ORDERABILITY OF SPACES HAVING ORDERED DECOMPOSITIONS

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Abstract. The following may be well-known:

- the subspace $(0,1)\cup\{2\}$ of the usual real numbers $\mathbb{R}$ is the topological sum of two linearly ordered spaces, and well-known that there is no linear ordering of X whose open interval topology coincides with the topology of X.

In this paper, we consider when the topological sum of a pairwise disjoint collection $\mathcal{X}$ of ordered spaces are orderable. As corollaries, we see:

- whenever $\mathcal{X}$ contains infinitely many singletons or contains an infinite discrete space, its topological sum is orderable,
- whenever $\mathcal{X}$ contains at least one ordered space with a maximal element but without minimal elements, its topological sum is orderable,
- whenever $\mathcal{X}$ does not contain ordered spaces with both a maximal element and a minimal element, its topological sum is orderable,
- whenever $\mathcal{X}$ contains infinitely many ordered spaces with both a maximal element and a minimal element, its topological sum is orderable,
- whenever $\mathcal{X}$ consists of suborderable spaces, its topological sum is suborderable.

Let $<$ be a linear order on a set X, see [3, page 4]. The pair $\langle X,<\rangle$ is said to be a linearly ordered set or an ordered set, and usually simply denoted by X. So when we say "let X be an ordered set", we mean that $X\neq\emptyset$ and a linear order $<$ on X is already given is tacit understanding. An ordered set $\langle X,<\rangle$ has a natural topology $\lambda_<$, which is called an interval topology, generated by $\{(\leftarrow,x)_< : x\in X\}\cup\{(x,\rightarrow)_< : x\in X\}$ as a subbase, that is, the smallest topology containing it, where $(\leftarrow,x)_< = \{y\in X : y<x\}$, also $(x,\rightarrow)_<$ is similarly defined. So we can also consider an ordered set as a topological space with the interval

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topology, we say that the triple $\langle X, <, \lambda_< \rangle$ is an ordered space and denoted simply by X. A topological space X with a topology τ, which is also simply denoted by X, is said to be orderable if there is an order $<$ on X with $\tau = \lambda_<$, and such an order $<$ is called a compatible order of τ. Note that an orderable space can have many compatible orders. A topological space X is said to be suborderable if it is a subspace (in the topological sense) of some orderable spaces.

Let $\mathcal{X}$ be a pairwise disjoint collection of topological spaces, that is, $X \cap Y = \emptyset$ whenever $X \neq Y \in \mathcal{X}$. The topological space $\bigcup \mathcal{X}$ with the topology $\bigoplus_{X \in \mathcal{X}} \tau_X$ generated by $\bigcup_{X \in \mathcal{X}} \tau_X$ as a subbase is said to be the topological sum of $\mathcal{X}$, where τ_X is the topology on X. In this case, the topological sum is simply denoted by $\bigoplus \mathcal{X}$, or $\bigoplus_{\alpha \in A} X_\alpha$ when $\mathcal{X}$ is written as $\mathcal{X} = \{X_\alpha : \alpha \in A\}$. Because we consider the topological sum of a collection $\mathcal{X}$ of spaces, throughout the paper, we assume that $\mathcal{X}$ is non-empty and pairwise disjoint.

We will consider both collections of orderable spaces and collections of ordered spaces. A collection of ordered spaces naturally can be considered as a collection of orderable spaces. On the other hand, a collection of orderable spaces can be considered as a collection of ordered spaces by giving compatible orders. In this case, it will be important how to choose compatible orders.

When a topological space X is represented as a topological sum $X = \bigoplus \mathcal{X}$ for some collection $\mathcal{X}$ of ordered spaces, we say "$\mathcal{X}$ is an ordered decomposition of X". Obviously, if X is an orderable space, then considering X as an ordered space having a compatible order, the singleton $\{X\}$ is one of ordered decompositions of X. Thus an orderable space has at least one ordered decomposition.

It may be well-known that the topological sum $X = (0,1)_{\mathbb{R}} \oplus \{2\}$ (that is, the subspace $(0,1) \cup \{2\}$ in $\mathbb{R}$) is not orderable. For a proof, see Corollary 14. With the usual order, $\{(0,1)_{\mathbb{R}}, \{2\}\}$ is an ordered decomposition of X.

Also it is well-known that whenever $\mathcal{X}$ is a collection of ordered spaces having minimal elements or maximal elements, its topological sum is orderable, see [6, Lemma 5]. In this paper, we will consider when the topological sum of orderable/ordered spaces are orderable.

Definition 1. An ordered set X is said to be type 0 if it has neither minimal elements nor maximal elements with respect to the given order. An ordered set X is said to be type 1 if either it has a minimal element but not have maximal elements, or it has a maximal element but not have minimal elements. An ordered set X is said to be type 2 if it has both a minimal element and a maximal element. When X is a singleton, X is considered to be type 2. Note that the interval $(0,1)_{\mathbb{R}}$ above is type 0 and the singleton $\{2\}$ is type 2.

For a collection $\mathcal{X}$ of ordered spaces and $i \in 3$ $(= |0,1,2\})$, let

$$\mathcal{X}^i = \{X \in \mathcal{X} : X \text{ is type } i\ \}.$$

Then $\mathcal{X}$ is decomposed into $\mathcal{X}^0$, $\mathcal{X}^1$ and $\mathcal{X}^2$.

For an order $<$ on X, $<^{-1}$ denotes the reverse order on X, that is, $x <^{-1} y$ iff $y < x$. Note that the reverse order has the same type as the original type and does not change its interval topology.

Definition 2. Let $\mathcal{X}$ be a (pairwise disjoint, of course) collection of ordered spaces indexed as $\mathcal{X} = \{X_\alpha : \alpha < \kappa\}$ with a cardinal κ with $\kappa \geq 1$ and let $X = \bigcup_{\alpha<\kappa} X_\alpha$. Moreover let $<_\alpha$ be the order on X_α and $\lambda_{<_\alpha}$ its interval topology for each $\alpha < \kappa$. For each $x \in X$, let $\alpha(x)$ be the unique $\alpha < \kappa$ with $x \in X_\alpha$.

The symbol $\Sigma_{\alpha<\kappa} <_\alpha$ denotes the order $<$ on X defined by the following rule:

$$x < y \text{ iff } \begin{cases} x <_{\alpha(x)} y & \text{if } \alpha(x) = \alpha(y), \\ \alpha(x) < \alpha(y) & \text{otherwise.} \end{cases}$$

If $\kappa < \omega$, then $\Sigma_{\alpha<\kappa} <_\alpha$ is denoted by $<_0 + <_1 + \cdots + <_{\kappa-1}$. In particular, $<_0 + <_1$ denotes the resulting order on $X_0 \cup X_1$ adding the ordered space X_1 after the ordered space X_0. Similarly if $\kappa = \omega$, then $\Sigma_{\alpha<\kappa} <_\alpha$ is denoted by $<_0 + <_1 + <_2 + \cdots$. Moreover the ordered space $\langle X, \Sigma_{\alpha<\kappa} <_\alpha \rangle$ is also simply denoted by $\Sigma_{\alpha<\kappa} X_\alpha$ if contexts are clear.

The following lemma gives an equivalent condition of $\bigoplus_{\alpha<\kappa} X_\alpha = \Sigma_{\alpha<\kappa} X_\alpha$.

Lemma 3. *Let $\{X_\alpha : \alpha < \kappa\}$ be a collection of ordered spaces indexed by a cardinal κ with $\kappa \geq 1$. Let $<$ be the order $\Sigma_{\alpha<\kappa} <_\alpha$ defined above, where $<_\alpha$'s are orders on X_α's. Then $\lambda_< = \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$, that is, the topological sum $\bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$ is orderable by $<$, if and only if for every $\alpha < \kappa$, $X_\alpha \in \lambda_<$ holds.*

Proof. If $\lambda_< = \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$, then for every $\alpha < \kappa$, $X_\alpha \in \lambda_{<_\alpha} \subset \lambda_<$ holds.

Conversely assume that for every $\alpha < \kappa$, $X_\alpha \in \lambda_<$ holds. Note that the restriction $<\restriction X_\alpha$ of $<$ on X_α coincides with $<_\alpha$. To see $\lambda_< \subset \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$, let $x \in X$, where $X = \bigcup_{\alpha<\kappa} X_\alpha$. Then $(\leftarrow, x)_< = \bigcup_{\beta<\alpha(x)} X_\beta \cup (\leftarrow, x)_{<_{\alpha(x)}} \in \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$ and $(x, \rightarrow)_< = (x, \rightarrow)_{<_{\alpha(x)}} \cup \bigcup_{\alpha(x)<\beta} X_\beta \in \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$ hold. To see $\lambda_< \supset \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$, it suffices to see $\lambda_< \supset \lambda_{<_\alpha}$ for every $\alpha < \kappa$. Fix $x \in X_\alpha$. Then $(\leftarrow, x)_{<_\alpha} = (\leftarrow, x)_< \cap X_\alpha \in \lambda_<$ and $(x, \rightarrow)_{<_\alpha} = (x, \rightarrow)_< \cap X_\alpha \in \lambda_<$ hold. $\square$

Lemma 4. *The following hold.*

(1) *if* X_0 *and* X_1 *are two type* 0 *ordered spaces, then the topological sum* $X_0 \oplus X_1$ *is orderable by a type* 0 *order,*
(2) *if* X_0 *and* X_1 *are two type* 1 *ordered spaces, then the topological sum* $X_0 \oplus X_1$ *is orderable by a type* 0 *order,*
(3) *if* X_0 *and* X_1 *are two type* 2 *ordered spaces, then the topological sum* $X_0 \oplus X_1$ *is orderable by a type* 2 *order,*
(4) *if* X_0 *is a type* 0 *ordered space and* X_1 *is a type* 1 *ordered space, then the topological sum* $X_0 \oplus X_1$ *is orderable by a type* 1 *order,*
(5) *if* X_0 *is a type* 1 *ordered space and* X_1 *is a type* 2 *ordered space, then the topological sum* $X_0 \oplus X_1$ *is orderable by a type* 1 *order.*

Proof. (1) Let $<_0$ and $<_1$ be type 0 orders on X_0 and X_1 respectively, and let $<$ be the order $<_0 + <_1$. Obviously $<$ is type 0. By $X_0 = \bigcup_{x \in X_0} (\leftarrow, x)_< \in \lambda_<$ and $X_1 = \bigcup_{x \in X_1} (x, \rightarrow)_< \in \lambda_<$, we see that the topological sum $X_0 \oplus X_1$ is orderable by $<$.

(2) Let $<_0$ and $<_1$ be type 1 orders on X_0 and X_1 respectively. We consider $<_i^{-1}$ instead of $<_i$ if necessary, we may assume that both X_0 and X_1 have no minimal elements but has a maximal element. Let $<$ be the order $<_0 + <_1^{-1}$, then obviously $<$ is type 0. It follows from $X_0 = (\leftarrow, <_1^{-1}\text{-}\min X_1)_< \in \lambda_<$ and $X_1 = (<_0\text{-}\max X_0, \rightarrow)_< \in \lambda_<$ that the topological sum $X_0 \oplus X_1$ is orderable by $<$, where $<_1^{-1}\text{-}\min X_1$ and $<_0\text{-}\max X_0$ are the minimal element of X_1 with respect to the order $<_1^{-1}$ and the maximal element of X_0 with respect to the order $<_0$, respectively.

The remaining are similar, so we leave them to the reader. □

Remark 5. About (2) of the lemma, let $<_0$ be the usual order on the half open interval $X_0 = (0, 1]_{\mathbb{R}}$ in $\mathbb{R}$ and $<_1$ be the usual order on $X_1 = (2, 3]_{\mathbb{R}}$. Then both $<_0$ and $<_1$ are type 1, moreover the order $<$ defined by $<_0 + <_1$ is also type 1. However the order topology $\lambda_<$ does not induce the topological sum $X_0 \oplus X_1$, because the ordered space $\langle X_0 \cup X_1, < \rangle$ is homeomorphic to the interval $(0, 2]_{\mathbb{R}}$ in $\mathbb{R}$.

Note that the type 2 order $<_0^{-1} + <_1$ also induces the topological sum $X_0 \oplus X_1$, but in our discussion below, this order will not be so important.

Some pattern of the above lemma can be extended for further length.

Lemma 6. *Let* $\{X_\alpha : \alpha < \kappa\}$ *be a collection of type* 0 *ordered spaces indexed by a cardinal* κ *with* $\kappa \geq 1$*. Then the topological sum* $\bigoplus_{\alpha < \kappa} X_\alpha$ *is orderable by a type* 0 *order.*

Proof. Let $<$ be the order $\Sigma_{\alpha<\kappa} <_\alpha$, where $<_\alpha$ is the type 0 order on X_α. Then as in (1) of Lemma 4, we see $X_\alpha \in \lambda_<$ for every $\alpha < \kappa$ which shows $\lambda_< = \bigoplus_{\alpha<\kappa} \lambda_{<_\alpha}$. □

Lemma 7. *Let $\{X_\alpha : \alpha < \kappa\}$ be a collection of type 1 ordered spaces indexed by a cardinal κ with $\kappa \geq 1$. Then the following hold.*

(1) *if $\kappa = 2n+1$ for some $n \in \omega$, then the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ is orderable by a type 1 order,*

(2) *otherwise, the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ is orderable by a type 0 order,*

Proof. Let $<_\alpha$ be the type 1 order on X_α for each $\alpha < \kappa$, we may assume that each X_α has a $<_\alpha$-maximal element but not have $<_\alpha$-minimal elements.

(1) Let $<$ be the order $<_0 + <_1^{-1} + <_2 + \cdots + <_{2n-1}^{-1} + <_{2n}$. Since $<_{2k} + <_{2k+1}^{-1}$ is a type 0 order which induce the topological sum $X_{2k} \oplus X_{2k+1}$ for every $k < n$, the type 0 order $<_0 + <_1^{-1} + <_2 + \cdots + <_{2n-1}^{-1}$ induces the topological sum $X_0 \oplus X_1 \oplus \cdots \oplus X_{2n-1}$ by Lemma 6. Now by Lemma 4 (4), $<$ is a type 1 order which induces the topological sum $X_0 \oplus X_1 \oplus \cdots \oplus X_{2n}$.

(2) When $\kappa = \omega$, by Lemma 6, the type 0 order $\Sigma_{k<\omega}(<_{2k} + <_{2k+1}^{-1})$ induces the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$.

Assume $\kappa > \omega$. Note that every ordinal α can be represented as $\alpha = \omega \cdot \beta + n$ for a unique pair of ordinal β and $n \in \omega$, where $\omega \cdot \beta$ is the ordinal multiplication, that is, the order type of the lexicographic ordered set $\beta \times \omega$, see [5, I, Exercises (3)]. Obviously for each $\beta < \kappa$, the type 0 order $\Sigma_{k<\omega}(<_{\omega\cdot\beta+2k} + <_{\omega\cdot\beta+2k+1}^{-1})$ induces the topological sum $\bigoplus_{n\in\omega} X_{\omega\cdot\beta+n}$. Now it follows from Lemma 6 that the type 0 order $\Sigma_{\beta<\kappa}(\Sigma_{k<\omega}(<_{\omega\cdot\beta+2k} + <_{\omega\cdot\beta+2k+1}^{-1}))$ induces the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$. □

Lemma 8. *Let $\{X_\alpha : \alpha < \kappa\}$ be a collection of type 2 ordered spaces indexed by a cardinal κ with $\kappa \geq 1$. Then the following hold.*

(1) *if κ is finite, then the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ is orderable by a type 2 order,*

(2) *otherwise, the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ is orderable by a type 0 order, also is orderable by a type 1 order.*

Proof. (1) is similar to Lemma 4 (3).

(2) When $\kappa = \omega$, both the type 0 order $(\Sigma_{k<\omega} <_{2k})^{-1} + (\Sigma_{k<\omega} <_{2k+1})$ and the type 1 order $\Sigma_{n<\omega} <_n$ induce the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$.

Assume $\kappa > \omega$. As above, the type 0 order $(\Sigma_{k<\omega} <_{\omega\cdot\beta+2k})^{-1} + (\Sigma_{k<\omega} <_{\omega\cdot\beta+2k+1})$ induces the topological sum $\bigoplus_{n\in\omega} X_{\omega\cdot\beta+n}$ for each

$\beta < \kappa$. Now from Lemma 6 and Lemma 4, both the type 0 order $\Sigma_{\beta<\kappa}((\Sigma_{k<\omega} <_{\omega\cdot\beta+2k})^{-1} + (\Sigma_{k<\omega} <_{\omega\cdot\beta+2k+1}))$ and the type 1 order $\Sigma_{n<\omega} <_n +\Sigma_{1\leq\beta<\kappa}((\Sigma_{k<\omega} <_{\omega\cdot\beta+2k})^{-1} + (\Sigma_{k<\omega} <_{\omega\cdot\beta+2k+1}))$ induce the topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$. □

Lemma 9. *Let $\mathcal{X}$ be a non-empty collection of type 0 or 1 ordered spaces. Then the topological sum $\bigoplus \mathcal{X}$ is ordered by a type 0 or 1 order.*

Proof. For each $i \in 2 = \{0,1\}$, let enumerate $\mathcal{X}^i$ by a cardinal κ_i. If $\kappa_0 = 0$ or $\kappa_1 = 0$, then the conclusion is immediate, so we may assume $\kappa_0 \geq 1$ and $\kappa_1 \geq 1$. By Lemma 6, the topological sum $\bigoplus \mathcal{X}^0$ is ordered by a type 0 order. Moreover by Lemma 7, the topological sum $\bigoplus \mathcal{X}^1$ is ordered by a type 0 or 1 order. Now by Lemma 4 (1) or (4) , the topological sum $\bigoplus \mathcal{X} = \bigoplus \mathcal{X}^0 \oplus \bigoplus \mathcal{X}^1$ is ordered by a type 0 or 1 order. □

Theorem 10. *Let a space X can be written as a topological sum $X = \bigoplus \mathcal{Y}$ for some collection $\mathcal{Y}$ of orderable spaces. If $\mathcal{Y}$ satisfies either (1) or (2) below, then X is orderable,*

1. *there are $Y \in \mathcal{Y}$ and an ordered decomposition $\mathcal{Z}_Y$ of Y such that $\mathcal{Z}^1_Y \neq \emptyset$,*
2. *there is a sequence $\langle \mathcal{Z}_Y : Y \in \mathcal{Y}\rangle$ of ordered decompositions $\mathcal{Z}_Y$'s of Y's such that $\bigcup_{Y\in\mathcal{Y}} \mathcal{Z}^0_Y = \emptyset$, or $\bigcup_{Y\in\mathcal{Y}} \mathcal{Z}^2_Y$ is empty or infinite.*

Proof. Let $\mathcal{Y}$ be as above satisfying (1) or (2). For every $Y \in \mathcal{Y}$, by orderability of Y, one can fix a compatible order $<_Y$ on Y. Now $\mathcal{Y}$ can be considered as an ordered decomposition of X.

First assuming (1), take such $Y \in \mathcal{Y}$ and $\mathcal{Z}_Y$ with $\mathcal{Z}^1_Y \neq \emptyset$ in (1). Fix $Z^* \in \mathcal{Z}^1_Y$ and let $\mathcal{X} = (\mathcal{Y} \setminus \{Y\}) \cup (\mathcal{Z}_Y \setminus \{Z^*\})$. Then $\mathcal{X} \cup \{Z^*\}$ is an ordered decomposition of X. For each $i \in 3$, enumerate $\mathcal{X}^i$ by some cardinal. Lemmas 6, 7 and 8 show the following,

- $\bigoplus \mathcal{X}^0$ can be ordered by a type 0 order,
- $\bigoplus \mathcal{X}^1$ can be ordered by a type 0 order or a type 1 order,
- $\bigoplus \mathcal{X}^2$ can be ordered by a type 0 order or a type 2 order.

Then it follows from Lemma 4 (1) and (4) that $(\bigoplus \mathcal{X}^0) \oplus (\bigoplus \mathcal{X}^1)$ can be ordered by a type 0 order or a type 1 order.

Claim. $(\bigoplus \mathcal{X}^2) \oplus Z^*$ can be ordered by a type 1 order,

Proof. We may assume that Z^* has no minimal elments but has a maximal element. When $\bigoplus \mathcal{X}^2$ is type 0, add Z^* after $\bigoplus \mathcal{X}^2$. When $\bigoplus \mathcal{X}^2$ is type 2, add $\bigoplus \mathcal{X}^2$ after Z^*. Then in both cases, these odered spaces are type 1. This completes the proof of Claim.

Now since $(\bigoplus \mathcal{X}^0) \oplus (\bigoplus \mathcal{X}^1)$ is type 0 or 1 and $(\bigoplus \mathcal{X}^2) \oplus Z^*$ is type 1, Lemma 4 (2) (4) or Lemma 9 shows $X = ((\bigoplus \mathcal{X}^0) \oplus (\bigoplus \mathcal{X}^1)) \oplus ((\bigoplus \mathcal{X}^2) \oplus Z^*)$ is orderable.

Next assuming (2), let $\langle \mathcal{Z}_Y : Y \in \mathcal{Y} \rangle$ be a sequence of ordered decompositions in (2) and set $\mathcal{X} = \bigcup_{Y \in \mathcal{Y}} \mathcal{Z}_Y$. Then $\mathcal{X}$ is also an ordered decomposition of X and for every $i \in 3$, $\mathcal{X}^i = \bigcup_{Y \in \mathcal{Y}} \mathcal{Z}_Y^i$ holds. When $\mathcal{X}^1 \neq \emptyset$, by (1), we see that X is orderable. Therefore we assume $\mathcal{X}^1 = \emptyset$. We consider two cases.

Case 1. $\mathcal{X}^0 = \emptyset$.

In this case, since $\mathcal{X} = \mathcal{X}^2$, by Lemma 8, we see that X is orderable.

Case 2. $\mathcal{X}^0 \neq \emptyset$.

In this case, by (2), $\mathcal{X}^2$ is empty or infinite. When $\mathcal{X}^2$ is empty, by $\mathcal{X} = \mathcal{X}^0$, applying Lemma 6, see that X is orderable. Now we assume that $\mathcal{X}^2$ is infinite. It follows from Lemma 8 that $\bigoplus \mathcal{X}^2$ can be considered as a type 0 ordered space. Now $\mathcal{X}^0 \cup \{\bigoplus \mathcal{X}^2\}$ is considered as an ordered decomposition of X by type 0 ordered spaces, by Lemma 6, we see that X is orderable. $\square$

Applying the theorem above, we see the following, where note that discrete spaces are orderable.

Corollary 11. *Assume that a space X has an ordered decomposition $\mathcal{X}$. If $\mathcal{X}$ satisfies one of the following clauses, then X is orderable,*

(1) $\mathcal{X}^0 = \emptyset$, *see* [6, Lemma 5],
(2) $\mathcal{X}^1 \neq \emptyset$,
(3) $\mathcal{X}^2$ *is empty or infinite,*
(4) $\mathcal{X}$ *contains a type* 0 *ordered space with a jump,*
(5) $\mathcal{X}$ *contains a type* 2 *ordered space with a gap,*
(6) $\mathcal{X}$ *contains infinitely many singletons, or contains an infinite discrete space,*
(7) $\mathcal{X}$ *contains a space which is homeomorphic to either the rationals* $\mathbb{Q}$ *or the irrationals* $\mathbb{P}$.

Proof. (1) - (6) are easy. For (7), remark the following results:

• the space $\mathbb{Q}$ is characterized as the unique space that is non-empty, countable, metrizable, without isolated points [7],

• the space $\mathbb{P}$ is characterized as the unique space that is non-empty, zero-dimensional, separable, completely metrizable, nowhere locally compact (=no compact subset has non-empty interior) [1].

Using these characterizations, we see that $\mathbb{Q}$ and $\mathbb{P}$ are homeomorphic to the type 1 ordered spaces $(0,1]\cap\mathbb{Q}$ and $(0,\pi]\cap\mathbb{P}$ respectively. Now apply (2). $\square$

We will see that the reverse implication of Theorem 10 is true whenever X is locally connected. To see this, we need the following proposition which is a consequence of [2, Theorem II], however we present its easy and direct proof.

Proposition 12. [2] *Let $\langle X,<_0\rangle$ be an ordered set whose interval topology $\lambda_{<_0}$ is connected. Moreover let $<_1$ be an order on X whose interval topology $\lambda_{<_1}$ is weaker than $\lambda_{<_0}$, that is, $\lambda_{<_1}\subset\lambda_{<_0}$. Then the following hold.*

(1) *if there are $x_0,x_1\in X$ with $x_0<_0 x_1$ and $x_0<_1 x_1$, then $<_1=<_0$, that is, $x<_1 y$ iff $x<_0 y$,*
(2) *if there are $x_0,x_1\in X$ with $x_0<_0 x_1$ and $x_1<_1 x_0$, then $<_1=<_0^{-1}$.*

Thus the orders $<_0$ and $<_1$ have the same type and the topologies $\lambda_{<_0}$ and $\lambda_{<_1}$ coincide.

Proof. We only prove (1), because (2) is similar. Assume $x_0<_0 x_1$ and $x_0<_1 x_1$. For every $x\in X$, we prove the following Claims.

Claim 1. If $x<_0 x_0$, then $x<_1 x_0$.

Proof. Let $x<_0 x_0$. First assume $x_0<_1 x<_1 x_1$. From $X\setminus\{x\}=(\leftarrow,x)_{<_1}\oplus(x,\rightarrow)_{<_1}$ with $x_0\in(\leftarrow,x)_{<_1}$ and $x_1\in(x,\rightarrow)_{<_1}$, the connected set $[x_0,x_1]_{<_0}$ is covered by non-empty disjoint open sets $[x_0,x_1]_{<_0}\cap(\leftarrow,x)_{<_1}$ and $[x_0,x_1]_{<_0}\cap(x,\rightarrow)_{<_1}$, a contradiction.

Next assume $x_1<_1 x$. Then the connected set $[x,x_0]_{<_0}$ is covered by non-empty disjoint open sets $[x,x_0]_{<_0}\cap(\leftarrow,x_1)_{<_1}$ and $[x,x_0]_{<_0}\cap(x_1,\rightarrow)_{<_1}$, a contradiction. This completes the proof of Claim 1.

Similarly we see:

Claim 2. If $x_0<_0 x<_0 x_1$, then $x_0<_1 x<_1 x_1$.

Claim 3. If $x_1<_0 x$, then $x_1<_1 x$.

Moreover for every $x,y\in X$, similarly we can see:

Claim 4. If $x<_0 y<_0 x_0$, then $x<_1 y<_1 x_0$.

Claim 5. If $x_0<_0 x<_0 y<_0 x_1$, then $x_0<_1 x<_1 y<_1 x_1$.

Claim 6. If $x_1<_0 x<_0 y$, then $x_1<_1 x<_1 y$.

Now the claims above show $<_1=<_0$. $\square$

The above proposition says that a connected orederable space has two compatible orders so that one order is the reverse order of another one, so these types coincide.

Theorem 13. *Let a space X can be written as a topological sum $X = \bigoplus \mathcal{Y}$ for some collection $\mathcal{Y}$ of locally connected orderable spaces. Then X is orderable if and only if $\mathcal{Y}$ satisfies either* (1) *or* (2) *in Theorem 10.*

Proof. One direction follows from Theorem 10. So assume that X is orderable by an order $<_X$ and $\mathcal{Y}$ satisfies the negation of " (1) or (2)", we will get a contradiction. For each $Y \in \mathcal{Y}$, fixing a compatible order $<_Y$ on Y, let $\mathcal{Z}_Y$ be the set of all connected components of Y, that is, the set of all maximal connected subsets of Y. Since Y is locally connected, all members of $\mathcal{Z}_Y$ are clopen in Y.

Moreover fix $Z \in \mathcal{Z}_Y$. Let $<_Z$ be the restricted order $<_Y \restriction Z$ of the order $<_Y$ on Z. Since Z is convex in the ordered space $\langle Y, <_Y \rangle$ and clopen in X, the interval topology $\lambda_{<_Z}$ coincides with the subspace topology $\lambda_{<_Y} \restriction Z$ of the interval topology $\lambda_{<_Y}$ on Z. Also since Z is connected clopen subspace of X, by the convexity of Z in $\langle X, <_X \rangle$, we have $\lambda_{<_X \restriction Z} = \lambda_{<_X} \restriction Z = (\lambda_{<_X} \restriction Y) \restriction Z = \lambda_{<_Y} \restriction Z = \lambda_{<_Z}$. Since Z is connected, by the proposition above, we see that $<_X \restriction Z$ coincide with either $<_Z$ or $<_Z^{-1}$ and Z is a convex set in the ordered set $\langle X, <_X \rangle$. So the ordered sets $\langle Z, <_X \restriction Z \rangle$ and $\langle Z, <_Z \rangle$ have the same type.

Now let $\mathcal{Z} = \bigcup_{Y \in \mathcal{Y}} \mathcal{Z}_Y$, then $\mathcal{Z}$ is an ordered decomposition of X, where we consider the order $<_Z$ on Z for every $Z \in \mathcal{Z}$. By the negation of " (1) or (2)", we see that $\mathcal{Z}^0 \neq \emptyset$, $\mathcal{Z}^1 = \emptyset$ and $\mathcal{Z}^2$ is non-empty and finite, where $\mathcal{Z}^i = \{Z \in \mathcal{Z} : \langle Z, <_Z \rangle \text{ is type } i\}$. Enumerate $\mathcal{Z}^2$ as $\mathcal{Z}^2 = \{Z_k : k < n\}$ for some $n \in \omega$ with $1 \leq n$. Since it is finite, we may assume $Z_0 <_X Z_1 <_X \cdots <_X Z_{n-1}$, where $A <_X B$ means $a <_X b$ for every $a \in A$ and $b \in B$. For every $k < n$, because $\langle Z_k, <_{Z_k} \rangle$ is type 2, so is $\langle Z_k, <_X \restriction Z_k \rangle$, therefore Z_k has both a $<_X$-minimal element $<_X\text{-}\min Z_k$ and a $<_X$-maximal element $<_X\text{-}\max Z_k$ in X. Moreover fix $Z^* \in \mathcal{Z}^0$. Since $\{Z^*\} \cup \mathcal{Z}^2$ is a pairwise disjoint collection of convex sets in $\langle X, <_X \rangle$, we see either $Z^* <_X Z_{n-1}$ or $Z_0 <_X Z^*$. We may assume $Z^* <_X Z_{n-1}$. Take the smallest $k < n$ with $Z^* <_X Z_k$. It follows from $Z^* \in \mathcal{Z}^0$ that Z^* has neither $<_X$-minimal elements nor $<_X$-maximal elements. Now by the minimality of k, $(\leftarrow, <_X\text{-}\min Z_k)_{<_X}$ has no $<_X$-maximal elements in X. Since Z_k is clopen in the ordered space $\langle X, <_X \rangle$ with $<_X\text{-}\min Z_k \in Z_k$ and $(\leftarrow, <_X\text{-}\min Z_k)_{<_X} \neq \emptyset$, $(\leftarrow, <_X\text{-}\min Z_k)_{<_X}$ has to have a $<_X$-maximal element, a contradiction. $\square$

The above theorem yields the following.

Corollary 14. *Let X be a locally connected space. Then X is orderable if and only if the set $\mathcal{Y}$ of all connected components satisfies both* (A) *and* (B)*,*

(A) *for every $Y \in \mathcal{Y}$, Y is orderable,*
(B) *either* (1) *or* (2) *holds:*
  (1) $\mathcal{Y}^1 \neq \emptyset$,
  (2) $\mathcal{Y}^0 = \emptyset$ *or* $\mathcal{Y}^2$ *is empty or infinite.*

This corollary also shows that the subspace $(0,1)_{\mathbb{R}} \cup \{2\}$ of the real line $\mathbb{R}$ is not orderable.

Example 15. Applying the corollary above, we can see:

- when $1 \leq n < \omega$, the subspace $(-2,-1)_{\mathbb{R}} \cup \bigcup_{k<n}[k, k+\frac{1}{2}]_{\mathbb{R}}$ of $\mathbb{R}$ is not orderable,
- the subspace $(-2,-1)_{\mathbb{R}} \cup \bigcup_{k\in\omega}[k, k+\frac{1}{2}]_{\mathbb{R}}$ of $\mathbb{R}$ is orderable,
- when $1 \leq n < \omega$, the subspace $[-2,-1]_{\mathbb{R}} \cup \bigcup_{k<n}(k, k+\frac{1}{2})_{\mathbb{R}}$ of $\mathbb{R}$ is not orderable,
- the subspace $[-2,-1]_{\mathbb{R}} \cup \bigcup_{k\in\omega}(k, k+\frac{1}{2})_{\mathbb{R}}$ of $\mathbb{R}$ is not orderable,
- when $1 \leq n < \omega$, the subspace $(-4,-3]_{\mathbb{R}} \cup (-2,-1)_{\mathbb{R}} \cup \bigcup_{k<n}[k, k+\frac{1}{2}]_{\mathbb{R}}$ of $\mathbb{R}$ is orderable.

Finally we consider the suborderability of topological sums of suborderable spaces.

Corollary 16. *If a space X can be written as a topological sum $X = \bigoplus \mathcal{Y}$ for some collection $\mathcal{Y}$ of suborderable spaces, then X is suborderable. In particular, a topological sum of orderable spaces is suborderable.*

Proof. It is well-known that for every ordered space $\langle Y, <_Y\rangle$, there is an ordered space $\langle Z, <_Z\rangle$ such that $Y \subset Z$, $<_Y = <_Z \restriction Y$, $\lambda_{<_Y} = \lambda_{<_Z} \restriction Y$ and the space $\langle Z, \lambda_{<_Z}\rangle$ is compact, hence $\langle Z, <_Z\rangle$ is type 2, see [4].

Let $\mathcal{Y}$ be as above. For every $Y \in \mathcal{Y}$, take a compact ordered space $\tilde{Y}$ containing Y as a topological subspace. Let $\tilde{\mathcal{Y}}$ be the collection $\{\tilde{Y} : Y \in \mathcal{Y}\}$ of compact ordered spaces. Taking order isomorphic copies of $\tilde{Y}$'s if necessary, we may assume that the collection $\tilde{\mathcal{Y}}$ is pairwise disjoint. Now it follows from Lemma 8 that $\bigoplus \tilde{\mathcal{Y}}$ is orderable. Since X is a subspace of $\bigoplus \tilde{\mathcal{Y}}$, it is suborderable. □

As above $\mathbb{Q}$ and $\mathbb{P}$ can be ordered by a type 1 order and they are totally disconnected, where a space X is totally disconnected if X does not contain any connected subsets of cardinality larger than one. The referee of the present paper pointed out that the lexicographic product space $X = [0,1]_{\mathbb{R}} \times \{0,1\}$ is compact and totally disconnected, so it cannot be ordered by a type 1 order. We would like to ask:

Question 17. Under what conditions, a non-compact totally disconnected orderable space can be ordered by a type 1 order?

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