THE WEIGHT OF LEXICOGRAPHIC PRODUCTS

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ABSTRACT. We will calculate the weight of lexicographic products of GO-spaces, using this we will see:

- the assertion that the weight of the lexicographic product $2^\omega$ is $\aleph_1$ is equivalent to the Continuum Hypothesis (CH), that is, $2^{\aleph_0} = \aleph_1$,
- the assertion that the weight of both lexicographic products $2^\omega$ and $2^{\omega_1+1}$ coincide is equivalent to the assertion $2^{\aleph_0} = 2^{\aleph_1}$,
- the assertion that the lexicographic product $2^\gamma$ is homeomorphic to the usual Tychonoff product $2^\gamma$ is equivalent to $\gamma \leq \omega$.

1. Introduction

We will work on the usual ZFC-set theory including the Axiom of Choice (AC) [4, 11]. All spaces are assumed to be regular $T_1$ containing at least 2 points and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all $X_\alpha$’s are also assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminologies follow [4, 11, 1].

In [2], second countability of lexicographic products of LOTS’s is characterized. It is known in [2, p.78, Example 4] that the usual Tychonoff product $2^\omega$, which is homeomorphic to the Cantor set $\mathbb{C}$, is also homeomorphic to the lexicographic product $2^\omega$, where $2 = \{0, 1\}$ with $0 < 1$. So they are second countable, that is, the weight is at most countable. On the other hand, the weight of the usual Tychonoff product $2^{\omega_1}$ is easily seen to be $\aleph_1$. So it is natural to conjecture:

(1) the lexicographic product $2^{\omega_1}$ is homeomorphic to the usual Tychonoff product $2^{\omega_1}$,
(2) the weight of the lexicographic product $2^{\omega_1}$ is $\aleph_1$.

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Recently, the notion of lexicographic products of GO-spaces is introduced and discussed in [8, 9, 10], also see [3, 6, 7] for products of LOTS’s. In this paper, we will calculate the weight of lexicographic products of GO-spaces. As corollaries, we see:

- the conjecture (1) is false, in fact, the assertion that the lexicographic product $2^\gamma$ is homeomorphic to the usual Tychonoff product $2^\gamma$ is equivalent to $\gamma \leq \omega$,
- the conjecture (2) is equivalent to the Continuum Hypothesis (CH), that is, $2^{\aleph_0} = \aleph_1$.

Obviously the usual Tychonoff products $2^{\omega_1}$ and $2^{\omega_1+1}$ are homeomorphic, however we will also see:

- the assertion that the weight of both lexicographic products $2^{\omega_1}$ and $2^{\omega_1+1}$ coincide is equivalent to the assertion $2^{\aleph_0} = 2^{\aleph_1}$.

A linearly ordered set $\langle L; <_L \rangle$ has a natural topology $\lambda_L$, which is called an interval topology, generated by $f(x; y)_L = \{z \in L : x <_L z \leq_L y\}$ and so on. The triple $\langle L; <_L, \lambda_L \rangle$, which is simply denoted by $L$, is called a LOTS.

A triple $\langle X; <_X, \tau_X \rangle$ is said to be a GO-space, which is also simply denoted by $X$, if $\langle X; <_X \rangle$ is a linearly ordered set and $\tau_X$ is a $T_2$-topology on $X$ having a base consisting of convex sets, where a subset $C$ of $X$ is convex if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subset C$ holds. In this situation, $\langle X; <_X \rangle$ is called an underlying linearly ordered set of $X$. The symbols $\mathbb{R}$ and $\mathbb{Q}$ denote the reals and the rationals respectively. Note that they are LOTS’s. On the other hand, the Sorgenfrey line $\mathbb{S}$, whose underlying linearly ordered set is $\mathbb{R}$ and the sets of type $[a, b)$ are declared to be open, is known to be a GO-space but not a LOTS. For more information on LOTS’s or GO-spaces, see [12]. Usually $<_L$, $(x, y)_L$, $\lambda_L$ or $\tau_X$ are written simply $<, (x, y), \lambda$ or $\tau$ if contexts are clear.

$\omega$ and $\omega_1$ denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \delta \cdots$, are considered to be LOTS’s with the usual interval topology. $\text{cf} \alpha$ denotes the cofinality of the ordinal $\alpha$. When $\alpha$ is a successor ordinal, i.e., $\alpha = \delta + 1$ for some ordinal $\delta$, this $\delta$ is denoted by $\alpha - 1$. A non-zero ordinal which is not successor is said to be a limit ordinal.

An ordinal $\alpha$ is said to be a cardinal if $\alpha = |\alpha|$, where $|X|$ denotes the cardinality of a set $X$, that is, $|X|$ is the smallest ordinal $\delta$ such that there is a 1-1 map from $X$ onto $\delta$ [11, I Definition 10.3], where note that the existence of $|X|$ is ensured by AC. When we want to
emphasize that \( \omega \) and \( \omega_1 \) are cardinals, we write them by \( \aleph_0 \) and \( \aleph_1 \), respectively. Generally, the \( \alpha \)-th uncountable cardinal is denoted by \( \aleph_\alpha \) or \( \aleph_\lambda \). Cardinals are usually denoted by Greek letters \( \kappa, \lambda, \mu, \ldots \). For cardinals \( \kappa \) and \( \lambda \), \( \kappa^\lambda \) denotes the cardinal \( |X^Y| \) with \( |X| = \kappa \) and \( |Y| = \lambda \), where \( X^Y \) denotes the set of all functions on \( Y \) to \( X \).

It is well known that for a LOTS \( \langle Y, <_Y, \lambda_Y \rangle \), if \( X \subseteq Y \), then \( \langle X, <_X, \tau_X \rangle \) is a GO-space with \( <_X = <_Y \upharpoonright X \) and \( \tau_X \) is the subspace topology \( \lambda_Y \upharpoonright X \). For every GO-space \( X \), there is a LOTS \( X^* \) such that \( X \) is a dense subspace of \( X^* \) and \( X^* \) has the property that if \( L \) is a LOTS containing \( X \) as a dense subspace, then \( L \) also contains the LOTS \( X^* \) as a subspace, see [13]. Such a \( X^* \) is called the minimal \( d \)-extension of a GO-space \( X \). Indeed, the LOTS \( X^* \) is constructed as follows, see also [8]. Let

\[
X^+ = \{ x \in X : (\leftarrow, x) \in \tau_X \setminus \lambda_X \},
\]

\[
X^- = \{ x \in X : (x, \rightarrow) \in \tau_X \setminus \lambda_X \}.
\]

Then

\[
X^* = (X^* \times \{-1\}) \cup (X \times \{0\}) \cup (X^+ \times \{1\}),
\]

where the order \( <_{X^*} \) on \( X^* \) is the restriction of the usual lexicographic order on \( X \times \{-1, 0, 1\} \) with \(-1 < 0 < 1\). Also we identify \( X \times \{0\} \) with \( X \) in the obvious way. Obviously, we can see:

- if \( X \) is a LOTS, then \( X^* = X \),
- \( X \) has a maximal element \( \text{max} X \) if and only if \( X^* \) has a maximal element \( \text{max} X^* \), in this case, \( \text{max} X = \text{max} X^* \) (similarly for minimal elements).

For every \( \alpha < \gamma \), let \( X_\alpha \) be a LOTS and \( X = \prod_{\alpha < \gamma} X_\alpha \). Every element \( x \in X \) is identified with the sequence \( \langle x(\alpha) : \alpha < \gamma \rangle \). In the present paper, a sequence means a function whose domain is an ordinal. For notational convenience, \( \prod_{\alpha < \gamma} X_\alpha \) is considered as \( \{\emptyset\} \) whenever \( \gamma = 0 \), where \( \emptyset \) is considered to be a function whose domain is \( 0 \). When \( 0 \leq \beta < \gamma \), \( y_0 \in \prod_{\alpha < \beta} X_\alpha \) and \( y_1 \in \prod_{\beta \leq \alpha} X_\alpha \), \( y_0 \uparrow y_1 \) denotes the sequence \( y \in \prod_{\alpha < \gamma} X_\alpha \) defined by

\[
y(\alpha) = \begin{cases} 
y_0(\alpha) & \text{if } \alpha < \beta, 
y_1(\alpha) & \text{if } \beta \leq \alpha.
\end{cases}
\]

In this case, whenever \( \beta = 0 \), \( \emptyset \uparrow y_1 \) is considered as \( y_1 \). In case \( 0 \leq \beta < \gamma \), \( y_0 \in \prod_{\alpha < \beta} X_\alpha \), \( u \in X_\beta \) and \( y_1 \in \prod_{\beta \leq \alpha} X_\alpha \), \( y_0 \uparrow (u \uparrow) \uparrow y_1 \) denotes
the sequence $y \in \prod_{\alpha < \gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ u & \text{if } \alpha = \beta, \\ y_1(\alpha) & \text{if } \beta < \alpha. \end{cases}$$

More general cases are similarly defined.

The lexicographic order $<_X$ on $X = \prod_{\alpha < \gamma} X_\alpha$, where all $X_\alpha$’s are LOTS’s, is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma, x | \alpha = x' | \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x | \alpha = \langle x(\beta) : \beta < \alpha \rangle$ and $<_{X_\alpha}$ is the order on $X_\alpha$. Now for every $\alpha < \gamma$, let $X_\alpha$ be a GO-space and $X = \prod_{\alpha < \gamma} X_\alpha$. The subspace $X$ of the lexicographic product $\hat{X} = \prod_{\alpha < \gamma} X_\alpha^*$ is said to be the lexicographic product of GO-spaces $X_\alpha$’s, for more details see [8].

$\prod_{i \in \omega} X_i \prod_{i \leq n} X_i$, where $n \in \omega$ (where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha < \gamma} X_\alpha$ is also denoted by $X^\gamma$ whenever $X_\alpha = X$ for all $\alpha < \gamma$. When $X_\alpha$’s are GO-spaces, $\prod_{\alpha < \gamma} X_\alpha$ usually means the lexicographic product otherwise stated.

2. The weight of GO-spaces

Recall that the weight $w(X)$ and the density $d(X)$ of a topological space $X$ are defined as follows:

$$w(X) = \min \{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X \},$$

$$d(X) = \min \{|D| : D \text{ is dense in } X \}. $$

Because $X$ is regular $T_1$, note that if $X$ is infinite, then $\aleph_0 \leq d(X) \leq w(X)$ and $d(X) \leq |X|$ hold, and also that if $X$ is finite, then $d(X) = w(X) = |X|$. For a GO-space $X$, let

$$N_+^X = \{ x \in X : \text{ there is } y \in X \text{ with } x < y \text{ and } (x, y) = \emptyset \} ,$$

$$N_-^X = \{ x \in X : \text{ there is } y \in X \text{ with } y < x \text{ and } (y, x) = \emptyset \} .$$

In other words, an element of $N_+^X$ is called a left neighbor of neighbors in the sense of [2, p.6]. For every $x \in N_+^X$, assign $y \in X$ with $x < y$ and $(x, y) = \emptyset$. Then this assignment defines a 1-1 map on $N_+^X$ onto $N_-^X$, so we have $|N_+^X| = |N_-^X|$. Obviously we have :

- if $x \in N_+^X$, then $[\leftarrow, x] \in \lambda_X \subset \tau_X$, 
- $x \in X^+ \cup N_+^X$ iff $(x, \to)_X \neq \emptyset$ and $(\leftarrow, x)_X \in \tau_X$. 

Let $S$ of at most two members of $X$ is a subbase for $X$. Therefore, if $x \not\in X^+ \cup N_X^+$, $(x, \rightarrow)_X \neq \emptyset$, $D$ is dense in $X$ and $U$ is a neighborhood of $x$ in $X$, then there is $d \in D$ with $x < d$ and $[x, d]_X \subset U$.

Proof. The necessity is obvious. To see the the sufficiency, let $x \not\in X^+ \cup N_X^+$, $(x, \rightarrow)_X \neq \emptyset$. If $y_0 \in X^*$ were an immediate successor of $x$, then $y_0$ would not belong to $X$ otherwise $x \in N_X^+$ and $(\leftarrow, x) = (\leftarrow, y_0)_{X^*} \cap X$ would be open in $X$. Thus $x \in X^+$, a contradiction.

To see the latter half, let $x \not\in X^+ \cup N_X^+$, $(x, \rightarrow)_X \neq \emptyset$, $D$ be dense in $X$ and $U$ a neighborhood of $x$. Take $y \in X^*$ with $x < y$, $y$ and $[x, y)_{X^*} \cap X \subset U$. Then $y$ is not the immediate successor of $x$ in $X^*$, so $(x, y)_X \neq \emptyset$ and $(x, y)_X \cap X$ is a non-empty open set in $X$. Therefore there exists a $d \in D$ such that $d \in (x, y)_X \cap X$, then $[x, d]_X \subset U$. □

We can also verify an analogous Lemma above for $x \not\in X^- \cup N_X^-$. The weight $w(X)$ of a GO-space $X$ is decided from $d(X), |N_X^+|, |X^+|$ and $|X^-|$.

Lemma 2.2. Let $X$ be a GO-space. Then

$$w(X) = \max\{d(X), |N_X^+|, |X^+|, |X^-|\}.$$

Proof. If $X$ is finite, then we have $|N_X^+| \leq |X| = w(X) = d(X)$ and $X^+ = X^- = \emptyset$. So we assume that $X$ is infinite.

To see the inequality “$\geq$”, let $\kappa = w(X)$ and $B$ be a base for $X$ with $|B| = \kappa$. For each $x \in N_X^+$, assign $B_x \in B$ with $x \in B_x \subset (\leftarrow, x)_X$. Then this assignment defines an injective function on $N_X^+$ to $B$, so we have $|N_X^+| \leq \kappa$. Similarly we can see $|X^+| \leq \kappa$ and $|X^-| \leq \kappa$.

To see the other inequality, let $\kappa = \max\{d(X), |N_X^+|, |X^+|, |X^-|\}$ and fix a dense set $D$ in $X$ with $|D| = d(X)$. Since $|N_X^+| = |N_X^-|$ holds, it suffices to see the following claim.

Claim 1. The collection

$$\mathcal{S} := \{(\leftarrow, x) : x \in D\} \cup \{(x, \rightarrow) : x \in D\}
\cup\{(\leftarrow, x) : x \in N_X^+ \cup X^+\} \cup \{(x, \rightarrow) : x \in N_X^- \cup X^-\}$$

is a subbase for $X$, in fact, the collection of all non-empty intersections of at most two members of $\mathcal{S}$ is a base for $X$.

Proof. Let $U$ be a non-empty open set with $x \in U$. We consider several cases.

Case 1. $(\leftarrow, x) = \emptyset$, that is, $x = \min X$. 

Let \((x, \to) \neq \emptyset\). Whenever \(x \in N_X^+ \cup X^+\), we have \(x \in (\leftarrow, x] = \{x\} \subseteq U\). So let \(x \notin N_X^+ \cup X^+\). From Lemma 2.1, we can take \(d \in D\) with \(x < d\) and \([x, d] \subseteq U\). Then \(x \in (\leftarrow, d)_X \subseteq U\) with \((\leftarrow, d)_X \subseteq S\).

Similarly we see:

**Case 2.** \((x, \to) = \emptyset\), that is, \(x = \max X\).

**Case 3.** \((\leftarrow, x) \neq \emptyset\) and \((x, \to) \neq \emptyset\).

Whenever \(x \notin N_X^+ \cup X^+\) and \(x \notin N_X^- \cup X^-\), taking \(d', d \in D\) with \(d' < x < d\) and \([d', d] \subseteq U\) from Lemma 2.1, we have \(x \in (\leftarrow, d) \cap (d', \to) \subseteq U\) with \((\leftarrow, d), (d', \to) \subseteq S\). Whenever \(x \notin N_X^+ \cup X^+\) and \(x \in N_X^- \cup X^-\), taking \(d \in D\) with \(x < d\) and \([x, d] \subseteq U\) from Lemma 2.1, we have \(x \in (\leftarrow, d) \cap [x, \to) \subseteq U\) with \((\leftarrow, d), [x, \to) \subseteq S\). The case \(x \in N_X^+ \cup X^+\) and \(x \notin N_X^- \cup X^-\) is similar. Whenever \(x \in N_X^+ \cup X^+\) and \(x \in N_X^- \cup X^-\), we have \(x \in (\leftarrow, x] \cap [x, \to) = \{x\} \subseteq U\) with \((\leftarrow, x], [x, \to) \subseteq S\).

Remark that this lemma also shows the well-known fact \(w(X) \leq |X|\) about a GO-space \(X\).

### 3. The Weight of Lexicographic Products

In this section, we calculate the weight of the lexicographic products.

**Lemma 3.1.** Let \(X = X_0 \times X_1\) be a lexicographic product of GO-spaces with \(|X| \geq \omega\) and \(\kappa\) an infinite cardinal. Then \(w(X) \leq \kappa\) holds if and only if \(|X_0| \leq \kappa\) and \(w(X_1) \leq \kappa\).

**Proof.** Let \(\tilde{X} = X_0^* \times X_1^*\).

To see “only if” part, let \(w(X) \leq \kappa\) and \(\mathcal{B}\) be a base for \(X\) with \(|\mathcal{B}| = w(X)\). Take \(v', v \in X_1\) with \(v' < v\). Then for every \(u \in X_0\), \((\langle u, v'\rangle, \to)_{X}\) is a neighborhood of \(\langle u, v\rangle\). So for every \(u \in X_0\) assign \(B_u \in \mathcal{B}\) with \(\langle u, v\rangle \in B_u \subseteq (\langle u, v'\rangle, \to)_{X}\). Then this assignment witnesses \(|X_0| \leq |\mathcal{B}| \leq \kappa\). Now fix \(u_0 \in X_0\), then obviously \(X_1\) can be identified with the subspace \(\{u_0\} \times X_1\), see also [9, Lemma 3.4]. Then we have \(w(X_1) = w(\{u_0\} \times X_1) \leq w(X) \leq \kappa\).

To see “if” part, let \(|X_0| \leq \kappa\) and \(w(X_1) \leq \kappa\). Then by Lemma 2.2, we see \(d(X_1) \leq \kappa\), \(|N_{X_1}^+| \leq \kappa\), \(|X_1^+| \leq \kappa\) and \(|X_1^-| \leq \kappa\). So we can fix a dense set \(D_1\) in \(X_1\) with \(|D_1| = d(X_1)\). Let

\[
M = \{v \in X_1 : (\leftarrow, v) = \emptyset \text{ or } (v, \to) = \emptyset\},
\]

that is, \(M\) is the set of a maximal element and a minimal element if exists, so \(|M| \leq 2\). From Lemma 2.2, it suffices to see the following claims.
Claim 1. \(d(X) \leq \kappa.\)

Proof. Let \(D = X_0 \times (D_1 \cup M).\) The assumption ensures \(|D| \leq \kappa,\)
so it suffices to see that \(D\) is dense in \(X.\) Let \(x \in X\) and \(U\) be a
neighborhood of \(x\) in \(X,\) say \(x = \langle u, v \rangle.\) When \(v \in M,\) obviously \(U \cap D\)
is non-empty. So assume \(v \notin M,\) then we can take \(v'_0, v'_1 \in X_i\) with \(v'_0 < v < v'_1\)
and \((\langle u, v'_0 \rangle, \langle u, v'_1 \rangle)_X \cap X \subseteq U.\) Since \((v'_0, v'_1)_X \cap X_1\) is
non-empty open set in \(X_1,\) we can find \(d \in D_1 \cap ((v'_0, v'_1)_X \cap X_1).\) Now
we have \(\langle u, d \rangle \in U \cap D.\)

Claim 2. \(|N_X^+| \leq \kappa.\)

Proof. It suffices to see \(N_X^+ \subseteq X_0 \times (N_X^+ \cup M).\) Let \(x \in N_X^+,\) say
\(x = \langle u, v \rangle.\) As above, we may assume \(v \notin M.\) From \(x \in N_X^+,\) we can
find \(y \in X\) with \(x <_X y\) and \((x, y)_X = \emptyset.\) By \((v, \rightarrow)_X \neq \emptyset,\) \(y\) has to be
\(\langle u, v' \rangle\) for some \(v' \in X_1\) with \(v <_{X_1} v'.\) Then we have \((v, v')_X = 0,\)
therefore \(v \in N_{X_1}^+,\) so \(x \in X_0 \times (N_{X_1}^+ \cup M).\)

Claim 3. \(|X^+| \leq \kappa.\)

Proof. It suffices to see \(X^+ \subseteq X_0 \times (X_{1}^+ \cup M).\) Let \(x \in X^+,\) say
\(x = \langle u, v \rangle.\) As above, we may assume \(v \notin M.\) From \(x \in X^+,\) note \((\langle \leftarrow, x \rangle)_X \in \tau_X \setminus \lambda_X.\) If \((\langle \leftarrow, v \rangle)_X \in \lambda_X\) were true, then there is \(v' \in X_1\) with \(v <_{X_1} v'\) and \((v, v')_X \neq 0.\) Then we have \((\langle \leftarrow, x \rangle)_X = (\langle \leftarrow, (u, v') \rangle)_X \in \lambda_X,\) a contradiction. So we have \((\langle \leftarrow, v \rangle)_X \notin \lambda_X.\) By \((\langle \leftarrow, x \rangle)_X \in \tau_X,\) we
find \(y \in X\) with \(x <_{X} y\) and \((x, y)_X \cap X = \emptyset.\) Since \((v, \rightarrow)_X \neq \emptyset\) holds, \(y\) can be represented as \(\langle u, v^* \rangle\) with \(v <_{X^*} v^* \in X_1^+.\) Then we have \((v, v^*)_X = 0,\) otherwise \((x, y)_X \cap X \neq \emptyset.\) Now we have \((\langle \leftarrow, v \rangle)_X = (\langle \leftarrow, v^* \rangle)_X \cap X_1 \in \tau_{X_1},\) which implies, by \((\langle \leftarrow, v \rangle)_X \notin \lambda_{X_1},\)
\(v \in X_{1}^+\) thus \(x \in X_0 \times (X_{1}^+ \cup M).\)

Similarly we see the following.

Claim 4. \(|X^-| \leq \kappa.\)

Lemma 3.2. Let \(X = \prod_{\alpha < \gamma} X_\alpha\) be a lexicographic product of GO-
spaces and \(\kappa\) an infinite cardinal. Assume that \(\gamma\) is a limit ordinal.
Then \(w(X) \leq \kappa\) holds if and only if \(\gamma \leq \kappa\) holds and for every \(\beta < \gamma,\)
\(|\prod_{\alpha \leq \beta} X_\alpha| \leq \kappa\) holds.

Proof. Let \(\hat{X} = \prod_{\alpha < \gamma} X_\alpha.\)

Assume \(w(X) \leq \kappa\) and \(\beta < \gamma.\) It follows from \(X = (\prod_{\alpha \leq \beta} X_\alpha) \times (\prod_{\beta < \alpha} X_\alpha),\) see [8, Lemma 1.5], that \(|\prod_{\alpha \leq \beta} X_\alpha| \leq \kappa\) has to be true
from the lemma above. If \(\gamma > \kappa\) were true, then by \(X = (\prod_{\alpha < \kappa} X_\alpha) \times\)
\((\prod_{\alpha \leq \alpha} X_\alpha)\), applying the lemma above, we see \(\kappa < 2^\kappa \leq |\prod_{\alpha < \kappa} X_\alpha| \leq \kappa\), a contradiction.

To see the other direction, let \(\gamma \leq \kappa\) and for every \(\beta < \gamma\), \(|\prod_{\alpha \leq \beta} X_\alpha| \leq \kappa\). Define
\[
J^+ = \{\alpha < \gamma : X_\alpha \text{ has no maximal element.}\},
\]
\[
J^- = \{\alpha < \gamma : X_\alpha \text{ has no minimal element.}\}.
\]

By Lemma 2.2, it suffices to see the following claims.

**Claim 1.** \(d(X) \leq \kappa\).

**Proof.** Fix \(x_0 \in X\) and let
\[
D = \bigcup_{\beta < \gamma} \{y^\wedge(x_0 \upharpoonright (\beta, \gamma)) : y \in \prod_{\alpha \leq \beta} X_\alpha\}.
\]
The assumption ensures \(|D| \leq \kappa\), so it suffices to see that \(D\) is dense in \(X\). Let \(x \in X\) and \(U\) be a neighborhood of \(x\) in \(X\). We consider some cases.

**Case 1.** \((\leftarrow, x) = \emptyset\).

Because of \(x = \min X\) and \((x, \rightarrow) \neq \emptyset\), we can find \(b \in \hat{X}\) with \(x <^\hat{X} b\) and \([x, b]\hat{X} \cap X \subset U\). Set \(\alpha_0 = \min\{\alpha < \gamma : a(\alpha) \neq b(\alpha)\}\). Then \((x \uparrow (\alpha_0 + 1))^\wedge(x_0 \uparrow (\alpha_0, \gamma)) \in D \cap U\).

Similarly we see the following case.

**Case 2.** \((x, \rightarrow) = \emptyset\).

**Case 3.** \((\leftarrow, x) \neq \emptyset\) and \((x, \rightarrow) \neq \emptyset\).

Take \(a, b \in \hat{X}\) with \(a <^\hat{X} x <^\hat{X} b\) and \([a, b]\hat{X} \cap X \subset U\). Set \(\alpha_0 = \min\{\alpha < \gamma : a(\alpha) \neq b(\alpha)\}\). By \(a < x < b\), we have \(a \uparrow \alpha_0 = x \uparrow \alpha_0 = b \uparrow \alpha_0\). We consider further 3 cases.

**Case 3-1.** \(x \uparrow (\alpha_0 + 1) = a \uparrow (\alpha_0 + 1)\).

Let \(\alpha_1 = \min\{\alpha < \gamma : a(\alpha) \neq x(\alpha)\}\). Then noting \(\alpha_0 < \alpha_1\), we see \((x \uparrow (\alpha_1 + 1))^\wedge(x_0 \uparrow (\alpha_1, \gamma)) \in D \cap U\).

Similarly we see the following case.

**Case 3-2.** \(x \uparrow (\alpha_0 + 1) = b \uparrow (\alpha_0 + 1)\).

**Case 3-3.** \(x \uparrow (\alpha_0 + 1) \neq a \uparrow (\alpha_0 + 1)\) and \(x \uparrow (\alpha_0 + 1) \neq b \uparrow (\alpha_0 + 1)\).

In this case, we have \(a \uparrow (\alpha_0 + 1) = x \uparrow (\alpha_0 + 1) = b \uparrow (\alpha_0 + 1)\) and \(a(\alpha_0) < x(\alpha_0) < b(\alpha_0)\). Therefore we have \((x \uparrow (\alpha_0 + 1))^\wedge(x_0 \uparrow (\alpha_0, \gamma)) \in D \cap U\).

**Claim 2.** \(|N^+_X| \leq \kappa|\).
Proof. We consider 2 cases.

Case 1. sup $J^- = \gamma$ or sup $J^+ = \gamma$.

We will see $N^+_X = \emptyset$. Assuming $N^+_X \neq \emptyset$, take $x \in N^+_X$. Then we can take $y \in X$ with $x < y$ and $(x, y) = \emptyset$. Let $\alpha_0 = \min\{\alpha < \gamma : x(\alpha) \neq y(\alpha)\}$. We consider further 2 subcases.

Case 1-1. sup $J^+ = \gamma$.

Let $\alpha_1 = \min(J^+ \cap (\alpha_0, \gamma))$ and take $u \in X_{\alpha_1}$ with $x(\alpha_1) < u$. Then we have $(x \upharpoonright \alpha_1) \upharpoonright (u \upharpoonright (\alpha_1, \gamma)) \in (x, y)$, a contradiction.

Case 1-2. sup $J^- < \gamma$ and sup $J^+ < \gamma$.

Let $\alpha_0 = \max\{\sup J^-, \sup J^+\}$. We consider 2 subcases.

Case 2-1. sup $J^- = \gamma$.

In this case, as above, we can see $N^+_X \subseteq \bigcup_{\alpha_0 \leq \beta < \gamma} (\prod_{\alpha} X_{\alpha}) \times \{\max X_{\alpha} : \alpha > \beta\}$, because the cardinality of the right hand set is of $\leq \kappa$. To see this, let $x \in N^+_X$. Then there is $y \in X$ with $x < y$ and $(x, y) = \emptyset$. Let $\alpha_1 = \min\{\alpha < \gamma : x(\alpha) \neq y(\alpha)\}$. Then for every $\alpha < \gamma$ with $\alpha_1 < \alpha$, $X_{\alpha}$ has a maximal element and $x(\alpha) = \max X_{\alpha}$, otherwise, taking some $\alpha > \alpha_1$ and $u \in X_{\alpha}$ with $x(\alpha) < u$, we see $(x \upharpoonright \alpha) \upharpoonright (u \upharpoonright (\alpha, \gamma)) \in (x, y)$, a contradiction. Therefore $\alpha_0 \leq \alpha_1$ and $x \in (\prod_{\alpha_0 \leq \alpha_1} X_{\alpha}) \times \{\max X_{\alpha} : \alpha > \alpha_1\}$.

Case 2-2. sup $J^- = \alpha_0$.

In this case, as above, we can see $N^+_X \subseteq \bigcup_{\alpha_0 \leq \beta \leq \gamma} (\prod_{\alpha} X_{\alpha}) \times \{\min X_{\alpha} : \alpha > \beta\}$.

Then we see $|N^+_X| = |N^-_X| \leq \kappa$. \hfill $\square$

Claim 3. $|X^+| \leq \kappa$.

Proof. We consider 2 cases.

Case 1. sup $J^+ = \gamma$.

In this case, we prove $X^+ = \emptyset$. Assume $x \in X^+$, that is, $(-, x)_X \in \tau_X \setminus \lambda_X$. We will get a contradiction. Note $(x, \rightarrow)_X \neq \emptyset$ because
of $(\langle \cdot, x \rangle)_X \not\in \lambda_X$. Since $(\langle \cdot, x \rangle)_X$ is open in $X$ and $X$ is a subspace of $\hat{X}$, we can find $b \in \hat{X}$ with $x < x_b$ and $(x, b)_X \cap X = \emptyset$. Let $\alpha_0 = \min \{ \alpha < \gamma : x(\alpha) \neq b(\alpha) \}$. Then we have $x \uparrow \alpha_0 = b \uparrow \alpha_0$ and $X_{\alpha_0} \ni x(\alpha_0) < x_{\alpha_0} b(\alpha_0) \in X_{\alpha_0}$. Further let $\alpha_1 = \min (J^+ \cap (\alpha_0, \gamma))$ and take $u \in X_{\alpha_1}$ with $x(\alpha_1) < u$. Then $(x \uparrow \alpha_1)^{\langle u \rangle^{\langle x \uparrow (\alpha_1, \gamma) \rangle}} \in (x, b)_X \cap X$, a contradiction. Thus we have $X^+ = \emptyset$.

Case 2. $\sup J^+ < \gamma$.

Let $\alpha_0 = \sup J^+$. As in Case 2-1 of Claim 2, it suffices to see

$$X^+ \subset \bigcup_{\alpha_0 \leq \beta < \gamma} \{ \langle \max X_\alpha : \alpha > \beta \rangle \}.$$

Let $x \in X^+$. As in Case 1 above, take $b \in \hat{X}$ with $x < x_b$ and $(x, b)_X \cap X = \emptyset$. Let $\alpha_1 = \min \{ \alpha < \gamma : x(\alpha) \neq b(\alpha) \}$. Then for every $\alpha > \alpha_1$, a maximal element of $X_\alpha$ exists and $x(\alpha) = \max X_\alpha$, otherwise for some $\alpha > \alpha_1$ and $u \in X_{\alpha_1}$, $x(\alpha) < u$ holds, now $(x \uparrow \alpha)^{\langle u \rangle^{\langle x \uparrow (\alpha, \gamma) \rangle}} \in (x, b)_X \cap X$, a contradiction. Thus we have $\alpha_0 \leq \alpha_1$ and $x \in \langle \prod_{\alpha \leq \alpha_1} X_\alpha \rangle \times \{ \langle \max X_\alpha : \alpha > \alpha_1 \rangle \}$. □

Similarly we see the following and the proof is complete.

Claim 4. $|X^-| \leq \kappa$. □

**Theorem 3.3.** Let $X = \prod_{\alpha \leq \gamma} X_\alpha$ be a lexicographic product of GO-spaces with $|X| \geq \omega$. Then

$$w(X) = \begin{cases} \sup \{ |\prod_{\alpha \leq \beta} X_\alpha| : \beta < \gamma \} & \text{if } \gamma \text{ is limit}, \\ \max \{ |\prod_{\alpha < \gamma^{-}} X_\alpha|, w(X_{\gamma^{-}}) \} & \text{if } \gamma \text{ is successor}. \end{cases}$$

**Proof.** First assume that $\gamma$ is limit. The inequality \(\geq\) is obvious from Lemma 3.2. To see the inequality \(\leq\), let $\kappa = \sup \{ |\prod_{\alpha \leq \beta} X_\alpha| : \beta < \gamma \}$. If $\gamma > \kappa$ were true, then we have $\kappa < 2^\kappa \leq |\prod_{\alpha \leq \kappa} X_\alpha| \leq \kappa$, a contradiction. So we have $\gamma \leq \kappa$. Now Lemma 3.2 shows $w(X) \leq \kappa$.

Next let $\gamma$ be a successor. Because of $X = \prod_{\alpha < \gamma^{-}} X_\alpha \times X_{\gamma^{-}}$, Lemma 3.1 directly shows $w(X) = \max \{ |\prod_{\alpha < \gamma^{-}} X_\alpha|, w(X_{\gamma^{-}}) \}$. □

**Example 3.4.** Applying $\gamma = 2$ in the theorem above, we see $w(Q \times \mathbb{R}) = \aleph_0$ but $w(\mathbb{R} \times Q) = 2^{\aleph_0}$. This fact is also directly checked by the fact that $Q \times \mathbb{R}$ is the topological sum of $|Q|$-many $\mathbb{R}$’s but $\mathbb{R} \times Q$ is the topological sum of $|\mathbb{R}|$-many $Q$’s. Also note $w(\omega \times [0, 1]) = \aleph_0$ but $w([0, 1]_\mathbb{R} \times \omega) = 2^{\aleph_0}$, where $[0, 1]_\mathbb{R}$ denotes the interval $[0, 1)$ in $\mathbb{R}$.

The theorem above extends Theorem 4.3.1 in [2] for lexicographic products of GO-spaces.
Corollary 3.5. [2, Theorem 4.3.1] Let $X = \prod_{\alpha<\gamma} X_\alpha$ be an infinite lexicographic product of GO-spaces. Then $X$ is second countable if and only if the following clauses hold.

(1) $\gamma \leq \omega$,
(2) if $\gamma = \omega$, then for every $\alpha < \gamma$, $X_\alpha$ is countable,
(3) if $\gamma < \omega$, then the GO-space $X_{\gamma-1}$ is second countable and for every $\alpha < \gamma - 1$, $X_\alpha$ is countable.

4. Applications

For a cardinal $\mu$, $\mu^+$ denotes the the smallest cardinal greater than $\mu$. An uncountable cardinal $\lambda$ with $\lambda = \mu^+$ for some cardinal $\mu$ is said to be a successor cardinal. A limit cardinal is an uncountable cardinal which is not a successor cardinal. For a cardinal $\kappa$ and a limit cardinal $\lambda$, the cardinal function $\kappa^{<\lambda}$ is defined as follows:

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \lambda \},$$

see [4, p.52, (5.10)]. However this cardinal function can be further extended as follows, for a cardinal $\kappa$ and an ordinal $\gamma$,

$$\kappa^{<\gamma} = \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \gamma \}, \text{ equivalently,}$$

$$\kappa^{<\gamma} = \sup\{\kappa^{[\alpha]} : \alpha \text{ is an ordinal and } \alpha < \gamma \}.$$ 

Note that under this definition, whenever $\mu^+$ is a successor cardinal, we have $\kappa^{<\mu^+} = \kappa^\mu$. Obviously, whenever $\omega \leq \kappa < \gamma$, $\kappa^{<\gamma} = 2^{<\gamma}$ holds because of $\kappa^\mu = 2^\mu$ for $\omega \leq \kappa \leq \mu$. For every infinite cardinal $\kappa$, also note that $\kappa \leq 2^{<\kappa} \leq \kappa^{<\kappa}$ and $\kappa < \kappa^{\text{cf}\kappa}$ hold, and that whenever the Generalized Continuum Hypothesis (GCH) is assumed, $\kappa$ is a regular cardinal (that is, $\text{cf}\kappa = \kappa$) if and only if $2^{<\kappa} = \kappa^{<\kappa}$, see [4, Theorem 5.15]. Moreover for example, we see $2^{<\omega} = \aleph_0^\omega = \aleph_0$, $2^{<\omega+1} = \aleph_0^{<\omega+1} = 2^{\aleph_0}$, $\aleph_1^{<\omega} = \aleph_1$, $\aleph_1^{<\omega+1} = \aleph_1^{\omega+1} = 2^{\aleph_0}$, $\aleph_1^{<\omega_1+1} = 2^{\aleph_1}$, etc.

In this section, using this cardinal function, we will calculate the weight of special types of lexicographic products.

Corollary 4.1. Let $\gamma$ be an infinite ordinal, then the weight of the lexicographic product $2^\gamma$ is the cardinality $2^{<\gamma}$, that is, $w(2^\gamma) = 2^{<\gamma}$.

Proof. When $\gamma$ is limit, from Theorem 3.3, we see $w(2^\gamma) = \sup\{|2^{\beta+1}| : \beta < \gamma\} = \sup\{|2^{[\beta]} : \beta < \gamma\} = 2^{<\gamma}$. When $\gamma$ is successor, from Theorem 3.3, we see $w(2^\gamma) = \max\{|2^{\gamma-1}|, w(2)\} = 2^{\gamma-1} = 2^{<\gamma}$. □

Example 4.2. Applying the corollary above, we see $w(2^\omega) = \aleph_0$, $w(2^{\omega+1}) = w(2^\omega) = 2^{\aleph_0}$, $w(2^{\omega+1}) = w(2^{\omega^2}) = 2^{\aleph_1}$, $w(2^{\omega^2}) = 2^{<\aleph_\omega}$, $\aleph_\omega$, more generally for infinite cardinal $\kappa$, $w(2^\kappa) = 2^{<\kappa} \geq \kappa$ and $w(2^\kappa) = 2^\kappa$ whenever $\kappa < \gamma \leq \kappa^+$. 
So we have:

**Corollary 4.3.** The following hold.

1. the assertion \( w(2^\omega) = \aleph_1 \) is equivalent to the Continuum Hypothesis (CH), that is, \( 2^{\aleph_0} = \aleph_1 \).
2. the assertion \( w(2^\omega) = w(2^{\omega+1}) \) is equivalent to the assertion \( 2^{\aleph_0} = 2^{\aleph_1} \).
3. \( w(2^{\omega_1}) > \aleph_\omega \) is equivalent to the assertion that \( \aleph_\omega < 2^{\aleph_0} \) holds for some \( n \in \omega \).

Corollary 4.3 (1) shows that if the negation of CH is assumed, then the lexicographic product \( 2^{\omega_1} \) and the usual Tychonoff product \( 2^\omega \) are not homeomorphic. However, we will see in the next section that they are not homeomorphic without additional set theoretical assumptions.

Next we calculate the weight of lexicographic product \( \prod_{\alpha < \gamma} X_\alpha \), where all \( X_\alpha \)'s have the same infinite cardinality \( \kappa \).

**Corollary 4.4.** Let \( X = \prod_{\alpha < \gamma} X_\alpha \) be a lexicographic product of GO-spaces and \( \kappa \) an infinite cardinal. If for every \( \alpha < \gamma \), the cardinality of \( X_\alpha \) is \( \kappa \), then the weight of \( X \) is the cardinality \( \kappa^{<\gamma} \).

**Proof.** Noting \( w(X_{\gamma-1}) \leq |X_{\gamma-1}| = \kappa \leq \kappa^{<\gamma} \), the proof is similar to Corollary 4.1. \( \square \)

**Example 4.5.** Note that the weight of the real line \( \mathbb{R} \) and the Sorgenfrey line \( \mathbb{S} \) are \( \aleph_0 \) and \( 2^{\aleph_0} \) respectively. Applying the corollary above, we see \( w(\mathbb{R}^2) = w(\mathbb{S}^2) = (2^{\aleph_0})^{<2} = 2^{\aleph_0} \), \( w(\mathbb{R}^\omega) = w(\mathbb{S}^\omega) = (2^{\aleph_0})^{<\omega} = 2^{\aleph_0} \), \( w(\mathbb{R}^{\omega+1}) = w(\mathbb{S}^{\omega+1}) = (2^{\aleph_0})^{<\omega+1} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} \), \( w(\mathbb{R}^{\omega_1}) = w(\mathbb{S}^{\omega_1}) = (2^{\aleph_0})^{<\omega_1} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} \), \( w(\mathbb{R}^{\omega_1+1}) = w(\mathbb{S}^{\omega_1+1}) = (2^{\aleph_0})^{<\omega_1+1} = (2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1} \), etc., whereas \( w(\mathbb{Q}^2) = w(\mathbb{Q}^\omega) = \aleph_0 \), \( w(\mathbb{Q}^{\omega+1}) = w(\mathbb{Q}^{\omega_1}) = 2^{\aleph_0} \) and \( w(\mathbb{Q}^{\omega_1+1}) = 2^{\aleph_1} \).

For ordinal spaces, we see \( w(\omega^2) = w(\omega^\omega) = \aleph_0 \), \( w(\omega^{\omega+1}) = w(\omega^{\omega_1}) = 2^{\aleph_0} \), \( w(\omega^{\omega_1+1}) = w(\omega^{\omega_2}) = 2^{\aleph_1} \), \( w(\omega^{\omega_1}) = w(\omega^{\omega_2}) = 2^{\aleph_1} \), \( w(\omega^{\omega_1}+1) = w(\omega^{\omega_2}) = 2^{\aleph_1} \), etc. Also note \( w((\omega_\omega)^{\omega_\omega}) = (\aleph_\omega)^{<\omega_\omega} = (\aleph_\omega)^{\aleph_0} > \aleph_\omega \), but note that if GCH is assumed, then \( w(2^{\omega_\omega}) = 2^{<\omega_\omega} = \aleph_\omega \).

Similar to Corollary 4.3, we see:

**Corollary 4.6.** The following hold.

1. the assertions \( w(\mathbb{S}^2) = \aleph_1 \), \( w(\mathbb{S}^{\omega_1}) = \aleph_1 \), \( w(\omega^{\omega_1}) = \aleph_1 \) and \( w(\omega^{\omega_1}) = \aleph_1 \) are equivalent to CH.
2. the assertions \( w(\mathbb{R}^{\omega_1}) = w(\mathbb{R}^{\omega_1+1}), w(\mathbb{S}^{\omega_1}) = w(\mathbb{S}^{\omega_1+1}), w(\omega^{\omega_1}) = w(\omega^{\omega_1+1}) \) and \( w(\omega^{\omega_1}) = w(\omega^{\omega_1+1}) \) are equivalent to the assertion \( 2^{\aleph_0} = 2^{\aleph_1} \).
Finally, we calculate the weight of other types of lexicographic products.

**Corollary 4.7.** Let $\gamma$ be an infinite ordinal. Then the weight of the lexicographic product $\prod_{\alpha<\gamma} \alpha$ is the cardinality $2^{<\gamma}$.

**Proof.** First let $\gamma$ be limit. For every $\beta < \gamma$ with $2 \leq \beta$, note $2^{[2,\beta]} = |\prod_{\alpha<\beta} \alpha| \leq |\prod_{\alpha<\beta} \beta| \leq |\beta|^{[2,\beta]}$. Therefore moreover if we assume $\omega \leq \beta$, then we have $|\prod_{\alpha<\beta} \alpha| = 2^{[\beta]}$. So, whenever $\gamma = \omega$, we have $w(\prod_{\alpha<\gamma} \alpha) = \sup\{|\prod_{\alpha<\beta} \alpha| : 2 \leq \beta < \gamma\} = \omega = 2^{<\gamma}$. Whenever $\gamma > \omega$, we have $w(\prod_{\alpha<\gamma} \alpha) = \sup\{|\prod_{\alpha<\beta} \alpha| : 2 \leq \beta < \gamma\} = \sup\{2^{[\beta]} : \omega \leq \beta < \gamma\} = 2^{<\gamma}$.

Next let $\gamma$ be successor. From $\gamma > \omega$, we have $2^{[\beta]} = \prod_{\alpha<\gamma-1} 2 \leq \prod_{\alpha<\gamma-1} \alpha \leq \prod_{\alpha<\gamma-1} (\gamma - 1) = 2^{[\gamma]}$, thus $|\prod_{\alpha<\gamma-1} \alpha| = 2^{[\gamma]} = 2^{<\gamma}$. Moreover by $w(\gamma - 1) \leq |\gamma - 1| \leq |\gamma| < 2^{[\gamma]} = 2^{<\gamma}$, we also have $w(\prod_{\alpha<\gamma} \alpha) = \max\{|\prod_{\alpha<\gamma-1} \alpha|, w(\gamma - 1)\} = 2^{<\gamma}$.

**Example 4.8.** Using the corollary above, we see $w(\prod_{\alpha<\omega} \alpha) = \aleph_0$, $w(\prod_{\alpha<\omega+1} \alpha) = w(\prod_{\alpha<\omega_1} \alpha) = 2^{\aleph_0}$, $w(\prod_{\alpha<\omega_1+1} \alpha) = 2^{\aleph_0}$, etc. Also we remark $w(\prod_{\alpha<\omega} \alpha) = \sup\{|\prod_{\alpha<\beta} \omega_{\alpha}| : \beta < \omega\} = \sup\{\aleph_{\alpha} : \beta < \omega\} = \aleph_\omega$ and $w(\prod_{\alpha<\omega+1} \alpha) = \max\{|\prod_{\alpha<\omega} \alpha|, w(\omega)\} = |\prod_{\alpha<\omega} \alpha| = (\sup\{\aleph_{\alpha} : \alpha < \omega\})^{\aleph_0} = \aleph_\omega^{\aleph_0} > \aleph_\omega$, where for $|\prod_{\alpha<\omega} \alpha| = (\sup\{\aleph_{\alpha} : \alpha < \omega\})^{\aleph_0}$, use [4, Lemma 5.9].

5. **The Lexicographic Products versus the Tychonoff Products**

In this section, we compare the lexicographic product $2^\gamma$ with the usual Tychonoff product $2^\gamma$.

First recall that a topological space $X$ is said to be homogeneous if for every $x, y \in X$, there is a homeomorphism $h$ from $X$ onto $X$ with $h(x) = y$. Obviously:

- if topological spaces $X_\alpha$’s ($\alpha \in \Lambda$) are homogeneous, then the usual Tychonoff product $\prod_{\alpha \in \Lambda} X_\alpha$ is also homogeneous,
- if a topological space $X$ is homogeneous, then there is a unique cardinal number $\kappa$ such that $\chi(x, X) = \kappa$ for every $x \in X$, where $\chi(x, X) = \min\{|U| : U$ is a neighborhood base at $x\}$, which is called the character at $x$, see [1],
• if a topological space $X$ is homogeneous with an isolated point, then it is discrete, thus whenever $\Lambda$ is infinite, the usual Tychonoff product $2^\Lambda$ is homogeneous without isolated points.

Next we remember the cofinality of a compact LOTS discussed in [5]. Let $L$ be a compact LOTS and $x \in L$. Note that every subset $A$ of $L$ has a least upper bound $\sup_L A$ (and greatest lower bound $\inf_L A$), see [1, 3.12.3 (a)]. A subset $A$ of $(\leftrightarrow, x)_L$ is said to be 0-unbounded for $x$ in $L$ if for every $y < x$, there is $a \in A$ with $y \leq a$. Let $0\text{-}cf_L x = \min\{|A| : A \text{ is 0-unbounded for } x\}$. Obviously $0\text{-}cf_L x$ can be 0, 1 or an infinite regular cardinal, also $0\text{-}cf_L x = 1$ (0-\(\text{cf}_L x = 0\) means that $x$ is the minimal element of $L$ ($x$ has an immediate predecessor in $L$, respectively). Usually $0\text{-}cf_L x$ is denoted by $0\text{-}cf x$. Since $L$ is a compact LOTS, for every $x \in L$, there is a sequence $f x : \alpha < 0\text{-}cf x g$ such that:

- if $\beta < \alpha < 0\text{-}cf x$, then $x_\beta < L x_\alpha$,
- if $\alpha < 0\text{-}cf x$ and $\alpha$ is limit, then $x_\alpha = \sup_L \{x_\beta : \beta < \alpha\}$,
- the set $\{x_\alpha : \alpha < 0\text{-}cf x\}$ is 0-unbounded.

Analogous notions “1-\(\text{cf} x\)”, “1-normal sequence for $x$”..., etc can be defined, see [5, section 3]. Note $(x; L) = \max\{0\text{-}cf x, 1\text{-}cf x\}$ for every $x \in L$ and also note that the lexicographic product $2^\gamma$ is a compact LOTS.

**Lemma 5.1.** Let $2^\gamma$ be a lexicographic product and $x \in 2^\gamma$. Then the following hold:

1. If $x^{-1}[\{1\}]$ has no maximal element, say $\delta = \sup x^{-1}[\{1\}]$, then $0\text{-}cf x = \text{cf} \delta$, where we consider as $\sup \emptyset = 0$ and $\text{cf} 0 = 0$,
2. If $x^{-1}[\{1\}]$ has a maximal element, say $\delta = \max x^{-1}[\{1\}]$, then $0\text{-}cf x = 1$.

**Proof.** (1) Assume that $x^{-1}[\{1\}]$ has no maximal element and let $\delta = \sup x^{-1}[\{1\}]$. Fix a strictly increasing sequence $\{\delta_\xi : \xi < \text{cf} \delta\}$ in $\delta$ such that:

- $(\delta_\xi, \delta_{\xi+1}) \cap x^{-1}[\{1\}] \neq \emptyset$ for every $\xi < \text{cf} \delta$,
- $\delta_\xi = \sup\{\delta_\zeta : \zeta < \xi\}$ if $\xi$ is limit,
- $\{\delta_\xi : \xi < \text{cf} \delta\}$ is (0-)unbounded in $\delta$.

Now for every $\xi < \text{cf} \delta$, let $x_\xi = (x | \delta_\xi)^\gamma(0 : \delta_\xi \leq \alpha < \gamma)$. Then obviously $\{x_\xi : \xi < \text{cf} \delta\}$ is a 0-normal sequence for $x$ in $2^\gamma$, therefore $0\text{-}cf x = \text{cf} \delta$. 

Let assume that there is a homeomorphism $f$. If $(1)$ let $x^{-1}\{1\}$ has a maximal element $\delta$. Let $y = (x \upharpoonright \delta)^\upharpoonright \{0\}^\upharpoonright (1 : \delta < \alpha < \gamma)$, then $y < x$ and $(y, x) = \emptyset$, which shows $0$-cf $x = 1$. \hfill \Box

Changing 0 and 1 by 1 and 0, respectively, in the lemma above, we can get an analogous result for $1$-cf $x$. For example, if $x$ is an element of $2^{\omega_1}$ so that both $x^{-1}\{1\}$ and $x^{-1}\{0\}$ are unbounded in $\omega_1$, then we have $0$-cf $x = 1$-cf $x = \omega_1$, thus $\chi(x, 2^{\omega_1}) = \aleph_1$.

**Definition 5.2.** Let $L$ be a compact LOTS. A point $x$ in $L$ is said to have type $I$ if $\min\{0$-cf $x, 1$-cf $x\} \leq 1$. Otherwise, we say that $x$ has type $II$, that is, $\omega \leq 0$-cf $x$ and $\omega \leq 1$-cf $x$.

**Lemma 5.3.** Let $L$ be a compact LOTS. If there are a type $I$ point $x$ with $\omega_1 \leq \max\{0$-cf $x, 1$-cf $x\}$ and a type $II$ point $y$ in $L$, then $L$ is not homogeneous.

**Proof.** Assume that there is a homeomorphism $h : X \rightarrow X$ with $h(y) = x$, we may assume $\omega_1 \leq 0$-cf $x$ and $1$-cf $x \leq 1$. For each $i \in 2$, fix an $i$-normal sequence $A_i := \{y_i(\alpha) : \alpha < i$-cf $y\}$ for $y$. Since $y$ has type $II$, $A_0$ and $A_1$ are infinite and $\{y\} = \text{Cl}_L A_0 \cap \text{Cl}_L A_1$. Since $h$ is a homeomorphism, we have $\{x\} = \text{Cl}_L h[A_0] \cap \text{Cl}_L h[A_1]$. It follows from $1$-cf $x \leq 1$ that $\{x\} = \text{Cl}_L B_0 \cap \text{Cl}_L B_1$, where $B_i = h[A_i] \cap (\epsilon, x)$. Fix a $0$-normal sequence $\{x(\alpha) : \alpha < 0$-cf $x\}$ for $x$. Since $B_i$’s are $0$-unbounded for $x$, by induction, for every $i \in 2$ and $n \in \omega$, we can fix $b_{in} \in B_i$ and $\alpha_n < 0$-cf $x$ with $b_{in} < b_{in} < x(\alpha_n) < b_{in+1}$. Then by letting $\alpha = \sup\{\alpha_n : n \in \omega\}$, we see $x(\alpha) \in \text{Cl}_L B_0 \cap \text{Cl}_L B_1$ and $x(\alpha) < x$, which contradicts $\{x\} = \text{Cl}_L B_0 \cap \text{Cl}_L B_1$. \hfill \Box

**Lemma 5.4.** The following hold:

1. if $\gamma$ is a successor ordinal with $\gamma > \omega$, then the lexicographic product $2^\gamma$ is not homogeneous,

2. if $\gamma$ is a limit ordinal with $\gamma \geq \omega_1$, then the lexicographic product $2^\gamma$ is not homogeneous.

**Proof.** (1) Let $\gamma$ be a successor ordinal with $\gamma > \omega$. Then the maximal element $x = \langle 1 : \alpha < \gamma \rangle$ is isolated, because of $0$-cf $x = 1$ and $1$-cf $x = 0$, see Lemma 5.1. On the other hand, the element $y = \langle 1 : \alpha < \omega \rangle^\upharpoonright \{0 : \omega \leq \alpha < \gamma \rangle$ is not isolated, in fact, $0$-cf $y = \omega$. Since $y$ is not isolated, $2^\gamma$ is not homogeneous.

(2) Let $\gamma$ be a limit ordinal with $\gamma \geq \omega_1$. First assume $\text{cf} \gamma > \omega$. Let $y$ be an element of $2^\gamma$ such that both $y^{-1}\{1\}$ and $y^{-1}\{0\}$ are unbounded in $\gamma$. Moreover let $x = \langle 0 \rangle^\upharpoonright (1 : 0 < \alpha < \gamma)$. Then Lemma 5.1 shows $0$-cf $y = 1$-cf $y = \text{cf} \gamma > \omega$, $0$-cf $x = \text{cf} \gamma > \omega$ and $1$-cf $x = 1$.\hfill \Box
Thus $x$ has type I with $0$-$\text{cf } x \geq \omega_1$ and $y$ has type II, now Lemma 5.3 shows that $2^\gamma$ is not homogeneous.

Next assume $\text{cf } \gamma = \omega$ and $\gamma \geq \omega_1$. Note $\gamma > \omega_1$. Let $x = \langle 1 : \alpha < \omega_1 \rangle \wedge (0 : \omega_1 \leq \alpha < \gamma)$ and $y = \langle 0 : \alpha \leq \omega_1 \rangle \wedge (1 : \omega_1 < \alpha < \gamma)$. Then Lemma 5.1 shows $0$-$\text{cf } x = \omega_1$, $1$-$\text{cf } x = \omega$, $0$-$\text{cf } y = \omega$ and $1$-$\text{cf } y = 1$, thus we we have $\chi(x, 2^\gamma) = \aleph_1$ and $\chi(y, 2^\gamma) = \aleph_0$. So $2^\gamma$ is not homogeneous.

□

Theorem 5.5. Let $\gamma$ be an ordinal, then the following are equivalent:

(1) the lexicographic product $2^\gamma$ and the usual Tychonoff product $2^\gamma$ are homeomorphic,

(2) the identity map from the lexicographic product $2^\gamma$ onto the usual Tychonoff product $2^\gamma$ is a homeomorphism,

(3) the lexicographic product $2^\gamma$ is homeomorphic to the usual Tychonoff product $2^\Lambda$ for some $\Lambda$,

(4) $\gamma \leq \omega$.

Proof. The implication (2) $\Rightarrow$ (1) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) Assume $\gamma > \omega$ and that the lexicographic product $2^\gamma$ is homeomorphic to the usual Tychonoff product $2^\Lambda$ for some $\Lambda$. Note $2^{|\Lambda|} = 2^{|\aleph_1|}$. Since the usual Tychonoff product $2^\Lambda$ is homogeneous, from Lemma 5.4, we see that $\gamma$ is limit with $\omega < \gamma < \omega_1$. It follows from Corollary 4.1 that the weight of the the lexicographic product $2^\gamma$ is $2^{<\gamma} = 2^{\aleph_1}$. On the other hand, the weight of the product $2^\Lambda$ is at most $|\Lambda|$, which contradicts $|\Lambda| < 2^{|\Lambda|} = 2^{\aleph_1}$.

(4) $\Rightarrow$ (2) Assume $\gamma \leq \omega$. Since the case “$\gamma < \omega$” is obvious, we may assume $\gamma = \omega$. Let $L$ and $T$ be the lexicographic product and the usual Tychonoff product $2^\omega$ respectively, and $id : L \rightarrow T$ be the identity map. The following claims complete the proof. Also note that in [2, p78, Example 4], the fact that $L$ and $T$ are homeomorphic is proved by using a characterization theorem of the Cantor set.

Claim 1. $id$ is continuous.

Proof. For every $n \in \omega$ and $i \in 2$, let $U_{ni} := \{x \in T : x(n) = i\}$. Since $\{U_{ni} : n \in \omega, i \in 2\}$ is a subbase for $T$, it suffices to see that each $U_{ni}$ is open in $L$. So let $x \in U_{ni}$. We may assume $i = 0$, then note $(x, \rightarrow) \neq \emptyset$.

Fact 1. If $(\leftarrow, x) \neq \emptyset$, then there is $a \in L$ with $a < x$ and $(a, x) \subset U_{n0}$.

Proof. Let $(\leftarrow, x) \neq \emptyset$, then note $x^{-1}\{1\} \neq \emptyset$. Whenever $x^{-1}\{1\}$ has a maximal element $m_0$, let $a = (x \upharpoonright m_0) \wedge (0) \wedge (1 : m_0 < m < \omega)$. Whenever $x^{-1}\{1\}$ has no maximal element, putting $m_0 = \min(x^{-1}\{1\} \cap (n, \omega))$, let $a = (x \upharpoonright m_0) \wedge (0) \wedge (1 : m_0 < m < \omega)$. Then $a$ is the required, see Lemma 5.1 □
Similarly we see:

**Fact 2.** There is $b \in L$ with $x < b$ and $[x, b) \subset U_{n0}$.

Now let

$$V = \begin{cases} [x, b) & \text{if } (\leftarrow, x) = \emptyset, \\ (a, b) & \text{if } (\leftarrow, x) \neq \emptyset. \end{cases}$$

Then $V$ is a neighborhood of $x$ in $L$ contained in $U_{n0}$, so $U_{n0}$ is open in $L$. □

**Claim 2.** $id^{-1}$ is continuous.

*Proof.* Since $\{(a, \rightarrow) : a \in 2^\omega\} \cup \{(-, a) : a \in 2^\omega\}$ is a subbase for $L$, it suffices to see that $(a, \rightarrow)$ and $(-, a)$ are open in $T$ for every $a \in 2^\omega$. We check the former, because the latter is similar. Let $a \in 2^\omega$ and $x \in (a, \rightarrow)$. Putting $m_0 = \min\{m \in \omega : x(m) \neq a(m)\}$, let $V = \{y \in 2^\omega : y \upharpoonright (m_0 + 1) = x \upharpoonright (m_0 + 1)\}$. Then $V$ is a neighborhood of $x$ in $T$ contained in $(a, \rightarrow)$, so $(a, \rightarrow)$ is open in $T$. □

**References**


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