CARDINAL FUNCTIONS ON LEXICOGRAPHIC PRODUCTS

YASUSHI HIRATA AND NOBUYUKI KEMOTO

Abstract. We will calculate the density, the spread and related cardinal functions on lexicographic products of GO-spaces, and give their applications.

1. Introduction

The notion of a lexicographic product of GO-spaces was introduced in [14], and their weight was calculated in [6]. Let $d(X)$, $s(X)$ and $w(X)$ denote the density, the spread and the weight of a space X, respectively. In [6], it is proved that whenever $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic product of GO-spaces, the weight of X is represented as

$$w(X) = \begin{cases} \sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma\} & \text{if } \gamma \text{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, w(X_{\gamma-1})\} & \text{if } \gamma \text{ is successor.} \end{cases}$$

Also, some cardinal functions of lexicographic products of LOTS are considered in [1]. In this paper, we will calculate the density, the spread and related cardinal functions on lexicographic products of GO-spaces. We will see that whenever $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic product of

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GO-spaces,

$$d(X) = \begin{cases} \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\} \\ \quad \text{if } \gamma \text{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, d(X_{\gamma-1})\} \\ \quad \text{if } \gamma \text{ is successor and } |X_{\gamma-1}| > 2, \\ \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma - 1\} \\ \quad \text{if } \gamma \text{ is successor, } |X_{\gamma-1}| = 2 \text{ and } \gamma - 1 \text{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-2} X_\alpha|, w(X_{\gamma-2})\} \\ \quad \text{if } \gamma \text{ is successor, } |X_{\gamma-1}| = 2 \text{ and } \gamma - 1 \text{ is successor.} \end{cases}$$

As applications, for example, we see:

- $d(2^\omega) = d(3^\omega) = d((\omega+1)^\omega) = d(2^{\omega+1}) = \aleph_0$ and $d(3^{\omega+1}) = d((\omega+1)^{\omega+1}) = 2^{\aleph_0}$, whereas $w(2^\omega) = w(3^\omega) = w((\omega+1)^\omega) = \aleph_0$ and $w(2^{\omega+1}) = w(3^{\omega+1}) = w((\omega+1)^{\omega+1}) = 2^{\aleph_0}$.

Modifying the spread $s(X)$, we will define, in section 3, an additional cardinal function 2-$s(X)$ for each GO-space X. By using it, we will see that whenever $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic product of GO-spaces,

$$s(X) = \begin{cases} \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\} \\ \quad \text{if } \gamma \text{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, s(X_{\gamma-1})\} \\ \quad \text{if } \gamma \text{ is successor and } |X_{\gamma-1}| > 2, \\ \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma - 1\} \\ \quad \text{if } \gamma \text{ is successor, } |X_{\gamma-1}| = 2 \text{ and } \gamma - 1 \text{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-2} X_\alpha|, 2\text{-}s(X_{\gamma-2})\} \\ \quad \text{if } \gamma \text{ is successor, } |X_{\gamma-1}| = 2 \text{ and } \gamma - 1 \text{ is successor.} \end{cases}$$

2. Preliminaries

All topological spaces are assumed to be regular T_2 containing at least 2 points and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, *all X_α's are also assumed to have cardinality at least* 2 *with* $\gamma \ge 2$.

The symbol $|x|$ denotes the cardinality of a set x. Usually the symbols $\alpha, \beta, \gamma, \cdots$ denote ordinals. An ordinal α satisfying $\alpha = |\alpha|$ is called a cardinal. Also usually the symbols $\kappa, \lambda, \mu, \cdots$ denote cardinals. The symbols ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal respectively. An infinite cardinal κ is *regular* if $\mathrm{cf}\kappa = \kappa$, where $\mathrm{cf}\kappa$ denotes the cofinality of κ, otherwise *singular*. For a cardinal κ, the symbol κ^+, which is called the *successor of* κ, denotes the smallest cardinal greater than κ. An uncountable cardinal

λ is called a *successor cardinal* if $\lambda = \kappa^+$ for some cardinal κ, otherwise *limit cardinal*. The symbol κ^λ denotes the cardinality of the set of all functions from λ to κ. When we want to infer that ω (ω_1) is a cardinal, it is written as $\aleph_0$ ($\aleph_1$ respectively). Ordinals have the usual order topology. The symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{I}$ denote the reals, the rationals, the irrationals and the unit interval $[0,1]$ in $\mathbb{R}$, which is also denoted by $[0,1]_{\mathbb{R}}$, respectively.

A lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of GO-spaces X_α's is defined in [14] as a subspace of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ of LOTS X^*_α's, where X^*_α is a LOTS with $X_\alpha \subset X^*_\alpha$ which is called the minimal *d*-extension of X_α. For readers' convenience, we recall here outlines of the concepts which are used in this paper.

LOTS and GO-spaces: A linearly ordered set $\langle X, <_X\rangle$, see [2, page 4], has a natural topology λ_X, which is called an *interval topology*, generated by

$$\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\}$$

as a subbase, where $(x, \rightarrow)_X = \{z \in X : x <_X z\}$, $(x, y)_X = \{z \in X : x <_X z <_X y\}$, $(x, y]_X = \{z \in X : x <_X z \leq_X y\}$ and so on. The triple $\langle X, <_X, \lambda_X\rangle$, which is simply denoted by X, is called a *LOTS*.

A triple $\langle X, <_X, \tau_X\rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X\rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subset C$ holds. In this situation, the pair $\langle X, <_X\rangle$ is called the *underlying linearly ordered set* of X, and the triple $\langle X, <_X, \lambda_X\rangle$, which is denoted by L_X, is called the *underlying LOTS* of X. Obviously, the GO-space topology τ_X is stronger than the interval topology λ_X, that is, $\lambda_X \subset \tau_X$. For more GO-spaces, see [4, 20]. Usually $<_X$, $(x, y)_X$, λ_X and τ_X are written simply $<$, (x, y), λ and τ if contexts are clear.

Lexicographic products of LOTS: The lexicographic product of a sequence of LOTS is a classic concept, although the lexicographic product of a sequence of GO-spaces was defined recently [14]. For every $\alpha < \gamma$, let Y_α be a LOTS and $Y = \prod_{\alpha<\gamma} Y_\alpha$. Every element $y \in Y$ is identified with the sequence $\langle y(\alpha) : \alpha < \gamma\rangle$, where a sequence means a function whose domain is an ordinal. For notational convenience, $\prod_{\alpha<\gamma} Y_\alpha$ is considered as $\{\emptyset\}$ whenever $\gamma = 0$, where $\emptyset$ is considered to be a function whose domain is 0. When $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} Y_\alpha$ and

$y_1 \in \prod_{\beta \leq \alpha} Y_\alpha$, $y_0 \ {}^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} Y_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ y_1(\alpha) & \text{if } \beta \leq \alpha. \end{cases}$$

In this case, whenever $\beta = 0$, $\emptyset \ {}^\wedge y_1$ is considered as y_1. In case $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} Y_\alpha$, $u \in Y_\beta$ and $y_1 \in \prod_{\beta<\alpha} Y_\alpha$, $y_0 \ {}^\wedge\langle u\rangle^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} Y_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ u & \text{if } \alpha = \beta, \\ y_1(\alpha) & \text{if } \beta < \alpha. \end{cases}$$

More general cases are similarly defined.

The lexicographic order $<_Y$ on $Y = \prod_{\alpha<\gamma} Y_\alpha$, where all Y_α's are LOTS's, is defined as follows: for every $y, y' \in Y$,

$$y <_Y y' \text{ iff for some } \alpha < \gamma,\ y \upharpoonright \alpha = y' \upharpoonright \alpha \text{ and } y(\alpha) <_{Y_\alpha} y'(\alpha),$$

where $y \upharpoonright \alpha = \langle y(\beta) : \beta < \alpha\rangle$ and $<_{Y_\alpha}$ is the order on Y_α.

The minimal d-extension of a GO-space: If $Y = \langle Y, <_Y, \tau_Y\rangle$ is a GO-space, then for each subset X of Y, the subspace $X = \langle X, <_X, \tau_X\rangle$ is defined and it is also a GO-space, where $<_X$ is the restriction $<_Y \upharpoonright X \times X$ and τ_X is the subspace topology $\{U \cap X : U \in \tau_Y\}$ of τ_Y. In particular, each subspace X of a LOTS Y is a GO-space since every LOTS is a GO-space. Conversely, for every GO-space X, there is a LOTS X^* such that

- X is a dense subspace of X^* (as a GO-space),
- if L is a LOTS containing X as a dense subspace, then L also contains the LOTS X^* as a subspace (in the sense of the identification).

Such an X^* is called the *minimal d-extension of a GO-space* X, see [21]. Indeed, for a GO-space $X = \langle X, <_X, \tau_X\rangle$ with $L_X = \langle X, <_X, \lambda_X\rangle$, the LOTS X^* is constructed as

$$X^* = (X^- \times \{-1\}) \cup (X \times \{0\}) \cup (X^+ \times \{1\}),$$

where

$$X^+ = \{x \in X : (\leftarrow, x] \in \tau_X \setminus \lambda_X\},$$
$$X^- = \{x \in X : [x, \rightarrow) \in \tau_X \setminus \lambda_X\},$$

and the order $<_{X^*}$ on X^* is the restriction of the usual lexicographic order on $X \times \{-1, 0, 1\}$ with $-1 < 0 < 1$, also we identify $X \times \{0\}$ with X in the obvious way. Obviously, we can see:

- if X is a LOTS, then $X^* = X$,

- X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

Lexicographic products of GO-spaces: Now we are ready to define the lexicographic product of GO-spaces. For every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. Take the minimal d-extension X_α^* of X_α for each $\alpha < \gamma$, and let $\hat{X} = \prod_{\alpha<\gamma} X_\alpha^*$ be the lexicographic product of the LOTS X_α^*'s. Then X is a subset of the LOTS $\hat{X}$. Considered as a GO-subspace of $\hat{X}$, we call $X = \prod_{\alpha<\gamma} X_\alpha$ the *lexicographic product of GO-spaces* X_α's, for more details see [14]. $\prod_{i\in\omega} X_i$ ($\prod_{i\leq n} X_i$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$. When X_α's are GO-spaces, $\prod_{\alpha<\gamma} X_\alpha$ usually means the lexicographic product unless otherwise stated. Moreover we assume that $\prod_{\alpha<\gamma} X_\alpha$ *is infinite,* $\gamma \geq 2$ *and* $|X_\alpha| \geq 2$ *for every* $\alpha < \gamma$. Therefore when $\gamma < \omega$, for at least one $\alpha < \gamma$, X_α is infinite.

We remark that when $\delta < \gamma$, a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is regarded as the lexicographic product $\prod_{\alpha<\delta} X_\alpha \times \prod_{\delta\leq\alpha<\gamma} X_\alpha$ of two lexicographic products, where the lexicographic product $\prod_{\delta\leq\alpha<\gamma} X_\alpha$ is considered in the natural way, see [14, Lemma 1.5]. About lexicographic products of GO-spaces, see [7, 12, 15, 16, 17, 18]. Also about Tychonoff products of GO-spaces, see [5, 8, 11, 13].

3. Cardinal functions on GO-spaces

Recall the following cardinal functions on a topological space X, see [2].

- $s(X) = \sup\{|H| : H$ is a relatively discrete subset in X $\}$, where H is *relatively discrete* in X if every element x in H is an isolated point in H, that is, for every $x \in H$, there is a neighborhood U of x with $U \cap H = \{x\}$,
- $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a pairwise disjoint collection of non-empty open sets of X $\}$.

The following relationships about a topological space X are well-known and easy to prove.

- $w(X) \geq s(X) \geq c(X)$,
- $w(X) \geq d(X) \geq c(X)$,
- $|X| \geq d(X)$.

Since we are assuming that spaces are regular T_2, a space X is discrete whenever X is finite, where X is *discrete* if all points in X are isolated in X. Therefore all cardinal functions above on X coincide with $|X|$ whenever X is finite. So hereafter we assume that *all spaces are infinite* unless otherwise stated.

Unfortunately, we cannot determine only from the equation $\kappa = s(X)$ whether a space X has a relatively discrete subset of cardinality κ in case κ is a limit cardinal. Similar phenomenon occurs for $c(X)$. So we consider further two cardinal functions. For a topological space X, let

- $ss(X) = \min\{\kappa :$ there are no relatively discrete subspaces of cardinality $\kappa\}$,
- $cc(X) = \min\{\kappa :$ there are no pairwise disjoint collections of κ-many non-empty open sets$\}$.

It is trivial that $cc(X) \leq ss(X)$ and $cc(X) \leq d(X)^+$. Since we are assuming that all spaces are infinite and T_2, we have $\omega_1 \leq cc(X)$. It is known that $cc(X)$ has to be a regular uncountable cardinal, see [3] or [9, Theorem 12.2]. Note that

- $ss(X) = s(X)$ holds whenever $ss(X)$ is a limit cardinal, $ss(X) = s(X)^+$ holds whenever $ss(X)$ is a successor cardinal,
- $cc(X) = c(X)$ holds whenever $cc(X)$ is a limit cardinal, $cc(X) = c(X)^+$ holds whenever $cc(X)$ is a successor cardinal.

Using the hereditary collectionwise Hausdorffness of GO-spaces, we can prove the following easily, see also [10, 2.23 (a)].

Lemma 3.1 (folklore)**.** *Let H be a relatively discrete subspace of a GO-space X. Then there is a pairwise disjoint collection $\{U(x) : x \in H\}$ of open sets in X with $x \in U(x)$ for every $x \in H$. Hence, $c(X) = s(X)$ and $cc(X) = ss(X)$ hold whenever X is a GO-space.*

For a GO-space X, since it is well known $|X| \geq w(X)$ (see below), we have:

- $|X| \geq w(X) \geq d(X) \geq s(X)$,
- $s(X)^+ \geq ss(X) \geq s(X)$.

Let $X = \langle X, <_X, \tau_X\rangle$ be a GO-space, define

$$N_X^+ = \{x \in X : \exists y \in X(x < y, (x, y) = \emptyset)\},$$

$$N_X^- = \{x \in X : \exists y \in X(y < x, (y, x) = \emptyset)\}.$$

For every $x \in N_X^+$, the element $y \in X$ with $x < y$ and $(x, y) = \emptyset$ is denoted by x^+, similarly for every $x \in N_X^-$, we can assign $x^- \in X$ with $x^- < x$ and $(x^-, x) = \emptyset$. Since $x \mapsto x^+$ is a one-to-one onto map from N_X^+ to N_X^-, we see $|N_X^+| = |N_X^-|$.

N_X^+ and the other sets listed are not properties but sets that are defined, in the case of the sets N_X^+ and N_X^-, from the underlying LOTS, and for X^+ and X^- from the GO-space. Obviously, the topology τ_X is generated by $\mathcal{S} = \{(\leftarrow, x) : x \in X\} \cup \{(x, \rightarrow) : x \in X\} \cup \{(\leftarrow, x] : x \in X^+\} \cup \{[x, \rightarrow) : x \in X^-\}$ as a subbase, which also shows $|X| \geq w(X)$. Thus a GO-space X is completely determined by reserving its underlying LOTS L_X, X^+ and X^- with $X^+ \subset \{x \in X : (\leftarrow, x] \notin \lambda_X\}$ and $X^- \subset \{x \in X : [x, \rightarrow) \notin \lambda_X\}$. For example,

- the Sorgenfrey line $\mathbb{S}$ is determined from $L_{\mathbb{S}} = \mathbb{R}$, $\mathbb{S}^+ = \emptyset$ and $\mathbb{S}^- = \mathbb{R}$,
- the Michael line $\mathbb{M}$ is determined from $L_{\mathbb{M}} = \mathbb{R}$, $\mathbb{M}^+ = \mathbb{P}$ and $\mathbb{M}^- = \mathbb{P}$.

The cardinal functions $w(X), d(X), s(X), ss(X)$ of a GO-space X can be described using the terms L_X, N_X^+, X^+ and X^-. The equality (4) in the following lemma is essentially proved in [6, Lemma 2.2], so we give here proofs of (1), (2) and (3).

Lemma 3.2. *Let X be a GO-space. Then*

(1) $d(X) = \max\{d(L_X), |X^+ \cap X^-|, |X^+ \cap N_X^-|, |X^- \cap N_X^+|\}$,
(2) $s(X) = \max\{s(L_X), |X^+ \cap X^-|, |X^+ \cap N_X^-|, |X^- \cap N_X^+|\}$,
(3) $ss(X) = \max\{ss(L_X), |X^+ \cap X^-|^+, |X^+ \cap N_X^-|^+, |X^- \cap N_X^+|^+\}$,
(4) $w(X) = \max\{d(L_X), |N_X^+|, |X^+|, |X^-|\}$.

Although we are assuming that spaces are infinite and T_2, note that all equalities above hold whenever X is finite, because of $X = L_X$, $|X| = w(L_X) = d(L_X) = s(L_X)$ and $|X|^+ = ss(L_X)$ by the discreteness of X.

Proof. Obviously, $d(X)$, $d(L_X)$, $s(X)$, and $s(L_X)$ are infinite, also $ss(X)$ and $ss(L_X)$ are uncountable. Let

$$D_0 = (X^+ \cap X^-) \cup (X^+ \cap N_X^-) \cup (X^- \cap N_X^+),$$

then all members of D_0 are isolated points of X, so $d(X) \geq s(X) \geq |D_0|$ and $ss(X) > |D_0|$. And we have $d(X) \geq d(L_X)$, $s(X) \geq s(L_X)$ and $ss(X) \geq ss(L_X)$ because of $\lambda_X \subset \tau_X$. Therefore the inequality "$\geq$" in (1), (2) and (3) holds. Let

$$E_0 = \{x \in X : \ (x, \rightarrow) = \emptyset \text{ or } (\leftarrow, x) = \emptyset \ \},$$

then it is trivial that $|E_0| \leq 2$.

Claim. If U is a non-empty open set in X, then $U \cap (D_0 \cup E_0) \neq \emptyset$ or $\mathrm{Int}_{L_X} U \neq \emptyset$.

Proof. We may assume that U is a convex set. If there are $x, y, z \in U$ with $x < y < z$, then the interval (x, z) is an open set in L_X, so $y \in (x, z) \subset \mathrm{Int}_{L_X} U$. In the other case, we have $1 \leq |U| \leq 2$, so U contains an isolated point d in X. It follows that $(\leftarrow, d] \in \tau_X$ and $[d, \rightarrow) \in \tau_X$. If $d \in N_X^+ \cap N_X^-$, then it is also isolated in L_X, so $d \in \mathrm{Int}_{L_X} U$. If $d \notin N_X^+$, then $d = \max X \in E_0$ or $d \in X^+$. If $d \notin N_X^-$, then $d = \min X \in E_0$ or $d \in X^-$. Hence, $d \notin N_X^+ \cap N_X^-$ implies $d \in U \cap (D_0 \cup E_0)$. This completes the proof of the claim.

(1) To see "$\leq$", let

$$\kappa_1 = \max\{d(L_X), |X^+ \cap X^-|, |X^+ \cap N_X^-|, |X^- \cap N_X^+|\}.$$

Take a subset D of X with $D_0 \cup E_0 \subset D$ and $|D| = \kappa_1$ which is dense in L_X. Let U be an arbitrary non-empty open set of X. By Claim, we see that $\emptyset \neq (\mathrm{Int}_{L_X} U) \cap D \subset U \cap D$ or $\emptyset \neq U \cap (D_0 \cup E_0) \subset U \cap D$. Hence, D is dense in X, and we have $d(X) \leq |D| = \kappa_1$.

(2) and (3)
To see "$\leq$", let

$$\kappa_2 = \max\{s(L_X), |X^+ \cap X^-|, |X^+ \cap N_X^-|, |X^- \cap N_X^+|\},$$

$$\kappa_3 = \max\{ss(L_X), |X^+ \cap X^-|^+, |X^+ \cap N_X^-|^+, |X^- \cap N_X^+|^+\}.$$

And let $\mathcal{U}$ be an arbitrary pairwise disjoint collection of non-empty open sets in X. Put $\mathcal{U}_0 = \{U \in \mathcal{U} : U \cap (D_0 \cup E_0) \neq \emptyset\}$ and $\mathcal{U}_1 = \{U \in \mathcal{U} : \mathrm{Int}_{L_X} U \neq \emptyset\}$. Then an assignment $\mathcal{U}_0 \ni U \mapsto x(U) \in U \cap (D_0 \cup E_0)$ is one-to-one since $\mathcal{U}$ is pairwise disjoint. So we see $|\mathcal{U}_0| \leq |D_0 \cup E_0| \leq \max\{\aleph_0, |X^+ \cap X^-|, |X^+ \cap N_X^-|, |X^- \cap N_X^+|\}$ and $\max\{\aleph_0, |X^+ \cap X^-|, |X^+ \cap N_X^-|, |X^- \cap N_X^+|\}$ is $\leq \kappa_2$ and $< \kappa_3$. Since $\{\mathrm{Int}_{L_X} U : U \in \mathcal{U}_1\}$ is a pairwise disjoint collection of non-empty open sets in L_X, we see that $|\mathcal{U}_1| \leq c(L_X) = s(L_X) \leq \kappa_2$ and $|\mathcal{U}_1| < cc(L_X) = ss(L_X) \leq \kappa_3$. By Claim, we have $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$, so $|\mathcal{U}| \leq \kappa_2$ and $|\mathcal{U}| < \kappa_3$. Hence $s(X) = c(X) \leq \kappa_2$ and $ss(X) = cc(X) \leq \kappa_3$. $\square$

Example 3.3. Applying the lemma above with $d(\mathbb{R}) = s(\mathbb{R}) = \aleph_0$, we can calculate the well-known cardinal functions on $\mathbb{S}$ and $\mathbb{M}$,

- noting $L_{\mathbb{S}} = \mathbb{R}$, $N_{\mathbb{S}}^+ = N_{\mathbb{S}}^- = \emptyset$, $\mathbb{S}^+ = \emptyset$ and $\mathbb{S}^- = \mathbb{R}$, we see $d(\mathbb{S}) = s(\mathbb{S}) = \aleph_0$ and $w(\mathbb{S}) = 2^{\aleph_0}$,
- noting $L_{\mathbb{M}} = \mathbb{R}$, $N_{\mathbb{M}}^+ = N_{\mathbb{M}}^- = \emptyset$, $\mathbb{M}^+ = \mathbb{P}$ and $\mathbb{M}^- = \mathbb{P}$, we see $d(\mathbb{M}) = s(\mathbb{M}) = w(\mathbb{M}) = 2^{\aleph_0}$.

In the next section, we calculate cardinal functions of lexicographic products. To describe the spread of lexicographic products, we further need some fine cardinal functions.

Definition 3.4. Let X be a GO-space and $H \subset X$. A point x in X is a 0-*cluster* (1-*cluster*) *point of* H if for every neighborhood U of x, $H \cap (U \cap (\leftarrow, x)) \neq \emptyset$ ($H \cap (U \cap (x, \rightarrow)) \neq \emptyset$ respectively) holds. We say that H is *relatively* 0-*discrete* if H does not have a 0-cluster point of H, that is, for every $x \in H$, there is a neighborhood U of x with $H \cap (U \cap (\leftarrow, x)) = \emptyset$. The *relatively* 1-*discreteness* is similarly defined. Note that H is relatively discrete if and only if it is relatively 0-discrete and relatively 1-discrete, that is, for every $x \in H$, there is a neighborhood U of x with $H \cap (U \cap (\leftarrow, x)) = \emptyset$ *and* $H \cap (U \cap (x, \rightarrow)) = \emptyset$.

Further we give an additional notion. A point x in X is a 2-*cluster point of* H if it is 0-cluster and 1-cluster. We say that H is *relatively* 2-*discrete* if H does not have a 2-cluster point of H, that is, for every $x \in H$, there is a neighborhood U of x with $H \cap (U \cap (\leftarrow, x)) = \emptyset$ *or* $H \cap (U \cap (x, \rightarrow)) = \emptyset$.

Obviously,

- if H is relatively discrete, then it is both relatively 0-discrete and relatively 1-discrete,
- if H is relatively 0-discrete (or relatively 1-discrete), then it is relatively 2-discrete.

Now we can define corresponding cardinal functions on GO-spaces.

Definition 3.5. Let X be a GO-space. For $i \in 3$ $(= \{0, 1, 2\})$, let

- $i\text{-}s(X) = \sup\{|H| : H \text{ is relatively } i\text{-discrete }\}$,
- $i\text{-}ss(X) = \min\{\kappa : \text{there are no relatively } i\text{-discrete subspaces of cardinality } \kappa\}$.

Obviously, every relatively 2-discrete subset H of a GO-space X is expressed as $H_0 \cup H_1$ for some relatively 0-discrete subset H_0 and relatively 1-discrete subset H_1, so we see

- $2\text{-}s(X) = \max\{0\text{-}s(X), 1\text{-}s(X)\}$,
- $2\text{-}ss(X) = \max\{0\text{-}ss(X), 1\text{-}ss(X)\}$,
- $\min\{0\text{-}s(X), 1\text{-}s(X)\} \geq s(X)$,
- $\min\{0\text{-}ss(X), 1\text{-}ss(X)\} \geq ss(X)$.

Let $X = \langle X, <, \tau\rangle$ be a GO-space. Then we can define GO-spaces $X_0 = \langle X, <, \tau_0\rangle$ and $X_1 = \langle X, <, \tau_1\rangle$ which have the following bases, respectively:

$$\{I \cap (\leftarrow, a] : I \text{ is an open convex in } X,\ a \in I\},$$

$$\{I \cap [a, \rightarrow) : I \text{ is an open convex in } X,\ a \in I\}.$$

For each $i \in 2$ and for each subset H of X, it is easily seen that H is relatively i-discrete in X if and only if H is relatively discrete in X_i.

Hence, i-$ss(X) = ss(X_i) = cc(X_i)$ is a regular uncountable cardinal. And 2-$ss(X) = \max\{0\text{-}ss(X), 1\text{-}ss(X)\}$ is also a regular uncountable cardinal.

As in Lemma 3.2, we can describe these cardinal functions using the terms L_X, N_X^+, X^+ and X^- as follows. By (3) of the following lemma with Lemma 3.2, we also see:

- $w(X) \geq 2\text{-}s(X)$.

Lemma 3.6. *Let X be a GO-space. Then*

(1) $0\text{-}s(X) = \max\{s(L_X), |N_X^+|, |X^-|\}$,
$0\text{-}ss(X) = \max\{ss(L_X), |N_X^+|^+, |X^-|^+\}$,
(2) $1\text{-}s(X) = \max\{s(L_X), |N_X^+|, |X^+|\}$,
$1\text{-}ss(X) = \max\{ss(L_X), |N_X^+|^+, |X^+|^+\}$,
(3) $2\text{-}s(X) = \max\{s(L_X), |N_X^+|, |X^-|, |X^+|\}$,
$2\text{-}ss(X) = \max\{ss(L_X), |N_X^+|^+, |X^-|^+, |X^+|^+\}$.

Proof. (2) is similar to (1) and (3) follows from (1) and (2), so we only show (1). Noting $|N_X^+| = |N_X^-|$, let $\kappa = \max\{s(L_X), |N_X^-|, |X^-|\}$ and $\kappa^* = \max\{ss(L_X), |N_X^-|^+, |X^-|^+\}$. Since $0\text{-}s(X) \geq s(X) \geq s(L_X)$, $0\text{-}ss(X) \geq ss(X) \geq ss(L_X)$ and both N_X^- and X^- are relatively 0-discrete, it is obvious that $0\text{-}s(X) \geq \kappa$ and $0\text{-}ss(X) \geq \kappa^*$.

To see $0\text{-}s(X) \leq \kappa$ and $0\text{-}ss(X) \leq \kappa^*$, let H be a relatively 0-discrete subspace of X, we will see $|H| \leq \kappa$ and $|H| < \kappa^*$. Let

$$H_0 = \{x \in H : x \in N_X^- \cup X^- \text{ or } (\leftarrow, x) = \emptyset\}.$$

Obviously, $|H_0| \leq \kappa$ and $|H_0| < \kappa^*$. Let $H_1 = H \setminus H_0$.

Claim. $|H_1| \leq \kappa$ and $|H_1| < \kappa^*$.

Proof. Note that $x \in \mathrm{Cl}_X(\leftarrow, x)$ for every $x \in H_1$. Since H is relatively 0-discrete, for every $x \in H_1$, we can fix an open convex neighborhood B_x of x in X with $H \cap (B_x \cap (\leftarrow, x)) = \emptyset$. Also from $x \in \mathrm{Cl}_X(\leftarrow, x)$, we can fix $y_x \in B_x \cap (\leftarrow, x)$ for every $x \in H_1$. Then $\{(y_x, x) : x \in H_1\}$ is a pairwise disjoint collection of non-empty intervals, that is, non-empty L_X-open sets. Thus $|H_1| \leq s(L_X) \leq \kappa$ and $|H_1| < ss(L_X) \leq \kappa^*$. This completes the proof of Claim.

By the claim, we see $|H| = |H_0 \cup H_1| \leq \kappa$ and $< \kappa^*$. □

Applying the lemma above, we see $2\text{-}s(\mathbb{R}) = \aleph_0$, $1\text{-}s(\mathbb{S}) = \aleph_0$, $0\text{-}s(\mathbb{S}) = 2\text{-}s(\mathbb{S}) = 2^{\aleph_0}$ and $0\text{-}s(\mathbb{M}) = 1\text{-}s(\mathbb{M}) = 2\text{-}s(\mathbb{M}) = 2^{\aleph_0}$.

4. Cardinal functions on lexicographic products

In [6], the weight of lexicographic products of GO-spaces was calculated. In this section, we calculate the density and the spread of lexicographic products of GO-spaces. First we consider cardinal functions of lexicographic products of two GO-spaces, which extend Theorem 2.2 and 2.3 in [1].

Lemma 4.1. *Let $X = X_0 \times X_1$ be a lexicographic products of two GO-spaces. Then the following hold:*

(1)
$$d(X) = \begin{cases} w(X_0) & \text{if } |X_1| = 2, \\ \max\{|X_0|, d(X_1)\} & \text{if } |X_1| > 2, \end{cases}$$

(2)
$$s(X) = \begin{cases} 2\text{-}s(X_0) & \text{if } |X_1| = 2, \\ \max\{|X_0|, s(X_1)\} & \text{if } |X_1| > 2, \end{cases}$$

(3)
$$ss(X) = \begin{cases} 2\text{-}ss(X_0) & \text{if } |X_1| = 2, \\ \max\{|X_0|^+, ss(X_1)\} & \text{if } |X_1| > 2, \end{cases}$$

(4) $i\text{-}s(X) = \max\{|X_0|, i\text{-}s(X_1)\}$ *for every* $i \in 3$ $(= \{0, 1, 2\})$,
(5) $i\text{-}ss(X) = \max\{|X_0|^+, i\text{-}ss(X_1)\}$ *for every* $i \in 3$.

Proof. Let $\hat{X} = X_0^* \times X_1^*$. Of course, $X_i = \langle X_i, <_{X_i}, \tau_{X_i} \rangle$ is implicitly understood for $i \in 2$ $(= \{0, 1\})$.

(3) and (5) for $i \in 2$: To see "$\geq$", three claims below suffice.

Claim 1. Let $H \subset X_0$, $v \in X_1$ and $K = H \times \{v\}$.

- If $(\leftarrow, v)_{X_1} \neq \emptyset$, then K is relatively 0-discrete in X.
- If $(v, \rightarrow)_{X_1} \neq \emptyset$, then K is relatively 1-discrete in X.
- If H is relatively 0-discrete in X_0, then K is relatively 0-discrete in X.
- If H is relatively 1-discrete in X_0, then K is relatively 1-discrete in X.

Proof. We prove the first and the third ones, because the remaining are similar. To see that K is relatively 0-discrete in X, let $x \in H$ and put $K_0 = K \cap (\leftarrow, \langle x, v \rangle)_X$. Then we have $K_0 = (H \cap (\leftarrow, x)_{X_0}) \times \{v\}$. It suffices to find an open neighborhood V of $\langle x, v \rangle$ in X which is disjoint from K_0.

If $(\leftarrow, v)_{X_1} \neq \emptyset$, then take a $v' \in X_1$ with $v' <_{X_1} v$. Obviously, $V = (\langle x, v' \rangle, \rightarrow)_{\hat{X}} \cap X$ is a required one. If $(\leftarrow, x)_{X_0} = \emptyset$, then $K_0 = \emptyset$, so

$V = X$ works. If $(\leftarrow, x)_{X_0} \neq \emptyset$ and H is relatively 0-discrete in X_0, then we can take a $y^* \in X_0^*$ with $y^* <_{X_0^*} x$ such that $H \cap ((y^*, x)_{X_0^*} \cap X_0) = \emptyset$. Then $V = (\langle y^*, v\rangle, \rightarrow)_{\hat{X}} \cap X$ is a required neighborhood of $\langle x, v\rangle$. This completes the proof of Claim 1.

Claim 2. The following hold.

- $ss(X) \geq 2\text{-}ss(X_0)$.
- $i\text{-}ss(X) \geq |X_0|^+$ for each $i \in 2$.
- If $|X_1| > 2$, then $ss(X) \geq |X_0|^+$.

Proof. First, fix a $v \in X_1$ with $(\leftarrow, v)_{X_1} \neq \emptyset$. Applying Claim 1 for $H = X_0$, we see that $K = X_0 \times \{v\}$ is relatively 0-discrete in X. Hence $0\text{-}ss(X) > |K| = |X_0|$, and so $0\text{-}ss(X) \geq |X_0|^+$. If H is an arbitrary relatively 1-discrere subset of X_0, then we see from Claim 1 that $K = H \times \{v\}$ is both relatively 0-discrete and relatively 1-discrete, so it is relatively discrete in X. Hence $ss(X) > |K| = |H|$, and we have $ss(X) \geq 1\text{-}ss(X_0)$.

Next, by fixing a $v \in X_1$ with $(v, \rightarrow)_{X_1} \neq \emptyset$, we see in a similar way that $X_0 \times \{v\}$ is relatively 1-discrete in X, $1\text{-}ss(X) \geq |X_0|^+$ and $ss(X) \geq 0\text{-}ss(X_0)$.

Then we also have $ss(X) \geq \max\{0\text{-}ss(X_0), 1\text{-}ss(X_0)\} = 2\text{-}ss(X_0)$.

If $|X_1| > 2$, then we can take a $v \in X_1$ such that $(\leftarrow, v)_{X_1} \neq \emptyset$ and $(v, \rightarrow)_{X_1} \neq \emptyset$. For such v, we see that $K = X_0 \times \{v\}$ is both relatively 0-discrete and relatively 1-discrete, so it is relatively discrete in X. Hence $ss(X) > |K| = |X_0|$, and we have $ss(X) \geq |X_0|^+$. This completes the proof of Claim 2.

Claim 3. $ss(X) \geq ss(X_1)$ and $i\text{-}ss(X) \geq i\text{-}ss(X_1)$ for each $i \in 2$.

Proof. Fix $u \in X_0$. Let H be a relatively 0-discrete subset of X_1 and $K = \{u\} \times H$. To see that K is relatively 0-discrete in X, let $v \in H$ with $(\leftarrow, \langle u, v\rangle)_{\hat{X}} \cap K \neq \emptyset$. Then by $(\leftarrow, v)_{X_1} \neq \emptyset$ and the relative 0-discreteness of H in X_1, we can find an element $w^* \in X_1^*$ with $w^* <_{X_1^*} v$ and $((w^*, v)_{X_1^*} \cap X_1) \cap H = \emptyset$. Now we have $((\langle u, w^*\rangle, \langle u, v\rangle)_{\hat{X}} \cap X) \cap K = \emptyset$, thus K is relatively 0-discrete in X. Therefore we have $|H| = |K| < 0\text{-}ss(X)$. We have shown $0\text{-}ss(X) \geq 0\text{-}ss(X_1)$.

Similarly, we see that $K = \{u\} \times H$ is relatively 1-discrete in X, for each relatively 1-discrete subset H of X_1, so $|H| = |K| < 1\text{-}ss(X)$. We have shown $1\text{-}ss(X) \geq 1\text{-}ss(X_1)$.

If H is a relatively discrete subset of X_1, then it is both relatively 0-discrete and relatively 1-discrete. Since $K = \{u\} \times H$ is both relatively 0-discrete and relatively 1-discrete in X, it is relatively discrete in X. Therefore we have $|H| = |K| < ss(X)$. We have shown $ss(X) \geq ss(X_1)$. This completes the proof of Claim 3.

To see "$\leq$", two claims below suffice.

Claim 4. If $|X_1| = 2$, then $ss(X) \leq 2\text{-}ss(X_0)$.

Proof. Let $X_1 = \{v_0, v_1\}$ with $v_0 <_{X_1} v_1$. Since $X = X_0 \times X_1$ is assumed to be infinite, we have $\omega \leq |X_0|$, so $\omega_1 \leq 2\text{-}ss(X_0)$. Let H be a relatively discrete subset of X and put $K_i = \{u \in X_0 : \langle u, v_i\rangle \in H\}$ and $H_i = K_i \times \{v_i\}$ for each $i \in 2$. Then we have $H = H_0 \cup H_1$. To see that K_0 is relatively 0-discrete in X_0, let $u \in K_0$ with $(\leftarrow, u)_{X_0} \neq \emptyset$. Since H is relatively discrete in X, we have $H \cap ((\langle u^*, v\rangle, \langle u, v_1\rangle)_{\hat{X}} \cap X) = \{\langle u, v_0\rangle\}$ for some $u^* \in X_0^*$ with $u^* <_{X_0^*} u$ and $v \in X_1$. Now we obtain an open neighborhood $(u^*, \rightarrow)_{X_0^*} \cap X_0$ of u in X_0 which is disjoint from $K_0 \cap (\leftarrow, u)_{X_0}$, so K_0 is relatively 0-discrete in X_0. Therefore $|H_0| = |K_0| < 0\text{-}ss(X_0) \leq 2\text{-}ss(X_0)$. Similarly, we see $|H_1| < 1\text{-}ss(X_0) \leq 2\text{-}ss(X_0)$, and so $|H| < 2\text{-}ss(X_0)$. Hence, $ss(X) \leq 2\text{-}ss(X_0)$ holds. This completes the proof of Claim 4 .

Claim 5. The following hold.

- $ss(X) \leq \max\{|X_0|^+, ss(X_1)\}$.
- $i\text{-}ss(X) \leq \max\{|X_0|^+, i\text{-}ss(X_1)\}$ for each $i \in 2$.

Proof. Let $\kappa = \max\{|X_0|^+, ss(X_1)\}$ and $\kappa_i = \max\{|X_0|^+, i\text{-}ss(X_1)\}$ for each $i \in 2$. We would like to show that $ss(X) \leq \kappa$ and $i\text{-}ss(X) \leq \kappa_i$. The lexicographic product $X = X_0 \times X_1$ is assumed to be infinite, so either $|X_0| \geq \omega$ or $|X_1| \geq \omega$. In the latter case, $ss(X_1)$ and $i\text{-}ss(X_1)$, for each $i \in 2$, are regular uncountable cardinals. Hence, κ and κ_i are regular uncountable cardinals in any case.

Let H be a subset of X and set $K_u = \{v \in X_1 : \langle u, v\rangle \in H\}$ and $H_u = \{u\} \times K_u$ for every $u \in X_0$. Then $H = \bigcup_{u \in X_0} H_u$.

Assume that H is relatively 0-discrete in X. To see that K_u is relatively 0-discrete in X_1, let $v \in K_u$ with $(\leftarrow, v)_{X_1} \neq \emptyset$. We have $H \cap ((\langle u, v^*\rangle, \langle u, v\rangle)_{\hat{X}} \cap X) = \emptyset$ for some $v^* \in X_1^*$ with $v^* <_{X_1^*} v$. Now we obtain an open neighborhood $(v^*, \rightarrow)_{X_1^*} \cap X_1$ of v in X_1 which is disjoint from $K_u \cap (\leftarrow, v)_{X_1}$, so K_u is relatively 0-discrete in X_1. Therefore $|H_u| = |K_u| < 0\text{-}ss(X_1) \leq \kappa_0$. By $|X_0| < \kappa_0$, we have $|H| < \kappa_0$. Hence, $0\text{-}ss(X) \leq \kappa_0$ holds.

It had been seen that if H is relatively 0-discrete in X, then K_u is relatively 0-discrete in X_1 for every $u \in X_0$, and $0\text{-}ss(X) \leq \kappa_0$. Similarly, we see that if H is relatively 1-discrete in X, then K_u is relatively 1-discrete in X_1 for every $u \in X_0$, and $1\text{-}ss(X) \leq \kappa_1$.

If H is relatively discrete in X, then it is both relatively 0-discrete and relatively 1-discrete in X, so K_u is both relatively 0-discrete and relatively 1-discrete, thus relatively discrete in X_1 for every $u \in X$.

Hence, $|H_u| = |K_u| < ss(X_1) \leq \kappa$. By $|X_0| < \kappa$, we have $|H| < \kappa$. Hence, $ss(X) \leq \kappa$ holds. This completes the proof of Claim 5.

(2) and (4) for $i \in 2$: We can see (2) and (4) from (3) and (5), respectively. We only prove (4) because the remaining is similar.

First we consider the case $i\text{-}ss(X_1) \geq |X_0|^+$. Since $X = X_0 \times X_1$ is assumed to be infinite, we have $i\text{-}ss(X_1) \geq \omega_1$. By (5), either $i\text{-}ss(X) = i\text{-}ss(X_1) = \kappa$ for some limit cardinal κ, or $i\text{-}ss(X) = i\text{-}ss(X_1) = \kappa^+$ for some infinte cardinal κ. In any case, $i\text{-}s(X) = i\text{-}s(X_1) = \kappa \geq |X_0|$ holds, so we have $i\text{-}s(X) = \max\{|X_0|, i\text{-}s(X_1)\}$.

Next we consider the case $i\text{-}ss(X_1) < |X_0|^+$. We have $|X_0| \geq \omega$ in this case. By (5), we see $i\text{-}ss(X) = |X_0|^+$, so $i\text{-}s(X) = |X_0| \geq i\text{-}ss(X_1) \geq i\text{-}s(X_1)$. Hence, $i\text{-}s(X) = \max\{|X_0|, i\text{-}s(X_1)\}$.

(4) and (5) for $i = 2$: We only prove (5) because the remaining is similar. We had already seen that

$$0\text{-}ss(X) = \max\{|X_0|^+, 0\text{-}ss(X_1)\},$$

$$1\text{-}ss(X) = \max\{|X_0|^+, 1\text{-}ss(X_1)\}.$$

So we have

$$\begin{aligned} 2\text{-}ss(X) &= \max\{0\text{-}ss(X), 1\text{-}ss(X)\} \\ &= \max\{|X_0|^+, \max\{0\text{-}ss(X_1), 1\text{-}ss(X_1)\}\} \\ &= \max\{|X_0|^+, 2\text{-}ss(X_1)\}. \end{aligned}$$

(1): To see "$\geq$", let $\kappa = d(X)$. Since we are assuming that X is infinite, $\kappa \geq \omega$ holds. Take a dense subset D in X with $|D| = \kappa$. Take an elementary submodel M of $H(\theta)$, where θ is large enough, with $\langle X_0, <_{X_0}, \tau_{X_0}\rangle, \langle X_1, <_{X_1}, \tau_{X_1}\rangle \in M$, $D \subset M$ and $|M| = \kappa$. Also note that by the definability, almost all objects such as $X_0^*, X_1^*, \hat{X}, \cdots$, etc., belong to M.

First we consider the case $|X_1| = 2$. Let $X_1 = \{v_0, v_1\}$ with $v_0 <_{X_1} v_1$. To see $\kappa \geq w(X_0)$, from $|M| = \kappa$, it suffices to see that $\tau_{X_0} \cap M$ is a base for X_0.

Claim 6. For every $u \in X_0$ and every open neighborhood W of u in X_0, the following hold:

(1) there is an open set $U \in M$ in X_0 with $u \in U \subset (\leftarrow, u)_{X_0} \cup W$,
(2) there is an open set $V \in M$ in X_0 with $u \in V \subset (u, \rightarrow)_{X_0} \cup W$.

Proof. Let $u \in X_0$ and W be an open neighborhood of u in X_0. When $(u, \rightarrow)_{X_0} = \emptyset$ (i.e., $u = \max X_0$), set $U = X_0 \in M$, then this U works. So let $(u, \rightarrow)_{X_0} \neq \emptyset$. Since W is an open neighborhood of u in X_0 and X_0^* is a LOTS containing X_0 as a dense subset, there is $w^* \in X_0^*$ with $u <_{X_0^*} w^*$ such that $(u, w^*)_{X_0^*} \cap X_0 \subset W$. We consider two cases.

Case 1. $(u, w^*)_{X_0^*} \neq \emptyset$.

In this case, since $(\langle u, v_1\rangle, \langle w^*, v_0\rangle)_{\hat{X}} \cap X$ is a non-empty open set in X, by the density of D, we can fix a point $x \in ((\langle u, v_1\rangle, \langle w^*, v_0\rangle)_{\hat{X}} \cap X) \cap D$, say $x = \langle x(0), x(1)\rangle \in X_0 \times X_1$. Noting $x \in D \subset M$, by the definability, we see $x(0) \in M$ and $u <_{X_0} x(0) <_{X_0^*} w^*$. So by letting $U = (\leftarrow, x(0))_{X_0}$, obviously $U \in M$ and U is the desired one.

Case 2. $(u, w^*)_{X_0^*} = \emptyset$.

In this case, from $\{\langle u, v_1\rangle\} = (\langle u, v_0\rangle, \langle w^*, v_0\rangle)_{\hat{X}} \cap X$, $\langle u, v_1\rangle$ is an isolated point in X, therefore $\langle u, v_1\rangle \in D \subset M$ because of the density of D. Then by the definability, we see $u \in M$ and $U = (\leftarrow, u]_{X_0} \in M$ is the desired one.

(2) is similar to (1). This completes the proof of Claim 6.

Now by the claim above, $U \cap V$ is an open neighborhood of u with $U \cap V \subset W$ and $U \cap V \in M$. Therefore $\tau_{X_0} \cap M$ is a base for X_0.

Next we consider the case $|X_1| > 2$. We see from (2) that $|X_0| \leq s(X) \leq \kappa$. The following claim shows $d(X_1) \leq \kappa$, hence we see $\max\{|X_0|, d(X_1)\} \leq \kappa$.

Claim 7. $X_1 \cap M$ is dense in X_1.

Proof. Let V be a non-empty open set in X_1. It suffices to find an element of $V \cap M$. Fix $v \in V$. We may assume $(\leftarrow, v)_{X_1} \neq \emptyset$ and $(v, \rightarrow)_{X_1} \neq \emptyset$, otherwise v belongs to M by the definability so $v \in V \cap M$. Therefore there are $v_0^*, v_1^* \in X_1^*$ with $v_0^* <_{X_1^*} v <_{X_1^*} v_1^*$ and $(v_0^*, v_1^*)_{X_1^*} \cap X_1 \subset V$. Fix $u \in X_0$. Since $(\langle u, v_0^*\rangle, \langle u, v_1^*\rangle)_{\hat{X}} \cap X$ is non-empty open set in X, by the density of D, we can find a point $x \in D \cap ((\langle u, v_0^*\rangle, \langle u, v_1^*\rangle)_{\hat{X}} \cap X)$, say $x = \langle x(0), x(1)\rangle$. Now we have $x \in D \subset M$, $x(0) = u$, $x(1) \in (v_0^*, v_1^*)_{X_1^*} \cap X_1 \cap M \subset V \cap M$. This completes the proof of Claim 7.

Now we check "$\leq$". Two claims below suffice.

Claim 8. If $|X_1| = 2$, then $d(X) \leq w(X_0)$.

Proof. Let $\kappa = w(X_0)$. Since $X = X_0 \times X_1$ is assumed to be infinite, we have $|X_0| \geq \omega$, so $\kappa \geq \omega$. Noting $\kappa = \max\{d(L_{X_0}), |N_{X_0}^+|, |X_0^+|, |X_0^-|\}$, we can take a dense set D in L_{X_0} with $|D| = \kappa$ such that

- $N_{X_0}^+ \cup N_{X_0}^- \cup X_0^+ \cup X_0^- \subset D$,
- $\max X_0 \in D$ if X_0 has a maximal element,
- $\min X_0 \in D$ if X_0 has a minimal element.

By $|X_1| = 2$, we have $|D \times X_1| = \kappa$. So it suffices to show that $D \times X_1$ is dense in X.

Let U be a non-empty open set in X. We would like to find an element of $U \cap (D \times X_1)$. Fix $x \in U$, say $x = \langle x(0), x(1)\rangle$. We may assume $x(0) \notin D$, otherwise x is a desired one. Either $(\leftarrow, x(1))_{X_1} = \emptyset$ or $(x(1), \rightarrow)_{X_1} = \emptyset$ holds since $|X_1| = 2$. We may assume the former, because the latter is similar. We have $(\leftarrow, x(0))_{X_0} \neq \emptyset$ since $x(0) \notin D$. By $x \in U$ and $(\leftarrow, x)_X \neq \emptyset$, we can find $y \in \hat{X}$, with $y <_{\hat{X}} x$ and $(y, x)_{\hat{X}} \cap X \subset U$, say $y = \langle u^*, v^*\rangle$ for some $u^* \in X_0^*$ and $v^* \in X_1^* = X_1$. From $(\leftarrow, x(1))_{X_1} = \emptyset$ and $y <_{\hat{X}} x$, we have $u^* <_{X_0^*} x(0)$. Moreover from $x(0) \notin N_{X_0}^- \cup X_0^-$, we can take $u_0, u_1 \in X_0$ with $u^* < u_0 < u_1 < x(0)$. Since D is dense in L_{X_0} and $(u_0, x(0))_{X_0}$ is a non-empty open set in L_{X_0}, there is a point $u \in D \cap (u_0, x(0))_{X_0}$. By $u^* < u_0 < u < x(0)$, we have $y = \langle u^*, v^*\rangle <_{\hat{X}} \langle u, v^*\rangle <_{\hat{X}} x$, so $\langle u, v^*\rangle \in U \cap (D \times X_1)$. This completes the proof of Claim 8.

Claim 9. $d(X) \leq \max\{|X_0|, d(X_1)\}$.

Proof. Let $\kappa = \max\{|X_0|, d(X_1)\}$. Since $X = X_0 \times X_1$ is assumed to be infinite, we have $\kappa \geq \omega$. By $d(X_1) \leq \kappa$, we can take a dense subset D in X_1 with $|D| \leq \kappa$ such that

- $\max X_1 \in D$ if X_1 has a maximal element,
- $\min X_1 \in D$ if X_1 has a minimal element.

By $|X_0| \leq \kappa$, we have $|X_0 \times D| = \kappa$. So it suffices to show that $X_0 \times D$ is dense in X.

Let U be a non-empty open set in X. We would like to find an element of $U \cap (X_0 \times D)$. Fix $x \in U$, say $x = \langle x(0), x(1)\rangle$. We may assume $x(1) \notin D$, otherwise x is a desired one. Then we have $(\leftarrow, x(1))_{X_1} \neq \emptyset$ and $(x(1), \rightarrow)_{X_1} \neq \emptyset$. Since U is an open set with $x \in U$, we can find $v_0^*, v_1^* \in X_1^*$ with $v_0^* <_{X_1^*} x(1) <_{X_1^*} v_1^*$ and $(\langle x(0), v_0^*\rangle, \langle x(0), v_1^*\rangle)_{\hat{X}} \cap X \subset U$. Because $(v_0^*, v_1^*)_{X_1^*} \cap X_1$ is a non-empty open set in X_1 and D is dense in X_1, we can find a point $v \in ((v_0^*, v_1^*)_{X_1^*} \cap X_1) \cap D$. We see $\langle x(0), v\rangle \in U \cap (X_0 \times D)$. This completes the proof of Claim 9.

□

Now the following result in [6] is an easy consequence of Lemmas 3.2, 3.6 and 4.1. We leave its proof to the reader.

Corollary 4.2. [6, Lemma 3.1] *Let $X = X_0 \times X_1$ be a lexicographic products of two GO-spaces. Then $w(X) = \max\{|X_0|, w(X_1)\}$ holds.*

Example 4.3. Using the lemmas and corollary above with Example 3.3, for example, we can see the following:

(1) if $X = \mathbb{R} \times 2$, then $w(X) = 2\text{-}s(X) = 1\text{-}s(X) = 0\text{-}s(X) = 2^{\aleph_0}$, $d(X) = s(X) = \aleph_0$, note that the subspace $\mathbb{R} \times \{1\}$ is identified with $\mathbb{S}$,
(2) if $X = \mathbb{R} \times 3$, then $w(X) = s(X) = 2^{\aleph_0}$,
(3) if $X = 2 \times \mathbb{R}$, then $w(X) = s(X) = \aleph_0$,
(4) if $X = \mathbb{S} \times 2$, then $w(X) = s(X) = 2^{\aleph_0}$,
(5) if $X = 2 \times \mathbb{S}$, then $w(X) = 2\text{-}s(X) = 0\text{-}s(X) = 2^{\aleph_0}$, $1\text{-}s(X) = d(X) = s(X) = \aleph_0$,
(6) if $X = \mathbb{S} \times \mathbb{S}$, then $w(X) = s(X) = 2^{\aleph_0}$,
(7) if $X = \mathbb{M} \times \mathbb{P}$, then $w(X) = s(X) = 2^{\aleph_0}$,
(8) if $X = \mathbb{P} \times \mathbb{M}$, then $w(X) = s(X) = 2^{\aleph_0}$,
(9) if $X = \mathbb{R} \times \mathbb{Q}$, then $w(X) = s(X) = 2^{\aleph_0}$,
(10) if $X = \mathbb{Q} \times \mathbb{R}$, then $w(X) = s(X) = \aleph_0$.

Using the lemmas and corollary above repeatedly, we can also calculate such cardinal functions on lexicographic products of finite length. Next we consider lexicographic products of infinite length. The following lemma extends Theorem 3.1 in [1].

Lemma 4.4. *Let γ be a limit ordinal and $X = \prod_{\alpha<\gamma} X_\alpha$ a lexicographic product of GO-spaces. Then $w(X) = s(X) = \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\}$ and $2\text{-}ss(X) = ss(X) = s(X)^+$.*

Proof. Letting $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, set $\kappa = \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\}$. For each $\beta < \gamma$, let $\kappa_\beta = |\prod_{\alpha\le\beta} X_\alpha|$, then $\kappa = \sup\{\kappa_\beta : \beta < \gamma\}$.

Claim 1. $\gamma \le \kappa$. And $\kappa_\beta < ss(X)$ for every $\beta < \gamma$.

Proof. Since X_α is assumed to have at least two points for each $\alpha < \gamma$, we have $\beta < 2^{|\beta|} \le \kappa_\beta$ for each $\beta < \gamma$. Hence $\gamma \le \kappa$ holds. Let $\beta < \gamma$. As GO-spaces, the lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ can be identified with the lexicographic product $Z_0 \times Z_1$ of two GO-spaces $Z_0 = \prod_{\alpha\le\beta} X_\alpha$ and $Z_1 = \prod_{\beta<\alpha} X_\alpha$ with $|Z_1| > 2$, by Lemma 4.1 (3), we see $ss(X) > |Z_0| = \kappa_\beta$. This completes the proof of Claim 1.

Claim 2. $\kappa < ss(X)$.

Proof. In the case that the sequence $\{\kappa_\beta : \beta < \gamma\}$ is eventually constant, i.e. there is $\beta_0 < \gamma$ such that for every $\beta < \gamma$ with $\beta_0 \le \beta$, $\kappa_\beta = \kappa_{\beta_0}$, we have $\kappa = \kappa_{\beta_0} < ss(X)$.

Next we consider the case that the sequence $\{\kappa_\beta : \beta < \gamma\}$ is not eventually constant, then note $\mathrm{cf}\kappa = \mathrm{cf}\gamma \le \gamma \le \kappa \le ss(X)$ by Claim 1. When κ is singular, we see $\kappa < ss(X)$ since $ss(X)$ cannot be singular. When κ is regular, we have $\gamma = \kappa$. For every $\alpha < \gamma$, fix two points

$v_\alpha, w_\alpha \in X_\alpha$ with $v_\alpha <_{X_\alpha} w_\alpha$. For every $\beta < \gamma$, define a point $x_\beta \in \prod_{\alpha<\gamma} X_\alpha$ by

$$x_\beta(\alpha) = \begin{cases} w_\alpha & \text{if } \alpha = \beta, \\ v_\alpha & \text{if } \alpha \neq \beta. \end{cases}$$

Then obviously the sequence $\{x_\beta : \beta < \gamma\}$ is strictly decreasing in X, therefore $H = \{x_{\beta+1} : \beta < \gamma\}$ is relatively discrete in X and of size κ, so we see $\kappa < ss(X)$.

Claim 3. $w(X) \leq \kappa$.

Proof. Take an elementary submodel M of $H(\theta)$, where θ is large enough, with $\langle\langle X_\alpha, <_{X_\alpha}, \tau_{X_\alpha}\rangle : \alpha < \gamma\rangle, \langle\langle X^*_\alpha, <_{X^*_\alpha}\rangle : \alpha < \gamma\rangle \in M$, $\gamma \subset M$, $\bigcup_{\beta<\gamma} \prod_{\alpha\leq\beta} X^*_\alpha \subset M$ and $|M| \leq \kappa$. It suffices to show that $\tau_X \cap M$ is a base for X, where τ_X is the topology of X.

Let $x \in X$ and W be an open set in X with $x \in W$.

Fact. The following hold:

(1) there is an open set $U \in M$ in X with $x \in U \subset (\leftarrow, x)_X \cup W$,
(2) there is an open set $V \in M$ in X with $x \in V \subset (x, \rightarrow)_X \cup W$.

Proof. Since (2) is similar to (1), we prove (1). When $(x, \rightarrow)_X = \emptyset$, set $U = X$. By the difinability, we have $X \in M$, so this U works. Let $(x, \rightarrow)_X \neq \emptyset$. Since W is open in X and $x \in W$, we can find $x^* \in \hat{X}$ with $x <_{\hat{X}} x^*$ and $(x, x^*)_{\hat{X}} \cap X \subset W$. We consider two cases.

Case 1. $(x, x^*)_{\hat{X}} = \emptyset$.

In this case, set $\alpha_0 = \min\{\alpha < \gamma : x(\alpha) \neq x^*(\alpha)\}$. Then we have $x \restriction \alpha_0 = x^* \restriction \alpha_0$ and $x(\alpha_0) <_{X^*_{\alpha_0}} x^*(\alpha_0)$, $(x(\alpha_0), x^*(\alpha_0))_{X^*_{\alpha_0}} = \emptyset$, moreover for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $x^*(\alpha) = \min X_\alpha$ and $x(\alpha) = \max X_\alpha$ hold. Note $\alpha_0 \in \gamma \subset M$ and $x^* \restriction (\alpha_0 + 1) \in \prod_{\alpha\leq\alpha_0} X^*_\alpha \subset M$, also note by the definability, we have $x^* = x^* \restriction (\alpha_0 + 1)^\wedge\langle \min X_\alpha : \alpha_0 < \alpha < \gamma\rangle \in M$. Now let $U = (\leftarrow, x^*)_{\hat{X}} \cap X$, then obviously $U \in M$ and $x \in U \subset (\leftarrow, x)_X \cup W$.

Case 2. $(x, x^*)_{\hat{X}} \neq \emptyset$.

Take an element $z \in (x, x^*)_{\hat{X}}$, then there is $\alpha_0 < \gamma$ such that in $\prod_{\alpha\leq\alpha_0} X^*_\alpha$, $x \restriction (\alpha_0 + 1) < z \restriction (\alpha_0 + 1) < x^* \restriction (\alpha_0 + 1)$ holds. Note $\alpha_0 \in M$ and $z \restriction (\alpha_0 + 1) \in \prod_{\alpha\leq\alpha_0} X^*_\alpha \subset M$. By the elementarity, we obtain a $z' \in \hat{X} \cap M$ with $z' \restriction (\alpha_0 + 1) = z \restriction (\alpha_0 + 1)$. Now letting $U = (\leftarrow, z')_{\hat{X}} \cap X \in M$, this U is the desired one. This completes the proof of Fact.

Now from this fact, we see $U \cap V \in M$ and $x \in U \cap V \subset W$. Hence $\tau_X \cap M$ is a base for X. This completes the proof of Claim 3.

From Claim 2 and 3, we see $s(X) \leq w(X) \leq \kappa < ss(X) \leq s(X)^+$ and $2\text{-}s(X) \leq w(X) \leq \kappa < ss(X) \leq 2\text{-}ss(X) \leq 2\text{-}s(X)^+$. It follows that $w(X) = s(X) = \kappa$ and $ss(X) = s(X)^+ = 2\text{-}ss(X) = \kappa^+$. $\square$

Note that in the lemma above, $ss(X)$ cannot be a limit cardinal.

Now we can prove our main result. For an ordinal γ, set

$$\gamma - 1 = \begin{cases} \delta & \text{if } \gamma = \delta + 1 \text{ for an ordinal } \delta, \text{ i.e., } \gamma \text{ is a successor,} \\ \gamma & \text{if } \gamma \text{ is 0 or a limit,} \end{cases}$$

see [19, I, Definition 9.4]. The symbol $\gamma - 2$ denotes $(\gamma - 1) - 1$.

Theorem 4.5. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following hold:*

(1) *for every $i \in 3$,*

$$i\text{-}s(X) = \begin{cases} \sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma\} & \textit{if } \gamma \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, i\text{-}s(X_{\gamma-1})\} & \textit{if } \gamma \textit{ is successor.} \end{cases}$$

$$i\text{-}ss(X) = \begin{cases} (\sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma\})^+ & \textit{if } \gamma \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|^+, i\text{-}ss(X_{\gamma-1})\} & \textit{if } \gamma \textit{ is successor.} \end{cases}$$

(2)

$$s(X) = \begin{cases} \sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma\} \\ \qquad \textit{if } \gamma \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, s(X_{\gamma-1})\} \\ \qquad \textit{if } \gamma \textit{ is successor and } |X_{\gamma-1}| > 2, \\ \sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma - 1\} \\ \qquad \textit{if } \gamma \textit{ is successor, } |X_{\gamma-1}| = 2 \textit{ and } \gamma - 1 \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-2} X_\alpha|, 2\text{-}s(X_{\gamma-2})\} \\ \qquad \textit{if } \gamma \textit{ is successor, } |X_{\gamma-1}| = 2 \textit{ and } \gamma - 1 \textit{ is successor.} \end{cases}$$

$$ss(X) = \begin{cases} (\sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\})^+ \\ \quad \textit{if } \gamma \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|^+, ss(X_{\gamma-1})\} \\ \quad \textit{if } \gamma \textit{ is successor and } |X_{\gamma-1}| > 2, \\ (\sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma - 1\})^+ \\ \quad \textit{if } \gamma \textit{ is successor, } |X_{\gamma-1}| = 2 \textit{ and } \gamma - 1 \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-2} X_\alpha|^+, 2\text{-}ss(X_{\gamma-2})\} \\ \quad \textit{if } \gamma \textit{ is successor, } |X_{\gamma-1}| = 2 \textit{ and } \gamma - 1 \textit{ is successor.} \end{cases}$$

(3)

$$d(X) = \begin{cases} \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\} \\ \quad \textit{if } \gamma \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, d(X_{\gamma-1})\} \\ \quad \textit{if } \gamma \textit{ is successor and } |X_{\gamma-1}| > 2, \\ \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma - 1\} \\ \quad \textit{if } \gamma \textit{ is successor, } |X_{\gamma-1}| = 2 \textit{ and } \gamma - 1 \textit{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-2} X_\alpha|, w(X_{\gamma-2})\} \\ \quad \textit{if } \gamma \textit{ is successor, } |X_{\gamma-1}| = 2 \textit{ and } \gamma - 1 \textit{ is successor.} \end{cases}$$

Proof. When γ is limit, we see from Lemma 4.4 that

$$s(X) = w(X) = \sup\{|\textstyle\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\},$$

$$ss(X) = 2\text{-}ss(X) = (\sup\{|\textstyle\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\})^+.$$

By $s(X) \le d(X) \le w(X)$, $s(X) \le i\text{-}s(X) \le w(X)$ and $ss(X) \le i\text{-}ss(X) \le 2\text{-}ss(X)$, for each $i \in 3$, we also have

$$d(X) = i\text{-}s(X) = \sup\{|\textstyle\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\},$$

$$i\text{-}ss(X) = (\sup\{|\textstyle\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\})^+.$$

Hence all equations in the lemma hold in this case.

When γ is successor, the lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ can be identified with the lexicographic product $Y \times X_{\gamma-1}$ of two GO-spaces $Y = \prod_{\alpha<\gamma-1} X_\alpha$ and $X_{\gamma-1}$. Since $|Y| = |\prod_{\alpha<\gamma-1} X_\alpha|$ and $|Y|^+ = |\prod_{\alpha<\gamma-1} X_\alpha|^+$, we obtain the equations in (1) by applying Lemma 4.1 (4) and (5). In case $|X_{\gamma-1}| > 2$, we obtain the equations in (2) and (3) by applying Lemma 4.1 (2), (3) and (1). In case $|X_{\gamma-1}| = 2$ and $\gamma - 1$ is limit, we see, by the similar argument above, that

$$s(Y) = i\text{-}s(Y) = d(Y) = w(Y) = \sup\{|\textstyle\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma - 1\},$$

$$ss(Y) = i\text{-}ss(Y) = (\sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma - 1\})^+$$

for every $i \in 3$. Hence we also obtain the equations in (2) and (3) by applying Lemma 4.1 (2), (3) and (1).

Let us consider the remaining case, that is, γ is successor, $|X_{\gamma-1}| = 2$ and $\gamma - 1$ is successor. Then the lexicographic product $Y = \prod_{\alpha<\gamma-1} X_\alpha$ can be identified with the lexicographic product $Z \times X_{\gamma-2}$ of two GO-spaces $Z = \prod_{\alpha<\gamma-2} X_\alpha$ and $X_{\gamma-2}$. Applying Lemma 4.1 (2) and (4), we have

$$s(X) = s(Y \times X_{\gamma-1}) = 2\text{-}s(Y) = 2\text{-}s(Z \times X_{\gamma-2}) = \max\{|Z|, 2\text{-}s(X_{\gamma-2})\}.$$

Similarly we obtain from Lemma 4.1 (3) and (5) that

$$ss(X) = 2\text{-}ss(Y) = \max\{|Z|^+, 2\text{-}ss(X_{\gamma-2})\}.$$

By Lemma 4.1 (1) and Corollary 4.2, we have

$$d(X) = d(Y \times X_{\gamma-1}) = w(Y) = w(Z \times X_{\gamma-2}) = \max\{|Z|, w(X_{\gamma-2})\}.$$

It had been seen that all equations in the present lemma hold in each case. □

For comparison, we state a result in [6] which can be proved similarly as (1) of the above theorem using Corollary 4.2.

Proposition 4.6. [6, Theorem 3.2] *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then*

$$w(X) = \begin{cases} \sup\{|\prod_{\alpha\le\beta} X_\alpha| : \beta < \gamma\} & \text{if } \gamma \text{ is limit,} \\ \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, w(X_{\gamma-1})\} & \text{if } \gamma \text{ is successor.} \end{cases}$$

5. Applications

About the hereditarily Lindelöf property of lexicographic products of LOTS's, Faber essentially proved the following result [4, Theorem 4.33] which is written in our terminology. Note that a GO-space X is hereditarily Lindelöf iff $s(X) \le \aleph_0$.

Proposition 5.1. [4, Theorem 4.3.3] *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of LOTS's. Then $s(X) \le \aleph_0$ if and only if the following clauses hold.*

1. *$\gamma \le \omega + 1$,*
2. *if $\gamma = \omega + 1$, then $|X_\omega| = 2$ and for every $\alpha < \omega$, $|X_\alpha| \le \aleph_0$,*
3. *if $\gamma = \omega$, then for every $\alpha < \omega$, $|X_\alpha| \le \aleph_0$,*
4. *if $2 \le \gamma < \omega$ and $|X_{\gamma-1}| > 2$, then $s(X_{\gamma-1}) \le \aleph_0$ and for every $\alpha < \gamma - 1$, $|X_\alpha| \le \aleph_0$,*

(5) *if* $2 \leq \gamma < \omega$ *and* $|X_{\gamma-1}| = 2$, *then* $s(X_{\gamma-2}) \leq \aleph_0$, $|N^+_{X_{\gamma-2}}| \leq \aleph_0$ *and for every* $\alpha < \gamma - 2$, $|X_\alpha| \leq \aleph_0$.

However, in the proposition above, "LOTS's" cannot be replaced by "GO-spaces". The lexicographic product $\mathbb{S} \times 2$ is such an example. Because, note that $s(\mathbb{S}) \leq \aleph_0$ and $N^+_{\mathbb{S}} = \emptyset$ hold. On the other hand, we see $s(\mathbb{S} \times 2) = 2^{\aleph_0}$ because $\mathbb{S} \times \{0\}$ is relatively discrete in $\mathbb{S} \times 2$, compare with (5) in the proposition above.

We can improve as follows. The proof is an easy consequence of (2) of Theorem 4.5.

Corollary 5.2. *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be a lexicographic product of GO-spaces. Then* $s(X) \leq \aleph_0$ *if and only if the following clauses hold.*

(1) $\gamma \leq \omega + 1$,
(2) *if* $\gamma = \omega + 1$, *then* $|X_\omega| = 2$ *and for every* $\alpha < \omega$, $|X_\alpha| \leq \aleph_0$,
(3) *if* $\gamma = \omega$, *then for every* $\alpha < \omega$, $|X_\alpha| \leq \aleph_0$,
(4) *if* $2 \leq \gamma < \omega$ *and* $|X_{\gamma-1}| > 2$, *then* $s(X_{\gamma-1}) \leq \aleph_0$ *and for every* $\alpha < \gamma - 1$, $|X_\alpha| \leq \aleph_0$,
(5) *if* $2 \leq \gamma < \omega$ *and* $|X_{\gamma-1}| = 2$, *then* 2-$s(X_{\gamma-2}) \leq \aleph_0$ *and for every* $\alpha < \gamma - 2$, $|X_\alpha| \leq \aleph_0$.

Proof. Assume $s(X) \leq \aleph_0$. If $\omega + 1 < \gamma$ were true, then noting $|\prod_{\omega\leq\alpha<\gamma} X_\alpha| > 3$ and by (2) of Theorem 4.5, we have $s(X) = s(\prod_{\alpha<\omega} X_\alpha \times \prod_{\omega\leq\alpha<\gamma} X_\alpha) \geq |\prod_{\alpha<\omega} X_\alpha| \geq 2^{\aleph_0} > \aleph_0$, a contradiction. We have shown (1). The remaining are left to the reader. $\square$

Similarly we can see the following which is a direct extension of [4, Theorem 4.3.2] for "GO-spaces" .

Corollary 5.3. *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be a lexicographic product of GO-spaces. Then* $d(X) \leq \aleph_0$ *if and only if the following clauses hold.*

(1) $\gamma \leq \omega + 1$,
(2) *if* $\gamma = \omega + 1$, *then* $|X_\omega| = 2$ *and for every* $\alpha < \omega$, $|X_\alpha| \leq \aleph_0$,
(3) *if* $\gamma = \omega$, *then for every* $\alpha < \omega$, $|X_\alpha| \leq \aleph_0$,
(4) *if* $2 \leq \gamma < \omega$ *and* $|X_{\gamma-1}| > 2$, *then* $d(X_{\gamma-1}) \leq \aleph_0$ *and for every* $\alpha < \gamma - 1$, $|X_\alpha| \leq \aleph_0$,
(5) *if* $2 \leq \gamma < \omega$ *and* $|X_{\gamma-1}| = 2$, *then* $w(X_{\gamma-2}) \leq \aleph_0$ *and for every* $\alpha < \gamma - 2$, $|X_\alpha| \leq \aleph_0$.

For a cardinal κ and an ordinal γ, define

$$\kappa^{<\gamma} = \sup\{\kappa^\mu : \ \mu \text{ is a cardinal and } \mu < \gamma\ \}, \text{ equivalently,}$$
$$\kappa^{<\gamma} = \sup\{\kappa^{|\alpha|} : \ \alpha \text{ is an ordinal and } \alpha < \gamma\ \},$$

see [6]. Using this cardinal function, we can easily calculate cardinal functions of lexicographic products of type Y^γ.

Corollary 5.4. *Let κ be a cardinal with $\kappa \geq 2$ and $X = \prod_{\alpha<\gamma} X_\alpha$ a lexicographic product of GO-spaces with $|X_\alpha| = \kappa$ for every $\alpha < \gamma$. Then the following hold.*

(1)
$$d(X) = s(X) = \begin{cases} \kappa^{<\gamma-1} & \text{if } \kappa = 2 \text{ and } \gamma \text{ is successor,} \\ \kappa^{<\gamma} & \text{otherwise,} \end{cases}$$

(2) $w(X) = 2\text{-}s(X) = 1\text{-}s(X) = 0\text{-}s(X) = \kappa^{<\gamma}$, *see* [6, Corollary 4.4] *for the weight.*

(3)
$$ss(X) = \begin{cases} (\kappa^{<\gamma-1})^+ & \text{if } \kappa = 2 \text{ and } \gamma \text{ is successor,} \\ (\kappa^{<\gamma})^+ & \text{otherwise,} \end{cases}$$

(4) $2\text{-}ss(X) = 1\text{-}ss(X) = 0\text{-}ss(X) = (\kappa^{<\gamma})^+$.

Proof. (1) First we consider the case $\kappa = 2$. Since we are assuming that X is infinite, we have $\gamma \geq \omega$. When γ is limit, it follows from Lemma 4.4 that $d(X) = s(X) = 2^{<\gamma}$. So let γ be successor. When $\gamma - 1$ is limit, it follows from $\gamma > \gamma - 1 \geq |\gamma| \geq \omega$ and Theorem 4.5 (2) and (3) that $d(X) = s(X) = \sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma - 1\} = 2^{<\gamma-1}$. When $\gamma - 1$ is successor, it follows from $\gamma - 1 > \gamma - 2 \geq |\gamma| \geq \omega$, $|X_{\gamma-2}| = 2$ and Theorem 4.5 (2) and (3) that $s(X) = \max\{|\prod_{\alpha<\gamma-2} X_\alpha|, 2\text{-}s(X_{\gamma-2})\} = |\prod_{\alpha<\gamma-2} X_\alpha| = 2^{|\gamma-2|} = 2^{<\gamma-1}$ and $d(X) = \max\{|\prod_{\alpha<\gamma-2} X_\alpha|, w(X_{\gamma-2})\} = |\prod_{\alpha<\gamma-2} X_\alpha| = 2^{|\gamma-2|} = 2^{<\gamma-1}$.

Next we consider the case $\kappa > 2$. When γ is limit, Lemma 4.4 shows $d(X) = s(X) = \kappa^{<\gamma}$. So let γ be successor. Then Theorem 4.5 (2) and (3) show $s(X) = \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, s(X_{\gamma-1})\}$ and $d(X) = \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, d(X_{\gamma-1})\}$. Now when $2 < \kappa < \omega$, by $\aleph_0 \leq |X|$, we have $\omega \leq \gamma$, therefore $|\prod_{\alpha<\gamma-1} X_\alpha| \geq \kappa^\omega = 2^\omega > \omega$, and so $s(X) = d(X) = |\prod_{\alpha<\gamma-1} X_\alpha| = \kappa^{|\gamma-1|} = \kappa^{<\gamma}$. When $\kappa \geq \omega$, it follows from $|\prod_{\alpha<\gamma-1} X_\alpha| \geq |X_0| = \kappa = |X_{\gamma-1}| \geq \max\{s(X_{\gamma-1}), d(X_{\gamma-1})\}$ that $s(X) = d(X) = |\prod_{\alpha<\gamma-1} X_\alpha| = \kappa^{|\gamma-1|} = \kappa^{<\gamma}$.

(2) We only prove the case $0\text{-}s(X)$, because the other are similar. When γ is limit, as above, use Lemma 4.4. Let γ be successor, then $0\text{-}s(X) = \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, 0\text{-}s(X_{\gamma-1})\}$. Because of $\gamma \geq 2$, we have $|\prod_{\alpha<\gamma-1} X_\alpha| \geq |X_0| = \kappa = |X_{\gamma-1}| \geq 0\text{-}s(X_{\gamma-1})$, therefore $0\text{-}s(X) = |\prod_{\alpha<\gamma-1} X_\alpha| = \kappa^{|\gamma-1|} = \kappa^{<\gamma}$.

(3) and (4) are similar. □

Example 5.5. By the corollary above, when Y is a GO-space with $|Y| > 2$, the weight, the density and the spread of the lexicographic product Y^γ coincide. For example: .

(1) $w(3^\omega) = s(3^\omega) = \aleph_0$, $w(3^{\omega+1}) = s(3^{\omega+1}) = w(3^{\omega+2}) = s(3^{\omega+2}) = \cdots = w(3^{\omega_1}) = s(3^{\omega_1}) = 2^{\aleph_0}$, $w(3^{\omega_1+1}) = s(3^{\omega_1+1}) = 2^{\aleph_1}$,
(2) $w((\omega+1)^\omega) = s((\omega+1)^\omega) = \aleph_0$, $w((\omega+1)^{\omega+1}) = s((\omega+1)^{\omega+1}) = w((\omega+1)^{\omega+2}) = s((\omega+1)^{\omega+2}) = \cdots = w((\omega+1)^{\omega_1}) = s((\omega+1)^{\omega_1}) = 2^{\aleph_0}$, $w((\omega+1)^{\omega_1+1}) = s((\omega+1)^{\omega_1+1}) = 2^{\aleph_1}$,
(3) $w(\mathbb{Q}^\omega) = s(\mathbb{Q}^\omega) = \aleph_0$, $w(\mathbb{Q}^{\omega+1}) = s(\mathbb{Q}^{\omega+1}) = w(\mathbb{Q}^{\omega+2}) = s(\mathbb{Q}^{\omega+2}) = \cdots = w(\mathbb{Q}^{\omega_1}) = s(\mathbb{Q}^{\omega_1}) = 2^{\aleph_0}$, $w(\mathbb{Q}^{\omega_1+1}) = s(\mathbb{Q}^{\omega_1+1}) = 2^{\aleph_1}$,
(4) $w(\mathbb{S}^\omega) = s(\mathbb{S}^\omega) = w(\mathbb{S}^{\omega+1}) = s(\mathbb{S}^{\omega+1}) = w(\mathbb{S}^{\omega+2}) = s(\mathbb{S}^{\omega+2}) = \cdots = w(\mathbb{S}^{\omega_1}) = s(\mathbb{S}^{\omega_1}) = 2^{\aleph_0}$, $w(\mathbb{S}^{\omega_1+1}) = s(\mathbb{S}^{\omega_1+1}) = 2^{\aleph_1}$.

Example 5.6. Compared with the example above, the case $|Y| = 2$ is somewhat strange.

(1) $w(2^\omega) = s(2^\omega) = \aleph_0$,
(2) $w(2^{\omega+1}) = 2\text{-}s(2^{\omega+1}) = 1\text{-}s(2^{\omega+1}) = 0\text{-}s(2^{\omega+1}) = 2^{\aleph_0}$,
(3) $d(2^{\omega+1}) = s(2^{\omega+1}) = \aleph_0$,
(4) $w(2^{\omega+2}) = s(2^{\omega+2}) = 2^{\aleph_0}$,
(5) $w(2^{\omega_1}) = s(2^{\omega_1}) = 2^{\aleph_0}$,
(6) $w(2^{\omega_1+1}) = 2\text{-}s(2^{\omega_1+1}) = 1\text{-}s(2^{\omega_1+1}) = 0\text{-}s(2^{\omega_1+1}) = 2^{\aleph_1}$,
(7) $d(2^{\omega_1+1}) = s(2^{\omega_1+1}) = 2^{\aleph_0}$,
(8) $w(2^{\omega_1+2}) = s(2^{\omega_1+2}) = 2^{\aleph_1}$.

It is known that if Y is a ω_1-Suslin line, that is, a LOTS Y with $\aleph_1 = d(Y) = cc(Y)$, then the Tychonoff product space Y^2 cannot be ccc, i.e., $c(Y^2) > \aleph_0$, in fact $c(Y^2) = d(Y^2)$ $(= \aleph_1)$, e.g., see [19, II Lemma 4.3]. Using (1) of the corollary above, we have a similar result about lexicographic products.

Corollary 5.7. *Let Y be a GO-space with $|Y| \geq 2$. For every $\gamma \geq 2$, the density $d(Y^\gamma)$ and the spread $s(Y^\gamma)$ $(= c(Y^\gamma))$ of the lexicographic product Y^γ coincide.*

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References

[1] D. Buhagiar, E. Chetcuti and H. Weber, *Some cardinal functions in lexicographic products of LOTS*, Eur. J. Math., **4** (2018), 1505-1514.

[2] R. Engelking, *General Topology-Revised and completed ed.*. Heldermann Verlag, Berlin (1989).
[3] P. Erdös and A. Tarski, *On the fomilies of mutually exclusive sets.* Ann. of Math., **44** (1943), 315-329.
[4] M. J. Faber, *Metrizability in generalized ordered spaces*, Mathematical Centre Tracts, No. 53. Mathematisch Centrum, Amsterdam, 1974.
[5] Y. Hirata and N. Kemoto, *Countable metacompactness of products of LOTS'*, Top. Appl., 178 (2014) 1-16.
[6] Y. Hirata and N. Kemoto, *The weight of lexicographic products*, Top. Appl., 284 (2020) Article 107357.
[7] Y. Hirata and N. Kemoto, *A characterization of paracompactness of lexicographic products*, Top. Proc., 56 (2020) 85-95.
[8] Y. Hirata N. Kemoto and Y. Yajima, *Products of monotonically normal spaces with various special factors*, Top. Appl., **164** (2014) 45–86.
[9] R. Hodel, *Cardinal functions I*, Handbook of set-theoretic topology, 1-62, North-Holland, Amsterdam, 1984.
[10] I. Juhász, *Cardinal functions in topology-ten years later. Second edition*, Mathematical Centre Tracts, 123. Mathematisch Centrum, Amsterdam (1980).
[11] N. Kemoto, *Normality of products of GO-spaces and cardinals*, Top. Proc. 18 (1993) 133-142.
[12] N. Kemoto, *The lexicographic ordered products and the usual Tychonoff products*, Top. Appl., 162 (2014) 20-33.
[13] N. Kemoto, *Orderability of products*, Top. Proc., 50 (2017) 67-78.
[14] N. Kemoto, *Lexicographic products of GO-spaces*, Top. Appl., 232 (2017), 267-280.
[15] N. Kemoto, *Paracompactness of Lexicographic products of GO-spaces*, Top. Appl., 240 (2018) 35-58.
[16] N. Kemoto, *Hereditary paracompactness of lexicographic products*, Top. Proc., 53 (2019) 301-317.
[17] N. Kemoto, *Countable compactness of lexicographic products of GO-spaces*, Comment. Math. Univ. Carol., 60 (2019) 421-439.
[18] N. Kemoto, *Completeness of lexicographic products*, Top. Proc., 56 (2020) 85-95.
[19] K. Kunen, *Set Theory. An Introduction to Independence Proofs, Studies in Logic and the Foundations of Mathematics*, vol. 102, North-Holland, Amsterdam, 1980.
[20] D.J. Lutzer, *On generalized ordered spaces*, Dissertationes Math. Rozprawy Mat. 89 (1971).
[21] T. Miwa and N. Kemoto, *Linearly ordered extensions of GO-spaces*, Top. Appl., 54 (1993), 133-140.

Department of Mathematics, Kanagawa University, Yokohama, 221-8686 Japan
E-mail address: `hirata-y@kanagawa-u.ac.jp`

Department of Mathematics, Oita University, Oita, 870-1192 Japan
E-mail address: `nkemoto@oita-u.ac.jp`