $y \in X$. We may assume $y \notin W_n \supset X^n$, thus $|\mathrm{supt}(y)| \ge n+1$. For each $A \in [\mathrm{supt}(y)]^{\le n}$, by $y_A \in X^n \subset W_n$, we can find a basic open neighborhood $O(A)$ of y_A in X with $O(A) \subset W_n$. We may assume $y_A \in O(A) = p_{B(A)}^{-1}(\prod_{i\in B(A)} O_i(A))$ for some finite set $B(A) \subset \kappa$ with $\mathrm{supt}(y) \subset B(A)$ and some open set $O_i(A)$ in X_i, $i \in B(A)$. Let $C = \bigcup\{B(A) : A \in [\mathrm{supt}(y)]^{\le n}\}$, moreover define for each $i \in C$,

$$O_i = \begin{cases} \bigcap\{O_i(A) : i \in A \in [\mathrm{supt}(y)]^{\le n}\}, & \text{if } i \in \mathrm{supt}(y), \\ \bigcap\{O_i(A) : A \in [\mathrm{supt}(y)]^{\le n}, i \in B(A)\}, & \text{if } i \in C \setminus \mathrm{supt}(y) \end{cases}$$

Then $p_C^{-1}(\prod_{i\in C} O_i)$ is a neighborhood of y.

On the other hand, for each $B \in [\mathrm{supt}(y)]^{n+1}$, it follows from $p_B(y) \in X(B)$ and the local finiteness of $\mathcal{V}(B)$ in $X(B)$ that we can find a basic open neighborhood $\prod_{i\in B} O'_i(B)$ of $p_B(y)$ such that $\mathcal{F}_B = \{F \in \mathcal{F} : \prod_{i\in B} O'_i(B) \cap V(F,B) \ne \emptyset\}$ is finite. We will show that the neighborhood $O = p_C^{-1}(\prod_{i\in C} O_i) \cap \bigcap\{p_B^{-1}(\prod_{i\in B} O'_i(B)) : B \in [\mathrm{supt}(y)]^{n+1}\}$ of y witnesses the local finiteness of $\mathcal{V}_{n+1}$ at y. It suffice to show $\{F \in \mathcal{F} : O \cap V_{n+1}(F) \ne \emptyset\} \subset \bigcup\{\mathcal{F}_B : B \in [\mathrm{supt}(y)]^{n+1}\}$. Assume $O \cap V_{n+1}(F) \ne \emptyset$ and pick a point $x \in O \cap V_{n+1}(F)$. Then $x \in O \cap p_B^{-1}(V(F,B))$ for some $B \in [\kappa]^{n+1}$. Now we have $B \in [\mathrm{supt}(y)]^{n+1}$. To show this, assume $B \not\subset \mathrm{supt}(y)$. Then $A = B \cap \mathrm{supt}(y) \in [\mathrm{supt}(y)]^{\le n}$. It follows from $x_B \in Z(B)$ and $p_B(x) = p_B(x_B) \in V(F,B)$ that by (2), $x_B \notin \mathrm{cl}W_n$. Now we will show $x_B \in O(A)$. Let $i \in B(A)$.

By $x \in O \subset p_C^{-1}(\prod_{i\in C} O_i)$, if $i \in A$, then $x_B(i) = x(i) \in O_i \subset O_i(A)$. If $i \in B \setminus A$, then it follows from $i \in C \setminus \mathrm{supt}(y)$, $i \in B(A)$ and $A \in [\mathrm{supt}(y)]^{\le n}$ that $x_B(i) = x(i) \in O_i \subset O_i(A)$. Finally if $i \in B(A) \setminus B$, then $x_B(i) = s(i) = y_A(i) \in O_i(A)$. So we have $x_B \in O(A) \subset W_n$, this contradicts $x_B \notin \mathrm{cl}W_n$. Therefore we have $B \subset \mathrm{supt}(y)$.

Since

$$\begin{aligned} & x \in O \cap p_B^{-1}(V(F,B)) \\ & \subset p_B^{-1}(\prod_{i\in B} O'_i(B)) \cap p_B^{-1}(V(F,B)) = p_B^{-1}(\prod_{i\in B} O'_i(B) \cap V(F,B)), \end{aligned}$$

we have $F \in \mathcal{F}_B$. This completes the proof of Claim 3.

Let $G_{n+1}(F) = G_n(F) \cup V_{n+1}(F)$ for each $F \in \mathcal{F}$. By (c) and the Claim above, $\mathcal{G}_{n+1} = \{G_{n+1}(F) : F \in \mathcal{F}\}$ is also locally finite.

Claim 4. $(F \setminus U_n) \cap X^{n+1} \subset V_{n+1}(F)$ for each $F \in \mathcal{F}$.

Proof. Let $x \in (F \setminus U_n) \cap X^{n+1}$. It follows from $x \notin U_n \supset X^n$ that $B = \mathrm{supt}(x) \in [\kappa]^{n+1}$. Therefore by (1), we have

$$x \in (F \setminus U_n) \cap Z(B) \subset p_B^{-1}(V(F,B)).$$

Moreover by $x \notin U_n \supset \mathrm{cl}W_n$, we have $x \in V_{n+1}(F)$.

Now fix $F \in \mathcal{F}$. By (b) and Claim 4, we have

$F \cap X^{n+1} \subset [F \cap (X^{n+1} \cap U_n)] \cup [F \cap (X^{n+1} \setminus U_n)] \subset G_n(F) \cup V_{n+1}(F) = G_{n+1}(F)$.

Therefore

$$\begin{aligned} F \cap (X^{n+1} \cup \mathrm{cl}W_n) &\subset (F \cap X^{n+1}) \cup (F \cap \mathrm{cl}W_n) \subset (F \cap X^{n+1}) \cup (F \cap U_n)) \\ &\subset G_{n+1}(F) \cup G_n(F) = G_{n+1}(F). \end{aligned}$$

So we have

$$(F \setminus G_{n+1}(F)) \cap (X^{n+1} \cup \mathrm{cl}W_n) = \emptyset.$$

Now since $H = \bigcup_{F \in \mathcal{F}} (F \setminus G_{n+1}(F))$ is a closed set disjoint from $X^{n+1} \cup \mathrm{cl}W_n$, by the normality of X, we can find open sets W_{n+1} and U_{n+1} such that

$$X^{n+1} \cup \mathrm{cl}W_n \subset W_{n+1} \subset \mathrm{cl}W_{n+1} \subset U_{n+1} \subset \mathrm{cl}U_{n+1} \subset X \setminus H.$$

Obviously we have $F \cap U_{n+1} \subset G_{n+1}(F)$ for each $F \in \mathcal{F}$. This completes the construction of U_{n+1}, W_{n+1}, $V_{n+1}(F)$'s and $G_{n+1}(F)$'s.

Finally for each $F \in \mathcal{F}$ define $G(F) = \bigcup_{n\in\omega} G_n(F) = \bigcup_{n\in\omega} V_n(F)$. It follows from (a) and (b) that $F \subset G(F)$ for each $F \in \mathcal{F}$. The following claim completes the proof.

Claim 5. $\mathcal{G} = \{G(F) : F \in \mathcal{F}\}$ is locally finite.

Proof. Let $x \in X$. Since by (a), $\{W_n : n \in \omega\}$ is an increasing open cover of X, we can find $n \in \omega$ with $x \in W_n$. By (b), we have $V_m(F) \cap W_n = \emptyset$ for each $m > n$ and $F \in \mathcal{F}$. Since $G(F) = G_n(F) \cup \bigcup_{m>n} V_m(F)$ and $\mathcal{G}_n$ is locally finite, $\mathcal{G}$ is also locally finite. □

3. κ-normality and strong zero-dimensionality

First we fix notations throughout this section: Let κ be a cardinal and let $\langle \alpha_i : i \in \kappa\rangle$ be a sequence of ordinals. For each $i \in \kappa$, let $Y_i = \{\beta < \alpha_i : \mathrm{cf}\beta \le \omega\}$ and fix a point $s \in \prod_{i\in\kappa} Y_i$. Moreover let $X = \sigma(\prod_{i\in\kappa} \alpha_i, s)$ and $Y = \sigma(\prod_{i\in\kappa} Y_i, s)$. Observe that Y_i is ω-bounded (i.e., each countable subset has a compact closure) whenever $\mathrm{cf}\alpha_i \neq \omega$ and that each Y_i is first countable hence Y has countable tightness (i.e., for each point x and subset A with $x \in \mathrm{cl}A$, there is a countable subset $A' \subset A$ such that $x \in \mathrm{cl}A'$, see [10] Proposition 1).

For each subset $B \subset \kappa$, let $X(B) = \sigma(\prod_{i\in B} \alpha_i, s \restriction B)$ and $Y(B) = \sigma(\prod_{i\in B} Y_i, s \restriction B)$, moreover let $p_B : X \to X(B)$ and $\pi_B : Y \to Y(B)$ denote the projection maps. For $x \in X$, define $x_B \in X$ by

$$x_B(i) = \begin{cases} x(i), & \text{if } i \in B, \\ s(i), & \text{otherwise.} \end{cases}$$

For each $n \in \omega$, Y^n denotes the set $\{y \in Y : |\mathrm{supt}(y)| \le n\}$. Similarly $Y(B)^n$ denotes the set $\{y \in Y(B) : |\mathrm{supt}(y)| \le n\}$.

For notational conveniences, -1 is considered as the immediate successor of the ordinal 0.

We will use the facts: For each finite subset B of κ,

- $Y(B)$ is normal, see [8] Theorem 5-Claim 4,
- $Y(B)$ is strongly zero-dimensional, see [6] Theorem 5.1.

The following lemma is a modification of Theorem 1 of [13].

Lemma 3.1. *Y is normal and strongly zero-dimensional.*

Proof. First we show the following claim.

Claim 1. For each $n \in \omega$, the following statement (S_n) hold.

(S_n): If F is closed subset of Y which is disjoint from Y^n, then there is a clopen set U in Y such that $Y^n \subset U$ and $U \cap F = \emptyset$.

Proof. We will prove this claim by induction on $n \in \omega$. Since Y is a subspace of the product space $\prod_{i\in\kappa} Y_i$ of the zero-dimensional spaces Y_i's, it is also zero-dimensional. Therefore (S_0) holds.

Assume that (S_n) holds for every such a space Y and that F is closed and disjoint from Y^{n+1}. Since $s \notin F$, there are a finite set $B \subset \kappa$ and $t \in \prod_{i\in B}(s(i) \cup \{-1\})$ such that $\pi_B^{-1}(\prod_{i\in B}(t(i), s(i)]) \cap F = \emptyset$. Observe that $Y = Y(\kappa \setminus \{i\}) \times Y_i$ for each $i \in B$. Fix $i \in B$. Now we will prove:

Fact. There is a clopen set U_i in Y such that $\pi_{\kappa\setminus\{i\}}^{-1}(Y(\kappa \setminus \{i\})^n) \subset U_i$ and $U_i \cap F = \emptyset$.

Proof. We divide into two cases.

Case 1. $\mathrm{cf}\alpha_i \neq \omega$.

It is known from [11] Lemma 3 that if A has countable tightness and B is ω-bounded, then the projection $\pi : A \times B \to A$ is closed. Therefore the projection $\pi_{k\setminus\{i\}} : Y = Y(\kappa \setminus \{i\}) \times Y_i \to Y(\kappa \setminus \{i\})$ is closed. It follows from $F \cap Y^{n+1} = \emptyset$ that $\pi_{k\setminus\{i\}}(F) \cap Y(\kappa \setminus \{i\})^n = \emptyset$. By the inductive assumption, there is a clopen set U_i' of $Y(\kappa \setminus \{i\})$ such that $Y(\kappa \setminus \{i\})^n \subset U_i'$ and $U_i' \cap \pi_{k\setminus\{i\}}(F) = \emptyset$. Then $U_i = \pi_{k\setminus\{i\}}^{-1}(U_i')$ is the desired one.

Case 2. $\mathrm{cf}\alpha_i = \omega$.

Let $\langle \alpha_i(k) : k \in \omega\rangle$ be a strictly increasing sequence cofinal in α_i with $s(i) < \alpha_0(i)$ and set $Y_i(k) = Y_i \cap (\alpha_i(k-1), \alpha_i(k)]$ for each $k \in \omega$, where $\alpha_i(-1) = -1$. Then each $Y_i(k)$ is ω bounded and $Y = Y(\kappa \setminus \{i\}) \times Y_i = \bigoplus_{k\in\omega} Y(\kappa \setminus \{i\}) \times Y_i(k)$. Applying the argument in Case 1, we can find a clopen set $U_i(k) \subset Y(\kappa \setminus \{i\}) \times Y_i(k)$ such that $\pi_{\kappa\setminus\{i\}}^{-1}(Y(\kappa \setminus \{i\})^n) \cap Y(\kappa \setminus \{i\}) \times Y_i(k) \subset U_i(k)$ and $U_i(k) \cap F = \emptyset$ for each $k \in \omega$. Then $U_i = \bigcup_{k\in\omega} U_i(k)$ is the desired one. This completes the proof of the fact.

Now consider the clopen set $U = \pi_B^{-1}(\prod_{i\in B}(t(i), s(i)]) \cup \bigcup_{i\in B} U_i$. $F \cap U = \emptyset$ is evident. Assume $y \in Y^{n+1} \setminus \pi_B^{-1}(\prod_{i\in B}(t(i), s(i)])$. Then there is $i \in B$ such that $y(i) \neq s(i)$. Therefore $\pi_{\kappa\setminus\{i\}}(y) \in Y(\kappa \setminus \{i\})^n$, thus $y \in U_i$. This shows $Y^{n+1} \subset U$ and completes the proof of Claim 1.

To show normality and strong zero-dimensionality simultaneously, let F_0 and F_1 be disjoint closed sets in Y. It suffices to find disjoint clopen sets V_0 and V_1 including F_0 and F_1 respectively. We may assume $s \notin F_1$. Fix a clopen set V_{00} such that $s \in V_{00}$ and $V_{00} \cap F_1 = \emptyset$. Set $V_{10} = \emptyset$. Now by induction on $n \in \omega$, we will define clopen sets V_{0n} and V_{1n} in Y such that

(a) $Y^n \subset V_{0n} \cup V_{1n}$,
(b) $V_{jn} \cap F_{1-j} = \emptyset$ for each $j \in 2$.

Assume that V_{jk} has been defined for each $j \in 2$ and $k \leq n$. Let $V_n = V_{0n} \cup V_{1n}$ and $Z(B) = \{y \in Y : \mathrm{supt}(y) \subset B\}$ for each $B \in [\kappa]^{n+1}$. Observe that $\pi_B \restriction$

$Z(B)$ is a homeomorphism between the closed subspace $Z(B)$ of Y and the σ-subproduct $Y(B)$. Since $\pi_B(F_0 \cap Z(B) \setminus V_n)$ and $\pi_B(F_1 \cap Z(B) \setminus V_n)$ are disjoint closed sets in the clopen subspace $\pi_B(Z(B) \setminus V_n)$ of the normal strongly zero-dimensional space $Y(B)$, there are disjoint clopen sets $W_0(B)$ and $W_1(B)$ of $Y(B)$ such that $F_0 \cap Z(B) \setminus V_n \subset \pi_B^{-1}(W_0(B))$, $F_1 \cap Z(B) \setminus V_n \subset \pi_B^{-1}(W_1(B))$ and $\pi_B^{-1}(W_0(B) \cup W_1(B)) \cap Z(B) = Z(B) \setminus V_n$. Set $W_j = \bigcup\{\pi_B^{-1}(W_j(B)) : B \in [\kappa]^{n+1}\}$ for each $j \in 2$.

Claim 2. W_j is clopen in Y for each $j \in 2$.

Proof. By the continuity of π_B, W_j is evidently open. Let $y \in \mathrm{cl}_Y W_j$ and let $A \in [\mathrm{supt}(y)]^{\leq n}$. Then y_A, as defined above, is an element of $Z(A)$. Thus $y_A \in Y^n \subset V_n$. Take a basic clopen neighborhood $V(A)$ of y_A in Y with $V(A) \subset V_n$. We may assume that there is a finite subset $B(A)$ of κ with $\mathrm{supt}(y) \subset B(A)$ such that $y_A \in V(A) = \pi_{B(A)}^{-1}(\prod_{i \in B(A)} V_i(A))$, where $V_i(A)$ is clopen in Y_i for each $i \in B(A)$. Set $C = \bigcup\{B(A) : A \in [\mathrm{supt}(y)]^{\leq n}\}$ and for each $i \in C$, set

$$V_i = \begin{cases} \bigcap\{V_i(A) : i \in A \in [\mathrm{supt}(y)]^{\leq n}\}, & \text{if } i \in \mathrm{supt}(y), \\ \bigcap\{V_i(A) : A \in [\mathrm{supt}(y)]^{\leq n}, i \in B(A)\}, & \text{if } i \in C \setminus \mathrm{supt}(y). \end{cases}$$

Then $V = \pi_C^{-1}(\prod_{i \in C} V_i)$ is a neighborhood of y in Y. Since $\pi_B^{-1}(W_j(B))$'s are clopen, it suffices to show that V meets $\pi_B^{-1}(W_j(B))$ for at most finitely many $B \in [\kappa]^{n+1}$. To show this, assume that there is $B \in [\kappa]^{n+1}$ such that $B \not\subset \mathrm{supt}(y)$ and $V \cap \pi_B^{-1}(W_j(B)) \neq \emptyset$. Fix $x \in V \cap \pi_B^{-1}(W_j(B))$ and let $A = B \cap \mathrm{supt}(y) \in [\mathrm{supt}(y)]^{\leq n}$. The x_B defined above is in $Z(B)$ and $\pi_B(x_B) = \pi_B(x) \in W_j(B) \subset \pi_B(Z(B) \setminus V_n)$. Since $\pi_B \restriction Z(B)$ is a homeomorphism between $Z(B)$ and $Y(B)$, we have $x_B \notin V_n$. Now we will show $x_B \in V(A)$, let $i \in B(A)$. By $x \in V$, if $i \in A$, then $x_B(i) = x(i) \in V_i \subset V_i(A)$. If $i \in B \setminus A$, then it follows from $i \in C \setminus \mathrm{supt}(y)$, $i \in B(A)$ and $A \in [\mathrm{supt}(y)]^{\leq n}$ that $x_B(i) = x(i) \in V_i \subset V_i(A)$. Finally if $i \in B(A) \setminus B$, then $x_B(i) = s(i) = y_A(i) \in V_i(A)$. Thus $x_B \in V(A) \subset V_n$, a contradiction. Therefore V meets $\pi_B^{-1}(W_j(B))$ only for $B \subset \mathrm{supt}(y)$. This completes the proof of Claim 2.

Let $W'_j = W_j \setminus V_n$ for each $j \in 2$. By Claim 2, W'_j is clopen.

Claim 3. $W'_j \cap (F_{1-j} \cap Y^{n+1}) = \emptyset$ for each $j \in 2$.

Proof. Let $y \in W'_j \cap (F_{1-j} \cap Y^{n+1})$. Since $y \in Y^{n+1} \setminus V_n \subset Y^{n+1} \setminus Y^n$, we have $B = \mathrm{supt}(y) \in [\kappa]^{n+1}$. Then $y \in (F_{1-j} \cap Z(B)) \setminus V_n \subset \pi_B^{-1}(W_{1-j}(B))$. On the other hand, by $y \in W_j$, there is $B' \in [\kappa]^{n+1}$ such that $y \in \pi_{B'}^{-1}(W_j(B'))$. Since $W_0(B)$ and $W_1(B)$ are disjoint, we have $B \neq B'$. Since $\pi_{B'}(y_{B'}) = \pi_{B'}(y) \in W_j(B')$, we have $y_{B'} \in \pi_{B'}^{-1}(W_j(B')) \cap Z(B') \subset Z(B') \setminus V_n$. It follows from $Y^n \subset V_n$ that $|\mathrm{supt}(y_{B'})| = |B'| = n+1$. But this is a contradiction, because by $B \neq B'$, $|\mathrm{supt}(y_{B'})| \leq |B \cap B'| \leq n$. The proof of Claim 3 is complete.

Claim 4. $Y^{n+1} \setminus V_n \subset W'_0 \cup W'_1$.

Proof. Let $y \in Y^{n+1} \setminus V_n$ and $B = \mathrm{supt}(y) \in [\kappa]^{n+1}$. Then $y \in Z(B) \setminus V_n \subset \pi_B^{-1}(W_0(B) \cup W_1(B)) \subset W_0 \cup W_1$, therefore $y \in W'_0 \cup W'_1$.

Now let $F = (F_0 \cap W'_1) \cup (F_1 \cap W'_0)$. By Claims 3 and 4, we have $F \cap Y^{n+1} = \emptyset$. Applying Claim 1 for (S_{n+1}), we can find a clopen set W such that $Y^{n+1} \subset W$ and $W \cap F = \emptyset$. Then obviously we have that $V_{jn+1} = V_{jn} \cup (W'_j \cap W)$'s satisfy the conditions (a) and (b) for $n+1$. It is straightforward to show that $V_0 = \bigcup_{n\in\omega}(V_{0n} \setminus \bigcup_{m\le n} V_{1m})$ and $V_1 = \bigcup_{n\in\omega}(V_{1n} \setminus \bigcup_{m\le n} V_{0m})$ are clopen sets separating F_0 and F_1 respectively. Hence Y is normal and strongly zero-dimensional. □

Theorem 3.2. *If $Y = \sigma(\prod_{i\in\kappa} Y_i, s) \subset Z \subset X = \sigma(\prod_{i\in\kappa} \alpha_i, s)$, where $Y_i = \{\beta < \alpha_i : \mathrm{cf}\beta \le \omega\}$ and $s \in \prod_{i\in\kappa} Y_i$, then Z is κ-normal and strongly zero-dimensional.*

Proof. Remember that if a space has a dense C^*-embedded κ-normal (strongly zero-dimensional) subspace, then it is also κ-normal (strongly zero-dimensional, see [5] 7.1.17). So it suffices to show that Y is C^*-embedded in X. Let F_0 and F_1 be disjoint zero-sets in Y. We will show that F_0 and F_1 have disjoint closures in X, see [5] 3.2.1. By Lemma 3.1, we can find disjoint clopen sets U_0 and U_1 in Y separating F_0 and F_1 with $U_0 \cup U_1 = Y$.

Claim 1. $B = \{i \in \kappa : \exists y_0, y_1 \in Y(\pi_{\kappa\setminus\{i\}}(y_0) = \pi_{\kappa\setminus\{i\}}(y_1), y_0 \in U_0, y_1 \in U_1)\}$ is countable.

Proof. Assume that B is uncountable. For each $i \in B$, fix $y_0^i, y_1^i \in Y$ such that $\pi_{\kappa\setminus\{i\}}(y_0^i) = \pi_{\kappa\setminus\{i\}}(y_1^i), y_0^i \in U_0$ and $y_1^i \in U_1$. Since each finite subproduct of Y is ω_1-compact ([9] Corollary 2.2), Y is also ω_1-compact (apply $\triangle$-system lemma to $\{\mathrm{supt}(x_\alpha) : \alpha \in \omega_1\}$ assuming the existence of a closed discrete subspace $\{x_\alpha : \alpha \in \omega_1\}$), where a space is ω_1-compact if there does not exist an uncountable closed discrete subspace.

So there is a cluster point y of $\{y_0^i : i \in B\}$ in Y. By $y \in \mathrm{cl}_Y\{y_0^i : i \in B\} \subset U_0$, we may fix a basic open neighborhood W of y in Y with $W \subset U_0$. Since $C = \{i \in B : y_0^i \in W\}$ is infinite, we can pick $i \in C \setminus \mathrm{supt}(W)$. It follows from $\pi_{\kappa\setminus\{i\}}(y_0^i) = \pi_{\kappa\setminus\{i\}}(y_1^i)$ that $y_1^i \in W \subset U_0$, a contradiction.

Claim 2. $\pi_B(U_0) \cap \pi_B(U_1) = \emptyset$.

Proof. Assume $x \in \pi_B(U_0) \cap \pi_B(U_1)$ and fix $y_j \in U_j$ with $x = \pi_B(y_j)$ for each $j \in 2$. Then $A = \{i \in \kappa : y_0(i) \ne y_1(i)\}$ is finite and disjoint from B. Order A as $A = \{i(k) : 0 \le k < l\}$ and for every m with $0 \le m \le l$, define $z_m \in Y$ by

$$z_m(i) = \begin{cases} y_1(i), & \text{if } i \in \{i(k) : k < m\}, \\ y_0(i), & \text{otherwise.} \end{cases}$$

Then $y_0 = z_0$, $y_1 = z_l$, moreover $\pi_{\kappa\setminus\{i(m)\}}(z_m) = \pi_{\kappa\setminus\{i(m)\}}(z_{m+1})$ and $i(m) \notin B$ for each $m < l$. Since $z_0 = y_0 \in U_0$, $\pi_{\kappa\setminus\{i(0)\}}(z_0) = \pi_{\kappa\setminus\{i(0)\}}(z_1)$ and $i(0) \notin B$, it follows, by definition of B, that $z_1 \in U_0$. By induction we can show $y_0 = z_0, z_1, .., z_l = y_1 \in U_0$, a contradiction.

Since π_B is an open map, $\{\pi_B(U_0), \pi_B(U_1)\}$ is a disjoint clopen cover of $Y(B)$.

Claim 3. $\mathrm{cl}_{X(B)}\pi_B(U_0) \cap \mathrm{cl}_{X(B)}\pi_B(U_1) = \emptyset$.

Proof. Assume $z \in \mathrm{cl}_{X(B)}\pi_B(U_0) \cap \mathrm{cl}_{X(B)}\pi_B(U_1)$ and let $A = \{i \in B : \mathrm{cf}z(i) > \omega\}$. Note $A \subset \mathrm{supt}(z)$. For each $i \in B - A$, since $\mathrm{cf}z(i) \le \omega$, one can fix a strictly increasing sequence $\{z_n(i) : n \in \omega\}$ cofinal in $z(i)$ when $\mathrm{cf}z(i) = \omega$, and set $z_n(i) = z(i)-1$ when $\mathrm{cf}z(i) < \omega$. Moreover fix an increasing sequence $\{H_n : n \in \omega\}$ of finite sets with $B \setminus A = \bigcup_{n\in\omega} H_n$. Inductively we will define $\{x_j^n : n \in \omega\} \subset \pi_B(U_j)$ for each $j \in 2$ as follows. Since

$$V_0 = \{x \in X(B) : \forall i \in A(x(i) \le z(i)), \forall i \in H_0(z_0(i) < x(i) \le z(i))\}$$

is a neighborhood of z in $X(B)$, we can fix, for $j \in 2$, $x_j^0 \in \pi_B(U_j) \cap V_0$. Observe that for $i \in A$, $x_j^0(i) < z(i)$ holds, because of $\mathrm{cf}x_j^0(i) \le \omega$ and $\mathrm{cf}z(i) > \omega$. Now assume that points x_j^k's and open neighborhoods V_k's for $k < n$ and $j \in 2$ are defined such that $x_j^k(i) < z(i)$ for each $i \in A$. Set

$$V_n = \{x \in X(B) : \forall i \in A(\max\{x_0^{n-1}(i), x_1^{n-1}(i)\} < x(i) \leq z(i)),$$
$$\forall i \in H_n(z_n(i) < x(i) \leq z(i))\}$$

and fix $x_j^n \in \pi_B(U_j) \cap V_n$ for each $j \in 2$. Define $x \in \prod_{i\in B} \alpha_i$ by

$$x(i) = \begin{cases} \sup\{x_0^n(i) : n \in \omega\}, & \text{if } i \in A, \\ z(i), & \text{if } i \in B \setminus A. \end{cases}$$

Then obviously $x \in Y(B)$ and $x \in \mathrm{cl}_{Y(B)}\{x_0^n : n \in \omega\} \cap \mathrm{cl}_{Y(B)}\{x_1^n : n \in \omega\} \subset \mathrm{cl}_{Y(B)}\pi_B(U_0) \cap \mathrm{cl}_{Y(B)}\pi_B(U_1) = \pi_B(U_0) \cap \pi_B(U_1) = \emptyset$, a contradiction.

Clearly $F_0 \subseteq U_0 \subseteq p_B^{-1}(\pi_B(U_0))$. Thus

$$\mathrm{cl}_X F_0 \subseteq \mathrm{cl}_X p_B^{-1}(\pi_B(U_0)) \subseteq p_B^{-1}(\mathrm{cl}_{X(B)}\pi_B(U_0)).$$

Similarly, we have that

$$\mathrm{cl}_X F_1 \subseteq p_B^{-1}(\mathrm{cl}_{X(B)}\pi_B(U_1)).$$

Now by Claim 3,

$$p_B^{-1}(\mathrm{cl}_{X(B)}\pi_B(U_0)) \cap p_B^{-1}(\mathrm{cl}_{X(B)}\pi_B(U_1)) = \emptyset.$$

Therefore, F_0 and F_1 have disjoint closures in X. Thus, Y is C^*-embedded in X. □

Finding a proof, which does not depend on elementary submodel techniques or some kind of closing off argument, of countable paracompactness of σ-product of ordinals with the base point $\mathbf{0}$ seems to be strangely difficult.

4. Problems

Recall that the space X of Example 2.1 is strongly zero-dimensional assuming the negation of CH. So we ask:

Question 4.1. *Is every σ-product of ordinals at arbitrary base point strongly zero-dimensional? In particular, is the space in Example 2.1 strongly zero-dimensional in ZFC?*

Question 4.2. *Can normality of a σ-product depend on the base point?*

[9] Corollary 1.11 shows that in the notation of section 3, if $\Sigma(\prod_{i\in\kappa} Y_i, s) \subset Z \subset \prod_{i\in\kappa} \alpha_i$, then Z is κ-normal and strongly zero-dimensional. In contrast to Theorem 3.2:

Question 4.3. *Is a space Z satisfying $\sigma(\prod_{i\in\kappa} Y_i, s) \subset Z \subset \prod_{i\in\kappa} \alpha_i$, where $s \in \prod_{i\in\kappa} Y_i$, κ-normal and strongly zero-dimensional?*

In connection with Proposition 2.3:

Question 4.4. *Let κ be an uncountable cardinal, X_i be a space with $|X_i| \geq 2$ for each $i \in \kappa$ and $s \in \prod_{i\in\kappa} X_i$. If $X = \sigma(\prod_{i\in\kappa} X_i, s)$ is κ-expandable and each finite subproduct of X is normal, then is X normal?*

References

[1] O.T. Alas *On a characterization of collectionwise normality*, Canad. Math. Bull. **14** (1971) 13–15.

[2] K. Chiba *Pseudonormality and starcompactness of σ-products*, Proc. Amer. Math. Soc. **131** 1 (2002) 319–327.

[3] K. Chiba *On the $\mathcal{D}$-property of σ-products*, Math. Japonica **132** (1987) 5–10.

[4] H.H. Corson *Normality of subsets in product spaces* Amer. J. Math. **81** (1959) 785-796.

[5] R. Engelking *General Topology* (Revised and completed edition), Heldermann Verlag, Berlin 1989.

[6] W. Fleissner, N. Kemoto and J. Terasawa *Strong zero-dimensionality of products of ordinals* Top. Appl. **132** (2003) 109–127.

[7] M. Katětov *Extension of locally finite covering* Colloq. Math. **6** (1958) 145–151.

[8] L. Kalantan and P.J. Szeptycki *κ-normality and products of ordinals* Top. Appl. **123** (2002) 537–545.

[9] N. Kemoto and P.J. Szeptycki *Topological properties of products of ordinals* Top. Appl. **143** (2004) 257–277.

[10] A. P. Kombarov and V. I. Malyhin *On Σ-products* Soviet Math. Dokl. **14** (1973) 1780–1783.

[11] A. P. Kombarov *On the product of normal spaces. Uniformities on Σ-products* Soviet Math. Dokl. **13** (1972) 1068–1071.

[12] A. P. Kombarov *On the normality of Σ_m-products* Soviet Math. Dokl. **14** (1973) 1050–1054

[13] A. P. Kombarov *Normal σ-products* Soviet Math. Dokl. **18** (1977) 184–188

[14] P. Nyikos *Problem Section* Top. Proc **7** no.2 (1982) 381–393 .

[15] E. Pearl *Problems from Topology Proceedings*, Topology Atlas Document, URL: http://at.yorku.ca/i/a/a/z/10.htm

[16] H. Teng *On σ-product spaces. I* Math. Japon. **36** (1991) 515–522.

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