σ-COLLECTIONWISE HAUSDORFNESS AT SINGULAR STRONG LIMIT CARDINALS

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Abstract. Assuming the Singular Cardinals Hypothesis, we prove the following property:

σ-CWH: For every singular strong limit cardinal κ and ↗-normal space X such that for some χ < κ, every x ∈ X has a neighborhood base of size ≤ χ, if every closed discrete subspace of size < κ is σ-separated, then so is every closed discrete subspace of size κ.

So for getting a model of the negation of σ-CWH, we require a large cardinal.

Throughout the paper, spaces are regular $T_1$ topological spaces and generally $\alpha, \beta, \gamma, ..., (\kappa, \lambda, \mu, ..., k, l, m, ...)$. A closed discrete subspace $Y$ of a space $X$ is said to be separated if there is a pairwise disjoint collection $\{ U(y) : y \in Y \}$ of open sets such that $y \in U(y)$ for each $y \in Y$. A closed discrete subspace $Y$ of a space $X$ is said to be σ-separated if $Y$ can be represented as the countable sum $Y = \bigcup_{n \in \omega} Y_n$, where each $Y_n$ is separated.

For a cardinal $\kappa$, a space $X$ is $\kappa$-collectionwise Hausdorff (κ-σ-collectionwise Hausdorff) if every closed discrete subspace $Y \subset X$ of size $\kappa$ is separated. For short, CollectionWise Hausdorff is abbreviated as CWH. A space is $< \kappa$-CWH ($< \kappa$-σ-CWH) if it is $\lambda$-CWH ($\lambda$-σ-CWH) for every cardinal $\lambda < \kappa$. For a cardinal $\kappa$, we consider the following inductive type properties:

CWH($\kappa$): If $X$ is $< \kappa$-CWH, then it is $\kappa$-CWH.

σ-CWH($\kappa$): If $X$ is $< \kappa$-σ-CWH, then it is $\kappa$-σ-CWH.

For a regular cardinal $\kappa$ (i.e., $\text{cf} \kappa = \kappa$), Fleissner [2] proved:

- Under a $\Diamond$-like assumption that is a consequence of $V = L$, if $X$ is normal and each point $x \in X$ has a neighborhood base of size $\leq \kappa$, then $X$ has $	ext{CWH}(\kappa)$.

Balogh and Burke [1] generalized this result as:

- Under the same assumption, if $X$ is ↗-normal and each point $x \in X$ has a neighborhood base of size $\leq \kappa$, then $X$ has $\text{CWH}(\kappa)$.

Here a space is ↗-normal if for every pair of disjoint closed sets $H$ and $K$, there are sequences $\{ U_n : n \in \omega \}$ and $\{ V_n : n \in \omega \}$ of open sets such that $U_n \cap V_n = \emptyset$ for every $n \in \omega$, $\{ H \cap U_n : n \in \omega \} \nearrow H$ and $\{ K \cap V_n : n \in \omega \} \nearrow K$, where for a well-ordered set $A$, $\{ B_\alpha : \alpha \in A \}$ ↗ $B$ means that $\bigcup_{\alpha \in A} B_\alpha = B$ and $B_\alpha \subset B_\alpha'$ whenever $\alpha < \alpha'$. For later use, $\{ B_\alpha : \alpha \in A \}$ ↗ means that $B_\alpha \subset B_\alpha'$ whenever $\alpha < \alpha'$.

For a singular cardinal $\kappa$ (i.e., $\text{cf} \kappa < \kappa$), these papers also showed:

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• Assuming the Generalized Continuum Hypothesis (GCH), that is, $2^\lambda = \lambda^+$ for every cardinal $\lambda$, if $X$ is normal ($\mathcal{J}$-normal) and for some $\chi < \kappa$, each point $x \in X$ has a neighborhood base of size $\leq \chi$, then $X$ has CWH($\kappa$) (\sigma-CWH($\kappa$)), see [2] ([1], respectively).

Obviously assuming GCH, every singular cardinal $\kappa$ is strong limit, i.e., $2^{\kappa^{+}} < \kappa$ for each cardinal $\lambda < \kappa$. As a weakening of GCH, the following is well-known:

**The Singular Cardinals Hypothesis (SCH):** For every singular cardinal $\kappa$, if $2^{\kappa^{+}} < \kappa$, then $\kappa^{\omega} = \kappa^+$.

For SCH, see [3]. The author proved in [4]:

• Assuming SCH, for every singular strong limit cardinal $\kappa$, if $X$ is normal and for some $\chi < \kappa$, each point $x \in X$ has a neighborhood base of size $\leq \chi$, then $X$ has CWH($\kappa$).

In this paper, we will prove:

**Theorem 1.** Assuming SCH, for every singular strong limit cardinal $\kappa$, if $X$ is $\mathcal{J}$-normal and for some $\chi < \kappa$, each point $x \in X$ has a neighborhood base of size $\leq \chi$, then $X$ has $\sigma$-CWH($\kappa$).

However, the proof here is necessarily tedious and does not always follow predictable modifications of the proofs listed above. Now we prepare some notions.

**Definition 2.** Let $Y$ be a closed discrete subspace of a space $X$ with $\omega_1 \leq \text{cf}\kappa \leq \kappa = |Y|$ and $Y = \{Y_\alpha : \alpha < \text{cf}\kappa\}$ a partition of $Y$, that is, $Y$ is a disjoint cover of $Y$. Fix a neighborhood base $B_y$ of $y$ for each $y \in Y$. Define

$$F = \{\langle f_n : n \in \omega \rangle : \forall n \in \omega (\text{dom} \ f_n \subset Y), f_n \in \prod_{y \in \text{dom} \ f_n} B_y, \{\text{dom} \ f_n : n \in \omega\} \uparrow \bigcup_{\beta < \alpha} Y_\beta\},$$

$$F_\alpha = \{\langle f_n : n \in \omega \rangle \in F : \{\text{dom} \ f_n : n \in \omega\} \uparrow \bigcup_{\beta < \alpha} Y_\beta\},$$

for each $\alpha < \text{cf}\kappa$.

For each $\alpha < \text{cf}\kappa$ and $f \in \prod_{y \in \text{dom} \ f} B_y$ with $\text{dom} \ f \subset Y$, let

$$C(f, Y, \alpha) = \text{Cl}(\bigcup \{f(y) : y \in \text{dom} \ f \cap (\bigcup_{\beta < \alpha} Y_\beta)\}).$$

Then obviously $\text{dom} \ f \cap (\bigcup_{\beta < \alpha} Y_\beta) \subset C(f, Y, \alpha)$. Moreover for each $f = \langle f_n : n \in \omega \rangle \in F$ and $\alpha < \text{cf}\kappa$, let

$$S(f, Y, \alpha) = \{y \in Y : \{C(f_\alpha, Y, \alpha) : n \in \omega\} \text{ is not point-finite at } y\}.$$  

If $f = \langle f_n : n \in \omega \rangle \in F_{\text{cf}\kappa}$, then by $\{\text{dom} \ f_n : n \in \omega\} \uparrow Y$, we have $\bigcup_{\beta < \alpha} Y_\beta \subset S(f, Y, \alpha)$ for each $\alpha < \text{cf}\kappa$, and $S(f, Y, \alpha) : \alpha < \text{cf}\kappa \uparrow Y$.

**Definition 3.** Let $Y$ be a closed discrete subspace of a space $X$ with $\omega_1 \leq \text{cf}\kappa \leq \kappa = |Y|$. $Y$ has good partitions if there is a sequence $\{Y^m : m \in \omega\}$ of partitions of $Y$, where $Y^m = \{Y^m_\alpha : \alpha < \text{cf}\kappa\}$, such that

(a) $|Y^m_\alpha| < \kappa$ for each $m \in \omega$ and $\alpha < \text{cf}\kappa$,

(b) for each $m \in \omega$, there is $f^m = \langle f^m_n : n \in \omega \rangle \in F_{\text{cf}\kappa}$ such that $\{\alpha < \text{cf}\kappa : S(f^m, Y^m, \alpha) \subset \bigcup_{\beta < \alpha} Y^m_{\beta + 1}\}$ contains a closed unbounded (cub) set in $\text{cf}\kappa$.  

Lemma 4. Let $Y$ be a closed discrete subspace of a space $X$ with $\omega_1 \leq \text{cf} \kappa \leq \kappa = |Y|$ such that every subset $Y' \subset Y$ of size $< \kappa$ is $\sigma$-separated. If $Y$ has good partitions $\{Y^m : m \in \omega\}$, then $Y$ is $\sigma$-separated.

Proof. Let $\{Y^m : m \in \omega\}$ be good partitions and $f^m \in \mathcal{F}_{\text{cf} \kappa}$ and a cub set $C_m \subset \text{cf} \kappa (m \in \omega)$ ensure the clause (b) of the definition above. For notational simplicity, let $S^m_\alpha = S(f^m, Y^m, \alpha)$ for each $m \in \omega$ and $\alpha < \text{cf} \kappa$. Let $C = \{0\} \cup \bigcap_{m \in \omega} C_m$ and enumerate $C$ as $C = \{\alpha(\gamma) : \gamma < \text{cf} \kappa\}$ with the increasing order. For each $m \in \omega$ and $\gamma < \text{cf} \kappa$, set $Y(m, \gamma) = \bigcup\{Y^m_\beta : \alpha(\gamma) \leq \beta < \alpha(\gamma + 1)\}$. Since $C$ is cub, $\{Y(m, \gamma) : \gamma < \text{cf} \kappa\}$ is also a partition of $Y$. For each $y \in Y$ and $m \in \omega$, let $\gamma(m, y) < \text{cf} \kappa$ be the unique $\gamma$ with $y \in Y(m, \gamma)$. Since $\bigcup_{\beta<\alpha(\gamma)} Y^\beta_\beta \subset \bigcup_{\beta<\alpha(\gamma)} Y^\beta_{\beta+1}$ for each $\gamma < \text{cf} \kappa$ and $m \in \omega$, we have $\gamma(m, y) \geq \gamma(m+1, y)$ for each $m \in \omega$ and $y \in Y$. Therefore for each $y \in Y$, one can fix $m(y) \in \omega$ such that $\gamma(m(y), y) = \gamma(m, y)$ for each $m \geq m(y)$. Set for each $m, n \in \omega$,$$
 D^m_n = \{y \in \text{dom } f^m_n : y \notin C(f^m_n, Y^n, \alpha(\gamma(m, y)))\}.$$Claim. $Y = \bigcup_{n, m \in \omega} D^m_n.$

Proof. Let $y \in Y$ and $\gamma = \gamma(m(y), y)$. It follows from $\gamma = \gamma(m(y)+1, y)$ that $y \in Y((m(y)+1), \gamma)$. Since $Y(m(y)+1, \gamma)$ is disjoint from $\bigcup_{\beta<\alpha(\gamma)} Y^\beta_{\beta+1} \supset S^\beta_\alpha$, we have $y \notin S^m_\alpha(y)$. Therefore $\{C(f^m_n, Y^n, \alpha(\gamma)) : n \in \omega\}$ is point-finite at $y$. So by $\{\text{dom } f^m_n : n \in \omega\} \not\supset Y$, one can fix $n \in \omega$ with $y \in \text{dom } f^m_\gamma$ and $y \notin C(f^m_n, Y^n, \alpha(\gamma))$, thus $y \in D^m_n$.

Now it suffices to show that $D^m_n$'s are $\sigma$-separated. For each $\gamma < \text{cf} \kappa$, set $D^m_n(\gamma) = D^m_n \cap Y(m, \gamma)$, then obviously $D^m_n(\gamma) \cap C(f^m_n, Y^n, \alpha(\gamma)) = \emptyset$. It follows from $|D^m_n(\gamma)| < \kappa$ and the assumption that we get a partition $\{D^m_n(\gamma, k) : k \in \omega\}$ of $D^m_n(\gamma)$, where each $D^m_n(\gamma, k)$ is separated. By induction on $\gamma < \text{cf} \kappa$, one can define $g \in \prod_{y \in D^m_n} B_y$ satisfying:

(a) $g(y) \in f^m_n(y)$ for each $y \in D^m_n$.

(b) $\{g(y) : y \in D^m_n(\gamma, k)\}$ is pairwise disjoint for each $\gamma < \text{cf} \kappa$ and $k \in \omega$.

(c) $g(y) \cap C(f^m_n, Y^n, \alpha(\gamma)) = \emptyset$ for each $\gamma < \text{cf} \kappa$ and $y \in D^m_n(\gamma)$.

Then $\bigcup_{\gamma<\text{cf} \kappa} D^m_n(\gamma, k)$ is separated for each $k \in \omega$, thus $D^m_n$ is $\sigma$-separated. □

For a limit cardinal $\kappa$, we can always fix a normal sequence $\{\kappa_\alpha : \alpha < \text{cf} \kappa\}$ of cardinals, that is, it is strictly increasing and cofinal in $\kappa$ and $\kappa_\alpha = \sup\{\kappa_\beta : \beta < \alpha\}$ whenever $\alpha$ is limit. Moreover if $\kappa$ is strong limit, then we may fix a normal sequence $\{\kappa_\alpha : \alpha < \text{cf} \kappa\}$ so that $2^{\kappa_\alpha} \leq \kappa_{\alpha+1}$ for each $\alpha < \text{cf} \kappa$.

Lemma 5. Let $\kappa$ be a strong limit cardinal with $\omega_1 \leq \text{cf} \kappa$ and $\{\kappa_\alpha : \alpha < \text{cf} \kappa\}$ a normal sequence for $\kappa$ satisfying $2^{\kappa_\alpha} \leq \kappa_{\alpha+1}$ for each $\alpha < \text{cf} \kappa$. Moreover, let $Y$ be a closed discrete subspace of a $\mathcal{U}$-normal space $X$ such that $|Y| = \kappa$, and each $y \in Y$ has a neighborhood base $B_y$ at $y$ with $|B_y| \leq \chi$, where $\chi < \kappa$. Then for every partition $\mathcal{Y} = \{Y_\alpha : \alpha < \text{cf} \kappa\}$ of $Y$ with $|Y_\alpha| < \kappa$ ($\alpha < \text{cf} \kappa$), there is $\mathcal{f} = \{f_n : n \in \omega\} \in \mathcal{F}_{\text{cf} \kappa}$ such that $\{\alpha < \text{cf} \kappa : |S(\mathcal{f}, \mathcal{Y}, \alpha)| < 2^{\kappa_\alpha}\}$ contains a cub set in $\mathcal{F}_{\text{cf} \kappa}$.

Proof. Let $\mathcal{Y} = \{Y_\alpha : \alpha < \text{cf} \kappa\}$ be a partition of $Y$ with $|Y_\alpha| < \kappa$ for each $\alpha < \text{cf} \kappa$. Set $Z_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ for each $\alpha < \text{cf} \kappa$. Obviously for each $\alpha < \text{cf} \kappa$, $|Z_\alpha| < \kappa$, so one
can fix $b(\alpha) < \text{cf} \kappa$ with $|Z_{\alpha}| \leq \kappa_{b(\alpha)}$. Now we consider the cub set

$$C = \{ \alpha < \text{cf} \kappa : \forall \beta < \alpha (b(\beta) < \alpha), \alpha \text{ is limit, } \chi < \kappa_{\alpha} \}.$$  

Let $\alpha \in C$. Since $\alpha$ is limit, we have $Z_{\alpha} = \bigcup_{\beta < \alpha} Z_{\beta}$. Moreover since $|Z_{\beta}| \leq \kappa_{b(\beta)} < \kappa_{\alpha}$ for each $\beta < \alpha$, we have $|Z_{\alpha}| \leq \alpha \cdot \kappa_{\alpha} = \kappa_{\alpha}$. Noting $|F_{\alpha}| \leq (\chi^{\alpha})^\omega = 2^{\kappa_{\alpha}}$, we can enumerate $F_{\alpha} \times Z_{\alpha}$ 2 as

$$F_{\alpha} \times Z_{\alpha} = \{ \langle f_{\alpha}(i), g_{\alpha}(i) \rangle : 1 \leq \delta < 2^{\kappa_{\alpha}} \}.$$ 

Now enumerate the cub set $C$ as $\{ \alpha(\gamma) : \gamma < \text{cf} \kappa \}$ with the increasing order.

By induction on $\gamma < \text{cf} \kappa$ and $\delta < 2^{2^{\kappa_{\alpha}(\gamma)}}$, we will define a partial function $G_{\gamma,\delta}$ from $Y$ to $2 = \{ 0, 1 \}$ such that:

1. $G_{\gamma,\delta} \subseteq G_{\gamma',\delta'}$ for each $\gamma < \text{cf} \kappa$ and $\delta < \delta' < 2^{2^{\kappa_{\alpha}(\gamma)}}$
2. $G_{\gamma,0} = \bigcup \{ G_{\gamma',\delta} : \gamma' < \gamma, \delta' < 2^{2^{\kappa_{\alpha}(\gamma)}} \}$ for each $\gamma < \text{cf} \kappa$,
3. $|G_{\gamma,\delta}| \leq |\alpha(\gamma) + \delta|$ for each $\gamma < \text{cf} \kappa$ and $\delta < 2^{2^{\kappa_{\alpha}(\gamma)}}$.

First let $G_{00} = \emptyset$. We can show that, using the inductive assumption, the $G_{\gamma,0}$ defined by the clause (2) also satisfies the clause (3). So it remains to define $G_{\gamma,\delta}$ assuming that $G_{\gamma,\delta'}$ has been defined for all $\delta' < \delta$, where $\gamma < \text{cf} \kappa$ and $0 < \delta < 2^{2^{\kappa_{\alpha}(\gamma)}}$. Since $f_{\alpha(\gamma)} = \langle f_{\alpha(\gamma)}(n) : n \in \omega \rangle \in F_{\alpha(\gamma)}$, $S(f_{\alpha(\gamma)}, \gamma, \alpha(\gamma))$ is defined. We consider two cases.

Case 1. $S(f_{\alpha(\gamma)}, \gamma, \alpha(\gamma)) \cap (\bigcup_{\delta' < \delta} \text{dom} G_{\gamma,\delta'}) = \emptyset$.

In this case, fix a point $y(\gamma, \delta)$ in this set. Set

$$K = \{ n \in \omega : y(\gamma, \delta) \in C(f_{\alpha(\gamma)}(n), \gamma, \alpha(\gamma)) \}.$$ 

Then $K$ is infinite. For each $i \in 2$, set

$$C_{i}(f_{\alpha(\gamma)}(n), \gamma, \alpha(\gamma)) = \text{Cl} \bigcup \{ f_{\alpha(\gamma)}(n)(y) : y \in \text{dom} f_{\alpha(\gamma)}(n) \cap g_{\alpha(\gamma)}^{-1}(i) \}.$$ 

Then $C(f_{\alpha(\gamma)}(n), \gamma, \alpha(\gamma)) = \bigcup_{i \in 2} C_{i}(f_{\alpha(\gamma)}(n), \gamma, \alpha(\gamma))$, therefore for each $n \in K$, one can fix $i_{n} \in 2$ such that $y(\gamma, \delta) \in C_{i}(f_{\alpha(\gamma)}(n), \gamma, \alpha(\gamma))$. Since $K$ is infinite, there are an infinite subset $K(\gamma, \delta) \subseteq K$ and $i(\gamma, \delta) \in 2$ such that $i(\gamma, \delta) = i_{n}$ for each $n \in K(\gamma, \delta)$. Define

$$G_{\gamma,\delta} = \bigcup_{\delta' < \delta} G_{\gamma,\delta'},$$

Case 2. Otherwise.

In this case, let define $G_{\gamma,\delta} = \bigcup_{\delta' < \delta} G_{\gamma,\delta'}$.

In either cases, obviously $G_{\gamma,\delta}$ satisfies the clause (3). Let $G$ be a full function on $Y$ to 2 extending all $G_{\gamma,\delta}$'s. Since $X$ is $\gamma$-normal, there is $f = \langle f_{n} : n \in \omega \rangle \in F_{\text{cf} \kappa}$ such that for each $n \in \omega$, 

(*) $\bigcup \{ f_{n}(y) : y \in \text{dom} f_{n} \cap G^{-1}(0) \} \cap \bigcup \{ f_{n}(y) : y \in \text{dom} f_{n} \cap G^{-1}(1) \} = \emptyset$.

The following claim completes the proof.

Claim. $C \subseteq \{ \alpha < \text{cf} \kappa : |S(f, \gamma, \alpha(\gamma))| < 2^{2^{\kappa_{\alpha}(\gamma)}} \}.$

Proof. Assume not, then $|S(f, \gamma, \alpha(\gamma))| \geq 2^{2^{2^{\kappa_{\alpha}(\gamma)}}}$ for some $\gamma < \text{cf} \kappa$. Fix $\delta < 2^{2^{\kappa_{\alpha}(\gamma)}}$ with $1 \leq \delta$ such that

$$\langle f_{n} \upharpoonright Z_{\alpha}(\gamma) : n \in \omega \rangle, G \upharpoonright Z_{\alpha}(\gamma) \rangle = \langle f_{\alpha(\gamma)}, \gamma, \alpha(\gamma) \rangle.$$
It follows from $C(f_\alpha, Y, \alpha(\gamma)) = C(f_\alpha(\gamma) \delta_n, Y, \alpha(\gamma))$, $2^{\omega_\alpha} \leq |S(f, Y, \alpha(\gamma))| = |S(f_\alpha(\gamma) \delta, Y, \alpha(\gamma))|$ and $|\bigcup_{\beta < \delta} \text{dom} G_{\alpha(\gamma)} \cup Z_{\alpha(\gamma)}| < 2^{\omega_\alpha}$ that Case 1 happens. Fix $n \in \mathcal{K}(\gamma, \delta)$ with $y(\gamma, \delta) \in \text{dom} f_\alpha$. Then

$$y(\gamma, \delta) \in C(y(\gamma, \delta))(f_\alpha(\gamma) \delta_n, Y, \alpha(\gamma)) = \text{Cl}(\bigcup \{f_\alpha(y) : y \in \text{dom} f_\alpha(\gamma) \delta_n \cap g_\alpha^{-1}(i(\gamma, \delta))\}) \subset \text{Cl}(\bigcup \{f_\alpha(y) : y \in \text{dom} f_\alpha \cap G^{-1}(i(\gamma, \delta))\}).$$

Since $f_\alpha(y(\gamma, \delta))$ is a neighborhood of $y(\gamma, \delta)$, we have $f_\alpha(y(\gamma, \delta)) \cap f_\alpha(y) \neq \emptyset$ for some $y \in \text{dom} f_\alpha \cap G^{-1}(i(\gamma, \delta))$, this contradicts (*).

The following is an improvement of Lemma 2.5 in [4] and can be proved by some easy modifications.

**Lemma 6.** Let $Y$ be a set (need not be a (sub)space) with $\omega_1 \leq \text{cf} \kappa < \kappa = |Y|$, $\mathcal{Y} = \{Y_\alpha : \alpha < \text{cf} \kappa\}$ a partition of $Y$ and $\{\kappa_\alpha : \alpha < \text{cf} \kappa\}$ a normal sequence for $\kappa$. If $\{S_n : \alpha < \text{cf} \kappa\}/_{\beta \lessdot \alpha} Y_\beta \subset S_n$ for each $\alpha < \text{cf} \kappa$ and $\{\alpha < \text{cf} \kappa : \text{cf} S_n \leq \kappa_\alpha\}$ contains a club set in $\text{cf} \kappa$, then there is a partition $\mathcal{Y}' = \{Y_\alpha' : \alpha < \text{cf} \kappa\}$ of $\mathcal{Y}$ such that $|Y_\alpha'| < \kappa$ for each $\alpha < \text{cf} \kappa$ and $\{\alpha < \text{cf} \kappa : S_n \subset \bigcup_{\beta < \alpha} Y_\beta'\}$ contains a club set in $\text{cf} \kappa$.

**Lemma 7.** Let $\kappa$ be a singular strong limit cardinal of uncountable cofinality, and $Y$ a closed discrete subspace of a $\text{cf} \kappa$-normal space $X$ such that $|Y| = \kappa$ and for some $\chi < \kappa$, each $y \in Y$ has a neighborhood base $\mathcal{B}_y$ of size $\leq \chi$. If there is a normal sequence $\{\kappa_\alpha : \alpha < \text{cf} \kappa\}$ for $\kappa$ such that $\{\alpha < \text{cf} \kappa : 2^{\omega_\alpha} = \kappa_\alpha^+\}$ contains a club set in $\text{cf} \kappa$, then $Y$ has good partitions (therefore $Y$ is $\sigma$-separated if every subset $Y' \subset Y$ of size $\kappa$ is $\sigma$-separated).

**Proof.** We may assume $2^{\omega_\alpha} \leq \kappa_{\alpha+1}$ for each $\alpha < \text{cf} \kappa$. First fix a 1-1 onto function $f : Y \to \kappa$ and set $Y^0_\alpha = f^{-1}(\kappa_\alpha \uparrow \sup_{\beta < \alpha} \kappa_\beta)$ for each $\alpha < \text{cf} \kappa$. Then $\mathcal{Y}^0 = \{Y^0_\alpha : \alpha < \text{cf} \kappa\}$ is a partition of $Y$. Assume that a partition $\mathcal{Y}^m = \{Y^m_\alpha : \alpha < \text{cf} \kappa\}$ with $|Y^m_\alpha| < \kappa$ for each $\alpha < \text{cf} \kappa$ is defined. By Lemma 5, there is $\mathcal{F}^m \in \mathcal{F}_{\text{cf} \kappa}$ such that $\{\alpha < \text{cf} \kappa : |S(\mathcal{F}^m, Y^m_\alpha, \alpha)| < 2^{\omega_\alpha}\}$ contains a cub set, therefore by the assumption, $\{\alpha < \text{cf} \kappa : |S(\mathcal{F}^m, Y^m_\alpha, \alpha)| \leq \kappa_\alpha\}$ also contains a cub set. Since $\bigcup_{\beta < \alpha} Y^m_\beta \subset S(\mathcal{F}^m, Y^m_\alpha, \alpha)$ for each $\alpha < \text{cf} \kappa$ and $\{S(\mathcal{F}^m, Y^m_\alpha, \alpha) : \alpha < \text{cf} \kappa\}/_{\beta < \alpha} Y^m_\beta$ for $\mathcal{F}^m$, we get a partition $\mathcal{Y}^{m+1} = \{Y^{m+1}_\alpha : \alpha < \text{cf} \kappa\}$ of $Y$ such that $|Y^{m+1}_\alpha| < \kappa$ for each $\alpha < \text{cf} \kappa$ and $\{\alpha < \text{cf} \kappa : S(\mathcal{F}^m, Y^{m+1}_\alpha, \alpha) \subset \bigcup_{\beta < \alpha} Y^{m+1}_\beta\}$ contains a cub set in $\text{cf} \kappa$. By repeatedly applications, we have good partitions.

**Proof of Theorem 1.** Whenever $\text{cf} \kappa = \omega$, Theorem 1 is obviously true without assuming SCH or $\text{cf} \kappa$-normality of $X$. So let $\text{cf} \kappa \geq \omega_1$. Since assuming SCH, $\{\alpha < \text{cf} \kappa : 2^{\omega_\alpha} = \kappa_\alpha^+\}$ contains a cub set (see Lemma 2.11 of [4]), Lemma 7 completes the proof.

**References**