C^*-EMBEDDING AND P-EMBEDDING IN SUBSPACES OF PRODUCTS OF ORDINALS

NOBUYUKI KEMOTO AND TOSHIMICHI USUBA

Abstract. It is known that in $X = A \times B$, where A and B are subspaces of ordinals, all closed C^*-embedded subspaces of X are P-embedded. Also it is asked whether all closed C^*-embedded subspaces of X are P-embedded whenever X is a subspace of products of two ordinals.

In this paper, we prove that both of the following are consistent with ZFC:

- there is a subspace X of $(\omega + 1) \times \omega_1$ such that the closed subspace $X \cap (\{\omega\} \times \omega_1)$ is C^*-embedded in X but not P-embedded in X,
- for every subspace X of $(\omega + 1) \times \omega_1$, if the closed subspace $X \cap (\{\omega\} \times \omega_1)$ is C^*-embedded in X, then it is P-embedded in X.

1. Introduction

A subset F of a space X is *C^*-embedded* in X if every continuous function from F to the unit interval $\mathbb{I} := [0, 1]$ can be continuously extended over X. Also recall that a subset F of a space of X is *P-embedded* in X if every continuous function from F to a Banach space can be continuously extended over X. We remark that P-embedded subspaces are C^*-embedded and that a clopen subspace F' of a C^*-embedded (P-embedded) subspace F of X is also C^*-embedded (P-embedded, respectively) in X. Also it is well known that:

- a space X is normal if and only if all closed subspaces are C^*-embedded [4, Theorem 2.1.8],
- a space X is collectionwise normal if and only if all closed subspaces are P-embedded [2].

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Note that both Lindelöf spaces and metrizable spaces are collectionwise normal.

In [10], it is proved that in $X = A \times B$, where A and B are subspaces of ordinals, all closed C^*-embedded subspaces of X are P-embedded and it is asked whether all closed C^*-embedded subspaces of X are P-embedded whenever X is a subspace of the product of two ordinals [10, Question 2].

In this paper, we prove that both of the following are consistent with ZFC:

- there is a subspace X of $(\omega + 1) \times \omega_1$ such that the closed subspace $X \cap (\{\omega\} \times \omega_1)$ is C^*-embedded in X but not P-embedded in X,
- for every subspace X of $(\omega + 1) \times \omega_1$, if the closed subspace $X \cap (\{\omega\} \times \omega_1)$ is C^*-embedded in X, then it is P-embedded in X.

In the remainder of this section, we prepare basic notions and facts. Spaces are completely regular T_1 topological spaces. For a space X, a subset U is a *cozero-set in* X if $U = h^{-1}[(0, 1]]$ for some continuous function $h : X \to \mathbb{I}$, where $(0, 1]$ denotes the unit half open interval in $\mathbb{I}$.

A collection $\mathcal{U}$ of subsets of a space X is said to be *locally finite* (*discrete*) in X if every point in X has a neighborhood which meets at most finitely many (one, respectively) members of $\mathcal{U}$. A subset F of a space X is said to be *discrete* if the collection $\{\{x\} : x \in F\}$ is discrete. Note that the union of a locally finite collection of cozero sets is also a cozero set.

For a collection $\mathcal{U}$ of subsets of X and a subspace F of X, $\mathcal{U} \restriction F$ denotes the set $\{U \cap F : U \in \mathcal{U}\}$. Also $h \restriction F$ denotes the restriction of h to F whenever h is a function on X and $F \subset X$. An open cover $\mathcal{U}$ of subsets of a space X is called a *cozero cover* of X if each member of $\mathcal{U}$ is a cozero-set in X.

There are characterizations of C^*-embedding and P-embedding in terms of cozero covers. We will use these characterizations rather than original definitions.

Proposition 1.1. [28, Lemma 2.1] *Let F be a subspace of a space X. Then F is C^*-embedded in X if and only if for every finite (or two elements) cozero cover $\mathcal{U}$ of F, there is a locally finite cozero cover $\mathcal{V}$ of X such that $\mathcal{V} \restriction F$ refines $\mathcal{U}$, that is, for every $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ with $V \cap F \subset U$.*

Proposition 1.2. [1, Theorem 14.7] *Let F be a subspace of a space X. Then F is P-embedded in X if and only if for every locally finite* *cozero cover* $\mathcal{U}$ *of* F*, there is a locally finite cozero cover* $\mathcal{V}$ *of* X *such that* $\mathcal{V} \restriction F$ *refines* $\mathcal{U}$*.*

The symbols ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. The first cardinal exceeding ω_1 is denoted by ω_2. Also the symbol 2^ω (2^{ω_1}) denotes the cardinality of the collection of all subsets of ω (ω_1, respectively). The symbol $[\omega]^\omega$ denotes the set of all infinite subsets of ω. Ordinal numbers have the usual order topologies. A subset S of a regular uncountable cardinal κ is called *stationary* if it intersects all closed unbounded (abbreviated as club) subsets of κ. The Pressing Down Lemma (PDL) will be frequently used.

Lemma 1.3. [27, PDL, II Lemma 6.15] *Let* S *be a stationary set in a regular uncountable cardinal* κ*. If a function* $f : S \to \kappa$ *is regressive, that is* $f(\alpha) < \alpha$ *for every* $\alpha \in S$*, then there are a stationary set* $S' \subset S$ *and* $\gamma < \kappa$ *such that* $f(\alpha) = \gamma$ *for every* $\alpha \in S'$*.*

The following are easy consequences of the PDL.

Lemma 1.4. *Let* X *be a stationary set in a regular uncountable cardinal* κ*. If* $\mathcal{U}$ *is a locally finite collection of subsets of* X*, then there is* $\gamma < \kappa$ *such that* $\{U \in \mathcal{U} : (X \cap (\gamma, \kappa)) \cap U \neq \emptyset\}$ *is finite.*

Lemma 1.5. *Let* κ *be a regular uncountable cardinal and* X *a stationary set of* κ*. If* $h : X \to \mathbb{R}$ *is continuous, where* $\mathbb{R}$ *denotes the real line, then it is constant on some tail, that is, there is* $\alpha^* < \kappa$ *with* $h \restriction (X \cap (\alpha^*, \kappa))$ *is constant. Therefore a cozero-set of* X *is either bounded or contains some tail.*

Further we introduce technical notation which will be used frequently in our arguments. Let A be a subset of a regular uncountable cardinal κ. $\mathrm{Lim}_\kappa(A)$, which is usually written $\mathrm{Lim}(A)$, denotes the set $\{\alpha < \kappa : \alpha = \sup(A \cap \alpha)\}$, that is, the set of all cluster points of A in κ, where we define $\sup \emptyset = -1$. Note that $\mathrm{Lim}(A)$ is club whenever A is unbounded in κ.

Let C be a club set in a regular uncountable cardinal κ, then obviously $\mathrm{Lim}(C) \subset C$, in this case, we define $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}(C)$. Moreover let $p_C(\alpha) = \sup(C \cap \alpha)$ for $\alpha \in C$. Note that for each $\alpha \in C$, $p_C(\alpha) \in C \cup \{-1\}$ holds, in particular $p_C(\min C) = -1$, also $p_C(\alpha) < \alpha$ if and only if $\alpha \in \mathrm{Succ}(C)$. Intuitively, $p_C(\alpha)$ is the immediate predecessor of α in C whenever $\alpha \in \mathrm{Succ}(C)$. Observe that $\kappa \setminus C$ is the disjoint union of $\{(p_C(\alpha), \alpha) : \alpha \in \mathrm{Succ}(C)\}$ of open intervals. Also note that $\kappa \setminus \mathrm{Lim}(C)$ is the disjoint union of $\{(p_C(\alpha), \alpha] : \alpha \in \mathrm{Succ}(C)\}$. In particular, $\mathrm{Lim}(\omega_1)$ and $\mathrm{Succ}(\omega_1)$ are denoted by Lim and Succ respectively.

Note the following lemma.

Lemma 1.6. [27, II Lemma 6.13] *Let κ be a regular uncountable cardinal, $A \subset \kappa$ and $f : A \to \kappa$. Then the set $\{\alpha < \kappa : \forall \beta \in A \cap \alpha (f(\beta) < \alpha)\}$ is club in κ.*

We also use the following notation. Let μ and ν be ordinals and $X \subset (\mu+1) \times (\nu+1)$. For subsets $C \subset \mu+1$ and $D \subset \nu+1$, define

$$X_C = X \cap C \times (\nu+1), X^D = X \cap (\mu+1) \times D, X_C^D = X_C \cap X^D.$$

Moreover for $\alpha \leq \mu$ and $\beta \leq \nu$, $V_\alpha(X)$ denotes the vertical slice $\{\delta \leq \nu : \langle \alpha, \delta \rangle \in X\}$ of X and $H_\beta(X)$ denotes the horizontal slice $\{\gamma \leq \mu : \langle \gamma, \beta \rangle \in X\}$ of X.

For a space X, $X = \bigoplus_{\lambda \in \Lambda} X_\lambda$ means that the space X is the pairwise disjoint sum of (cl)open subspaces X_λ's.

For undefined topological or set theoretical notions, see [4, 11, 27]. For other researches about topological properties of products of ordinals and the above notation, see [5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

2. P-embedding in subspaces of $(\omega+1) \times \omega_1$

Two subsets F and H of a space X are said to be *separated* in X if there are disjoint open sets U and V in X with $F \subset U$ and $H \subset V$. In this section, we consider the P-embedding in subspaces of $(\omega+1) \times \omega_1$. First we prove:

Lemma 2.1. *Let X be a subspace of $(\omega+1) \times \omega_1$ and A denote the set $\{n \in \omega : V_n(X)$ is stationary in $\omega_1\}$. Then for every closed subset F in X with $F \subset X_{\{\omega\}}$, F is P-embedded in X if and only if, if $V_\omega(X)$ is not stationary in ω_1, then $\mathrm{Cl}_X X_A$ and F are almost separated, that is, there is $\alpha^* < \omega_1$ such that $X^{(\alpha^*, \omega_1)} \cap \mathrm{Cl}_X X_A$ and $X^{(\alpha^*, \omega_1)} \cap F$ are separated in $X^{(\alpha^*, \omega_1)}$, where Cl_X denotes the closure in X.*

Proof. "only if" part: Let F be P-embedded in X and suppose that $V_\omega(X)$ is not stationary in ω_1. Take a club set C in ω_1 disjoint from $V_\omega(X)$. For every $\alpha \in \mathrm{Succ}(C)$, set $F(\alpha) = F^{(p_C(\alpha), \alpha]}$ $(= F \cap X^{(p_C(\alpha), \alpha]})$ and $X(\alpha) = X^{(p_C(\alpha), \alpha]}$. Note that each $F(\alpha)$ is closed in X and each $X(\alpha)$ is countable and first countable, so metrizable. Since $\mathcal{U} = \{F(\alpha) : \alpha \in \mathrm{Succ}(C)\}$ is a disjoint clopen cover of F and F is P-embedded in X, we can find a locally finite cozero cover $\mathcal{V}$ of X such that $\mathcal{V} \restriction F$ refines $\mathcal{U}$. For each $\alpha \in \mathrm{Succ}(C)$, set $W(\alpha) = (\bigcup\{V \in \mathcal{V} : F(\alpha) \cap V \neq \emptyset\}) \cap X(\alpha)$. Then obviously $\mathcal{W} = \{W(\alpha) : \alpha \in \mathrm{Succ}(C)\}$ is a locally finite and pairwise disjoint collection of cozero sets in X with $F(\alpha) \subset W(\alpha) \subset X(\alpha)$ for every $\alpha \in \mathrm{Succ}(C)$ and also $\mathcal{W}$ covers F. For every $n \in A$, since $V_n(X)$ is stationary and $\mathcal{W}$ is locally finite in X with $W(\alpha) \subset X(\alpha)$ ($\alpha \in \mathrm{Succ}(C)$), we can find $\alpha_n < \omega_1$ with $X_{\{n\}}^{(\alpha_n,\omega_1)} \cap (\bigcup \mathcal{W}) = \emptyset$. Letting $\alpha^* = \sup\{\alpha_n : n \in A\}$, we see $X_A^{(\alpha^*,\omega_1)} \cap (\bigcup \mathcal{W}) = \emptyset$. For every $\alpha \in \mathrm{Succ}(C)$, using the normality of $X(\alpha)$, take an open set $U(\alpha)$ of $X(\alpha)$ with $F(\alpha) \subset U(\alpha) \subset \mathrm{Cl}_{X(\alpha)} U(\alpha) \subset W(\alpha)$. Since $\mathcal{W}$ is locally finite, we have $X_A^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X(\bigcup_{\alpha\in\mathrm{Succ}(C)} U(\alpha)) = \emptyset$. Now letting $U = X^{(\alpha^*,\omega_1)} \cap (\bigcup_{\alpha\in\mathrm{Succ}(C)} U(\alpha))$, U and $X^{(\alpha^*,\omega_1)} \setminus \mathrm{Cl}_{X^{(\alpha^*,\omega_1)}} U$ separate $X^{(\alpha^*,\omega_1)} \cap F$ and $X^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X X_A$.

"if" part: Assume that if $V_\omega(X)$ is not stationary in ω_1, then $\mathrm{Cl}_X X_A$ and F are almost separated. We will see that F is P-embedded in X.

First assume that $V_\omega(X)$ is stationary in ω_1 and $\mathcal{U}$ is a locally finite cozero cover of F. Since F is closed in the collectionwise normal space $X_{\{\omega\}}$, one can fix a locally finite cozero cover $\mathcal{U}'$ of $X_{\{\omega\}}$ such that $\mathcal{U}' \restriction F$ refines $\mathcal{U}$. Since $X_{\{\omega\}}$ is homeomorphic to the stationary set $V_\omega(X)$, it follows from Lemmas 1.4 and 1.5 that there are $U \in \mathcal{U}'$ and $\alpha^* < \omega_1$ with $X_{\{\omega\}}^{(\alpha^*,\omega_1)} \subset U$. On the other hand, since $X^{[0,\alpha^*]}$ is collectionwise normal, we can find a locally finite cozero cover $\mathcal{V}_0$ of $X^{[0,\alpha^*]}$such that $\mathcal{V}_0 \restriction X_{\{\omega\}}^{[0,\alpha^*]}$ refines $\mathcal{U}'$ hence $\mathcal{V}_0 \restriction F^{[0,\alpha^*]}$ refines $\mathcal{U}$. Now $\mathcal{V} := \mathcal{V}_0 \cup \{X^{(\alpha^*,\omega_1)}\}$ is a locally finite cozero cover of X such that $\mathcal{V} \restriction F$ refines $\mathcal{U}$.

Next assume that $V_\omega(X)$ is not stationary in ω_1. By the assumption, we can fix $\alpha^* < \omega_1$ and an open set W in $X^{(\alpha^*,\omega_1)}$ such that

$$X^{(\alpha^*,\omega_1)} \cap F \subset W \subset \mathrm{Cl}_{X^{(\alpha^*,\omega_1)}} W \subset X^{(\alpha^*,\omega_1)} \setminus \mathrm{Cl}_X X_A.$$

Since $X^{[0,\alpha^*]}$ is collectionwise normal, as above, it suffices to verify the following claim.

Claim. $F^{(\alpha^*,\omega_1)}$ is P-embedded in $X^{(\alpha^*,\omega_1)}$.

Proof. Let $\mathcal{U}$ be a locally finite cozero cover of $F^{(\alpha^*,\omega_1)}$ and take a club set C in ω_1 disjoint from $V_\omega(X) \cup (\bigcup_{n\in\omega\setminus A} V_n(X))$. For each $\alpha \in \mathrm{Succ}(C)$, set $F(\alpha) = F^{(\alpha^*,\omega_1)\cap(p_C(\alpha),\alpha]}$ and $X(\alpha) = X^{(\alpha^*,\omega_1)\cap(p_C(\alpha),\alpha]}$. Obviously $F^{(\alpha^*,\omega_1)}$ can be represented as the topological sum $F^{(\alpha^*,\omega_1)} = \bigoplus_{\alpha\in\mathrm{Succ}(C)} F(\alpha)$. For each $\alpha \in \mathrm{Succ}(C)$, since $F(\alpha)$ is a closed subspace of the collectionwise normal clopen subspace $X(\alpha)$ of X, we can take a locally finite cozero cover $\mathcal{V}_\alpha$ of $X(\alpha)$ such that $\mathcal{V}_\alpha \restriction F(\alpha)$ refines $\mathcal{U}$, moreover we can take a cozero set V_α in $X(\alpha)$ (hence in X) such that $F(\alpha) \subset V_\alpha \subset W \cap X_\alpha$ and let $\mathcal{V}'_\alpha = \{V \cap V_\alpha : V \in \mathcal{V}_\alpha\}$. Then $\mathcal{V}'_\alpha$ is a locally finite collection of cozero sets in X with $F(\alpha) \subset \bigcup \mathcal{V}'_\alpha \subset X(\alpha)$ such that $\mathcal{V}'_\alpha \restriction F(\alpha)$ refines $\mathcal{U}$. Now let $\mathcal{V}' = \bigcup_{\alpha\in\mathrm{Succ}(C)} \mathcal{V}'_\alpha$, then we see $\bigcup \mathcal{V}' \subset W$ and that $\mathcal{V}'$ is a collection of cozero sets in $X^{(\alpha^*,\omega_1)}$ with $F^{(\alpha^*,\omega_1)} \subset \bigcup \mathcal{V}'$. We show:

Fact. $\mathcal{V}'$ is locally finite in $X^{(\alpha^*,\omega_1)}$.

Proof. It suffices to see that $\{V_\alpha : \alpha \in \mathrm{Succ}(C)\}$ is locally finite in $X^{(\alpha^*,\omega_1)}$. So let $x \in X^{(\alpha^*,\omega_1)}$. When $x \in X^{(\alpha^*,\omega_1)}_{\{\omega\}}$, say $x = \langle \omega, \gamma \rangle$, we can fix $\alpha \in \mathrm{Succ}(C)$ with $p_C(\alpha) < \gamma < \alpha$. Then $X(\alpha)$ is a neighborhood of x meeting at most one member of $\{V_\alpha : \alpha \in \mathrm{Succ}(C)\}$.

Now assume $x \in X^{(\alpha^*,\omega_1)}_\omega$, say $x = \langle n, \gamma \rangle$ for some $n \in \omega$. When $n \in \omega \setminus A$, by $C \cap V_n(X) = \emptyset$, we can fix $\alpha \in \mathrm{Succ}(C)$ with $p_C(\alpha) < \gamma < \alpha$. Then as above, $X(\alpha)$ is a neighborhood of x meeting at most one member of $\{V_\alpha : \alpha \in \mathrm{Succ}(C)\}$. When $n \in A$, by $x \in X^{(\alpha^*,\omega_1)}_{\{n\}} \subset X^{(\alpha^*,\omega_1)}_A \subset X_A \subset \mathrm{Cl}_X X_A$, we see $x \notin \mathrm{Cl}_{X^{(\alpha^*,\omega_1)}} W$. It follows from $\bigcup \mathcal{V}' \subset W$ that $X^{(\alpha^*,\omega_1)} \setminus \mathrm{Cl}_{X^{(\alpha^*,\omega_1)}} W$ is a neighborhood of x meeting no members of $\{V_\alpha : \alpha \in \mathrm{Succ}(C)\}$. This completes the proof of Fact.

Now since $X^{(\alpha^*,\omega_1)}_\omega$ is cozero in $X^{(\alpha^*,\omega_1)}$, $\mathcal{V}' \cup \{X^{(\alpha^*,\omega_1)}_\omega\}$ witnesses the P-embedding of $F^{(\alpha^*,\omega_1)}$ in $X^{(\alpha^*,\omega_1)}$. This completes the proof of Claim. □

Corollary 2.2. *Let X be a subspace of $(\omega + 1) \times \omega_1$ and $A = \{n \in \omega : V_n(X)$ is stationary in $\omega_1\}$. Then $X_{\{\omega\}}$ is P-embedded in X if and only if, if $V_\omega(X)$ is not stationary in ω_1, then $V_\omega(\mathrm{Cl}_X X_A)$ is bounded in ω_1.*

Proof. "only if" part: Assume that $X_{\{\omega\}}$ is P-embedded in X and $V_\omega(X)$ is not stationary. By Lemma 2.1, $\mathrm{Cl}_X X_A$ and $X_{\{\omega\}}$ are almost separated, in particular, $X^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X X_A$ and $X^{(\alpha^*,\omega_1)} \cap X_{\{\omega\}}$ are separated in $X^{(\alpha^*,\omega_1)}$ for some $\alpha^* < \omega_1$, which implies that $V_\omega(\mathrm{Cl}_X X_A)$ is bounded in ω_1.

"if" part: Assume that if $V_\omega(X)$ is not stationary in ω_1, then $V_\omega(\mathrm{Cl}_X X_A)$ is bounded in ω_1. By Lemma 2.1, it suffices to check that if $V_\omega(X)$ is not stationary in ω_1, then $\mathrm{Cl}_X X_A$ and $X_{\{\omega\}}$ are almost separated. So assume that $V_\omega(X)$ is not stationary, then by our assumption, $X^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X X_A$ and $X^{(\alpha^*,\omega_1)}_{\{\omega\}}$ are disjoint for some $\alpha^* < \omega_1$. From $X^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X X_A = \mathrm{Cl}_{X^{(\alpha^*,\omega_1)}} X^{(\alpha^*,\omega_1)}_A$, it suffices to verify the following claim.

Claim. $X^{(\alpha^*,\omega_1)}_A$ is clopen in $X^{(\alpha^*,\omega_1)}$.

Proof. Since A is open in $\omega + 1$, $X^{(\alpha^*,\omega_1)}_A$ is open in $X^{(\alpha^*,\omega_1)}$. Let $y \in \mathrm{Cl}_{X^{(\alpha^*,\omega_1)}} X^{(\alpha^*,\omega_1)}_A$. Because $X^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X X_A$ and $X^{(\alpha^*,\omega_1)}_{\{\omega\}}$ are disjoint and $y \in X^{(\alpha^*,\omega_1)} \cap \mathrm{Cl}_X X_A$, we see $y \in X_\omega^{(\alpha^*,\omega_1)}$. Since each $X_{\{n\}}^{(\alpha^*,\omega_1)}$ is open in $X^{(\alpha^*,\omega_1)}$, so we must have $y \in X_A^{(\alpha^*,\omega_1)}$, thus $X_A^{(\alpha^*,\omega_1)}$ is closed. This completes the proof of the claim.

□

3. C^*-EMBEDDING IN SPECIAL SUBSPACES OF $(\omega+1)\times\omega_1$

In this section, we consider C^*-embedding in certain type of subspaces of $(\omega+1)\times\omega_1$.

Definition 3.1. Let $\mathcal{N}$ be a sequence in $[\omega]^\omega$ of length ω_1, say $\mathcal{N} = \{N(\alpha) : \alpha < \omega_1\}$. The subspace $X_\mathcal{N}$ of $(\omega+1)\times\omega_1$ is defined as follows:

$$X_\mathcal{N} = \omega \times \mathrm{Lim} \cup \Big(\bigcup_{\alpha<\omega_1} (N(\alpha) \cup \{\omega\}) \times \{\alpha+1\}\Big).$$

This type of subspaces of $(\omega+1)\times\omega_1$ is first discussed in [15]. Throughout the rest of this section, let $X = X_\mathcal{N}$ and $A = \{n \in \omega : V_n(X) \text{ is stationary in } \omega_1\}$. First note that the subspace $X_{\{\omega\}}$ $(= \{\omega\}\times \mathrm{Succ})$ of X is closed and discrete in X and $A = \omega$. So Corollary 2.2 yields:

Corollary 3.2. *$X_{\{\omega\}}$ is not P-embedded in X.*

Moreover we check:

Lemma 3.3. *X is not normal.*

Proof. Obviously $X_{\{\omega\}}$ and X^{Lim} are disjoint closed sets, we will see that they cannot be separated. So let U be an open set in X containing X^{Lim}. For every $n \in \omega$, by $\mathrm{Lim} \subset V_n(X)$ and the PDL, we can fix $\alpha_n < \omega_1$ with $X_{\{n\}}^{(\alpha_n,\omega_1)} \subset U$. Letting $\alpha^* = \sup\{\alpha_n : n \in \omega\}$, we see $X_\omega^{(\alpha^*,\omega_1)} \subset U$. Pick $\alpha < \omega_1$ with $\alpha^* \le \alpha$. Because $X_\omega^{\{\alpha+1\}} = N(\alpha) \times \{\alpha+1\} \subset X_\omega^{(\alpha^*,\omega_1)} \subset U$ holds and $N(\alpha)$ is infinite, we see $\langle\omega, \alpha+1\rangle \in \mathrm{Cl}_X U \cap X_{\{\omega\}}$. Therefore $X_{\{\omega\}}$ and X^{Lim} cannot be separated. □

The proof of the above lemma also shows that $X_{\{\omega\}} \cup X^{\mathrm{Lim}}$ is a closed subspace of X which is not C^*-embedded in X. In the following lemma, we characterize that the closed subspace $X_{\{\omega\}}$ is C^*-embedded in X.

Lemma 3.4. *For the space $X = X_\mathcal{N}$ above with $\mathcal{N} = \{N(\alpha) : \alpha < \omega_1\}$, the following are equivalent:*

(1) *the closed subspace $X_{\{\omega\}}$ is C^*-embedded in X,*

(2) *for every function $h : \omega_1 \to 2$, where $2 = \{0, 1\}$, there is a sequence $\{r_n : n \in \omega\} \subset \mathbb{I}$ such that for every $i \in 2$, except for countably many $\alpha \in h^{-1}[\{i\}]$, $\{r_n : n \in N(\alpha)\}$ converges to i in the usual sense, that is, for every $\varepsilon > 0$, there is $m \in \omega$ such that for every $n \in N(\alpha) \cap (m, \omega)$, $|r_n - i| < \varepsilon$ holds,*
(3) *for every function $h : \omega_1 \to 2$, there is a function $k : \omega \to 2$ such that for every $i \in 2$, except for countably many $\alpha \in h^{-1}[\{i\}]$, $N(\alpha) \subset^* k^{-1}[\{i\}]$ holds, that is, $N(\alpha) \setminus k^{-1}[\{i\}]$ is finite,*
(4) *for every $H \subset \omega_1$, there is $K \subset \omega$ such that both of the following hold:*
  (a) *except for countably many $\alpha \in H$, $N(\alpha) \subset^* K$ holds,*
  (b) *except for countably many $\alpha \in \omega_1 \setminus H$, $N(\alpha) \subset^* \omega \setminus K$ holds.*

Proof. (1) $\Rightarrow$ (2): Let $X_{\{\omega\}}$ be C^*-embedded in X and $h : \omega_1 \to 2$. The function $f : X_{\{\omega\}} \to 2$ defined by:

$$f(\langle \omega, \alpha + 1\rangle) = i \text{ if } \alpha \in h^{-1}[\{i\}]$$

is continuous, because $X_{\{\omega\}}$ is discrete. Since $X_{\{\omega\}}$ is C^*-embedded in X, we can find a continuous function $g : X \to \mathbb{I}$ with $g \restriction X_{\{\omega\}} = f$. For every $n \in \omega$, by the stationarity of $V_n(X)$, we can find $\alpha_n < \omega_1$ and $r_n \in \mathbb{I}$ such that the restriction $g \restriction X^{(\alpha_n,\omega_1)}_{\{n\}}$ is constant taking the value r_n. Let $\alpha^* = \sup\{\alpha_n : n \in \omega\}$. It suffices to see the following claim.

Claim 1. For every $i \in 2$ and $\alpha \in h^{-1}[\{i\}] \cap (\alpha^*, \omega_1)$, the sequence $\{r_n : n \in N(\alpha)\}$ converges to i.

Proof. Let $i \in 2$, $\alpha \in h^{-1}[\{i\}] \cap (\alpha^*, \omega_1)$ and $\varepsilon > 0$. Since g is continuous at $\langle \omega, \alpha+1\rangle$ and $g(\langle \omega, \alpha+1\rangle) = f(\langle \omega, \alpha+1\rangle) = i$, there is $m \in \omega$ with $g[((N(\alpha) \cap (m, \omega)) \cup \{\omega\})) \times \{\alpha + 1\}] \subset (i - \varepsilon, i + \varepsilon)$. Now for every $n \in N(\alpha) \cap (m, \omega)$, we have $|r_n - i| = |g(\langle n, \alpha + 1\rangle) - i| < \varepsilon$. This completes the proof of Claim 1.

(2) $\Rightarrow$ (3): Assume (2) and let $h : \omega_1 \to 2$. We can take a sequence $\{r_n : n \in \omega\} \subset \mathbb{I}$ and $\alpha^* < \omega_1$ such that for every $i \in 2$ and $\alpha \in h^{-1}[\{i\}] \cap (\alpha^*, \omega_1)$, $\{r_n : n \in N(\alpha)\}$ converges to i. Take a function $k : \omega \to 2$ with $\{n \in \omega : r_n < \frac{1}{3}\} \subset k^{-1}[\{0\}]$ and $\{n \in \omega : r_n > \frac{2}{3}\} \subset k^{-1}[\{1\}]$ Since for every $\alpha \in h^{-1}[\{0\}] \cap (\alpha^*, \omega_1)$, $\{r_n : n \in N(\alpha)\}$ converges to 0, we can find $m \in \omega$ such that $N(\alpha) \cap (m, \omega) \subset k^{-1}[\{0\}]$, thus we see $N(\alpha) \subset^* k^{-1}[\{0\}]$. Similarly for every $\alpha \in h^{-1}[\{1\}] \cap (\alpha^*, \omega_1)$, we see $N(\alpha) \subset^* k^{-1}[\{1\}]$.

The equivalence (3) $\Leftrightarrow$ (4) follows from the identifications $H = h^{-1}[\{1\}]$ and $K = k^{-1}[\{1\}]$.

(3) $\Rightarrow$ (1): Assume (3). To see (1), let $\mathcal{U}$ be a two elements cozero cover of $X_{\{\omega\}}$. Since $X_{\{\omega\}}$ is discrete, we may assume $\mathcal{U} = \{U_0, U_1\}$ with $U_i = \{\langle\omega, \alpha+1\rangle : \alpha \in h^{-1}[\{i\}]\}$ $(i \in 2)$ for some function $h : \omega_1 \to 2$. For this h, take a function $k : \omega \to 2$ in (3). Take $\alpha^* < \omega_1$ such that for every $i \in 2$ and $\alpha \in h^{-1}[\{i\}] \cap (\alpha^*, \omega_1)$, $N(\alpha) \subset^* k^{-1}[\{i\}]$ holds. Since $X^{[0,\alpha^*]}$ is collectionwise normal, it suffices to find a locally finite cozero cover $\mathcal{V}$ of $X^{(\alpha^*,\omega_1)}$ such that $\mathcal{V} \upharpoonright X^{(\alpha^*,\omega_1)}_{\{\omega\}}$ refines $\mathcal{U}$. Define a function $g : X^{(\alpha^*,\omega_1)} \to \mathbb{I}$ by the following clauses:

(a) if $x = \langle\omega, \alpha+1\rangle$ for some $\alpha \in h^{-1}[\{i\}] \cap [\alpha^*, \omega_1)$, then $g(x) = i$,
(b) if $x \in X^{(\alpha^*,\omega_1)}_{\{n\}}$ for some $n \in k^{-1}[\{0\}]$, then $g(x) = \frac{1}{n}$,
(c) if $x \in X^{(\alpha^*,\omega_1)}_{\{n\}}$ for some $n \in k^{-1}[\{1\}]$, then $g(x) = 1 - \frac{1}{n}$.

We check:

Claim 2. The above function g is continuous.

Proof. Let $x \in X^{(\alpha^*,\omega_1)}$ and $\varepsilon > 0$. Whenever $x \in X^{(\alpha^*,\omega_1)}_{\{n\}}$ for some $n \in \omega$, the set $X^{(\alpha^*,\omega_1)}_{\{n\}}$ is a clopen neighborhood of x on which g is constant. So we assume $x \in X^{(\alpha^*,\omega_1)}_{\{\omega\}}$. We may assume $x = \langle\omega, \alpha+1\rangle$ with $\alpha \in h^{-1}[\{0\}] \cap [\alpha^*, \omega_1)$. From $g(x) = 0$ and $N(\alpha) \subset^* k^{-1}[\{0\}]$, we can find $m' \in \omega$ with $N(\alpha) \cap (m', \omega) \subset k^{-1}[\{0\}]$. Taking $m \in \omega$ with $m' \le m$ and $\frac{1}{m} < \varepsilon$, let $W = ((N(\alpha) \cap (m, \omega) \cup \{\omega\})) \times \{\alpha+1\}$. Obviously W is a clopen neighborhood of x. If $y \in W \setminus \{x\}$, then $y = \langle n, \alpha+1\rangle$ holds for some $n \in N(\alpha) \cap (m, \omega)$. Now by $n \in k^{-1}[\{0\}]$, we see $g(y) = \frac{1}{n} < \frac{1}{m} < \varepsilon$, so g is continuous at x. This completes the proof of the claim.

Now the collection $\mathcal{V} = \{g^{-1}[[0, \frac{2}{3})], g^{-1}[(\frac{1}{3}, 1]]\}$ is the desired cozero cover of $X^{(\alpha^*,\omega_1)}$. □

Definition 3.5. Property (A) means the following property.

[Property (A)] There is a sequence $\mathcal{N} \subset [\omega]^\omega$ of length ω_1, say $\mathcal{N} = \{N(\alpha) : \alpha < \omega_1\}$, which satisfies (4) (equivalently (1), (2) and (3)) of Lemma 3.4, that is, for every $H \subset \omega_1$, there is $K \subset \omega$ such that both of the following hold:

(1) except for countably many $\alpha \in H$, $N(\alpha) \subset^* K$ holds,
(2) except for countably many $\alpha \in \omega_1 \setminus H$, $N(\alpha) \subset^* \omega \setminus K$ holds,

In the next section, we will discuss the consistency of Property (A).

4. Consistency

A subcollection $\mathcal{A}$ of the collection $[\omega]^\omega$ is called *almost disjoint* if for every distinct pair $A, B \in \mathcal{A}$, the intersection $A \cap B$ is finite, see [27, II, Definition 1.1]. Note that all maximal almost disjoint collections are uncountable, see [27, II, Theorem 1.2 (a)]. Dow and Shelah proved in [3] the following.

Proposition 4.1. [3, Theorem 2.1] *It is consistent with Martin's axiom and $2^\omega = \omega_2$ that there is a maximal almost disjoint collection $\mathcal{A}$ such that for every disjoint pair $\mathcal{A}_0$ and $\mathcal{A}_1$ with $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{A}$, there is $K \subset \omega$ satisfying both of the following:*
 (a) *$A \subset^* K$ holds for every $A \in \mathcal{A}_0$,*
 (b) *$A \subset^* \omega \setminus K$ holds for every $A \in \mathcal{A}_1$.*

We remark that if Martin's axiom and $2^\omega = \omega_2$ are assumed, then $2^\omega = 2^{\omega_1}$ holds, see [27, II, Theorem 2.18]. In the above proposition, taking a sequence $\mathcal{N}$ $(= \{N(\alpha) : \alpha < \omega_1\})$ of length ω_1 from $\mathcal{A}$, we see that $\mathcal{N}$ satisfies (4) of Lemma 3.4. Using Corollary 2.2, we have:

Theorem 4.2. *It is consistent with ZFC that Property (A) holds, therefore there is a subspace X of $(\omega+1) \times \omega_1$ such that the closed subspace $X_{\{\omega\}}$ of X is C^*-embedded in X but not P-embedded in X.*

We will see that the negation of Property (A) is also consistent with ZFC. In fact, we have the following.

Theorem 4.3. *If $2^\omega < 2^{\omega_1}$ is assumed, then for every subspace X of $(\omega + 1) \times \omega_1$, if $X_{\{\omega\}}$ is C^*-embedded in X, then it is P-embedded in X. Therefore $2^\omega < 2^{\omega_1}$ implies the negation of Property (A).*

Proof. Assume $2^\omega < 2^{\omega_1}$ and $X \subset (\omega + 1) \times \omega_1$. Moreover assume that $X_{\{\omega\}}$ is C^*-embedded in X. From Corollary 2.2, assuming that $V_\omega(X)$ is not stationary, it suffices to see that $V_\omega(\mathrm{Cl}_X X_A)$ is bounded in ω_1, where $A = \{n \in \omega : V_n(X) \text{ is stationary in } \omega_1\}$. So assume that $V_\omega(X)$ is not stationary but $V_\omega(\mathrm{Cl}_X X_A)$ is unbounded, we will derive a contradiction. Take a club set C' of ω_1 disjoint from $V_\omega(X) \cup (\bigcup_{n\in\omega\setminus A} V_n(X))$. For every $\alpha < \omega_1$, take $g(\alpha) \in V_\omega(\mathrm{Cl}_X X_A)$ with $\alpha < g(\alpha)$ and let $C = C' \cap \{\alpha < \omega_1 : \forall\beta < \alpha(g(\beta) < \alpha)\}$. Obviously from Lemma 1.6, C is also a club set in ω_1 having the following property:

- C is disjoint from $V_\omega(X) \cup (\bigcup_{n\in\omega\setminus A} V_n(X))$,
- for every $\alpha \in \mathrm{Succ}(C)$, $\mathrm{Cl}_X X_A \cap X_{\{\omega\}}^{(p_C(\alpha),\alpha]} \neq \emptyset$.

Then $X_{\{\omega\}}$ can be represented as $X_{\{\omega\}} = \bigoplus_{\alpha\in\mathrm{Succ}(C)} X_{\{\omega\}}^{(p_C(\alpha),\alpha]}$. For every $\alpha \in \mathrm{Succ}(C)$, fix $x_\alpha \in \mathrm{Cl}_X X_A \cap X_{\{\omega\}}^{(p_C(\alpha),\alpha]}$. Because of the first countability of X, for every $\alpha \in \mathrm{Succ}(C)$, we can fix a non-decreasing sequence $\{\gamma_\alpha(n) : n \in \omega\}$ in ω_1 with $p_C(\alpha) < \gamma_\alpha(0)$ and a strictly increasing sequence $N(\alpha) := \{m_\alpha(n) : n \in \omega\}$ in A such that $\{y_\alpha(n) : n \in \omega\}$ converges to x_α, where $y_\alpha(n) = \langle m_\alpha(n), \gamma_\alpha(n)\rangle$.

For every subset H of $\mathrm{Succ}(C)$, let $U(H) = \bigcup_{\alpha \in H} X_{\{\omega\}}^{(p_C(\alpha),\alpha]}$. Now fix $H \subset \mathrm{Succ}(C)$. Then since $\{U(H), U(\mathrm{Succ}(C) \setminus H)\}$ is a disjoint clopen cover of the C^*-embedded subspace $X_{\{\omega\}}$, we can find a continuous function $f_H : X \to \mathbb{I}$ such that $f_H(x) = 0$ ($f_H(x) = 1$) whenever $x \in U(H)$ ($x \in U(\mathrm{Succ}(C) \setminus H)$, respectively). For every $n \in A$, since $V_n(X)$ is stationary and f_H is continuous, we can fix $\beta_n < \omega_1$ and $r_n \in \mathbb{I}$ such that f_H is constant on $X_{\{n\}}^{(\beta_n,\omega_1)}$ taking value r_n. Letting $\beta^* = \sup\{\beta_n : n \in A\}$, for every $\alpha \in \mathrm{Succ}(C)$ with $\beta^* \le p_C(\alpha)$, since f_H is continuous at x_α and $\{y_\alpha(n) : n \in \omega\}$ converges to x_α, we can find $n_\alpha \in \omega$ such that:

(1) whenever $\alpha \in H$, $f_H(y_\alpha(n)) < \frac{1}{3}$ holds for every $n \in \omega$ with $n_\alpha \le n$,
(2) whenever $\alpha \in \mathrm{Succ}(C) \setminus H$, $f_H(y_\alpha(n)) > \frac{2}{3}$ holds for every $n \in \omega$ with $n_\alpha \le n$.

Because of $\beta^* \le p_C(\alpha)$, notice $f_H(y_\alpha(n)) = r_{m_\alpha(n)}$ in the clauses (1) and (2) above. Now let $K_H = \{m \in A : r_m < \frac{1}{3}\}$. Then from (1), we see that for every $\alpha \in \mathrm{Succ}(C)$ with $\beta^* \le p_C(\alpha)$, if $\alpha \in H$, then $N(\alpha) \subset^* K_H$ holds. Also from (2), we see that for every $\alpha \in \mathrm{Succ}(C)$ with $\beta^* \le p_C(\alpha)$, if $\alpha \in \mathrm{Succ}(C) \setminus H$, then $N(\alpha) \subset^* A \setminus K_H$ holds. Thus for every $H \subset \mathrm{Succ}(C)$, we have shown:

(a) except for countably many $\alpha \in H$, $N(\alpha) \subset^* K_H$ holds,
(b) except for countably many $\alpha \in \mathrm{Succ}(C) \setminus H$, $N(\alpha) \subset^* A \setminus K_H$ holds.

Now let $\{H_\xi : \xi < 2^{\omega_1}\}$ be an independent collection of subsets of $\mathrm{Succ}(C)$, that is, for distinct $\xi_1, \dots, \xi_m, \zeta_1, \dots, \zeta_n < 2^{\omega_1}$, the set

$$H_{\xi_1} \cap \cdots \cap H_{\xi_m} \cap (\mathrm{Succ}(C) \setminus H_{\zeta_1}) \cap \cdots \cap (\mathrm{Succ}(C) \setminus H_{\zeta_n})$$

is uncountable, see [27, VIII Exercises A6]. Letting $K_\xi = K_{H_\xi}$ for each $\xi < 2^{\omega_1}$, the following claim yields a contradiction, because A is countable and $2^\omega < 2^{\omega_1}$ is assumed.

Claim. If $\xi < \zeta < 2^{\omega_1}$, then $K_\xi \neq K_\zeta$.

Proof. Fix $\alpha^* < \omega_1$ with

$(\mathrm{a})_\xi$ $N(\alpha) \subset^* K_\xi$ holds for every $\alpha \in H_\xi$ with $\alpha^* \le \alpha$,
$(\mathrm{b})_\xi$ $N(\alpha) \subset^* A \setminus K_\xi$ holds for every $\alpha \in \mathrm{Succ}(C) \setminus H_\xi$ with $\alpha^* \le \alpha$,
$(\mathrm{a})_\zeta$ $N(\alpha) \subset^* K_\zeta$ holds for every $\alpha \in H_\zeta$ with $\alpha^* \le \alpha$,
$(\mathrm{b})_\zeta$ $N(\alpha) \subset^* A \setminus K_\zeta$ holds for every $\alpha \in \mathrm{Succ}(C) \setminus H_\zeta$ with $\alpha^* \le \alpha$.

Since $H_\xi \cap (\mathrm{Succ}(C) \setminus H_\zeta)$ is uncountable, take $\alpha \in H_\xi \setminus H_\zeta$ with $\alpha^* \le \alpha$. From $(\mathrm{a})_\xi$, we have $N(\alpha) \subset^* K_\xi$. Moreover from $(\mathrm{b})_\zeta$, we have $N(\alpha) \subset^* A \setminus K_\zeta$. Since $N(\alpha)$ is infinite, we have $K_\xi \neq K_\zeta$. This completes the proof of the claim.

Using Corollary 3.2 and Lemma 3.4, we see that $2^\omega < 2^{\omega_1}$ implies the negation of Property (A).

$\square$

Finally we ask:

Question 4.4. Is it consistent with ZFC that for every subspace X of $(\omega+1)\times\omega_1$ and every closed subspace F of X, F is P-embedded in X whenever it is C^*-embedded in X?

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Department of Mathematics, Oita University, Oita, 870-1192 Japan
E-mail address: nkemoto@cc.oita-u.ac.jp

Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan
E-mail address: usuba@waseda.jp