σ-COLLECTIONWISE HAUSDORFFNESS AT SINGULAR STRONG LIMIT CARDINALS

NOBUYUKI KEMOTO

ABSTRACT. Assuming the Singular Cardinals Hypothesis, we prove the following property:

σ-CWH : For every singular strong limit cardinal κ and $\nearrow$-normal space X such that for some $\chi < \kappa$, every $x \in X$ has a neighborhood base of size $\leq \chi$, if every closed discrete subspace of size $< \kappa$ is σ-separated, then so is every closed discrete subspace of size κ.

So for getting a model of the negation of σ-CWH, we require a large cardinal.

Throughout the paper, spaces are regular T_1 topological spaces and generally $\alpha, \beta, \gamma, ...(\kappa, \lambda, \mu, ..., k, l, m, ...)$ stand for ordinals (infinite cardinals, natural numbers). A closed discrete subspace Y of a space X is said to be separated if there is a pairwise disjoint collection $\{U(y) : y \in Y\}$ of open sets such that $y \in U(y)$ for each $y \in Y$. A closed discrete subspace Y of a space X is said to be σ-separated if Y can be represented as the countable sum $Y = \bigcup_{n\in\omega} Y_n$, where each Y_n is separated. For a cardinal κ, a space X is κ-collectionwise Hausdorff (κ-σ-collectionwise Hausdorff) if every closed discrete subspace $Y \subset X$ of size κ is separated (σ-separated). For short, CollectionWise Hausdorff is abbreviated as CWH. A space is $< \kappa$-CWH ($< \kappa$-σ-CWH) if it is λ-CWH (λ-σ-CWH) for every cardinal $\lambda < \kappa$. For a cardinal κ, we consider the following inductive type properties:

CWH(κ) : If X is $< \kappa$-CWH, then it is κ-CWH.

σ-CWH(κ) : If X is $< \kappa$-σ-CWH, then it is κ-σ-CWH.

For a regular cardinal κ (i.e., cf$\kappa = \kappa$), Fleissner [2] proved:

• Under a $\diamondsuit$-like assumption that is a consequence of $V = L$, if X is normal and each point $x \in X$ has a neighborhood base of size $\leq \kappa$, then X has CWH(κ).

Balogh and Burke [1] generalized this result as:

• Under the same assumption, if X is $\nearrow$-normal and each point $x \in X$ has a neighborhood base of size $\leq \kappa$, then X has σ-CWH(κ).

Here a space is $\nearrow$-normal if for every pair of disjoint closed sets H and K, there are sequences $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ of open sets such that $U_n \cap V_n = \emptyset$ for every $n \in \omega$, $\{H \cap U_n : n \in \omega\} \nearrow H$ and $\{K \cap V_n : n \in \omega\} \nearrow K$, where for a well-ordered set A, $\{B_\alpha : \alpha \in A\} \nearrow B$ means that $\bigcup_{\alpha\in A} B_\alpha = B$ and $B_\alpha \subset B_{\alpha'}$ whenever $\alpha < \alpha'$. For later use, $\{B_\alpha : \alpha \in A\} \nearrow$ means that $B_\alpha \subset B_{\alpha'}$ whenever $\alpha < \alpha'$.

For a singular cardinal κ (i.e., cf$\kappa < \kappa$), these papers also showed:

1991 *Mathematics Subject Classification.* 54D15, 03E55.

Key words and phrases. σ-collectionwise Hausdorff, $\nearrow$-normal, Singular Cardinals Hypothesis.

• Assuming the Generalized Continuum Hypothesis (GCH), that is, $2^\lambda = \lambda^+$ for every cardinal λ, if X is normal ($\nearrow$-normal) and for some $\chi < \kappa$, each point $x \in X$ has a neighborhood base of size $\leq \chi$, then X has CWH(κ) (σ-CWH(κ)), see [2] ([1], respectively).

Obviously assuming GCH, every singular cardinal κ is strong limit, i.e., $2^\lambda < \kappa$ for each cardinal $\lambda < \kappa$. As a weakening of GCH, the following is well-known:

The Singular Cardinals Hypothesis (SCH): For every singular cardinal κ, if $2^{\mathrm{cf}\kappa} < \kappa$, then $\kappa^{\mathrm{cf}\kappa} = \kappa^+$.

For SCH, see [3]. The author proved in [4]:

• Assuming SCH, for every singular strong limit cardinal κ, if X is normal and for some $\chi < \kappa$, each point $x \in X$ has a neighborhood base of size $\leq \chi$, then X has CWH(κ).

In this paper, we will prove:

Theorem 1. *Assuming SCH, for every singular strong limit cardinal κ, if X is $\nearrow$-normal and for some $\chi < \kappa$, each point $x \in X$ has a neighborhood base of size $\leq \chi$, then X has σ-CWH(κ).*

However, the proof here is necessarily tedious and does not always follow predictable modifications of the proofs listed above. Now we prepare some notions.

Definition 2. *Let Y be a closed discrete subspace of a space X with $\omega_1 \leq \mathrm{cf}\kappa \leq \kappa = |Y|$ and $\mathcal{Y} = \{Y_\alpha : \alpha < \mathrm{cf}\kappa\}$ a partition of Y, that is, $\mathcal{Y}$ is a disjoint cover of Y. Fix a neighborhood base $\mathcal{B}_y$ of y for each $y \in Y$. Define*

$$\mathcal{F} = \{\langle f_n : n \in \omega\rangle : \forall n \in \omega(\mathrm{dom} f_n \subset Y), f_n \in \prod_{y \in \mathrm{dom} f_n} \mathcal{B}_y, \{\mathrm{dom} f_n : n \in \omega\} \nearrow\},$$

$$\mathcal{F}_\alpha = \{\langle f_n : n \in \omega\rangle \in \mathcal{F} : \{\mathrm{dom} f_n : n \in \omega\} \nearrow \bigcup_{\beta<\alpha} Y_\beta\},$$

for each $\alpha \leq \mathrm{cf}\kappa$.

For each $\alpha < \mathrm{cf}\kappa$ and $f \in \prod_{y \in \mathrm{dom} f} \mathcal{B}_y$ with $\mathrm{dom} f \subset Y$, let

$$C(f, \mathcal{Y}, \alpha) = \mathrm{Cl}(\bigcup\{f(y) : y \in \mathrm{dom} f \cap (\bigcup_{\beta<\alpha} Y_\beta)\}).$$

Then obviously $\mathrm{dom} f \cap (\bigcup_{\beta<\alpha} Y_\beta) \subset C(f, \mathcal{Y}, \alpha)$. Moreover for each $\mathbf{f} = \langle f_n : n \in \omega\rangle \in \mathcal{F}$ and $\alpha < \mathrm{cf}\kappa$, let

$$S(\mathbf{f}, \mathcal{Y}, \alpha) = \{y \in Y : \{C(f_n, \mathcal{Y}, \alpha) : n \in \omega\} \text{ is not point-finite at } y\}.$$

If $\mathbf{f} = \langle f_n : n \in \omega\rangle \in \mathcal{F}_{\mathrm{cf}\kappa}$, then by $\{\mathrm{dom} f_n : n \in \omega\} \nearrow Y$, we have $\bigcup_{\beta<\alpha} Y_\beta \subset S(\mathbf{f}, \mathcal{Y}, \alpha)$ for each $\alpha < \mathrm{cf}\kappa$, and $\{S(\mathbf{f}, \mathcal{Y}, \alpha) : \alpha < \mathrm{cf}\kappa\} \nearrow Y$.

Definition 3. *Let Y be a closed discrete subspace of a space X with $\omega_1 \leq \mathrm{cf}\kappa \leq \kappa = |Y|$. Y has good partitions if there is a sequence $\{\mathcal{Y}^m : m \in \omega\}$ of partitions of Y, where $\mathcal{Y}^m = \{Y^m_\alpha : \alpha < \mathrm{cf}\kappa\}$, such that*

(a) *$|Y^m_\alpha| < \kappa$ for each $m \in \omega$ and $\alpha < \mathrm{cf}\kappa$,*
(b) *for each $m \in \omega$, there is $\mathbf{f}^m = \langle f^m_n : n \in \omega\rangle \in \mathcal{F}_{\mathrm{cf}\kappa}$ such that $\{\alpha < \mathrm{cf}\kappa : S(\mathbf{f}^m, \mathcal{Y}^m, \alpha) \subset \bigcup_{\beta<\alpha} Y^{m+1}_\beta\}$ contains a closed unbounded (cub) set in $\mathrm{cf}\kappa$.*

Lemma 4. *Let Y be a closed discrete subspace of a space X with $\omega_1 \leq \mathrm{cf}\kappa \leq \kappa = |Y|$ such that every subset $Y' \subset Y$ of size $< \kappa$ is σ-separated. If Y has good partitions $\{\mathcal{Y}^m : m \in \omega\}$, then Y is σ-separated.*

Proof. Let $\{\mathcal{Y}^m : m \in \omega\}$ be good partitions and $\mathbf{f}^m \in \mathcal{F}_{\mathrm{cf}\kappa}$ and a cub set $C_m \subset \mathrm{cf}\kappa$ ($m \in \omega$) ensure the clause (b) of the definition above. For notational simplicity, let $S^m_\alpha = S(\mathbf{f}^m, \mathcal{Y}^m, \alpha)$ for each $m \in \omega$ and $\alpha < \mathrm{cf}\kappa$. Let $C = \{0\} \cup \bigcap_{m\in\omega} C_m$ and enumerate C as $C = \{\alpha(\gamma) : \gamma < \mathrm{cf}\kappa\}$ with the increasing order. For each $m \in \omega$ and $\gamma < \mathrm{cf}\kappa$, set $Y(m,\gamma) = \bigcup\{Y^m_\beta : \alpha(\gamma) \leq \beta < \alpha(\gamma+1)\}$. Since C is cub, $\{Y(m,\gamma) : \gamma < \mathrm{cf}\kappa\}$ is also a partition of Y. So for each $y \in Y$ and $m \in \omega$, let $\gamma(m,y) < \mathrm{cf}\kappa$ be the unique γ with $y \in Y(m,\gamma)$. Since $\bigcup_{\beta<\alpha(\gamma)} Y^m_\beta \subset S^m_{\alpha(\gamma)} \subset \bigcup_{\beta<\alpha(\gamma)} Y^{m+1}_\beta$ for each $\gamma < \mathrm{cf}\kappa$ and $m \in \omega$, we have $\gamma(m,y) \geq \gamma(m+1,y)$ for each $m \in \omega$ and $y \in Y$. Therefore for each $y \in Y$, one can fix $m(y) \in \omega$ such that $\gamma(m(y),y) = \gamma(m,y)$ for each $m \geq m(y)$. Set for each $m,n \in \omega$,

$$D^m_n = \{y \in \mathrm{dom} f^m_n : y \notin C(f^m_n, \mathcal{Y}^m, \alpha(\gamma(m,y)))\}.$$

Claim. $Y = \bigcup_{n,m\in\omega} D^m_n$.

Proof. Let $y \in Y$ and $\gamma = \gamma(m(y),y)$. It follows from $\gamma = \gamma(m(y)+1,y)$ that $y \in Y(m(y)+1,\gamma)$. Since $Y(m(y)+1,\gamma)$ is disjoint from $\bigcup_{\beta<\alpha(\gamma)} Y^{m(y)+1}_\beta \supset S^{m(y)}_{\alpha(\gamma)}$, we have $y \notin S^{m(y)}_{\alpha(\gamma)}$. Therefore $\{C(f^{m(y)}_n, \mathcal{Y}^{m(y)}, \alpha(\gamma)) : n \in \omega\}$ is point-finite at y. So by $\{\mathrm{dom} f^{m(y)}_n : n \in \omega\} \nearrow Y$, one can fix $n \in \omega$ with $y \in \mathrm{dom} f^{m(y)}_n$ and $y \notin C(f^{m(y)}_n, \mathcal{Y}^{m(y)}, \alpha(\gamma))$, thus $y \in D^{m(y)}_n$.

Now it suffices to show that D^m_n's are σ-separated. For each $\gamma < \mathrm{cf}\kappa$, set $D^m_n(\gamma) = D^m_n \cap Y(m,\gamma)$, then obviously $D^m_n(\gamma) \cap C(f^m_n, \mathcal{Y}^m, \alpha(\gamma)) = \emptyset$. It follows from $|D^m_n(\gamma)| < \kappa$ and the assumtion that we get a partition $\{D^m_n(\gamma,k) : k \in \omega\}$ of $D^m_n(\gamma)$, where each $D^m_n(\gamma,k)$ is separated. By induction on $\gamma < \mathrm{cf}\kappa$, one can define $g \in \prod_{y\in D^m_n} \mathcal{B}_y$ satisfying:

(a) $g(y) \subset f^m_n(y)$ for each $y \in D^m_n$,
(b) $\{g(y) : y \in D^m_n(\gamma,k)\}$ is pairwise disjoint for each $\gamma < \mathrm{cf}\kappa$ and $k \in \omega$,
(c) $g(y) \cap C(f^m_n, \mathcal{Y}^m, \alpha(\gamma)) = \emptyset$ for each $\gamma < \mathrm{cf}\kappa$ and $y \in D^m_n(\gamma)$.

Then $\bigcup_{\gamma<\mathrm{cf}\kappa} D^m_n(\gamma,k)$ is separated for each $k \in \omega$, thus D^m_n is σ-separated. □

For a limit cardinal κ, we can always fix a normal sequence $\{\kappa_\alpha : \alpha < \mathrm{cf}\kappa\}$ of cardinals, that is, it is strictly increasing and cofinal in κ and $\kappa_\alpha = \sup\{\kappa_\gamma : \gamma < \alpha\}$ whenever α is limit. Moreover if κ is strong limit, then we may fix a normal sequence $\{\kappa_\alpha : \alpha < \mathrm{cf}\kappa\}$ so that $2^{\kappa_\alpha} \leq \kappa_{\alpha+1}$ for each $\alpha < \mathrm{cf}\kappa$.

Lemma 5. *Let κ be a strong limit cardinal with $\omega_1 \leq \mathrm{cf}\kappa$ and $\{\kappa_\alpha : \alpha < \mathrm{cf}\kappa\}$ a normal sequence for κ satisfying $2^{\kappa_\alpha} \leq \kappa_{\alpha+1}$ for each $\alpha < \mathrm{cf}\kappa$. Moreover, let Y be a closed discrete subspace of a $\nearrow$-normal space X such that $|Y| = \kappa$, and each $y \in Y$ has a neighoborhood base $\mathcal{B}_y$ at y with $|\mathcal{B}_y| \leq \chi$, where $\chi < \kappa$. Then for every partition $\mathcal{Y} = \{Y_\alpha : \alpha < \mathrm{cf}\kappa\}$ of Y with $|Y_\alpha| < \kappa$ ($\alpha < \mathrm{cf}\kappa$), there is $\mathbf{f} = \langle f_n : n \in \omega\rangle \in \mathcal{F}_{\mathrm{cf}\kappa}$ such that $\{\alpha < \mathrm{cf}\kappa : |S(\mathbf{f}, \mathcal{Y}, \alpha)| < 2^{\kappa_\alpha}\}$ contains a cub set in $\mathrm{cf}\kappa$.*

Proof. Let $\mathcal{Y} = \{Y_\alpha : \alpha < \mathrm{cf}\kappa\}$ be a partition of Y with $|Y_\alpha| < \kappa$ for each $\alpha < \mathrm{cf}\kappa$. Set $Z_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ for each $\alpha < \mathrm{cf}\kappa$. Obviously for each $\alpha < \mathrm{cf}\kappa$, $|Z_\alpha| < \kappa$, so one can fix $h(\alpha) < \mathrm{cf}\kappa$ with $|Z_\alpha| \le \kappa_{h(\alpha)}$. Now we consider the cub set

$$C = \{\alpha < \mathrm{cf}\kappa : \forall\beta < \alpha(h(\beta) < \alpha), \alpha \text{ is limit}, \chi < \kappa_\alpha\}.$$

Let $\alpha \in C$. Since α is limit, we have $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$. Moreover since $|Z_\beta| \le \kappa_{h(\beta)} < \kappa_\alpha$ for each $\beta < \alpha$, we have $|Z_\alpha| \le |\alpha| \cdot \kappa_\alpha = \kappa_\alpha$. Noting $|\mathcal{F}_\alpha| \le (\chi^{\kappa_\alpha})^\omega = 2^{\kappa_\alpha}$, we can enumerate $\mathcal{F}_\alpha \times {}^{Z_\alpha}2$ as

$$\mathcal{F}_\alpha \times {}^{Z_\alpha}2 = \{\langle \mathbf{f}_{\alpha\delta}, g_{\alpha\delta}\rangle : 1 \le \delta < 2^{\kappa_\alpha}\}.$$

Now enumerate the cub set C as $\{\alpha(\gamma) : \gamma < \mathrm{cf}\kappa\}$ with the increasing order.

By induction on $\gamma < \mathrm{cf}\kappa$ and $\delta < 2^{\kappa_{\alpha(\gamma)}}$, we will define a partial function $G_{\gamma\delta}$ from Y to $2 = \{0, 1\}$ such that :

(1) $G_{\gamma\delta} \subset G_{\gamma\delta'}$ for each $\gamma < \mathrm{cf}\kappa$ and $\delta < \delta' < 2^{\kappa_{\alpha(\gamma)}}$,
(2) $G_{\gamma 0} = \bigcup\{G_{\gamma'\delta'} : \gamma' < \gamma, \delta' < 2^{\kappa_{\alpha(\gamma')}}\}$ for each $\gamma < \mathrm{cf}\kappa$,
(3) $|G_{\gamma\delta}| \le |\kappa_{\alpha(\gamma)} + \delta|$ for each $\gamma < \mathrm{cf}\kappa$ and $\delta < 2^{\kappa_{\alpha(\gamma)}}$.

First let $G_{00} = \emptyset$. We can show that, using the inductive assumption, the $G_{\gamma 0}$ defined by the clause (2) also satisfies the clause (3). So it remains to define $G_{\gamma\delta}$ assuming that $G_{\gamma\delta'}$ has been defined for all $\delta' < \delta$, where $\gamma < \mathrm{cf}\kappa$ and $0 < \delta < 2^{\kappa_{\alpha(\gamma)}}$. Since $\mathbf{f}_{\alpha(\gamma)\delta} = \langle f_{\alpha(\gamma)\delta n} : n \in \omega\rangle \in \mathcal{F}_{\alpha(\gamma)}$, $S(\mathbf{f}_{\alpha(\gamma)\delta}, \mathcal{Y}, \alpha(\gamma))$ is defined. We consider two cases.

Case 1. $S(\mathbf{f}_{\alpha(\gamma)\delta}, \mathcal{Y}, \alpha(\gamma)) \setminus (\bigcup_{\delta'<\delta} \mathrm{dom} G_{\gamma\delta'} \cup Z_{\alpha(\gamma)}) \ne \emptyset$.

In this case, fix a point $y(\gamma, \delta)$ in this set. Set

$$K = \{n \in \omega : y(\gamma, \delta) \in C(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma))\}.$$

Then K is infinite. For each $i \in 2$, set

$$C_i(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma)) = \mathrm{Cl}(\bigcup\{f_{\alpha(\gamma)\delta n}(y) : y \in \mathrm{dom} f_{\alpha(\gamma)\delta n} \cap g^{-1}_{\alpha(\gamma)\delta}(i)\}).$$

Then $C(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma)) = \bigcup_{i\in 2} C_i(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma))$, therefore for each $n \in K$, one can fix $i_n \in 2$ such that $y(\gamma, \delta) \in C_{i_n}(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma))$. Since K is infinite, there are an infinite subset $K(\gamma, \delta) \subset K$ and $i(\gamma, \delta) \in 2$ such that $i(\gamma, \delta) = i_n$ for each $n \in K(\gamma, \delta)$. Define

$$G_{\gamma\delta} = (\bigcup_{\delta'<\delta} G_{\gamma\delta'}) \cup \{\langle y(\gamma, \delta), 1 - i(\gamma, \delta)\rangle\}.$$

Case 2. Otherwise.

In this case, let define

$$G_{\gamma\delta} = \bigcup_{\delta'<\delta} G_{\gamma\delta'}.$$

In either cases, obviously $G_{\gamma\delta}$ satisfies the clause (3). Let G be a full function on Y to 2 extending all $G_{\gamma\delta}$'s. Since X is $\nearrow$-normal, there is $\mathbf{f} = \langle f_n : n \in \omega\rangle \in \mathcal{F}_{\mathrm{cf}\kappa}$ such that for each $n \in \omega$,

(*) $\quad \bigcup\{f_n(y) : y \in \mathrm{dom} f_n \cap G^{-1}(0)\} \cap \bigcup\{f_n(y) : y \in \mathrm{dom} f_n \cap G^{-1}(1)\} = \emptyset.$

The following claim completes the proof.

Claim. $C \subset \{\alpha < \mathrm{cf}\kappa : |S(\mathbf{f}, \mathcal{Y}, \alpha)| < 2^{\kappa_\alpha}\}$.

Proof. Assume not, then $|S(\mathbf{f}, \mathcal{Y}, \alpha(\gamma))| \ge 2^{\kappa_{\alpha(\gamma)}}$ for some $\gamma < \mathrm{cf}\kappa$. Fix $\delta < 2^{\kappa_{\alpha(\gamma)}}$ with $1 \le \delta$ such that

$$\langle\langle f_n \restriction Z_{\alpha(\gamma)} : n \in \omega\rangle, G \restriction Z_{\alpha(\gamma)}\rangle = \langle \mathbf{f}_{\alpha(\gamma)\delta}, g_{\alpha(\gamma)\delta}\rangle.$$

It follows from $C(f_n, \mathcal{Y}, \alpha(\gamma)) = C(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma))$, $2^{\kappa_{\alpha(\gamma)}} \leq |S(\mathbf{f}, \mathcal{Y}, \alpha(\gamma))| = |S(\mathbf{f}_{\alpha(\gamma)\delta}, \mathcal{Y}, \alpha(\gamma))|$ and $|\bigcup_{\delta'<\delta} \mathrm{dom} G_{\gamma\delta'} \cup Z_{\alpha(\gamma)}| < 2^{\kappa_{\alpha(\gamma)}}$ that Case 1 happens. Fix $n \in K(\gamma, \delta)$ with $y(\gamma, \delta) \in \mathrm{dom} f_n$. Then

$$y(\gamma, \delta) \in C_{i(\gamma,\delta)}(f_{\alpha(\gamma)\delta n}, \mathcal{Y}, \alpha(\gamma)) =$$
$$\mathrm{Cl}(\bigcup\{f_{\alpha(\gamma)\delta n}(y) : y \in \mathrm{dom} f_{\alpha(\gamma)\delta n} \cap g^{-1}_{\alpha(\gamma)\delta}(i(\gamma, \delta))\}) \subset$$
$$\mathrm{Cl}(\bigcup\{f_n(y) : y \in \mathrm{dom} f_n \cap G^{-1}(i(\gamma, \delta))\}).$$

Since $f_n(y(\gamma, \delta))$ is a neighborhood of $y(\gamma, \delta)$, we have $f_n(y(\gamma, \delta)) \cap f_n(y) \neq \emptyset$ for some $y \in \mathrm{dom} f_n \cap G^{-1}(i(\gamma, \delta))$, this contradicts (*). □

The following is an improvement of Lemma 2.5 in [4] and can be proved by some easy modifications.

Lemma 6. *Let Y be a set (need not be a (sub)space) with $\omega_1 \leq \mathrm{cf}\kappa < \kappa = |Y|$, $\mathcal{Y} = \{Y_\alpha : \alpha < \mathrm{cf}\kappa\}$ a partition of Y and $\{\kappa_\alpha : \alpha < \mathrm{cf}\kappa\}$ a normal sequence for κ. If $\{S_\alpha : \alpha < \mathrm{cf}\kappa\} \nearrow$, $\bigcup_{\beta<\alpha} Y_\beta \subset S_\alpha$ for each $\alpha < \mathrm{cf}\kappa$ and $\{\alpha < \mathrm{cf}\kappa : |S_\alpha| \leq \kappa_\alpha\}$ contains a club set in $\mathrm{cf}\kappa$, then there is a partition $\mathcal{Y}' = \{Y'_\alpha : \alpha < \mathrm{cf}\kappa\}$ of Y such that $|Y'_\alpha| < \kappa$ for each $\alpha < \mathrm{cf}\kappa$ and $\{\alpha < \mathrm{cf}\kappa : S_\alpha \subset \bigcup_{\beta<\alpha} Y'_\beta\}$ contains a cub set in $\mathrm{cf}\kappa$.*

Lemma 7. *Let κ be a singular strong limit cardinal of uncountable cofinality, and Y a closed discrete subspace of a $\nearrow$-normal space X such that $|Y| = \kappa$ and for some $\chi < \kappa$, each $y \in Y$ has a neighborhood base $\mathcal{B}_y$ of size $\leq \chi$. If there is a normal sequence $\{\kappa_\alpha : \alpha < \mathrm{cf}\kappa\}$ for κ such that $\{\alpha < \mathrm{cf}\kappa : 2^{\kappa_\alpha} = \kappa_\alpha^+\}$ contains a cub set in $\mathrm{cf}\kappa$, then Y has good partitions (therefore Y is σ-separated if every subset $Y' \subset Y$ of size$< \kappa$ is σ-separated).*

Proof. We may assume $2^{\kappa_\alpha} \leq \kappa_{\alpha+1}$ for each $\alpha < \mathrm{cf}\kappa$. First fix a 1-1 onto function $f : Y \to \kappa$ and set $Y^0_\alpha = f^{-1}(\kappa_\alpha \setminus \sup_{\beta<\alpha} \kappa_\beta)$ for each $\alpha < \mathrm{cf}\kappa$. Then $\mathcal{Y}^0 = \{Y^0_\alpha : \alpha < \mathrm{cf}\kappa\}$ is a partition of Y. Assume that a partition $\mathcal{Y}^m = \{Y^m_\alpha : \alpha < \mathrm{cf}\kappa\}$ with $|Y^m_\alpha| < \kappa$ for each $\alpha < \mathrm{cf}\kappa$ is defined. By Lemma 5, there is $\mathbf{f}^m \in \mathcal{F}_{\mathrm{cf}\kappa}$ such that $\{\alpha < \mathrm{cf}\kappa : |S(\mathbf{f}^m, \mathcal{Y}^m, \alpha)| < 2^{\kappa_\alpha}\}$ contains a cub set, therefore by the assumption, $\{\alpha < \mathrm{cf}\kappa : |S(\mathbf{f}^m, \mathcal{Y}^m, \alpha)| \leq \kappa_\alpha\}$ also contains a cub set. Since $\bigcup_{\beta<\alpha} Y^m_\beta \subset S(\mathbf{f}^m, \mathcal{Y}^m, \alpha)$ for each $\alpha < \mathrm{cf}\kappa$ and $\{S(\mathbf{f}^m, \mathcal{Y}^m, \alpha) : \alpha < \mathrm{cf}\kappa\} \nearrow$, by Lemma 6, we get a partition $\mathcal{Y}^{m+1} = \{Y^{m+1}_\alpha : \alpha < \mathrm{cf}\kappa\}$ of Y such that $|Y^{m+1}_\alpha| < \kappa$ for each $\alpha < \mathrm{cf}\kappa$ and $\{\alpha < \mathrm{cf}\kappa : S(\mathbf{f}^m, \mathcal{Y}^m, \alpha) \subset \bigcup_{\beta<\alpha} Y^{m+1}_\beta\}$ contains a cub set in $\mathrm{cf}\kappa$. By repeatedly applications, we have good partitions. □

Proof of Theorem 1. Whenever $\mathrm{cf}\kappa = \omega$, Theorem 1 is obviously true without assuming SCH or $\nearrow$-normality of X. So let $\mathrm{cf}\kappa \geq \omega_1$. Since assuming SCH, $\{\alpha < \mathrm{cf}\kappa : 2^{\kappa_\alpha} = \kappa_\alpha^+\}$ contains a cub set (see Lemma 2.11 of [4]), Lemma 7 completes the proof. □

References

[1] Z.T. Balogh and D.K. Burke *On $\nearrow$-normal spaces*, Top. Appl.. **57** (1994) 71–85.
[2] W.G. Fleissner *Separating closed discrete collections of singular cardinality*, in: Set Theoretic Topology (ed. G.M. Reed), Academic Press, 1977, 135–140.
[3] T. Jech, *Set Theory* Academic Press, 1978.
[4] N. Kemoto, *Collectionwise Hausdorffness at limit cardinals*, Fund. Math., **138** (1991) 59–67.

Department of Mathematics, Faculty of Education, Oita University, Dannoharu, Oita, 870-1192, Japan