ANOTHER APPROACH FOR LEXICOGRAPHIC GO-SPACES

NOBUYUKI KEMOTO

Abstract. It is known that for a GO-space X, there is the smallest LOTS X^* containing X as a dense subspace, that is, if a LOTS L contains X as a dense subspace, then L contains X^*.

Also lexicographic products of LOTS', which is called lexicographic LOTS', have been well-discussed. Recently, the notion of lexicographic products of GO-spaces was defined as follows:

for a sequence $\{X_\alpha : \alpha < \gamma\}$ of GO-spaces, the lexicographic GO-space $X = \prod_{\alpha<\gamma} X_\alpha$ means the subspace X of the lexicographic LOTS $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$.

It is known that for a GO-space X, there is a well-known LOTS $X^\diamond$ containing X as a closed subspace. In this paper, first we show the LOTS $X^\diamond$ has the following nice property:

• if a LOTS L contains X as a closed subspace, then L contains $X^\diamond$.

Using this property, it is natural to define another notion of lexicographic GO-spaces as follows:

for a sequence $\{X_\alpha : \alpha < \gamma\}$ of GO-spaces, the lexicographic GO-space $X = \prod_{\alpha<\gamma} X_\alpha$ means the subspace X of the lexicographic LOTS $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$.

We will see:

• the GO-space X as a subspace of $\hat{X}$ coincides with the GO-space X as a subspace of $\check{X}$,

• $X^\diamond$ is contained in $\check{X}$.

• we characterize that the lexicographic GO-space X is closed in $\check{X}$.

1. Introduction

Lexicographic products of LOTS', which is called lexicographic LOTS', have been well-discussed, for instance see [2]. Recently, the notion of

Date: June 5, 2025.

2010 *Mathematics Subject Classification.* Primary 54F05, 54B10, 54B05 . Secondary 54C05.

Key words and phrases. lexicographic product, GO-space, LOTS, d-extension, c-extension.

This research was supported by Grant-in-Aid for Scientific Research (C) 21K03339.

lexicographic products of GO-spaces, which is called lexicographic GO spaces, was defined in [7] and has been developed in [3, 4, 5, 6, 8, 9, 10, 11]. For instance, the following strange result has been known:

• whenever γ is limit and all X_α's have a minimal element but not have maximal elements, "if the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is connected, then all X_α's are non-connected" [11].

In this paper, as in the abstract above, we will give another new approach for getting lexicographic GO-spaces and discuss relationships between these two notions.

We assume that all topological spaces have cardinality at least 2. A linearly ordered set $\langle X, <_X\rangle$ (see [1]) has a natural T_2-topology denoted by λ_X so called the *order topology* which is the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\}$ as a subbase, where $(x, \rightarrow)_X = \{w \in X : x <_X w\}$, $(x, y]_X = \{w \in X : x <_X w \leq_X y\}$, ..., etc. Here $w \leq_X x$ means $w <_X x$ or $w = x$. If the contexts are clear, we simply write $<$ and $(x, y]$ instead of $<_X$ and $(x, y]_X$ respectively. The triple $\langle X, <_X, \lambda_X\rangle$ is called a *LOTS* (= Linearly Ordered Topological Space) and simply denoted by LOTS X. In particular, an ordinal with the usual order is considered as a LOTS. The symbol ω denotes the set $\{0, 1, 2, 3, \cdots\}$ which is an ordinal, moreover the symbol $\mathbb{N}$ denotes the set of natural numbers, that is, $\{1, 2, 3, \cdots\}$. Note that for every $x \in X$, $(\leftarrow, x] \notin \lambda_X$ iff $(x, \rightarrow)$ is non-empty and has no minimal elements, also analogously $[x, \rightarrow) \notin \lambda_X$ iff $(\leftarrow, x)$ is non-empty and has no maximal elements. Let

$$X_R = \{x \in X : (\leftarrow, x] \notin \lambda_X\} \text{ and } X_L = \{x \in X : [x, \rightarrow) \notin \lambda_X\}.$$

A *generalized ordered space* (= GO-space) is a triple $\langle X, <_X, \tau_X\rangle$, where $<_X$ is linear order on X and τ_X is a T_1 topology on X which has a base consisting of convex sets, also simply denoted by a GO-space X, where a subset B of X is *convex* if for every $x, y \in B$ with $x <_X y$, $[x, y]_X \subset B$ holds. Such a topology τ_X is called a GO-topology on a linearly ordered set $\langle X, <_X\rangle$ and it is easy to verify that the GO-topology τ_X is stronger than the order topology λ_X, that is, $\tau_X \supset \lambda_X$. Also let

$$X^+_{\tau_X} = \{x \in X : (\leftarrow, x]_X \in \tau_X \setminus \lambda_X\},$$
$$X^-_{\tau_X} = \{x \in X : [x, \rightarrow)_X \in \tau_X \setminus \lambda_X\}.$$

Obviously $X^+_{\tau_X} \subset X_R$ and $X^-_{\tau_X} \subset X_L$. When contexts are clear, we usually simply write X^+ and X^- instead of $X^+_{\tau_X}$ and $X^-_{\tau_X}$, respectively. Note that X is a LOTS iff $X^+ \cup X^- = \emptyset$. For $A \subset X_R$ and $B \subset X_L$, let $\tau(A, B)$ be the GO-topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in A\} \cup \{[x, \rightarrow)_X : x \in B\}$ as a subbase. Obviously $\tau_X = \tau(X^+, X^-)$ whenever X is a GO-space, and also $\tau(A, B)$ defines a GO-topology on X whenever X is a LOTS with $A \subset X_R$ and $B \subset X_L$. The Sorgenfrey line $\mathbb{S}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\emptyset, \mathbb{R})\rangle$, where $<_{\mathbb{R}}$ is the usual order on the real line $\mathbb{R}$.

Let $X = \langle X, <_X, \tau_X\rangle$ be a GO-space and $Y \subset X$, then "the subspace Y of a GO-space X" means the GO-space $\langle Y, <_X\restriction Y, \tau_X \restriction Y\rangle$, where $<_X\restriction Y$ is the restricted order of $<_X$ on Y and $\tau_X \restriction Y := \{U \cap Y : U \in \tau_X\}$, that is, $\tau_X \restriction Y$ is the subspace topology of τ_X.

Let X and Y be LOTS' with the orders $<_X$ and $<_Y$ respectively. A map $f : X \to Y$ is said to be *order preserving* if, $f(x) <_Y f(x')$ whenever $x <_X x'$, in this case, f is 1-1. Moreover, f is said to be an *order isomorphism* if f is order preserving and onto, in this case f^{-1} is also order preserving. Obviously an oder isomorphism f between two LOTS' is a homeomorphism, that is, both f and f^{-1} are continuous. However, an oder isomorphism between two GO-spaces need not be a homeomorphism. For example, the identity map between the Sorgenfley line $\mathbb{S}$ and the real line $\mathbb{R}$ is an order isomorphism but not a homeomorphism. When there is an order isomorphism between two GO-spaces X and Y which is also a homeomorphism, then we can identify these GO-spaces and usually identify as $X = Y$. If there is an order preserving map f on a GO-space X onto a subspace Y of a GO-space Z which is homeomorphism, then we say that f *is an embedding of a GO-space X into a GO-space Z* or *a GO-space X is embeddable in a GO-space Z*.

Now for a given GO-space X, let consider the following well-known LOTS

$$X^* = X^- \times \{-1\} \cup X \times \{0\} \cup X^+ \times \{1\}$$

with the lexicographic order $<_{X^*}$ on X^* induced by the lexicographic order on $X \times \{-1, 0, 1\}$, here of course $-1 < 0 < 1$. We usually identify X as $X = X \times \{0\}$ in the obvious way (i.e., $x = \langle x, 0\rangle$), moreover for $x \in X^+$ ($x \in X^-$), put $x^+ := \langle x, 1\rangle \in X^+ \times \{1\}$ ($x^- := \langle x, -1\rangle \in X^- \times \{-1\}$, respectively). Then note that X^* can be identified as

$$X^* = \{x^- : x \in X^-\} \cup X \cup \{x^+ : x \in X^+\}.$$

Note $(\leftarrow, x]_X = (\leftarrow, x^+)_{X^*} \cap X \in \lambda_{X^*} \restriction X$ whenever $x \in X^+$, and also its analogy. Then the GO-space X is a dense subspace of the LOTS X^*, and X has a maximal element $\max X$ iff X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (and similarly for minimal elements). Note that X^* satisfies the property that if a LOTS L contains the GO-space X as a dense subspace, then there are a subspace Y of L which is a LOTS and an order isomorphism $f : X^* \to Y$ such that the restriction of f on X is the identity map on X, i.e., $f \restriction X = 1_X$, see [12]. Since X is dense in X^*, such a pair of Y and f is uniquely determined, so we can identify as $Y = X^*$. A LOTS L containing X as a dense subspace is called a *d-extension* of a GO-space X, where "*d*-" means "dense". So X^* is called the minimal d-extension of X.

Definition 1.1 (A lexicographic GO-space by d-extensions [7])**.** Let X_α be a LOTS for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$, where γ is an ordinal. When $\gamma = 0$, we consider as $\prod_{\alpha<\gamma} X_\alpha = \{\emptyset\}$, which is a trivial LOTS, for notational conveniences. When $\gamma > 0$, every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma\rangle$. Recall that the lexicographic order $<_X$ on X is defined as follows:
for $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha\rangle$ and $<_{X_\alpha}$ denotes the order on X_α. Then $X = \langle X, <_X, \lambda_X\rangle$ is a LOTS and called the lexicographic product of LOTS' X_α's.

Now let X_α be a GO-space for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$. Then the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, which is a LOTS, can be defined. The *lexicographic product of GO-spaces* X_α's defined by d-extensions is the GO-space $\langle X, <_{\hat{X}} \restriction X, \lambda_{\hat{X}} \restriction X\rangle$. Obviously this definition extends the lexicographic product of LOTS', and is reasonable because each X^*_α is the smallest LOTS which contains X_α as a dense subspace. When $n \in \omega$, then $\prod_{i<n} X_i$ is denoted by $X_0 \times \cdots \times X_{n-1}$. If all X_α's are X, then $\prod_{\alpha<\gamma} X_\alpha$ is denoted by X^γ. From now on, when we consider a lexicographic GO-space $X = \prod_{\alpha<\gamma} X_\alpha$, we assume $\gamma \geq 2$.

Next for a given GO-space X, let consider the following well-known LOTS

$$X^\diamond = X^- \times \{-n : n \in \mathbb{N}\} \cup \big(X \times \{0\}\big) \cup X^+ \times \{n : n \in \mathbb{N}\}$$

with the lexicographic order $<_{X^\diamond}$ on $X^\diamond$ induced by the lexicographic order on $X \times \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers with the usual order, that is, $\mathbb{Z} = \{\cdots, -3, -2, -1-, 0, 1, 2, 3, \cdots\}$. We usually identify X as $X = X \times \{0\}$ in the obvious way (i.e., $x = \langle x, 0\rangle$), moreover for $x \in X^+$ ($x \in X^-$) and $n \in \mathbb{N}$, put $x^{+n} := \langle x, n\rangle$ ($x^{-n} := \langle x, -n\rangle$, respectively). Then note that $X^\diamond$ can be represented as

$$X^\diamond = \{x^{-n} : x \in X^-, n \in \mathbb{N}\} \cup X \cup \{x^{+n} : x \in X^+, n \in \mathbb{N}\}.$$

For notational conveniences, we let $x^{+0} = x^{-0} = x$. Note $(\leftarrow, x]_X = (\leftarrow, x^{+1})_{X^\diamond} \cap X \in \lambda_{X^\diamond} \restriction X$ whenever $x \in X^+$, and also its analogy.

Then the GO-space X is a closed subspace of the LOTS $X^\diamond$, and X has a maximal element $\max X$ iff $X^\diamond$ has a maximal element $\max X^\diamond$, in this case, $\max X = \max X^\diamond$ (and similarly for minimal elements). In the next section, we will see that $X^\diamond$ satisfies the property that if a LOTS L contains the GO-space X as a closed subspace, then there are a subspace Y of L which is a LOTS and an order isomorphism $f : X^\diamond \to Y$ such that $f \restriction X = 1_X$. Unlike the case X^*, such a pair Y and f cannot be uniquely determined in this case, because, for example, $\{x^{-2n} : x \in X^-, n \in \mathbb{N}\} \cup X \cup \{x^{+2n} : x \in X^+, n \in \mathbb{N}\}$ is a proper subspace of $X^\diamond$ which is also order isomorphic to $X^\diamond$. A LOTS L containing X as a closed subspace is called a *c-extension* of a GO-space X, where "*c*-" means "closed". Although there are no minimal c-extensions, by the above property of a c-extension which will be proved in the next section, it is reasonable to define the following new notion of a lexicographic GO-space using c-extensions.

Definition 1.2 (A lexicographic GO-space by c-extensions)**.** Let X_α be a GO-space for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$. Then the lexicographic product $\check{X} = \prod_{\alpha<\gamma} X_\alpha^\diamond$, which is a LOTS, can be defined. The *lexicographic product of GO-spaces* X_α's defined by c-extensions is the GO-space $\langle X, <_{\check{X}}\restriction X, \lambda_{\check{X}} \restriction X\rangle$. Note that both orders $<_{\hat{X}}\restriction X$ and $<_{\check{X}}\restriction X$ on X coincide with the lexicographic order $<_X$ on $X = \prod_{\alpha<\gamma} X_\alpha$. So we can write these lexicographic GO-spaces $\langle X, <_{\hat{X}}\restriction X, \lambda_{\hat{X}} \restriction X\rangle$ and $\langle X, <_{\check{X}}\restriction X, \lambda_{\check{X}} \restriction X\rangle$ as $\langle X, <_X, \lambda_{\hat{X}} \restriction X\rangle$ and $\langle X, <_X, \lambda_{\check{X}} \restriction X\rangle$, respectively.

We frequently use the following notation. For instance (general cases are similarly defined), let $x_0 \in \prod_{\alpha<\alpha_0} X_\alpha, u \in X_{\alpha_0}, x_1 \in \prod_{\alpha_0<\alpha<\gamma} X_\alpha$ with $\alpha_0 < \gamma$, then the symbol $x_0{}^\wedge\langle u\rangle^\wedge x_1$ denotes the element $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} x_0(\alpha) & \text{if } \alpha < \alpha_0, \\ u & \text{if } \alpha = \alpha_0, \\ x_1(\alpha) & \text{if } \alpha_0 < \alpha < \gamma. \end{cases}$$

2. $X^\diamond$ can be embeddable into a LOTS containing X as a closed subspace

Since a GO-space X is dense in X^* and closed in $X^\diamond$, the following properties on a GO-space X which is not a LOTS are obvious:

- there is no embedding $f : X^* \to X^\diamond$ with $f \restriction X = 1_X$,
- there is no embedding $f : X^\diamond \to X^*$ with $f \restriction X = 1_X$.

In this section, we prove that if a GO-space X is contained in a LOTS L as a closed subspace, then there is an embedding $f : X^\diamond \to L$ with $f \restriction X = 1_X$. To do this, we need the following lemma.

Lemma 2.1. *If X is a GO-space, then the following hold.*

(1) *if $x \in X^+$, then the sequence $\{x^{+n} : n \in \omega\}$ is strictly increasing in $X^\diamond$ such that for every $n \in \mathbb{N}$, $x^{+n} \notin X$ and x^{+n} is the immediate successor of $x^{+(n-1)}$ in $X^\diamond$, that is, $(x^{+(n-1)}, x^{+n})_{X^\diamond} = \emptyset$, therefore x^{+n} is an isolated point of the LOTS $X^\diamond$,*
(2) *if $x \in X^-$, then the sequence $\{x^{-n} : n \in \omega\}$ is strictly decreasing in $X^\diamond$ such that for every $n \in \mathbb{N}$, $x^{-n} \notin X$ and x^{-n} is the immediate predecessor of $x^{-(n-1)}$ in $X^\diamond$, that is, $(x^{-n}, x^{-(n-1)})_{X^\diamond} = \emptyset$, therefore x^{-n} is an isolated point of the LOTS $X^\diamond$,*
(3) *if $x \in X^+$, $y \in X$ and $n \in \mathbb{N}$, then $x <_X y$ iff $x^{+n} <_{X^\diamond} y$ iff there is $z \in X$ with $x^{+n} <_{X^\diamond} z <_X y$,*
(4) *if $x \in X^-$, $y \in X$ and $n \in \mathbb{N}$, then $y <_X x$ iff $y <_{X^\diamond} x^{-n}$ iff there is $z \in X$ with $y <_X z <_{X^\diamond} x^{-n}$,*
(5) *if $x \in X^+$, $y \in X^-$ and $n, m \in \mathbb{N}$, then $x <_X y$ iff $x^{+n} <_{X^\diamond} y^{-m}$ iff there is $z \in X$ with $x^{+n} <_{X^\diamond} z <_{X^\diamond} y^{-m}$,*
(6) *for every $a \in X^\diamond$, there are $x, y \in X$ with $x \leq_{X^\diamond} a \leq_{X^\diamond} y$.*
(7) *if $x, y \in X$ with $x <_X y$ and $(x, y)_X = \emptyset$, then $(x, y)_{X^\diamond} = \emptyset$.*

Proof. (7) Let $x, y \in X$ with $x <_X y$ and $(x, y)_X = \emptyset$. Assuming $(x, y)_{X^\diamond} \neq \emptyset$, take $a \in (x, y)_{X^\diamond}$. We may assume that $a = z^{+n}$ for some $z \in X^+$ and $n \in \mathbb{N}$. Then obviously we have $x \leq_X z$. When $x <_X z$, we have $z \in (x, y)_X$, a contradiction. When $x = z$, since $(z, \to)_X$ is non-empty and has no minimal elements, we can take $z' \in (z, y)_X = (x, y)_X$, a contradiction.

Proofs of other clauses are left to the readers. □

Now we will show that if a LOTS L contains a closed subspace X, then $X^\diamond$ is also contained in L, more precisely:

Theorem 2.2. *If a LOTS L contains a closed subspace X, then there are a subspace Y of L and an order isomorphism $f : X^\diamond \to Y$ which is a homeomorphism with $f \restriction X = 1_X$.*

Proof. Let $L = \langle L, <_L, \lambda_L \rangle$, $X = \langle X, <_X, \tau_X \rangle$ and $X^\diamond = \langle X^\diamond, <_{X^\diamond}, \lambda_{X^\diamond} \rangle$. Then by the assumptions, note that $<_X = <_L \restriction X = <_{X^\diamond} \restriction X$, $\tau_X = \lambda_L \restriction X = \lambda_{X^\diamond} \restriction X$ and X is closed in both topological spaces $\langle L, \lambda_L \rangle$ and $\langle X^\diamond, \lambda_{X^\diamond} \rangle$. Let λ_X denote the order topology on the ordered set $\langle X, <_X \rangle$. First we show the following claim.

Claim 1. If $x \in X^+$ $(= X^+_{\tau_X})$, that is, $(\leftarrow, x]_X \in \tau_X \setminus \lambda_X$, then there is a strictly increasing sequence $\{x_n : n \in \omega\}$ in L such that $x_0 = x$ and for every $n \in \omega$, $x_n <_L (x, \rightarrow)_X$ holds, i.e., $x_n <_L y$ for every $y \in (x, \rightarrow)_X$.

Proof. Let $x \in X^+$, then it follows from $(\leftarrow, x]_X \in \tau_X \setminus \lambda_X$ that $(x, \rightarrow)_X$ is non-empty closed in X (i.e., in the space $\langle X, \tau_X\rangle$) and has no minimal elements in X (i.e., in the ordered set $\langle X, <_X\rangle$). Note that $(x, \rightarrow)_X$ is also closed in L, because X is closed in L.

First let $x_0 = x$. Since $(x, \rightarrow)_X$ is closed in L with $x \notin (x, \rightarrow)_X$, we can find $x_1 \in L$ with $x = x_0 <_L x_1$ and $(x, x_1)_L \cap (x, \rightarrow)_X = \emptyset$. Note $x_1 \notin X$. Assume that a sequence $x_0, x_1, x_2 \cdots, x_n \in L$ with $n \geq 1$ and $x = x_0 <_L x_1 <_L x_2 <_L \cdots <_L x_n <_L (x, \rightarrow)_X$ has been defied. Since $x_n \notin X$ and X is closed in L, there is $x_{n+1} \in L$ with $x_n <_L x_{n+1}$ such that $(x_n, x_{n+1})_L \cap X = \emptyset$. Then obviously $x_{n+1} <_L (x, \rightarrow)_X$. The proof of Claim 1 is complete.

Similarly we see:

Claim 2. If $x \in X^-$ $(= X^-_{\tau_X})$, then there is a strictly decreasing sequence $\{x_n : n \in \omega\}$ in L such that $x_0 = x$ and for every $n \in \omega$, $(\leftarrow, x)_X <_L x_n$ holds.

By Claim 1 (Claim 2), we can fix such a sequence $\{x_n : n \in \omega\} \subset L$ for $x \in X^+$ ($x \in X^-$, respectively). Using these sequences, define a map f on $X^\diamond$ as follows: for $a \in X^\diamond$,

$$f(a) = \begin{cases} a & \text{if } a \in X, \\ x_n & \text{if } \exists x \in X^+ \exists n \in \mathbb{N}(a = x^{+n}), \\ x_n & \text{if } \exists x \in X^- \exists n \in \mathbb{N}(a = x^{-n}), \end{cases}$$

Now letting $Y = f[X^\diamond]$, we will prove that f and Y are the desired. By the construction of f and Lemma 2.1, obviously by case-by-case, we can see $f : X^\diamond \to Y$ is an order isomorphism, so it suffices to see that f is an homeomorphism between $X^\diamond$ and Y, therefore the following claim completes the proof of this theorem.

Claim 3. $Y = \langle Y, <_L\restriction Y, \lambda_L \restriction Y\rangle$ is a LOTS.

Proof. Let $<_Y = <_L\restriction Y$, $\tau_Y = \lambda_L \restriction Y$ and λ_Y be the order topology on Y, of course, with respect to the order $<_Y$. To see that Y is a LOTS, it suffices to see $Y^+ = \emptyset$ and $Y^- = \emptyset$. Since the remaining is similar, we will only see $Y^- = \emptyset$, where $Y^- = \{y \in Y : [y, \rightarrow)_Y \in \tau_Y \setminus \lambda_Y\}$. So it suffices to see that for every $y \in Y$, if $[y, \rightarrow)_Y \in \tau_Y$, then $[y, \rightarrow)_Y \in \lambda_Y$. Let $[y, \rightarrow)_Y \in \tau_Y$, i.e., $(\leftarrow, y)_Y$ is closed in Y. We may assume $(\leftarrow, y)_Y \neq \emptyset$, otherwise obvious. Then it suffices to see that $(\leftarrow, y)_Y$ has a maximal element, say z, in Y, because of $[y, \rightarrow)_Y = (z, \rightarrow)_Y \in \lambda_Y$. Let $y = f(a)$ with $a \in X^\diamond$. We consider 3 cases.

Case 1. $a \in X$.

In this case, note $y = f(a) = a \in X$. Since $(\leftarrow, y)_Y$ is non-empty closed in $\langle Y, \tau_Y \rangle$, $y \notin (\leftarrow, y)_Y$ and $\tau_Y = \lambda_L \restriction Y$, we can find $t \in L$ with $t <_L y$ and

$(*)$ $\quad (t, y)_L \cap (\leftarrow, y)_Y = \emptyset$,

where $(t, y)_L$ denotes the interval in L. Whenever $t \in Y$, the element t is a maximal element of $(\leftarrow, y)_Y$. Therefore we may assume $t \notin Y$. Since X is closed in L and $t \notin Y \supset X$, we can find $s \in L$ with $s <_L t$ and $(s, t)_L \cap X = \emptyset$. Now by $t \notin Y$ and $(*)$, we see $(\leftarrow, y)_L \cap X \subset (\leftarrow, s]_L$. Note $(\leftarrow, y)_L \cap X = (\leftarrow, y)_X$ because of $y \in X$. We consider further 2 subcases.

Subcase 1 in Case 1. $(\leftarrow, y)_L \cap X$ has a maximal element x.

It follows from $x <_X y$, $(x, y)_X = \emptyset$ and Lemma 2.1 (7) that $(x, y)_{X^\diamond} = \emptyset$. Since f is an order isomorphism between $X^\diamond$ and Y, we see $(x, y)_Y = (f(x), f(y))_Y = \emptyset$, which means that x is a maximal element of $(\leftarrow, y)_Y$.

Subcase 2 in Case 1. $(\leftarrow, y)_L \cap X$ has no maximal elements.

In this case, since $(t, \rightarrow)_L$ is a neighborhood of y disjoint from $(\leftarrow, y)_Y \supset (\leftarrow, y)_X$, we see that $(\leftarrow, y)_X$ is closed in X. If $(\leftarrow, y)_X = \emptyset$ were true, then by $y = \min X = \min X^\diamond$, we see $(\leftarrow, y)_Y = (\leftarrow, f(y))_Y = \emptyset$, a contradiction. Thus $(\leftarrow, y)_X$ is non-empty and has no maximal elements, which shows $y \in X^-$. Then y^{-1} is the immediate predecessor of y in $X^\diamond$, therefore $f(y^{-1})$ is the immediate predecessor of $f(y) = y$ in Y, which means that $f(y^{-1})$ is a maximal element of $(\leftarrow, y)_Y$.

Case 2. $a = x^{+n}$ for some $x \in X^+$ and $n \in \mathbb{N}$.

In this case, $y = f(a) = f(x^{+n}) = x_n$ holds, where $\{x_n : n \in \mathbb{N}\}$ is the fixed sequence given in Claim 1. Then we see that $(\leftarrow, y)_Y = (\leftarrow, x_n)_Y = (\leftarrow, x_{n-1}]_Y$, therefore x_{n-1} is a maximal element of $(\leftarrow, y)_Y$.

Case 3. $a = x^{-n}$ for some $x \in X^-$ and $n \in \mathbb{N}$.

In this case, $y = f(a) = f(x^{-n}) = x_n$ holds, where $\{x_n : n \in \mathbb{N}\}$ is the fixed sequence given in Claim 2. Then x_{n+1} is a maximal element of $(\leftarrow, y)_Y$.

This completes the proof of Claim 3.

$\square$

3. Two lexicographic GO-spaces

Let $\{X_\alpha : \alpha < \gamma\}$ be a sequence of GO-spaces, set $X = \prod_{\alpha<\gamma} X_\alpha$ and let $<_X$ be the lexicographic order on X. Definitions 1.1 and 1.2 give two lexicographic GO-topologies $\lambda_{\hat{X}} \restriction X$ and $\lambda_{\check{X}} \restriction X$ on $\langle X, <_X \rangle$, where $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ and $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$. In this section, we will see that these two topologies on X coincide.

The following lemma is proved in [7, Lemma1.2].

Lemma 3.1. ([7, Lemma 1.2]) *Let $\{X_\alpha : \alpha < \gamma\}$ be a sequence of GO-spaces and $x \in X = \prod_{\alpha<\gamma} X_\alpha$. The following are equivalent:*

(1) *$x \in X^+_{\lambda_{\hat{X}} \restriction X}$, where $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ and $\lambda_{\hat{X}} \restriction X$ is the subspace topology of the order topology $\lambda_{\hat{X}}$,*
(2) *there is $\alpha_0 < \gamma$ such that:*
  (i) *$x(\alpha_0) \in X^+_{\alpha_0}$,*
  (ii) *for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has a maximal element $\max X_\alpha$ and $x(\alpha) = \max X_\alpha$.*

Also changing $+$ by $-$ in the lemma above, we get the analogous result Lemma 1.3 of [7], we refer this as an analogous lemma of Lemma 3.1. Now we will see that in Lemma 3.1, $\hat{X}$ and X^*_α can be changed by $\check{X}$ and $X^\diamond_\alpha$, respectively. Its proof is similar to Lemma 3.1, but since there are some difficulty in this case, we write down its proof.

Lemma 3.2. *Let $\{X_\alpha : \alpha < \gamma\}$ be a sequence of GO-spaces and $x \in X = \prod_{\alpha<\gamma} X_\alpha$. The following are equivalent:*

(1) *$x \in X^+_{\lambda_{\check{X}} \restriction X}$, where $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$ and $\lambda_{\check{X}} \restriction X$ is the subspace topology of the order topology $\lambda_{\check{X}}$,*
(2) *there is $\alpha_0 < \gamma$ such that:*
  (i) *$x(\alpha_0) \in X^+_{\alpha_0}$,*
  (ii) *for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has a maximal element $\max X_\alpha$ and $x(\alpha) = \max X_\alpha$.*

Proof. Let $<_X$ and $<_{\check{X}}$ be the lexicographic orders on $X = \prod_{\alpha<\gamma} X_\alpha$ and $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$ respectively, moreover λ_X and $\lambda_{\check{X}}$ the order topologies on the ordered sets $\langle X, <_X \rangle$ and $\langle \check{X}, <_{\check{X}} \rangle$ respectively. Furthermore let $\tau_X = \lambda_{\check{X}} \restriction X$ and for notational simplicity, put $X^+ = X^+_{\tau_X} = X^+_{\lambda_{\check{X}} \restriction X}$.

(1) $\Rightarrow$ (2) Let $x \in X^+$, then by $(\leftarrow, x]_X \in \tau_X \setminus \lambda_X$, $(x, \rightarrow)_X$ is non-empty and has no minimal elements with respect the order $<_X$. Moreover by $(\leftarrow, x]_X \in \tau_X = \lambda_{\check{X}} \restriction X$, we can find $y \in \check{X}$ with $x <_{\check{X}} y$ such

that $(x,y)_{\check{X}} \cap X = \emptyset$. It follows from $x <_{\check{X}} y$ that for some $\alpha_0 < \gamma$, $x \restriction \alpha_0 = y \restriction \alpha_0$ and $x(\alpha_0) <_{X^\diamond_{\alpha_0}} y(\alpha_0)$ hold.

Claim 1. For every $\alpha < \gamma$ with $\alpha_0 < \alpha$, a maximal element $\max X_\alpha$ exists and $x(\alpha) = \max X_\alpha$.

Proof. Assuming that there is $\alpha < \gamma$ with $\alpha_0 < \alpha$ such that $(x(\alpha), \rightarrow)_X$ is not empty, let $\alpha_1 = \min\{\alpha > \alpha_0 : (x(\alpha), \rightarrow)_X \neq \emptyset\}$. Taking $v \in (x(\alpha_1), \rightarrow)_X$, let $a = (x \restriction \alpha_1)^\wedge\langle v\rangle^\wedge(x \restriction (\alpha_1, \gamma))$. Then we have $a \in (x,y)_{\check{X}} \cap X$, a contradiction. This completes the proof of Claim 1.

Claim 2. $(x(\alpha_0), y(\alpha_0))_{X^\diamond_{\alpha_0}} \cap X_{\alpha_0} = \emptyset$ holds.

Proof. Assuming that there is $u \in (x(\alpha_0), y(\alpha_0))_{X^\diamond_{\alpha_0}} \cap X_{\alpha_0}$, let $a = (x \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha_0, \gamma))$. Then we have $a \in (x,y)_{\check{X}} \cap X$, a contradiction. This completes the proof of Claim 2.

From Claim 2, we see $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} = (\leftarrow, y(\alpha_0))_{X^\diamond_{\alpha_0}} \cap X_{\alpha_0} \in \lambda_{X^\diamond_{\alpha_0}} \restriction X_{\alpha_0} = \tau_{X_{\alpha_0}}$, where $\tau_{X_{\alpha_0}}$is the original GO-topology on X_{α_0} and $\lambda_{X^\diamond_{\alpha_0}}$ is the order topology on $X^\diamond_{\alpha_0}$.

Claim 3. $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$ holds, where $\lambda_{X_{\alpha_0}}$ is the order topology on X_{α_0}.

Proof. Assume $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, then there is $u \in X_{\alpha_0}$ with $x(\alpha_0) <_{X_{\alpha_0}} u$ such that $(x(\alpha_0), u)_{X_{\alpha_0}} = \emptyset$. Then by Lemma 2.1 (7), we have $(x(\alpha_0), u)_{X^\diamond_{\alpha_0}} = \emptyset$, thus by Claim 2, we see $y(\alpha_0) = u$.

We check the following fact.

Fact. For every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $(\leftarrow, y(\alpha))_{X^\diamond_\alpha} = \emptyset$, that is, $y(\alpha) = \min X^\diamond_\alpha = \min X_\alpha$.

Proof. Assuming $(\leftarrow, y(\alpha))_{X^\diamond_\alpha} \neq \emptyset$ for some $\alpha < \gamma$ with $\alpha_0 < \alpha$, let $\alpha_1 = \min\{\alpha > \alpha_0 : (\leftarrow, y(\alpha))_{X^\diamond_\alpha} \neq \emptyset\}$.

When X_{α_1} has a minimal element, let $v = \min X_{\alpha_1}$, then $v \in (\leftarrow, y(\alpha_1))_{X^\diamond_{\alpha_1}} \cap X_{\alpha_1}$. When X_{α_1} have no minimal elements, then by Lemma 2.1 (6), we can find $v' \in X_{\alpha_1}$ with $v' \leq_{X^\diamond_{\alpha_1}} y(\alpha_1)$. Then taking $v \in X_{\alpha_1}$ with $v <_{X_{\alpha_1}} v'$, we see $v \in (\leftarrow, y(\alpha_1))_{X^\diamond_{\alpha_1}} \cap X_{\alpha_1}$. In either cases, we see $v \in (\leftarrow, y(\alpha_1))_{X^\diamond_{\alpha_1}} \cap X_{\alpha_1}$. It follows from $x \in X$, $y(\alpha_0) = u \in X_{\alpha_0}$, $y(\alpha) = x(\alpha)$ for every $\alpha < \alpha_0$ and $y(\alpha) = \min X_\alpha$ for every $\alpha < \gamma$ with $\alpha_0 < \alpha < \alpha_1$ that $a = (y \restriction \alpha_1)^\wedge\langle v\rangle^\wedge(x \restriction (\alpha_1, \gamma)) \in X$. Then obviously $a \in (x,y)_{\check{X}} \cap X$, a contradiction. This completes the proof of Fact.

This fact shows that $y(\alpha) = \min X_\alpha$ for every $\alpha < \gamma$ with $\alpha_0 < \alpha$. It follows from $x \in X$, $x \restriction \alpha_0 = y \restriction \alpha_0$, $y(\alpha_0) = u \in X_{\alpha_0}$ and $y \restriction (\alpha_0, \gamma) = \langle \min X_\alpha : \alpha > \alpha_0\rangle$ that $x <_X y \in X$. Since by $x \in X^+$, $(x, \rightarrow)_X$

has no minimal elements, we can find $z \in (x,y)_X = (x,y)_{\check{X}} \cap X$, a contradiction.

This completes the proof of Claim 3.

Thus we have seen $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \in \tau_{X_{\alpha_0}} \setminus \lambda_{X_{\alpha_0}}$, that is, $x(\alpha_0) \in X^+_{\alpha_0}$. The proof of (1) $\Rightarrow$ (2) is complete.

(2) $\Rightarrow$ (1) Assume (2). We will see $x \in X^+$, i.e., $(\leftarrow, x]_X \in \tau_X \setminus \lambda_X$. It follows from (i) of (2) that $x(\alpha_0) <_{X^\diamond_{\alpha_0}} x(\alpha_0)^{+1} \in X^\diamond_{\alpha_0} \setminus X_{\alpha_0}$. Letting $y = (x \restriction \alpha_0)^\wedge \langle x(\alpha_0)^{+1}\rangle^\wedge (x \restriction (\alpha_0, \gamma))$, we see $x <_{\check{X}} y \in \check{X} \setminus X$. It follows from $x \restriction \alpha_0 = y \restriction \alpha_0$, $x(\alpha_0) <_{X^\diamond_{\alpha_0}} x(\alpha_0)^{+1} = y(\alpha_0) \notin X_{\alpha_0}$ and $x \restriction (\alpha_0, \gamma) = y \restriction (\alpha_0, \gamma) = \langle \max X_\alpha : \alpha > \alpha_0 \rangle$ that $(\leftarrow, x]_X = (\leftarrow, y)_{\check{X}} \cap X \in \lambda_{\check{X}} \restriction X = \tau_X$. Therefore the following claim completes the proof of the lemma.

Claim 4. $(\leftarrow, x]_X \notin \lambda_X$.

Proof. Assume $(\leftarrow, x]_X \in \lambda_X$. By $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$, $(x(\alpha_0), \rightarrow)_{X_{\alpha_0}}$ is non-empty, so take $u \in X_{\alpha_0}$ with $x(\alpha_0) <_{X_{\alpha_0}} u$. By letting $a = (x \restriction \alpha_0)^\wedge \langle u \rangle^\wedge (x \restriction (\alpha_0, \gamma))$, we see $a \in (x, \rightarrow)_X$, so $(x, \rightarrow)_X$ is non-empty. Therefore by $(\leftarrow, x]_X \in \lambda_X$, there is $x' \in X$ with $x <_X x'$ and $(x, x')_X = \emptyset$. Take $\alpha_1 < \gamma$ with $x \restriction \alpha_1 = x' \restriction \alpha_1$ and $x(\alpha_1) <_{X_{\alpha_1}} x'(\alpha_1)$. Because of $x(\alpha) = \max X_\alpha$ for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, we see $\alpha_1 \leq \alpha_0$. If $\alpha_1 < \alpha_0$ were true, then $(x \restriction \alpha_0)^\wedge \langle u \rangle^\wedge (x \restriction (\alpha_0, \gamma)) \in (x, x')_X$, a contradiction. Thus we have $\alpha_1 = \alpha_0$, so $x(\alpha_0) <_{X_{\alpha_1}} x'(\alpha_0)$. Since by $x(\alpha_0) \in X^+_{\alpha_0}$, $(x(\alpha_0), \rightarrow)_{X_{\alpha_0}}$ is non-empty and has no minimal elements, we can take $v \in (x(\alpha_0), x'(\alpha_0))_{X_{\alpha_0}}$. Then we see $(x \restriction \alpha_0)^\wedge \langle v \rangle^\wedge (x \restriction (\alpha_0, \gamma)) \in (x, x')_X$, a contradiction. $\square$

In Lemma 3.2, changing $+$ and $\max X_\alpha$ by $-$ and $\min X_\alpha$ respectively, we get an analogeous lemma of Lemma 3.2. Remark that (2) of Lemma 3.1 is the same as to (2) of Lemma 3.2, therefore we see that (1) of Lemma 3.1 is equivalent to (1) of Lemma 3.2, that is, $X^+_{\lambda_{\hat{X}} \restriction X} = X^+_{\lambda_{\check{X}} \restriction X}$. Similarly we see $X^-_{\lambda_{\hat{X}} \restriction X} = X^-_{\lambda_{\check{X}} \restriction X}$. Since a GO-topology τ_X on a ordered set X is determined by its order, $X^+_{\tau_X}$ and $X^-_{\tau_X}$, that is, $\tau_X = \tau(X^+_{\tau_X}, X^-_{\tau_X})$, we have $\lambda_{\check{X}} \restriction X = \tau(X^+_{\lambda_{\check{X}} \restriction X}, X^-_{\lambda_{\check{X}} \restriction X}) = \tau(X^+_{\lambda_{\hat{X}} \restriction X}, X^-_{\lambda_{\hat{X}} \restriction X}) = \lambda_{\hat{X}} \restriction X$. We have shown the following theorem.

Theorem 3.3. *Let $\{X_\alpha : \alpha < \gamma\}$ be a sequence of GO-spaces and consider $X = \prod_{\alpha<\gamma} X_\alpha$ as an ordered set with the lexicographic order $<_X$. Moreover let consider the two lexicographic LOTS' $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ and $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$.*

Then the subspace X of $\hat{X}$ is same as to the subspace X of $\check{X}$, that is, the GO-topology on X defined in Definition 1.1 is equal to the GO-topology on X defined in Definition 1.2.

4. $X^\diamond$ can be embedded in $\check{X}$

In this section, we shall prove that for a lexicographic GO-space $X = \prod_{\alpha<\gamma} X_\alpha$, $X^\diamond$ can be embedded in $\check{X}$, more precisely:

.

Theorem 4.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic GO-space. Then there are a subspace Y of the lexicographic LOTS $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$ with $X \subset Y$ and an order isomorphism f on $X^\diamond$ onto Y which is a homeomorphism satisfying $f \restriction X = 1_X$.*

Proof. It follows from Lemma 3.2 that for $x \in X^+$, we can fix $\alpha_x < \gamma$ such that $x(\alpha_x) \in X^+_{\alpha_x}$ and for every $\alpha < \gamma$ with $\alpha_x < \alpha$, $x(\alpha) = \max X_\alpha$ hold. Analogously for $x \in X^-$, we can fix $\alpha_x < \gamma$ such that $x(\alpha_x) \in X^-_{\alpha_x}$ and for every $\alpha < \gamma$ with $\alpha_x < \alpha$, $x(\alpha) = \min X_\alpha$ hold.

For $a \in X^\diamond$, define $f(a) \in \check{X}$ as follows:

$$\begin{cases} a & \text{if } a \in X, \\ (x \restriction \alpha_x)^\wedge \langle x(\alpha_x)^{+n} \rangle^\wedge (x \restriction (\alpha_x, \gamma)) & \text{if } \exists x \in X^+ \exists n \in \mathbb{N} (a = x^{+n}), \\ (x \restriction \alpha_x)^\wedge \langle x(\alpha_x)^{-n} \rangle^\wedge (x \restriction (\alpha_x, \gamma)) & \text{if } \exists x \in X^- \exists n \in \mathbb{N} (a = x^{-n}), \end{cases}$$

Now letting $Y = f[X^\diamond]$, we will prove that f and Y are the desired.

Claim 1. The following hold.

(1) if $a = x^{+n}$ for some $x \in X^+$ and $n \in \mathbb{N}$, then $x = f(x) <_{\check{X}} f(a)$ and $(f(x), f(a))_{\check{X}} \cap X = \emptyset$,

(2) if $a = x^{-n}$ for some $x \in X^-$ and $n \in \mathbb{N}$, then $f(a) <_{\check{X}} f(x) = x$ and $(f(a), f(x))_{\check{X}} \cap X = \emptyset$,

Proof. (1) Let $a = x^{+n}$ for some $x \in X^+$ and $n \in \mathbb{N}$. By the definition of f, $f(x) <_{\check{X}} f(a)$ is obvious. Assuming that there is $y \in (f(x), f(a))_{\check{X}} \cap X$, let $\alpha_0 = \min\{\alpha < \gamma : y(\alpha) \neq f(x)(\alpha)\}$. By $x \restriction \alpha_x = f(x) \restriction \alpha_x = f(a) \restriction \alpha_x$ and $f(x) \restriction (\alpha_x, \gamma) = x \restriction (\alpha_x, \gamma) = \langle \max X_\alpha : \alpha_x < \alpha < \gamma \rangle$, we see $\alpha_0 = \alpha_x$ and $x(\alpha_x) <_X y(\alpha_x) \in X_{\alpha_x}$. Now it follows from Lemma 2.1 (3) that $x(\alpha_x)^{+n} <_{X^\diamond_{\alpha_x}} y(\alpha_x)$. This says $f(a) <_{\check{X}} y$, a contradiction. Thus we see $(f(x), f(a))_{\check{X}} \cap X = \emptyset$.

(2) is similar.

This completes the proof of Claim 1.

Claim 2. $f : X^\diamond \to Y$ is order preserving.

Proof. Assuming $a, b \in X^\diamond$ with $a <_{X^\diamond} b$, we will show $f(a) <_{\check{X}} f(b)$ by case-by-case.

Case 1. $a, b \in X^\diamond$.

This case is obvious because of $f(a) = a <_X b = f(b)$.

Case 2. $a = x^{+n}$ for some $x \in X^+$ and $n \in \mathbb{N}$, and $b \in X$.

By $x^{+n} = a <_{X^\diamond} b$, $x \in X^+ \subset X$ and $b \in X$, applying Lemma 2.1 (3), we see $x <_X b$. Then there is $\alpha_0 < \gamma$ with $x \restriction \alpha_0 = b \restriction \alpha_0$ and $x(\alpha_0) <_{X_{\alpha_0}} b(\alpha_0)$. It follows from $x \restriction (\alpha_x, \gamma) = \langle \max X_\alpha : \alpha_x < \alpha < \gamma \rangle$ that $\alpha_0 \le \alpha_x$.

If $\alpha_0 < \alpha_x$ holds, then by $f(a) = f(x^{+n}) = (x \restriction \alpha_x)^\wedge \langle x(\alpha_x)^{+n} \rangle^\wedge (x \restriction (\alpha_x, \gamma))$, we have $f(a) \restriction \alpha_0 = x \restriction \alpha_0 = b \restriction \alpha_0 = f(b) \restriction \alpha_0$ and $f(a)(\alpha_0) = x(\alpha_0) <_{X_{\alpha_0}} b(\alpha_0)$, therefore we see $f(a) <_{\check{X}} b = f(b)$.

Let consider the remaining case "$\alpha_0 = \alpha_x$". It follows from $X^+_{\alpha_0} \ni x(\alpha_0) <_{X_{\alpha_0}} b(\alpha_0) \in X_{\alpha_0}$ and Lemma 2.1 (3) that $f(a)(\alpha_0) = x(\alpha_0)^{+n} <_{X_{\alpha_0}^\diamond} b(\alpha_0)$. Therefore by $f(a) \restriction \alpha_0 = x \restriction \alpha_0 = b \restriction \alpha_0$, we see $f(a) <_{\check{X}} b = f(b)$.

Using Lemma 2.1, we can also see other cases, so we leave it to the readers. This completes the proof Claim 2.

Since f is order preserving and onto Y, it is an order isomorphism. To see that f is homeomorphism, it suffices to see the subspace $Y = \langle Y, <_Y, \tau_Y \rangle$, where $<_Y = <_{\check{X}} \restriction Y$ and $\tau_Y = \lambda_{\check{X}} \restriction Y$, is, in fact, a LOTS.

Claim 3. The subspace $Y = \langle Y, <_Y, \tau_Y \rangle$ is a LOTS.

Proof. It suffices to see $Y^+_{\tau_Y} = \emptyset$ and $Y^-_{\tau_Y} = \emptyset$. We only show $Y^-_{\tau_Y} = \emptyset$. For short, let $Y^- = Y^-_{\tau_Y}$. Assume that an element $y \in Y^-$ exists, then note $[y, \rightarrow)_Y \in \tau_Y \setminus \lambda_Y$, where λ_Y denotes the order topology on Y with respect to the order $<_Y$. So the following properties hold.

[A] $(\leftarrow, y)_Y$ is non-empty and has no-maximal elements.

[B] there is $z \in \check{X}$ such that $z <_{\check{X}} y$ and $(z, y)_{\check{X}} \cap Y = \emptyset$.

We consider 3 cases.

Case 1. $f(x) = y$ for some $x \in X$, i.e., $x = y$.

In this case, from [B], take $\alpha_0 < \gamma$, $z \restriction \alpha_0 = y \restriction \alpha_0$ and $z(\alpha_0) <_{X_{\alpha_0}^\diamond} y(\alpha_0)$. We will prove several facts.

Fact 1. $(z(\alpha_0), y(\alpha_0))_{X_{\alpha_0}^\diamond} \cap X_{\alpha_0} = \emptyset$.

Proof. If there were $u \in (z(\alpha_0), y(\alpha_0))_{X^{\diamond}_{\alpha_0}} \cap X_{\alpha_0}$, then letting $a = (y \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(y \restriction (\alpha_0, \gamma))$, we see $a \in (z, y)_{\check{X}} \cap X \subset (z, y)_{\check{X}} \cap Y$, which contradicts [B]. This completes the proof of Fact 1.

Fact 2. $z \notin Y$.

Proof. If $z \in Y$ were true, then by $(z, y)_{\check{X}} \cap Y = \emptyset$, we see $z = \max(\leftarrow, y)_Y$, which contradicts [A]. This completes the proof of Fact 2.

Fact 3. For every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $(\leftarrow, y(\alpha))_{X_\alpha} = \emptyset$, that is, $y(\alpha) = \min X_\alpha$ $(= \min X^{\diamond}_{\alpha})$.

Proof. Assuming $(\leftarrow, y(\alpha))_{X_\alpha} \neq \emptyset$ for some $\alpha < \gamma$ with $\alpha_0 < \alpha$, let $\alpha_1 = \min\{\alpha > \alpha_0 : \exists u \in X_\alpha(u <_{X_\alpha} y(\alpha))\}$ and take $u \in X_{\alpha_1}$ with $u <_{X_{\alpha_1}} y(\alpha_1)$. Then letting $a = (y \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(y \restriction (\alpha_1, \gamma))$, we see $a \in (z, y)_{\hat{X}} \cap X \subset (z, y)_{\hat{X}} \cap Y$, which contradicts [B]. This completes the proof of Fact 3.

Fact 4. $z(\alpha_0) \notin X_{\alpha_0}$.

Proof. Assuming $z(\alpha_0) \in X_{\alpha_0}$, by $z \notin Y \supset X$, let $\alpha_1 = \min\{\alpha < \gamma : z(\alpha) \notin X_\alpha\}$. It follows from $y = x \in X, y \restriction \alpha_0 = z \restriction \alpha_0$ and $z(\alpha_0) \in X_{\alpha_0}$ that $\alpha_1 > \alpha_0$. Also it follows from $y(\alpha_1) = \min X_{\alpha_1}$ and $z(\alpha_1) \notin X_{\alpha_1}$ that $y(\alpha_1) <_{X^{\diamond}_{\alpha_1}} z(\alpha_1)$. Because of $z(\alpha_1) \in X^{\diamond}_{\alpha_1} \setminus X_{\alpha_1}$, we consider 2 cases in this fact.

Case 1 in Fact 4. There are $u \in X^{+}_{\alpha_1}$ and $n \in \mathbb{N}$ with $z(\alpha_1) = u^{+n}$.

In this case, by $(u, \rightarrow)_{X_{\alpha_1}} \neq \emptyset$, take $v \in (u, \rightarrow)_{X_{\alpha_1}}$. From Lemma 2.1 (3), we see $z(\alpha_1) = u^{+n} <_{X^{\diamond}_{\alpha_1}} v$. By letting $a = (z \restriction \alpha_1)^\wedge\langle v\rangle^\wedge(y \restriction (\alpha_1, \gamma))$. Then we see $a \in (z, y)_{\check{X}} \cap X \subset (z, y)_{\check{X}} \cap Y$, which contradicts [B].

Case 2 in Fact 4. There are $u \in X^{-}_{\alpha_1}$ and $n \in \mathbb{N}$ with $z(\alpha_1) = u^{-n}$.

In this case, let $a = (z \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(y \restriction (\alpha_1, \gamma))$. Then we see $a \in (z, y)_{\check{X}} \cap X \subset (z, y)_{\check{X}} \cap Y$, which contradicts [B].

This completes the proof of Fact 4.

Fact 5. There are $u \in X^{-}_{\alpha_0}$ and $n \in \mathbb{N}$ with $z(\alpha_0) = u^{-n}$.

Proof. From Fact 4, in the following, either (a) or (b) holds:
(a) there are $u \in X^{-}_{\alpha_0}$ and $n \in \mathbb{N}$ with $z(\alpha_0) = u^{-n}$,
(b) there are $u \in X^{+}_{\alpha_0}$ and $n \in \mathbb{N}$ with $z(\alpha_0) = u^{+n}$.
Assume (b) holds, then since $(u, \rightarrow)_{X_{\alpha_0}}$ has no minimal elements and $y(\alpha_0) \in (u, \rightarrow)_{X_{\alpha_0}}$, we can take $v \in (u, y(\alpha_0))_{X_{\alpha_0}}$. Then by Lemma

2.1 (3), we see $v \in (z(\alpha_0), y(\alpha_0))_{X_{\alpha_0}{}^\diamond} \cap X_{\alpha_0}$, which contradicts Fact 1. Therefore (a) holds. This completes the proof of Fact 5.

Take u and n in Fact 5, then by Fact 1, we see $u = y(\alpha_0)$ and $z(\alpha_0) = y(\alpha_0)^{-n}$. Thus $\alpha_0 = \alpha_y$, where α_y is the ordinal with $y(\alpha_y) \in X^-_{\alpha_y}$ and $y(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_y$, given in the analogous lemma of Lemma 3.2. Now by the definition of f, we have $f(y^{-n}) = (y \restriction \alpha_0)^\wedge\langle z(\alpha_0)\rangle^\wedge(y \restriction (\alpha_0, \gamma))$. Whenever $n \geq 2$, we see $f(y^{-1}) \in (z, y)_{\check{X}} \cap Y$ which contradicts [B]. Whenever $n = 1$, $f(y^{-1})$ is a maximal element of $(\leftarrow, y)_Y$ $(= (\leftarrow, f(y))_Y)$, which contradicts [A].

Case 1 is finished.

Case 2. $f(x^{+n}) = y$ for some $x \in X^+$ and $n \in \mathbb{N}$.

Let α_x be the ordinal with $x(\alpha_x) \in X^+_{\alpha_x}$ and $x(\alpha) = \max X_\alpha$ for every $\alpha > \alpha_x$, given in Lemma 3.2. Then obviously $f(x^{+(n-1)})$ is a maximal element of $(\leftarrow, y)_Y$ $(= (\leftarrow, f(x^{+n}))_Y)$ which contradicts [A]. Case 2 is finished.

Case 3. $f(x^{-n}) = y$ for some $x \in X^-$ and $n \in \mathbb{N}$.

As in Case 2, similarly we see that $f(x^{-(n+1)})$ is a maximal element of $(\leftarrow, y)_Y$ $(= (\leftarrow, f(x^{-n}))_Y)$ which contradicts [A]. Case 3 is finished. □

5. "DENSE" VERSUS "CLOSED"

Remember that every GO-space is dense (closed) in X^* ($X^\diamond$, respectively). When $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic GO-space, that X is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ was characterized as follows.

Lemma 5.1. [7, Theorem 3.2] *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic GO-space. Then X is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ if and only if for every $\alpha < \gamma$ with $\alpha + 1 < \gamma$, X_α is a LOTS.*

For instance, the lexicographic Sorgenfrey square $\mathbb{S} \times \mathbb{S}$ is not dense in $\mathbb{S}^* \times \mathbb{S}^*$. It is natural to ask whether that a lexicographic GO-space $X = \prod_{\alpha<\gamma} X_\alpha$ is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ is equivalent that X is closed in $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$, or not. In this section, we characterize that a lexicographic GO-space $X = \prod_{\alpha<\gamma} X_\alpha$ is closed in $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$.

Theorem 5.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic GO-space and $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$. Then the following are equivalent:*

(1) *Y is closed in $\check{X}$, where $Y = f[X^\diamond]$ is the one given in Theorem 4.1.*

(2) *X is closed in $\check{X}$,*
(3) *the following hold:*
 (a) *if* $\sup J^- < \sup J^+$*, then for every* $\alpha < \gamma$ *with* $\sup J^- \leq \alpha < \sup J^+$*,* $X^+_\alpha = \emptyset$ *holds,*
 (b) *if* $\sup J^+ < \sup J^-$*, then for every* $\alpha < \gamma$ *with* $\sup J^+ \leq \alpha < \sup J^-$*,* $X^-_\alpha = \emptyset$ *holds,*

 where $J^- = \{\alpha < \gamma : X_\alpha$ *has no minimal elements.*$\}$*,* $J^+ = \{\alpha < \gamma : X_\alpha$ *has no maximal elements.*$\}$ *and for notational conveniences, we let* $\sup\emptyset = -1$ *(= the immediate predecessor of* 0*).*

Proof. (1) $\Rightarrow$ (2) Assume that Y is closed in $\check{X}$. Since f is a homeomorphism with $f \restriction X = 1_X$ and X is closed in $X^\diamond$, X is also closed in $\check{X}$.

(2) $\Rightarrow$ (3) Assume that X is closed in $\check{X}$. Since (b) is similar, we only show (a). On the contrary, assume that $\sup J^- < \sup J^+$ holds but there is $\alpha_0 < \gamma$ with $\sup J^- \leq \alpha_0 < \sup J^+$ with $X^+_{\alpha_0} \neq \emptyset$. We will get a contradiction. Take $u \in X^+_{\alpha_0}$ and let $\alpha_1 = \min\{\alpha > \alpha_0 : X_\alpha$ has no maximal elements.$\}$, where note that the existence of α_1 is due to $\alpha_0 < \sup J^+$. Fix $t \in \prod_{\alpha<\alpha_0} X_\alpha$, then note that by $\sup J^- \leq \alpha_0$, for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has a minimal element $\min X_\alpha$. Let $b = t^\wedge\langle u^{+1}\rangle^\wedge\langle \min X_\alpha : \alpha > \alpha_0\rangle$, then by $u^{+1} \notin X_{\alpha_0}$, we see $b \notin X$. The following claim contradicts that X is closed in $\check{X}$.

Claim 1. $b \in \mathrm{Cl}_{\check{X}} X$, where $\mathrm{Cl}_{\check{X}}$ denotes the closure in $\check{X}$.

Proof. Let $c \in \check{X}$ with $c <_{\check{X}} b$. It suffices to see $(c, b)_{\check{X}} \cap X \neq \emptyset$. Take $\alpha_2 < \gamma$ with $c \restriction \alpha_2 = b \restriction \alpha_2$ and $c(\alpha_2) <_{X^\diamond_{\alpha_2}} b(\alpha_2)$. Since $b(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$, we see $\alpha_2 \leq \alpha_0$. We consider 2 cases.

Case 1. $\alpha_2 < \alpha_0$.

Letting $a = t^\wedge\langle u\rangle^\wedge(b \restriction (\alpha_0, \gamma))$, we see $a \in (c, b)_{\check{X}} \cap X$.

Case 2. $\alpha_2 = \alpha_0$.

In this case, by $t = b \restriction \alpha_0 = c \restriction \alpha_0$ and $c(\alpha_0) <_{X^\diamond_{\alpha_0}} b(\alpha_0) = u^{+1}$, we see $c(\alpha_0) \leq_{X^\diamond_{\alpha_0}} u$. Whenever $c(\alpha_0) <_{X^\diamond_{\alpha_0}} u$, letting $a = t^\wedge\langle u\rangle^\wedge(b \restriction (\alpha_0, \gamma))$, we see $a \in (c, b)_{\check{X}} \cap X$. So assume $c(\alpha_0) = u$. Then by $\alpha_0 < \alpha_1$, we can take $v \in X_{\alpha_1}$ with $c(\alpha_1) <_{X^\diamond_{\alpha_1}} v$. Then letting $a = t^\wedge\langle u\rangle^\wedge\langle \max X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle^\wedge\langle v\rangle^\wedge(b \restriction (\alpha_1, \gamma))$, we see $a \in (c, b)_{\check{X}} \cap X$.

(3) $\Rightarrow$ (1) Assume (3). To see that Y is closed in $\check{X}$, let $z \in \check{X} \setminus Y$. We will find a neighborhood of z in $\check{X}$ disjoint from Y. If for some $x \in X^+$ and $n \in \mathbb{N}$, the point z belongs to the interval $U := (f(x^{+(n-1)}), f(x^{+n}))_{\check{X}}$, then U is a neighborhood of z in $\check{X}$ disjoint

from Y. Similarly if for some $x \in X^-$ and $n \in \mathbb{N}$, the point z belongs to the interval $(f(x^{-n}), f(x^{-(n-1)}))_{\check{X}}$, then it is a neighborhood of z in $\check{X}$ disjoint from Y. Therefore we may assume the following property [C],

$$[C]\ z \notin \bigcup_{x \in X^+, n \in \mathbb{N}} (f(x^{+(n-1)}), f(x^{+n}))_{\check{X}} \cup \bigcup_{x \in X^+, n \in \mathbb{N}} (f(x^{-n}), f(x^{-(n-1)}))_{\check{X}}.$$

Now by $z \notin Y \supset X$, let $\alpha_0 = \min\{\alpha < \gamma : z(\alpha) \notin X_\alpha\}$. Then from $z(\alpha_0) \notin X_{\alpha_0}$, we may assume that there are $u \in X^+_{\alpha_0}$ and $n \in \mathbb{N}$ with $z(\alpha_0) = u^{+n}$.

Claim 2. There is $\alpha < \gamma$ with $\alpha_0 < \alpha$ such that X_α has no maximal elements.

Proof. Assume that for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has a maximal element $\max X_\alpha$. Let $x = (z \restriction \alpha_0)^\wedge\langle u\rangle^\wedge\langle \max X_\alpha : \alpha > \alpha_0\rangle$. Then by Lemma 3.2, we see $x \in X^+$, where X^+ means $X^+_{\lambda_{\check{X}} \restriction X}$. It follows from the definition f in Theorem 4.1 that $f(x^{+n}) = (z \restriction \alpha_0)^\wedge\langle z(\alpha_0)\rangle^\wedge\langle \max X_\alpha : \alpha > \alpha_0\rangle$. Now we see $z \in (f(x^{+(n-1)}), f(x^{+n}))_{\check{X}}$ by $z \notin Y$, which contradicts [C]. This completes the proof of Claim 2.

Using Claim2, let $\alpha_1 = \min\{\alpha > \alpha_0 : X_\alpha$ has no maximal elements.$\}$ $(= \min\{\alpha \in J^+ : \alpha > \alpha_0\})$ and fix $u_1 \in X_{\alpha_1}$ with $z(\alpha_1) <_{X^\diamond_{\alpha_1}} u_1$.

Claim 3. $\alpha_0 < \sup J^-$.

Proof. If $\sup J^- \leq \alpha_0$ were true, then we have $\alpha_0 < \alpha_1 \in J^+$ and $X^+_{\alpha_0} = \emptyset$, which contradicts (a). This completes the proof of Claim 3.

Using Claimn 3, let $\alpha_2 = \min\{\alpha \in J^- : \alpha > \alpha_0\}$. Fix $u_2 \in X_{\alpha_2}$ with $u_2 <_{X^\diamond_{\alpha_2}} z(\alpha_2)$. Let $z_1 = (z \restriction \alpha_1)^\wedge\langle u_1\rangle^\wedge(z \restriction (\alpha_1, \gamma))$ and $z_2 = (z \restriction \alpha_2)^\wedge\langle u_2\rangle^\wedge(z \restriction (\alpha_2, \gamma))$. Then we see $z \in (z_2, z_1)_{\check{X}}$.

Claim 4. $(z_2, z_1)_{\check{X}} \subset \prod_{\alpha < \alpha_0} X^\diamond_\alpha \times \{u^{+n}\} \times \prod_{\alpha_0 < \alpha} X^\diamond_\alpha$.

Proof. This Claim is obvious because of $z_1 \restriction \alpha_0 = z_2 \restriction \alpha_0$ and $z_1(\alpha_0) = z_2(\alpha_0) = z(\alpha_0) = u^{+n}$. This completes the proof of Claim 4.

The following claim with Claim 4 shows that $(z_2, z_1)_{\check{X}}$ is a neighborhood of z in $\check{X}$ disjoint from Y, thus the proof of (3) $\Rightarrow$ (1) is finished.

Claim 5. $Y \cap \prod_{\alpha < \alpha_0} X^\diamond_\alpha \times \{u^{+n}\} \times \prod_{\alpha_0 < \alpha} X^\diamond_\alpha = \emptyset$.

Proof. Assume that there is $a \in X^\diamond$ with $f(a) \in \prod_{\alpha < \alpha_0} X^\diamond_\alpha \times \{u^{+n}\} \times \prod_{\alpha_0 < \alpha} X^\diamond_\alpha$. We consider 3 cases, and in each case, we will get a contradiction.

Case 1. $a \in X$.

In this case, by $f(a) = a$, we see $X_{\alpha_0} \not\ni u^{+n} = f(a)(\alpha_0) = a(\alpha_0) \in X_{\alpha_0}$, a contradiction.

Case 2. $a = x^{+m}$ for some $x \in X^+$ and $m \in \mathbb{N}$.

It follows from Lemma 3.2 that there is $\alpha_x < \gamma$ such that $x(\alpha_x) \in X^+_{\alpha_x}$ and for every $\alpha < \gamma$ with $\alpha_x < \alpha$, $x(\alpha) = \max X_\alpha$ holds, which says $\sup J^+ \leq \alpha_x$. By $\alpha_0 < \alpha_1 \in J^+$, we see $\alpha_0 < \alpha_1 \leq \alpha_x$. It follows from $f(a) = f(x^{+m}) = (x \restriction \alpha_x)^\wedge\langle x(\alpha_x)^{+m}\rangle^\wedge(x \restriction (\alpha_x, \gamma))$, we see $X_{\alpha_0} \not\ni u^{+n} = f(a)(\alpha_0) = x(\alpha_0) \in X_{\alpha_0}$, a contradiction.

Case 3. $a = x^{-m}$ for some $x \in X^-$ and $m \in \mathbb{N}$.

As in Case 2, there is $\alpha_x < \gamma$ such that $x(\alpha_x) \in X^-_{\alpha_x}$ and for every $\alpha < \gamma$ with $\alpha_x < \alpha$, $x(\alpha) = \min X_\alpha$ holds, which says $\alpha_0 < \alpha_2 \leq \sup J^- \leq \alpha_x$. It follows from $f(a) = f(x^{-m}) = (x \restriction \alpha_x)^\wedge\langle x(\alpha_x)^{-m}\rangle\ {}^\wedge(x \restriction (\alpha_x, \gamma))$ that $X_{\alpha_0} \not\ni u^{+n} = f(a)(\alpha_0) = x(\alpha_0) \in X_{\alpha_0}$, a contradiction. This completes the proof of Claim 4

. $\square$

Applying the theorem above, we immediately see:

Corollary 5.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic GO-space. If $\sup J^- = \sup J^+$ holds, then X is closed in $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$. In particular, the following hold.*

(1) *if all X_α's have both a minimal element and a maximal element, then X is closed in $\check{X}$.*
(2) *if all X_α's have neither minimal elements nor maximal elements, then X is closed in $\check{X}$.*

For instance, let $X = \mathbb{S} \times \mathbb{S}$ be the lexicographic Sorgenfrey square. Then X is closed in $\check{X} = \mathbb{S}^\diamond \times \mathbb{S}^\diamond$, where remember that X is not dense in $\hat{X} = \mathbb{S}^* \times \mathbb{S}^*$. This shows that the reverse implication in the following corollary does not hold.

Corollary 5.4. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic GO-space. If X is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, then X is closed in $\check{X} = \prod_{\alpha<\gamma} X^\diamond_\alpha$.*

Proof. Assume that X is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, then by Lemma 5.1, for every $\alpha < \gamma$ with $\alpha + 1 < \gamma$, X_α is a LOTS, that is, $X^+_\alpha \cup X^-_\alpha = \emptyset$.

Case 1. γ is limit.

In this case, all X_α's are LOTS, therefore $X = \check{X}$.

Case 2. $\gamma = \delta + 1$ for some ordinal δ.

It suffices to see (a) and (b) of Theorem 5.2 (3). To see (a), assume $\sup J^- < \sup J^+$. Note $\sup J^+ \leq \delta < \gamma$. Now for every $\alpha < \gamma$ with $\sup J^- \leq \alpha < \sup J^+$, it follows from $\alpha + 1 \leq \delta < \gamma$ that $X^+_\alpha = \emptyset$ holds, thus (a) holds. (b) is similar. $\square$

Example 5.5. Let $[0,1)_\mathbb{R}$ be the interval in $\mathbb{R}$ and $<$ its usual order. Moreover let X_0 be the GO-space $\langle [0,1)_\mathbb{R}, <, \tau(\emptyset, \{\frac{1}{2}\})\rangle$, X_1 the usual LOTS $[0,1)_\mathbb{R}$ and consider lexicographic GO-space $X = X_0 \times X_1$. Then $X^+_0 = \emptyset$, $X^-_0 = \{\frac{1}{2}\}$, $X^+_1 = X^-_1 = \emptyset$, $J^- = \emptyset$ and $J^+ = \{0,1\}$, therefore $\sup J^- = -1$ and $\sup J^+ = 1$. Obviously (b) of Theorem 5.2 (3) holds. Let $\alpha < \gamma$ satisfy $\sup J^- \leq \alpha < \sup J^+$, then α has to be 0 and $X^+_0 = \emptyset$, this shows that (a) of Theorem 5.2 (3) holds. Therefore X is closed in $\check{X} = X^\diamond_0 \times X^\diamond_1$. Note that X is not dense in $\hat{X} = X^*_0 \times X^*_1$ by Lemma 5.1. On the other hand, remark that $Y = X_1 \times X_0$ is dense in $\hat{Y} = X^*_1 \times X^*_0$ and closed in $\check{Y} = X^\diamond_1 \times X^\diamond_0$

Example 5.6. In the above example, change $X_0 = \langle [0,1)_\mathbb{R}, <, \tau(\emptyset, \{\frac{1}{2}\})\rangle$ by $X_0 = \langle [0,1)_\mathbb{R}, <, \tau(\{\frac{1}{2}\}, \emptyset)\rangle$. Then we can easily check that X is not closed in $\check{X} = X^\diamond_0 \times X^\diamond_1$.

References

[1] R. Engelking, *General Topology-Revized and completed ed.*. Herdermann Verlag, Berlin (1989).

[2] M. J. Faber, *Metrizability in generalized ordered spaces.*, Mathematical Centre Tracts, No. 53. Mathematisch Centrum, Amsterdam, 1974.

[3] Y. Hirata and N. Kemoto, *The weight of lexicographic products*, Top. Appl., 284 (2020) Article 107357.

[4] Y. Hirata and N. Kemoto, *A characterization of paracompactness of lexicographic products*, Top. Proc., 56 (2020) 85-95.

[5] Y. Hirata and N. Kemoto, *Cardinal functions on lexicographic products*, Top. Appl., 352 (2024), Paper No. 108954, 19 pp.

[6] Y. Hirata and N. Kemoto, *The Lindelöf property of lexicographic products*, Top. Appl., 367 (2025), Paper No. 109338, 26 pp.

[7] N. Kemoto, *Lexicographic products of GO-spaces*, Top. Appl., 232 (2017), 267-280.

[8] N. Kemoto, *Paracompactness of Lexicographic products of GO-spaces*, Top. Appl., 240 (2018)35-58.

[9] N. Kemoto, *Hereditary paracompactness of lexicographic products*, Top. Proc., 53 (2019) 301-307.

[10] N. Kemoto, *Countable compactness of lexicographic products of GO-spaces*, Comment. Math. Univ. Carol., 60 (2019) 421-439.

[11] N. Kemoto, *Completeness of lexicographic products*, Top. Proc., 58 (2021) 105-123.

[12] T. Miwa and N. Kemoto, *Linearly ordered extensions of GO-spaces*, Top. Appl., 54 (1993), 133-140.

Department of Mathematics, Oita University, Oita, 870-1192 Japan
E-mail address: nkemoto@oita-u.ac.jp