# SEPARATING BY $G_\delta$-SETS IN FINITE POWERS OF $\omega_1$

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Abstract. It is known that all subspaces of $\omega_1^2$ have the property that every pair of disjoint closed sets can be separated by disjoint $G_\delta$-sets, see [4]. Moreover it is conjectured that all subspaces of $\omega_1^n$ have also this property for each $n < \omega$. In this paper, we give a subspace of $\{\langle\alpha,\beta,\gamma\rangle \in \omega_1^3 : \alpha \le \beta \le \gamma\}$ which does not have this property, disproving this conjecture. On the other hand, we prove that all subspaces of $\{\langle\alpha,\beta,\gamma\rangle \in \omega_1^3 : \alpha < \beta < \gamma\}$ have this property.

## 1. Introduction

A topological space $X$ is said to be *subnormal* if every pair of disjoint closed sets can be separated by disjoint $G_\delta$-sets. A *subshrinking* of an open cover $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ of $X$ is an $F_\sigma$-cover $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of $X$ which satisfies that $F_i \subseteq U_i$ for each $i \in \mathcal{I}$. A space $X$ is said to be *subshrinking* if every open cover has a subshrinking. It is easy to see that every subshrinking space is subnormal. For these properties, see [2] or [7].

It is well known that $\omega_1^2$ is normal but $\omega_1 \times (\omega_1+1)$ is not subnormal. Moreover it is known in [5] that there is a non-normal subspace of $\omega_1^2$. For example, $X = A \times B$, where $A$ and $B$ are disjoint stationary sets in $\omega_1$, is such a space. However in [4], it is proved an unexpected result that all subspaces of $\omega_1^2$ are subshrinking, so subnormal. It was conjectured that all subspaces of $\omega_1^n$ are subnormal for every $n < \omega$. In Section 4, we will give another unexpected result that this conjecture is false.

**Theorem 1.1.** *There exists a non-subnormal subspace of $\omega_1^3$.*

On the other hand, all subspaces of

$$\omega_1^n|_< = \{s \in \omega_1^n : s(i) < s(j) \text{ for each } i < j < n\}$$

are still subnormal for an arbitrary $n < \omega$. We will prove this in Section 5.

**Theorem 1.2.** *Every subspace of $\omega_1^n|_<$ is subshrinking, so subnormal, for every $n < \omega$.*

To prove these theorems, we show some combinatorial lemmas in Section 3. We use the concept of trees of finite sequences and state the Pressing Down Lemma in terms of trees. The Pressing Down Lemma in more general situation appears in [1].

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## 2. Preliminaries

We identify an ordinal $\alpha$ with the set of all ordinals less than $\alpha$. We do not distinguish nutural numbers from finite ordinals. Hence a natural number $n$ is the set $\{0, 1, \ldots, n-1\}$. A sequence $s$ of finite length $n$ is a function of domain $n$, so $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$. In particular, $A^n$ denotes the set of all functions from $\{0, 1, \ldots, n-1\}$ into $A$.

For each sequence $s$, $\mathrm{lh}(s)$ denotes the length of $s$, and $\mathrm{ran}(s)$ denotes the set $\{s(i) : i < \mathrm{lh}(s)\}$. Let $A$ be a set of sequences of ordinals. We use the following notations.

- $A|_< = \{s \in A : s(i) < s(j)$ for each $i < j < \mathrm{lh}(s)\}$.
- $A|_\leq = \{s \in A : s(i) \leq s(j)$ for each $i < j < \mathrm{lh}(s)\}$.
- Let $n < \omega$. $\alpha^{\leq n}$ and $\alpha^{<n}$ denote the sets $\bigcup_{k\leq n} \alpha^k$ and $\bigcup_{k<n} \alpha^k$ respectively.

Throughout this paper, each ordinal $\alpha$ is considered to be a space with order-topology and each subset of $\alpha^n$ is considered to be a subspace of the product space.

A family $\mathcal{A} = \langle A_i : i \in \mathcal{I}\rangle$ of subsets of a space is called $\sigma$-locally finite ($\sigma$-discrete) if $\mathcal{I}$ can be represented as $\bigcup_{j\in\mathcal{J}} \mathcal{I}_j$ for some $\mathcal{J}$ with $|\mathcal{J}| \leq \omega$ such that $\mathcal{A} \restriction \mathcal{I}_j = \langle A_i : i \in \mathcal{I}_j\rangle$ is locally finite (respectively discrete) for each $j \in \mathcal{J}$.

We will need the following two facts about $\sigma$-local finiteness and the subshrinking property. Their verification is routine.

**Lemma 2.1.** *Let $X$ be a topological space and $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ a $\sigma$-locally finite, closed cover of $X$ such that for each $i \in \mathcal{I}$, $F_i$ is subshrinking. Then $X$ is also subshrinking.*

**Lemma 2.2.** *Let $X$ be a topological space, $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ a point finite family of open sets, and $\mathcal{G} = \langle G_i : i \in \mathcal{I}\rangle$ a family of $G_\delta$-sets of $X$ such that $G_i \subseteq U_i$ for each $i \in \mathcal{I}$. Then the union of $\mathcal{G}$ is also a $G_\delta$-set.*

## 3. Trees and stationary sets

**Definition 3.1.** Let $\lambda$ be a regular uncountable cardinal and $n < \omega$. $X \subseteq \lambda^n$ is *stationary* if $C^n \cap X \neq \phi$ for every closed unbounded (club) set $C$ of $\lambda$.

We consider that the empty sequence $\phi$ is the unique sequence of length 0, and $\lambda^0 = \{\phi\}$. $X \subseteq \lambda^0$ is stationary if and only if $\phi \in X$.

**Definition 3.2.** A set $T$ of sequences of ordinals less than $\lambda$ is called a *tree of sequences on* $\lambda$ if $s \restriction k \in T$ for each $s \in T$ and $k \leq \mathrm{lh}(s)$. We use the concept of trees only where they are trees of finite sequences, so we will omit the word "of sequences" from now on.

Let $\lambda$ be a regular uncountable cardinal and $n < \omega$. For a tree $T \subseteq \lambda^{\leq n}$ and $j \leq n$, $T \cap \lambda^j$ ($T \cap \lambda^{<j}$, $T \cap \lambda^{\leq j}$) is denoted by $\mathrm{Lv}_j(T)$ (respectively $\mathrm{Lv}_{<j}(T)$, $\mathrm{Lv}_{\leq j}(T)$).

An *$n$-stationary tree* (*$n$-cofinal tree*) *on* $\lambda$ is a tree $T \subseteq \lambda^{\leq n}$ satisfying that $\phi \in T$ and $\{\alpha < \lambda : s\hat{}\langle\alpha\rangle \in T\}$ is stationary (respectively cofinal) in $\lambda$ for each $s \in \mathrm{Lv}_{<n}(T)$.

Let $X \subseteq \lambda^n$. A function $f : X \longrightarrow (\lambda \cup \{-\infty\})^n$ is called *regressive* if for each $s \in X$ and $k < n$, $f(s)(k) < s(k)$. ($-\infty$ is considered to be less than every ordinal.)

**Lemma 3.1.** *Let $\lambda$ be a regular uncountable cardinal and $n < \omega$.*

(1) *If $T$ is an $n$-stationary tree on $\lambda$, then $\mathrm{Lv}_k(T)|_<$ is stationary in $\lambda^k$ for all $k \le n$.*
(2) *If $X \subseteq \lambda^n|_<$ is stationary, then there exists an $n$-stationary tree $T$ on $\lambda$ such that $\mathrm{Lv}_n(T) \subseteq X$.*
(3) *(The Pressing Down Lemma) If $T$ is an $n$-stationary tree on $\lambda$ and $f : \mathrm{Lv}_n(T) \longrightarrow (\lambda \cup \{-\infty\})^n$ is a regressive function, then there exist an $n$-stationary subtree $U$ of $T|_<$ and a function $g : \mathrm{Lv}_{<n}(U) \longrightarrow \lambda \cup \{-\infty\}$ such that $f(s)(k) = g(s \restriction k)$ for each $s \in \mathrm{Lv}_n(U)$ and $k < n$.*

*Proof.* (1) is trivial.

(2) Define $X_k \subseteq \lambda^k|_<$ for each $k \le n$ inductively. Put $X_n = X$ and

$$X_k = \{s \in (\lambda^k|_<) : \{\alpha < \lambda : s\hat{\ }\langle\alpha\rangle \in X_{k+1}\} \textit{ is stationary } \}$$

if $k < n$. We show that $X_k$ is stationary inductively. $X_n$ is stationary by the assumption. Assume that $k < n$ and $X_{k+1}$ is stationary. For each $s \in \lambda^k|_< - X_k$, pick a club set $C_s$ disjoint from $\{\alpha < \lambda : s\hat{\ }\langle\alpha\rangle \in X_{k+1}\}$ and put

$$C = \{\alpha < \lambda : \alpha \in C_s \text{ for all } s \in (\alpha^k|_<) - X_k\}.$$

Note that $C$ is a club set of $\lambda$. If $D$ is a club set of $\lambda$, then there is an $s \in X_{k+1} \cap (C \cap D)^{k+1}$ since $X_{k+1}$ is stationary, and such an $s$ satisfies that $s \restriction k \in X_k$ since $s(k) \in C$ and $s \restriction k \in s(k)^k|_<$. Hence, $X_k$ is stationary.

$T = \{s \in (\lambda^{\le n}|_<) : s \restriction k \in X_k \text{ for all } k \le \mathrm{lh}(s)\}$ satisfies the required condition.

(3) Pick a regressive function $f_k : \mathrm{Lv}_k(T) \longrightarrow (\lambda \cup \{-\infty\})^k$ for each $k \le n$ as below inductively. Put $f_n = f$. Assume that $k < n$ and $f_{k+1}$ is regressive. For each $s \in \mathrm{Lv}_k(T)$, $A_s = \{\alpha < \lambda : s\hat{\ }\langle\alpha\rangle \in T\}$ is stationary and $f_{k+1}(s\hat{\ }\langle\alpha\rangle)(k) < \alpha$ for each $\alpha \in A_s$. By the Pressing Down Lemma for $\lambda$, there are a stationary set $B_s \subseteq A_s$ and a $\xi_s \in \lambda \cup \{-\infty\}$ such that $f_{k+1}(s\hat{\ }\langle\alpha\rangle)(k) = \xi_s$ for all $\alpha \in B_s$. Since $f_{k+1}$ is regressive, $|\{f_{k+1}(s\hat{\ }\langle\alpha\rangle) \restriction k : \alpha \in B_s\}| < \lambda$. By the completeness of the club filter, there are a stationary set $N_s \subseteq B_s$ and $f_k(s) \in (\lambda \cup \{-\infty\})^k$ such that $f_{k+1}(s\hat{\ }\langle\alpha\rangle) \restriction k = f_k(s)$ for all $\alpha \in N_s$. It is easily seen that $f_k$ is regressive.

Put $U = \{s \in (T|_<) : s(k) \in N_{s\restriction k} \text{ for all } k < \mathrm{lh}(s)\}$. Then $U$ is an $n$-stationary subtree of $T|_<$. Let $g(s) = \xi_s$ for each $s \in \mathrm{Lv}_{<n}(U)$. Inductively, $f_k(s)(i) = g(s \restriction i)$ for all $i < k \le n$ and $s \in \mathrm{Lv}_k(U)$. So $f(s)(k) = f_n(s)(k) = g(s \restriction k)$ for all $s \in \mathrm{Lv}_n(U)$ and $k < n$.

□

**Lemma 3.2.** *Let $\lambda$ be a regular uncountable cardinal, $n < \omega$, $T$ an $n$-cofinal tree on $\lambda$, and $\mathcal{H} = \langle H_i : i \in \mathcal{I}\rangle$ a family of subsets of $\mathrm{Lv}_n(T)$ such that $\bigcup \mathcal{H} = \mathrm{Lv}_n(T)$. Then there exist an $n$-cofinal subtree $U$ of $T$, $\mathcal{I}_0 \subseteq \mathcal{I}$, and a family $\langle t_i : i \in \mathcal{I}_0\rangle$ of elements of $U$ satisfying the following conditions.*

(a) *For each $t \in \mathrm{Lv}_n(U)$, there is unique $i \in \mathcal{I}_0$ such that $t_i \subseteq t$.*
(b) *For each $i \in \mathcal{I}_0$ and $t \in \mathrm{Lv}_n(U)$, if $t_i \subseteq t$ then $t \in H_i$.*

*Moreover, if $|\mathcal{I}| < \lambda$ then we can pick $\mathcal{I}_0$ as a singleton $\{i_0\}$ such that $t_{i_0} = \phi$.*

*Proof.* By induction on $n$. If $n = 0$ then it is trivial. Assume that $n = n' + 1$. Put $T' = \mathrm{Lv}_{\le n'}(T)$ and $A_i(t) = \{\alpha < \lambda : t\hat{\ }\langle\alpha\rangle \in H_i\}$ for each $t \in \mathrm{Lv}_{n'}(T)$ and $i \in \mathcal{I}$. $T'$ is an $n'$-cofinal tree. Define $\mathcal{H}' = \langle H'_i : i \in \mathcal{I}\rangle$ by

$$H'_i = \{t \in \mathrm{Lv}_{n'}(T) : A_i(t) \text{ is cofinal in } \lambda\}$$

for each $i \in \mathcal{I}$. Since $\mathrm{Lv}_{n'}(T') = \mathrm{Lv}_{n'}(T) = \bigcup \mathcal{H}' \cup (\mathrm{Lv}_{n'}(T) - \bigcup \mathcal{H}')$, there is an $n'$-cofinal subtree $T''$ of $T'$ such that $\mathrm{Lv}_{n'}(T'') \subseteq \bigcup \mathcal{H}'$ or $\mathrm{Lv}_{n'}(T'') \subseteq \mathrm{Lv}_{n'}(T) - \bigcup \mathcal{H}'$ by the 'moreover' part of the inductive hypothesis. Moreover, if $|\mathcal{I}| < \lambda$ then $\mathrm{Lv}_{n'}(T) = \bigcup \mathcal{H}'$, so the latter case does not happen.

(Case 1. $\mathrm{Lv}_{n'}(T'') \subseteq \bigcup \mathcal{H}'$.)

By the inductive hypothesis, there exist an $n'$-cofinal subtree $U'$ of $T''$, $\mathcal{I}_0 \subseteq \mathcal{I}$, and a family $\langle t_i : i \in \mathcal{I}_0 \rangle$ of elements of $U'$ satisfying the following conditions.

(a$'$) For each $t \in \mathrm{Lv}_{n'}(U')$, there is unique $i \in \mathcal{I}_0$ such that $t_i \subseteq t$.

(b$'$) For each $i \in \mathcal{I}_0$ and $t \in \mathrm{Lv}_{n'}(U')$, if $t_i \subseteq t$ then $t \in H'_i$.

Moreover, if $|\mathcal{I}| < \lambda$ then we can pick $\mathcal{I}_0$ as a singleton $\{i_0\}$ such that $t_{i_0} = \phi$.

Put

$$U = U' \cup \{t\hat{\ }\langle \alpha \rangle : t \in \mathrm{Lv}_{n'}(U'), \alpha \in A_i(t) \text{ for the } i \in \mathcal{I}_0 \text{ such that } t_i \subseteq t\}.$$

It is easy to check that $U$, $\mathcal{I}_0$, and $\langle t_i : i \in \mathcal{I}_0 \rangle$ satisfy the required contitions.

(Case 2. $\mathrm{Lv}_{n'}(T'') \subseteq \mathrm{Lv}_{n'}(T) - \bigcup \mathcal{H}'$.)

Fix a well ordering $\prec$ on $\mathrm{Lv}_{n'}(T'') \times \lambda$ of order type $\lambda$. For each $\langle t, \xi \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$, pick an $i(t,\xi) \in \mathcal{I}$ and an $\alpha(t,\xi) \in A_{i(t,\xi)}(t)$ inductively. Assume that $\langle t, \xi \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ and that $i(t',\xi') \in \mathcal{I}$ and $\alpha(t',\xi') \in A_{i(t',\xi')}(t')$ are defined for each $\langle t', \xi' \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ such that $\langle t', \xi' \rangle \prec \langle t, \xi \rangle$. $\bigcup_{i \in \mathcal{I}} A_i(t)$ is cofinal in $\lambda$ since $t \in \mathrm{Lv}_{n'}(T)$. On the other hand,

$$\bigcup \{A_{i(t',\xi')}(t) : \langle t', \xi' \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda \textit{ and } \langle t', \xi' \rangle \prec \langle t, \xi \rangle\}$$

is not cofinal in $\lambda$ since $t \notin H'_i$ for every $i \in \mathcal{I}$. So we can pick an $i(t,\xi) \in \mathcal{I}$ and $\alpha(t,\xi) \in A_{i(t,\xi)}(t)$ such that $\xi \leq \alpha(t,\xi)$ and $\alpha(t,\xi)$ does not belong to the set above.

Put $\mathcal{I}_0 = \{i(t,\xi) : \langle t, \xi \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda\}$ and

$$U = T'' \cup \{t\hat{\ }\langle \alpha(t,\xi) \rangle : t \in \mathrm{Lv}_{n'}(T''), \xi < \lambda\}.$$

If $\langle t, \xi \rangle$, $\langle t', \xi' \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ and $\langle t', \xi' \rangle \prec \langle t, \xi \rangle$, then $\alpha(t,\xi) \in A_{i(t,\xi)}(t)$ but $\alpha(t,\xi) \notin A_{i(t',\xi')}(t)$, so $i(t,\xi) \neq i(t',\xi')$. Hence $\langle t, \xi \rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ satisfying that $i = i(t,\xi)$ is unique for each $i \in \mathcal{I}_0$. For each $i \in \mathcal{I}_0$, put $t_i = t\hat{\ }\langle \alpha(t,\xi) \rangle$ where $\langle t, \xi \rangle$ is the element of $\mathrm{Lv}_{n'}(T'') \times \lambda$ such that $i = i(t,\xi)$.

It is easy to check that $U$, $\mathcal{I}_0$, and $\langle t_i : i \in \mathcal{I}_0 \rangle$ satisfy the required contitions.

□

**Definition 3.3.** Let $\lambda$ be a regular uncountable cardinal, $n < \omega$, $T \subseteq \lambda^{\leq n}$ a tree, and $g : T \longrightarrow \lambda$ a function. We say that $\langle \gamma, t \rangle$ is a *uniformly $n$-cofinal subtree of $T$ closed under $g$* if $\gamma = \langle \gamma_\xi : \xi < \lambda \rangle : \lambda \longrightarrow \lambda$ is a strictly increasing, continuous sequence, $t = \langle t(s) : s \in (\lambda^{\leq n}|_<) \rangle$ is a family of elements of $T$ such that:

(i) $\mathrm{lh}(t(s)) = \mathrm{lh}(s), t(s) \restriction k = t(s \restriction k)$ for each $s \in \lambda^{\leq n}|_<, k \leq \mathrm{lh}(s)$,

(ii) $g(t(s)) < \gamma_\xi$ for each $\xi < \lambda$ and $s \in \xi^{\leq n}|_<$,

(iii) $\gamma_{s(k)} \leq t(s)(k)$ for each $s \in \lambda^{\leq n}|_<$ and $k < \mathrm{lh}(s)$.

**Lemma 3.3.** *Let $\lambda$ be a regular uncountable cardinal, $n < \omega$, $T$ an $n$-cofinal tree on $\lambda$, $g : T \longrightarrow \lambda$ a function, and $E \subseteq \lambda$ a club set. Then there exists a uniformly $n$-cofinal subtree $\langle \gamma, t \rangle$ of $T$ closed under $g$ such that $\gamma_\xi \in E$ for all $\xi < \lambda$.*

*Proof.* We define $t(s) \in \mathrm{Lv}_{\mathrm{lh}(s)}(T)$ for $s \in \xi^{\leq n}|_<$ and $\gamma_\xi \in E$ by induction on $\xi < \lambda$. Assume that $\xi < \lambda$ and that $t(s) \in \mathrm{Lv}_{\mathrm{lh}(s)}(T)$ and $\gamma_\zeta \in E$ are defined for all $\zeta < \xi$ and $s \in \zeta^{\leq n}|_<$ .

First, we define $t(s) \in \mathrm{Lv}_{\mathrm{lh}(s)}(T)$ for $s \in \xi^{\leq n}|_< - \bigcup_{\zeta<\xi}(\zeta^{\leq n}|_<)$. In case $\xi$ is a limit ordinal, such an $s$ does not exist. In case $\xi = 0$, such an $s$ is only $\phi$, and put $t(\phi) = \phi$. In case $\xi = \zeta + 1$ for some $\zeta$, such an $s$ has the length $k + 1$ for some $k < n$, $s(k) = \zeta$, and $s \upharpoonright k \in \zeta^k|_<$; so $t(s \upharpoonright k) \in \mathrm{Lv}_k(T)$ is defined and we can pick $t(s) \in \mathrm{Lv}_{k+1}(T)$ such that $t(s) \upharpoonright k = t(s \upharpoonright k)$ and $\gamma_\zeta \leq t(s)(k)$ since $T$ is an $n$-cofinal tree.

Now, $t(s)$ is defined for all $s \in \xi^{\leq n}|_<$. We define $\gamma_\xi \in E$. In case $\xi$ is a limit ordinal, put $\gamma_\xi = \sup\{\gamma_\zeta : \zeta < \xi\}$; since $E$ is a club set, $\gamma_\xi \in E$. In the other case, pick $\gamma_\xi \in E$ such that $\gamma_\xi > \gamma_\zeta$ for every $\zeta < \xi$ and $\gamma_\xi > g(t(s))$ for every $s \in \xi^{\leq n}|_<$.

It is easy to check that $\langle \gamma, t\rangle$ defined as above satisfies the required conditions. □

## 4. Non-subnormal subspaces of $\omega_1^3$

In this section, we prove Theorem 1.1.

**Theorem 4.1.** *Let $X$ be a subspace of $\omega_1^3$. If $X|_<$ is stationary in $\omega_1^3$, $X_{0,1} = \{\langle \alpha, \beta\rangle \in (\omega_1^2|_<) : \langle \alpha, \alpha, \beta\rangle \in X\}$ and $X_{1,2} = \{\langle \alpha, \beta\rangle \in (\omega_1^2|_<) : \langle \alpha, \beta, \beta\rangle \in X\}$ are stationary in $\omega_1^2$, and $X_{0,1,2} = \{\alpha \in \omega_1 : \langle \alpha, \alpha, \alpha\rangle \in X\}$ is not stationary in $\omega_1$, then $X$ is not subnormal.*

*Proof.* Pick a club set $C$ of $\omega_1$ disjoint from $X_{0,1,2}$. Define $E$ and $F$ such that:

$E = \{\langle \alpha, \beta, \gamma\rangle \in X : \alpha = \beta \text{ and } \gamma \in C\}$,
$F = \{\langle \alpha, \beta, \gamma\rangle \in X : \beta = \gamma \text{ and } \alpha \in C\}$.

$E$ and $F$ are disjoint closed sets. We show that $E$ and $F$ cannot be separated by disjoint $G_\delta$-sets.

Assume that $P_i$ and $Q_i$ are open sets of $X$ such that $E \subseteq P_i, F \subseteq Q_i$ for each $i < \omega$. It suffices to show that $\bigcap_{i<\omega} P_i \cap \bigcap_{i<\omega} Q_i \neq \phi$.

Since $X|_<, X_{0,1} \cap C^2$ and $X_{1,2} \cap C^2$ are stationary, there are a 3-stationary tree $T$ and 2-stationary trees $U, V$ on $\omega_1$ such that $\mathrm{Lv}_3(T) \subseteq X|_<$, $\mathrm{Lv}_2(U) \subseteq X_{0,1} \cap C^2$, and $\mathrm{Lv}_2(V) \subseteq X_{1,2} \cap C^2$.

Let $i < \omega$. If $\langle \alpha, \beta\rangle \in \mathrm{Lv}_2(U)$, then it also belongs to $X_{0,1} \cap C^2$, so $\langle \alpha, \alpha, \beta\rangle \in E \subseteq P_i$. Since $P_i$ is open, we can pick a regressive function $e_i : \mathrm{Lv}_2(U) \longrightarrow (\omega_1 \cup \{-\infty\})^2$ satisfying that

$$X \cap ((e_i(u)(0), u(0)]^2 \times (e_i(u)(1), u(1)]) \subseteq P_i$$

for each $u \in \mathrm{Lv}_2(U)$. In the same way, we can pick a regressive function $f_i : \mathrm{Lv}_2(V) \longrightarrow (\omega_1 \cup \{-\infty\})^2$ satisfying that

$$X \cap ((f_i(v)(0), v(0)] \times (f_i(v)(1), v(1)]^2) \subseteq Q_i$$

for each $v \in \mathrm{Lv}_2(V)$. By the Pressing Down Lemma, there are 2-stationary subtrees $U_i$ of $U$, $V_i$ of $V$ , and functions $g_i : \mathrm{Lv}_{<2}(U_i) \longrightarrow (\omega_1 \cup \{-\infty\})$, $h_i : \mathrm{Lv}_{<2}(V_i) \longrightarrow (\omega_1 \cup \{-\infty\})$ such that $e_i(u)(k) = g_i(u \upharpoonright k)$, $f_i(v)(k) = h_i(v \upharpoonright k)$ for every $u \in \mathrm{Lv}_2(U_i), v \in \mathrm{Lv}_2(V_i)$, and $k < 2$.

Pick $t \in \mathrm{Lv}_3(T), u_i \in \mathrm{Lv}_2(U_i)$ , and $v_i \in \mathrm{Lv}_2(V_i)$ for $i < \omega$ such that:

(i) $g_i(\phi), h_i(\phi) < t(0)$ for all $i < \omega$,

(ii) $t(0) \leq v_i(0)$ for all $i < \omega$,
(iii) $t(0) \leq t(1)$ and $h_i(v_i \restriction 1) < t(1)$ for all $i < \omega$,
(iv) $t(1) \leq u_i(0)$ for all $i < \omega$,
(v) $t(1) \leq t(2)$ and $g_i(u_i \restriction 1) < t(2)$ for all $i < \omega$,
(vi) $t(2) \leq u_i(1), v_i(1)$ for all $i < \omega$.

It follows from $e_i(u_i)(0) = g_i(u_i \restriction 0) = g_i(\phi) < t(0) \leq t(1) \leq u_i(0)$ and $e_i(u_i)(1) = g_i(u_i \restriction 1) < t(2) \leq u_i(1)$ that

$$t \in X \cap ((e_i(u_i)(0), u_i(0)]^2 \times (e_i(u_i)(1), u_i(1)]) \subseteq P_i.$$

Since $f_i(v_i)(0) = h_i(v_i \restriction 0) = h_i(\phi) < t(0) \leq v_i(0)$ and $f_i(v_i)(1) = h_i(v_i \restriction 1) < t(1) \leq t(2) \leq v_i(1)$, we have

$$t \in X \cap ((f_i(v_i)(0), v_i(0)] \times (f_i(v_i)(1), v_i(1)]^2) \subseteq Q_i.$$

Hence, $t \in \bigcap_{i<\omega} P_i \cap \bigcap_{i<\omega} Q_i$.

□

For instance, $X = \{\langle \alpha, \beta, \gamma \rangle \in \omega_1^3 : \alpha \leq \beta < \gamma \text{ or } \alpha < \beta \leq \gamma\}$ satisfies the assumption of the theorem above. So Theorem 1.1 holds.

## 5. Canonical subnormal subspaces of $\omega_1^n$

The purpose of this section is to prove Theorem 1.2. We start with an easy fact.

**Fact 5.1.** *If $X \subseteq \omega_1$ is nonstationary in $\omega_1$, then there is a pairwise disjoint family of clopen, bounded subsets of $\omega_1$ which covers $X$.*

We show two ways deriving the subshrinking property of some spaces from ones of simpler spaces.

**Lemma 5.1.** *Let $m \leq n < \omega$ and $X \subseteq \omega_1^n|_\leq$. If $X_m = \{s \restriction m : s \in X\}$ is not stationary in $\omega_1^m$, then there exists a $\sigma$-discrete, closed cover $\mathcal{F} = \langle F_i : i \in \mathcal{I} \rangle$ of $X$ such that for each $i \in \mathcal{I}$, there is a $k < m$ such that $\{s(k) : s \in F_i\}$ is bounded in $\omega_1$.*

*Proof.* By induction on $m$. Fix an $n$. In case $m = 0$, if $X_m$ is not stationary, then $X_m = \phi$, so $X = \phi$. The empty family satisfies the required condition.

Assume that $m < n$ and the statement holds for $m$. Let $X \subseteq \omega_1^n|_\leq$ with $X_{m+1} = \{(s \restriction m+1) : s \in X\}$ nonstationary. There is a club set $C$ of $\omega_1$ such that $C^{m+1} \cap X_{m+1} = \phi$. Put $Y = \{s \in X : s \restriction m \in C^m\}$. $Y$ is a closed subset of $X$. $\{s(m) : s \in Y\}$ is nonstationary in $\omega_1$ since it is disjoint from $C$. Hence it is covered by a pairwise disjoint family of clopen bounded sets of $\omega_1$ by Fact 5.1. By pulling back this family by the projection, we obtain a pairwise disjoint family $\mathcal{P} = \langle P_j : j \in \mathcal{J} \rangle$ of clopen sets of $X$, covering $Y$, such that for each $j \in \mathcal{J}$, $\{s(m) : s \in P_j\}$ is bounded in $\omega_1$. For each $j \in \mathcal{J}$, $Y \cap P_j$ is a $G_\delta$-set because

$$Y \cap P_j = \bigcap_{\xi \in \mu - C} \bigcap_{k<m} \{s \in X : s(k) \neq \xi\} \cap P_j$$

where $\mu < \omega_1$ satisfies that $\{s(m) : s \in P_j\} \subseteq \mu$. By Lemma 2.2, $Y = \bigcup_{j \in \mathcal{J}} (Y \cap P_j)$ is a $G_\delta$-set of $X$. Hence, there are a closed cover $\mathcal{E} = \langle E_i : i < \omega \rangle$ of $X$ such that $E_0 = Y$, and $E_i \cap Y = \phi$ for every $i < \omega$ except 0. If $i \neq 0$, then $\{s \restriction m : s \in E_i\}$ is disjoint from $C^m$, so nonstationary, hence the inductive hypothesis can be applied to $E_i$. On the other hand, $\langle Y \cap P_j : j \in \mathcal{J} \rangle$ is a discrete, closed cover of $E_0$. In any case, there exists a $\sigma$-discrete, closed cover $\mathcal{F}_i = \langle F_{i,j} : j \in \mathcal{J}_i \rangle$ of $E_i$, for every $i < \omega$, such that for each $j \in \mathcal{J}_i$, there is a $k < m+1$ such that $\{s(k) : s \in F_{i,j}\}$ is bounded in $\omega_1$. $\mathcal{F} = \langle F_{i,j} : i < \omega, j \in \mathcal{J}_i \rangle$ is a $\sigma$-discrete, closed cover satisfying the required condition. Hence the statement also holds for $m+1$. □

**Corollary 5.1.** *Let $n < \omega$ and $X$ a nonstationary subset of $\omega_1^n$. Then there exists a $\sigma$-discrete closed cover $\mathcal{F} = \langle F_i : i \in \mathcal{I} \rangle$ of $X$ such that for each $i \in \mathcal{I}$, there is a $k < n$ such that $\{s(k) : s \in F_i\}$ is bounded in $\omega_1$.*

For the next 3 lemmas, let $T$ be a fixed $n$-cofinal tree on $\omega_1$ with $n < \omega$, $g : T \longrightarrow \omega_1$ a function such that $u(k) \le g(u)$ for each $u \in T$ and $k < \mathrm{lh}(u)$, and $\langle \gamma, t \rangle$ a uniformly $n$-cofinal subtree of $T$ closed under $g$. From these objects define $l(s,m,k)$, $r(s,m,k)$, $Z(s,m)$, and $Z(\tilde{s})$ for $s, \tilde{s} \in {\omega_1}^{\le n}|_<$, $m \le \mathrm{lh}(s)$, and $k < \mathrm{lh}(s)$ as follows:

- $l(s,m,k) = g(t(s) \restriction k)$ if $k \le m$, $l(s,m,k) = \gamma_{s(k-1)+1}$ if $m < k$,
- $r(s,m,k) = t(s)(k)$ if $k < m$, $r(s,m,k) = \gamma_{s(k)}$ if $m \le k$,
- $Z(s,m) = \Pi_{k<\mathrm{lh}(s)}(l(s,m,k), r(s,m,k)]$,
- $Z(\tilde{s}) = \bigcup\{Z(s, \mathrm{lh}(\tilde{s})) : s \in ({\omega_1}^n|_<), \tilde{s} \subseteq s\}$.

**Lemma 5.2.** *If $s \in {\omega_1}^{\le n}|_<, m \le \mathrm{lh}(s)$, and $s(k)$ is a limit ordinal for every $m < k < \mathrm{lh}(s)$, then $\langle \gamma_{s(k)} : k < \mathrm{lh}(s) \rangle \in Z(s,m)$.*

*Proof.* Let (i), (ii), and (iii) mean the conditions in Definition 3.3.

Let $k < \mathrm{lh}(s)$. $l(m,s,k) = g(t(s \restriction k)) < \gamma_{s(k)}$ by (i) and (ii) where $k \le m$. If $m < k$ then $s(k-1) < s(k)$ and $s(k)$ is a limit ordinal, hence $s(k-1)+1 < s(k), l(s,m,k) = \gamma_{s(k-1)+1} < \gamma_{s(k)}$.

$\gamma_{s(k)} \le r(s,m,k)$ by (iii) where $k < m$, and it is trivial that $\gamma_{s(k)} \le r(s,m,k)$ where $m \le k$.

Therefore $\langle \gamma_{s(k)} : k < \mathrm{lh}(s) \rangle \in Z(s,m)$. □

**Lemma 5.3.** *If $s \in {\omega_1}^{\le n}|_<$ and $m \le \mathrm{lh}(s)$, then*

$$Z(s,m) \subseteq \Pi_{k<\mathrm{lh}(s)}(g(t(s) \restriction k), t(s)(k)].$$

*Proof.* It suffices to show that $g(t(s) \restriction k) \le l(s,m,k)$ and $r(s,m,k) \le t(s)(k)$ for each $k < \mathrm{lh}(s)$. $g(t(s) \restriction k) = l(s,m,k)$ if $k \le m$, and $r(s,m,k) = t(s)(k)$ if $k < m$ by the definition. If $m < k$ then $g(t(s) \restriction k) = g(t(s \restriction k)) < \gamma_{s(k-1)+1} = l(s,m,k)$ by (i) and (ii). And if $m \le k$ then $r(s,m,k) = \gamma_{s(k)} \le t(s)(k)$ by (iii). □

**Lemma 5.4.** *If $\tilde{s} \in {\omega_1}^{\le n}|_<$ and $X \subseteq \omega_1^n|_<$, then $X \cap Z(\tilde{s})$ is an open, $F_\sigma$-set of $X$.*

*Proof.* For each limit ordinal $\xi < \omega_1$, let $\langle e(\xi,i) : i < \omega \rangle : \omega \longrightarrow \gamma_\xi$ be a strictly increasing, cofinal sequence. For each $i < \omega$ and $0 < k < n$, put

$$E_{i,k} = \{x \in \omega_1^n|_< : (x(k-1) \notin (e(\xi,i), \gamma_\xi]) \text{ or } (x(k) \notin (\gamma_\xi, \gamma_{\xi+1}]) \text{ for every limit ordinal } \xi < \omega_1\}, \text{ and}$$
$$F_{i,k} = \{x \in E_{i,k} : (x(k-1) \le \gamma_\xi) \text{ and } (\gamma_{\xi+1} < x(k)) \text{ for some } \xi < \omega_1\}.$$

**Claim.** For each $i < \omega$ and $0 < k < n$, $E_{i,k}$ and $F_{i,k}$ are closed in $\omega_1^n|_<$.

$\{x \in \omega_1^n|_< : x(k-1) \in (e(\xi,i), \gamma_\xi]\}$ and $\{x \in \omega_1^n|_< : x(k) \in (\gamma_\xi, \gamma_{\xi+1}]\}$ are clopen for every $\xi < \omega_1$. So $E_{i,k}$ is closed. To see that $F_{i,k}$ is closed in $E_{i,k}$, let $x \in E_{i,k}$

and $\xi_0 < \omega_1$ the least ordinal such that $x(k-1) \le \gamma_{\xi_0}$. If $\xi_0$ is a limit ordinal, then $\gamma_{\xi_0} = x(k-1) < x(k)$ and $x(k-1) = \gamma_{\xi_0} \in (e(\xi_0, i), \gamma_{\xi_0}]$, so $\gamma_{\xi_0+1} < x(k)$ and $x \in F_{i,k}$ by $x \in E_{i,k}$. Hence, if $x \in E_{i,k} - F_{i,k}$ then $\xi_0$ is not a limit ordinal and $x(k) \le \gamma_{\xi_0+1}$. If $\xi_0 = 0$ then $\{x \in E_{i,k} : x(k) \le \gamma_1\}$ is a neighborhood of $x$ in $E_{i,k}$ disjoint from $F_{i,k}$. If $\xi_0 = \xi + 1$ then

$$\{x \in E_{i,k} : x(k-1) > \gamma_\xi \text{ and } x(k) \le \gamma_{\xi+2}\}$$

is a neighborhood of $x$ in $E_{i,k}$ disjoint from $F_{i,k}$. So $F_{i,k}$ is closed in $E_{i,k}$.

Now we go back to the proof of the lemma. It is trivial that $X \cap Z(\tilde{s})$ is open. We prove that $X \cap Z(\tilde{s})$ is $F_\sigma$. Put $\tilde{m} = \mathrm{lh}(\tilde{s})$ and

$$\tilde{Z} = \{x \in X : g(t(\tilde{s}) \restriction k) < x(k) \text{ for every } k \in n \cap (\tilde{m}+1), \text{ and } x(k) \le t(\tilde{s})(k) \text{ for every } k < \tilde{m}\}.$$

Then $\tilde{Z}$ is closed in $X$. It suffices to show the following.

**Claim.** $X \cap Z(\tilde{s}) = \tilde{Z} \cap (\bigcup_{i<\omega} \bigcap_{\tilde{m}<k<n} F_{i,k})$.

Assume that $x \in X \cap Z(\tilde{s})$. There is an $s \in \omega_1^n|_<$ such that $\tilde{s} \subsetneq s$ and $x \in Z(s, \tilde{m})$. $x \in \tilde{Z}$ holds from the definition immediately. For each $\tilde{m} < k < n$, let $\xi(k) < \omega_1$ be the least limit ordinal such that $x(k) \le \gamma_{\xi(k)+1}$. Since $x(k-1) \le r(s, \tilde{m}, k-1) = \gamma_{s(k-1)} < \gamma_{s(k-1)+1} = l(s, \tilde{m}, k) < x(k) \le \gamma_{\xi(k)+1}$, we have $s(k-1) < \xi(k)$ and $x(k-1) < \gamma_{\xi(k)}$. So there is an $i < \omega$ such that $x(k-1) \le e(\xi(k), i)$ for all $\tilde{m} < k < n$. Let $\tilde{m} < k < n$ and $\xi < \omega_1$ a limit ordinal. If $\xi < \xi(k)$ then $x(k) \notin (\gamma_\xi, \gamma_{\xi+1}]$ by the minimality of $\xi(k)$. If $\xi = \xi(k)$ then, by $x(k-1) \le e(\xi(k), i) = e(\xi, i)$, $x(k-1) \notin (e(\xi, i), \gamma_\xi]$. If $\xi > \xi(k)$ then, by $x(k) \le \gamma_{\xi(k)+1} < \gamma_\xi$, $x(k) \notin (\gamma_\xi, \gamma_{\xi+1}]$. So $x \in E_{i,k}$. Since $x(k-1) \le r(s, \tilde{m}, k-1) = \gamma_{s(k-1)} < \gamma_{s(k-1)+1} = l(s, \tilde{m}, k) < x(k)$, $x \in F_{i,k}$. So $x \in \bigcap_{\tilde{m}<k<n} F_{i,k}$. $X \cap Z(\tilde{s}) \subseteq \tilde{Z} \cap (\bigcup_{i<\omega} \bigcap_{\tilde{m}<k<n} F_{i,k})$.

On the other hand, assume that $x \in \tilde{Z} \cap (\bigcup_{i<\omega} \bigcap_{\tilde{m}<k<n} F_{i,k})$. Pick an $i < \omega$ such that $x \in \bigcap_{\tilde{m}<k<n} F_{i,k}$. Let $s$ be the sequence of length $n$ such that $s \restriction \tilde{m} = \tilde{s}$ and for each $\tilde{m} \le k < n$, $s(k) < \omega_1$ is the least ordinal satisfying that $x(k) \le \gamma_{s(k)}$. If $\tilde{m} < k < n$ then $x \in F_{i,k}$, so there is a $\xi' < \omega_1$ such that $x(k-1) \le \gamma_{\xi'}$ and $\gamma_{\xi'+1} < x(k)$, and such $\xi$ has to be $s(k-1) \le \xi'$, hence $\gamma_{s(k-1)+1} \le \gamma_{\xi'+1} < x(k) \le \gamma_{s(k)}$ and $s(k-1) < s(k)$. If $k < \tilde{m} < n$ then, by $x \in \tilde{Z}$, $\gamma_{\tilde{s}(k)} \le t(\tilde{s})(k) \le g(t(\tilde{s})) < x(\tilde{m}) \le \gamma_{s(\tilde{m})}$, so $\tilde{s}(k) < s(\tilde{m})$. Hence $s \in \omega_1^n|_<$ and $x \in Z(s, \tilde{m}) \subseteq Z(\tilde{s})$. Therefore $\tilde{Z} \cap (\bigcup_{i<\omega} \bigcap_{\tilde{m}<k<n} F_{i,k}) \subseteq X \cap Z(\tilde{s})$. □

**Lemma 5.5.** *Let $n < \omega$ and $X \subseteq \omega_1^n|_<$. For each open cover $\mathcal{U} = \langle U_i : i \in \mathcal{I} \rangle$ of $X$, there exist a family $\mathcal{F} = \langle F_i : i \in \mathcal{I} \rangle$ of open, $F_\sigma$-sets of $X$ such that:*

(i) *$F_i \subseteq U_i$ for every $i \in \mathcal{I}$,*
(ii) *$X - \bigcup \mathcal{F}$ is nonstationary,*

*Proof.* If $X$ is not stationary, then $\mathcal{F} = \langle F_i : i \in \mathcal{I} \rangle$, where $F_i = \phi$ for each $i \in \mathcal{I}$, satisfies the required condition.

Assume that $X$ is stationary. There is an $n$-stationary tree $T''$ on $\omega_1$ such that $\mathrm{Lv}_n(T'') \subseteq X$ by Lemma 3.1(2). Pick a pairwise disjoint family $\mathcal{H} = \langle H_i : i \in \mathcal{I} \rangle$ such that $H_i \subseteq U_i$ for every $i \in \mathcal{I}$ and $\bigcup \mathcal{H} = \mathrm{Lv}_n(T'')$. Since $\mathcal{U}$ is an open cover of $X$, there is a regressive function $f : \mathrm{Lv}_n(T'') \longrightarrow (\omega_1 \cup \{-\infty\})^n$ such that for each $i \in \mathcal{I}$ and $t \in H_i, X \cap \Pi_{k<n}(f(t)(k), t(k)] \subseteq U_i$. By the Pressing Down Lemma, there are an $n$-stationary subtree $T'$ of $T''$ and a function $g' : \mathrm{Lv}_{<n}(T') \longrightarrow \omega_1$

such that $f(t)(k) = g'(t \restriction k)$ for every $t \in \mathrm{Lv}_n(T')$ and $k < n$. By Lemma 3.2, there exist an $n$-cofinal subtree $T$ of $T'$, $\mathcal{I}_0 \subseteq \mathcal{I}$, and a family $\langle t_i : i \in \mathcal{I}_0\rangle$ of elements of $T$ satisfying (a), (b). Pick a function $g : T \longrightarrow \omega_1$ such that $t(k) \le g(t)$ for every $t \in T$ and $k < \mathrm{lh}(t)$, and that $g'(t) \le g(t)$ for every $t \in \mathrm{Lv}_{<n}(T)$. By Lemma 3.3, there exists a uniformly $n$-cofinal subtree $\langle \gamma, t\rangle$ of $T$ closed under $g$.

Put $\mathcal{I}_1 = \{i \in \mathcal{I}_0 : \text{for some } s_i \in {\omega_1}^{\le n}|_<, t(s_i) = t_i\}$. For each $i \in \mathcal{I}_1$, there is unique $s_i$ witnessing $i \in \mathcal{I}_1$. Actually, if $s \in {\omega_1}^{\le n}|_<$ and $t(s) = t_i$, then $\mathrm{lh}(s) = \mathrm{lh}(t_i)$ and for each $k < \mathrm{lh}(t_i)$, $\gamma_{s(k)} \le t(s)(k) = t_i(k) \le g(t(s) \restriction k+1) = g(t(s \restriction k+1)) < \gamma_{s(k)+1}$. Such an $s$ is unique.

Apply Lemma 5.2, 5.3, and 5.4 to $T, g$, and $\langle \gamma, t\rangle$. Put $F_i = X \cap Z(s_i)$ for each $i \in \mathcal{I}_1$ and put $F_i = \phi$ for each $i \in \mathcal{I} - \mathcal{I}_1$. Let $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$.

By Lemma 5.4, $F_i$ is an open, $F_\sigma$-set for every $i \in \mathcal{I}$ in $X$. We show that the conditions (i) and (ii) hold for $\mathcal{F}$.

For each $i \in \mathcal{I}_1$ and $s \in \omega_1^n|_<$ with $s_i \subseteq s$, $t(s) \in \mathrm{Lv}_n(T)$ and $t_i = t(s_i) \subseteq t(s)$, so $t(s) \in H_i$ by the condition (b) in Lemma 3.2. Since $t(s) \in \mathrm{Lv}_n(T) \subseteq \mathrm{Lv}_n(T') \subseteq \mathrm{Lv}_n(T'')$, $f(t(s))(k) = g'(t(s) \restriction k) \le g(t(s) \restriction k)$ for each $k < n$, and $X \cap \Pi_{k<n}(f(t(s))(k), t(s)(k)] \subseteq U_i$. By Lemma 5.3,

$$\begin{aligned} X \cap Z(s, \mathrm{lh}(s_i)) &\subseteq X \cap \Pi_{k<n}(g(t(s) \restriction k), t(s)(k)] \\ &\subseteq X \cap \Pi_{k<n}(f(t(s))(k), t(s)(k)] \subseteq U_i. \end{aligned}$$

Hence (i) holds.

$D = \{\gamma_\xi : \xi \text{ is a limit ordinal } < \omega_1\}$ is a club set of $\omega_1$. To see that (ii) holds, it suffices to show that $X \cap D^n \subseteq \bigcup \mathcal{F}$. Let $x \in X \cap D^n$, say $x = \langle \gamma_{s(k)} : k < n\rangle$ for some $s \in {\omega_1}^n|_<$. Then $s(k)$ is a limit ordinal for each $k < n$ and $t(s) \in \mathrm{Lv}_n(T)$. By (a) in Lemma 3.2, there is unique $i \in \mathcal{I}_0$ such that $t_i \subseteq t(s)$. Since $t(s \restriction \mathrm{lh}(t_i)) = t(s) \restriction \mathrm{lh}(t_i) = t_i$, $i \in \mathcal{I}_1$ and $s_i = s \restriction \mathrm{lh}(t_i) \subseteq s$. By Lemma 5.2, $x \in X \cap Z(s, \mathrm{lh}(s_i)) \subseteq X \cap Z(s_i) = F_i$. Hence $X \cap D^n \subseteq \bigcup \mathcal{F}$, so (ii) holds. □

**Lemma 5.6.** *Assume that $n < \omega$, $X \subseteq \omega_1^n$, and $X_{k,\alpha} = \{s\hat{\ }t : s \in \omega_1^k, t \in {\omega_1}^{n-(k+1)}, s\hat{\ }\langle\alpha\rangle\hat{\ }t \in X\}$ is subshrinking for each $k < n$ and $\alpha < \omega_1$.*

(1) *If $X$ is nonstationary, then $X$ is subshrinking.*

(2) *If $X \subseteq \omega_1^n|_<$ then $X$ is subshrinking.*

*Proof.* (1) Let $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ be a closed cover obtained by the Corollary 5.1. By Lemma 2.1, it suffices to show that $F_i$ is subshrinking for every $i \in \mathcal{I}$. Fix an $i \in \mathcal{I}$. $\{s(k) : s \in F_i\} \subseteq \mu$ for some $k < n$ and $\mu < \omega_1$. For $\alpha < \mu$, put $C_\alpha = \{s \in F_i : s(k) = \alpha\}$. $\langle C_\alpha : \alpha < \mu\rangle$ is a closed cover of $F_i$. By Lemma 2.1 again, it suffices to show that $C_\alpha$ is subshrinking for each $\alpha < \mu$. In $X_{k,\alpha}$, $\{s\hat{\ }t : s \in \omega_1^k, t \in {\omega_1}^{n-(k+1)}, s\hat{\ }\langle\alpha\rangle\hat{\ }t \in C_\alpha\}$ is closed and homeomorphic to $C_\alpha$, hence $C_\alpha$ is subshrinking.

(2) Let $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ be an open cover of $X$. Pick a family $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of open, $F_\sigma$-sets of $X$ obtained by Lemma 5.5, and put $G = \bigcup \mathcal{F}$ . For each $k < n$ and $\alpha < \omega_1$, $\{s\hat{\ }t : s \in \omega_1^k, t \in {\omega_1}^{n-(k+1)}, s\hat{\ }\langle\alpha\rangle\hat{\ }t \in X - G\}$ is a closed subset of $X_{k,\alpha}$, so subshrinking. Since $X - G$ is nonstationary, we can apply (1) to $X - G$, so $X - G$ is subshrinking. Hence, there is a subshrinking $\mathcal{M} = \langle M_i : i \in \mathcal{I}\rangle$ of $\langle U_i \cap (X - G) : i \in \mathcal{I}\rangle$ in $X - G$. Since $X - G$ is closed in $X$, $\mathcal{M}$ is a family of $F_\sigma$-sets also in $X$. $\langle M_i \cup F_i : i \in \mathcal{I}\rangle$ is a subshrinking of $\mathcal{U}$ in $X$. Hence $X$ is subshrinking. □

Now we can prove Theorem 1.2.

*Proof.* Apply Lemma 5.6 (2) inductively. Then the statement is proved immediately.

□

Since all subspaces of $\omega_1^2$ are subshrinking (see [4]), the following holds by Lemma 5.6 (1).

**Corollary 5.2.** *All nonstationary subspaces of $\omega_1^3$ are subshrinking.*

$\omega_1^3$ in the corollary above cannot be changed with $\omega_1^4$. Because, there is a non-subnormal subspace $X$ of $\omega_1^3$ by Theorem 1.1. Therefore $\{0\} \times X$, which is homeomorphic to $X$, is a nonstationary and non-subnormal subspace of $\omega_1^4$.

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