THE WEIGHT OF LEXICOGRAPHIC PRODUCTS

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Abstract. We will calculate the weight of lexicographic products of GO-spaces, using this we will see:

- the assertion that the weight of the lexicographic product 2^{ω_1} is $\aleph_1$ is equivalent to the Continuum Hypothesis (CH), that is, $2^{\aleph_0} = \aleph_1$,
- the assertion that the weight of both lexicographic products 2^{ω_1} and 2^{ω_1+1} coincide is equivalent to the assertion $2^{\aleph_0} = 2^{\aleph_1}$,
- the assertion that the lexicographic product 2^γ is homeomorphic to the usual Tychonoff product 2^γ is equivalent to $\gamma \leq \omega$.

1. Introduction

We will work on the usual ZFC-set theory including the Axiom of Choice (AC) [4, 11]. All spaces are assumed to be regular T_1 containing at least 2 points and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α's are also assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminologies follow [4, 11, 1].

In [2], second countability of lexicographic products of LOTS's is characterized. It is known in [2, p.78, Example 4] that the usual Tychonoff product 2^ω, which is homeomorphic to the Cantor set $\mathbb{C}$, is also homeomorphic to the lexicographic product 2^ω, where $2 = \{0, 1\}$ with $0 < 1$. So they are second countable, that is, the weight is at most countable. On the other hand, the weight of the usual Tychonoff product 2^{ω_1} is easily seen to be $\aleph_1$. So it is natural to conjecture:

(1) the lexicographic product 2^{ω_1} is homeomorphic to the usual Tychonoff product 2^{ω_1},
(2) the weight of the lexicographic product 2^{ω_1} is $\aleph_1$.

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Recently, the notion of lexicographic products of GO-spaces is introduced and discussed in [8, 9, 10], also see [3, 6, 7] for products of LOTS's. In this paper, we will calculate the weight of lexicographic products of GO-spaces. As corollaries, we see:

- the conjecture (1) is false, in fact, the assertion that the lexicographic product 2^γ is homeomorphic to the usual Tychonoff product 2^γ is equivalent to $\gamma \leq \omega$,
- the conjecture (2) is equivalent to the Continuum Hypothesis (CH), that is, $2^{\aleph_0} = \aleph_1$.

Obviously the usual Tychonoff products 2^{ω_1} and 2^{ω_1+1} are homeomorphic, however we will also see:

- the assertion that the weight of both lexicographic products 2^{ω_1} and 2^{ω_1+1} coincide is equivalent to the assertion $2^{\aleph_0} = 2^{\aleph_1}$.

A linearly ordered set $\langle L, <_L \rangle$ has a natural topology λ_L, which is called an *interval topology*, generated by $\{(\leftarrow, x)_L : x \in L\} \cup \{(x, \rightarrow)_L : x \in L\}$ as a subbase, where $(x, \rightarrow)_L = \{z \in L : x <_L z\}$, $(x, y)_L = \{z \in L : x <_L z <_L y\}$, $(x, y]_L = \{z \in L : x <_L z \leq_L y\}$ and so on. The triple $\langle L, <_L, \lambda_L \rangle$, which is simply denoted by L, is called a *LOTS*.

A triple $\langle X, <_X, \tau_X \rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X \rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subset C$ holds. In this situation, $\langle X, <_X \rangle$ is called an *underlying linearly ordered set* of X. The symbols $\mathbb{R}$ and $\mathbb{Q}$ denote the reals and the rationals respectively. Note that they are LOTS's. On the other hand, the Sorgenfrey line $\mathbb{S}$, whose underlying linearly ordered set is $\mathbb{R}$ and the sets of type $[a, b)$ are declared to be open, is known to be a GO-space but not a LOTS. For more information on LOTS's or GO-spaces, see [12]. Usually $<_L$, $(x, y)_L$, λ_L or τ_X are written simply $<$, (x, y), λ or τ if contexts are clear.

ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \delta \cdots$, are considered to be LOTS's with the usual interval topology. $\mathrm{cf}\alpha$ denotes the cofinality of the ordinal α. When α is a successor ordinal, i.e., $\alpha = \delta + 1$ for some ordinal δ, this δ is denoted by $\alpha - 1$. A non-zero ordinal which is not successor is said to be a limit ordinal.

An ordinal α is said to be a *cardinal* if $\alpha = |\alpha|$, where $|X|$ denotes the cardinality of a set X, that is, $|X|$ is the smallest ordinal δ such that there is a 1-1 map from X onto δ [11, I Definition 10.3], where note that the existence of $|X|$ is ensured by AC. When we want to emphasize that ω and ω_1 are cardinals, we write them by $\aleph_0$ and $\aleph_1$, respectively. Generally, the α-th uncountable cardinal is denoted by ω_α or $\aleph_\alpha$. Cardinals are usually denoted by Greek letters $\kappa, \lambda, \mu, \cdots$. For cardinals κ and λ, κ^λ denotes the cardinal $|X^Y|$ with $|X| = \kappa$ and $|Y| = \lambda$, where X^Y denotes the set of all functions on Y to X.

It is well known that for a LOTS $\langle Y, <_Y, \lambda_Y\rangle$, if $X \subset Y$, then $\langle X, <_X, \tau_X\rangle$ is a GO-space with $<_X = <_Y \restriction X$ and τ_X is the subspace topology $\lambda_Y \restriction X$. For every GO-space X, there is a LOTS X^* such that X is a dense subspace of X^* and X^* has the property that if L is a LOTS containing X as a dense subspace, then L also contains the LOTS X^* as a subspace, see [13]. Such a X^* is called the *minimal d-extension of a GO-space* X. Indeed, the LOTS X^* is constructed as follows, see also [8]. Let

$$X^+ = \{x \in X : (\leftarrow, x] \in \tau_X \setminus \lambda_X\},$$

$$X^- = \{x \in X : [x, \rightarrow) \in \tau_X \setminus \lambda_X\}.$$

Then

$$X^* = (X^- \times \{-1\}) \cup (X \times \{0\}) \cup (X^+ \times \{1\}),$$

where the order $<_{X^*}$ on X^* is the restriction of the usual lexicographic order on $X \times \{-1, 0, 1\}$ with $-1 < 0 < 1$. Also we identify $X \times \{0\}$ with X in the obvious way. Obviously, we can see:

- if X is a LOTS, then $X^* = X$,
- X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

For every $\alpha < \gamma$, let X_α be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma\rangle$. In the present paper, a sequence means a function whose domain is an ordinal. For notational convenience, $\prod_{\alpha<\gamma} X_\alpha$ is considered as $\{\emptyset\}$ whenever $\gamma = 0$, where $\emptyset$ is considered to be a function whose domain is 0. When $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$ and $y_1 \in \prod_{\beta\leq\alpha} X_\alpha$, $y_0 {}^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ y_1(\alpha) & \text{if } \beta \leq \alpha. \end{cases}$$

In this case, whenever $\beta = 0$, $\emptyset {}^\wedge y_1$ is considered as y_1. In case $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$, $u \in X_\beta$ and $y_1 \in \prod_{\beta<\alpha} X_\alpha$, $y_0 {}^\wedge\langle u\rangle{}^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ u & \text{if } \alpha = \beta, \\ y_1(\alpha) & \text{if } \beta < \alpha. \end{cases}$$

More general cases are similarly defined.

The lexicographic order $<_X$ on $X = \prod_{\alpha<\gamma} X_\alpha$, where all X_α's are LOTS's, is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha \rangle$ and $<_{X_\alpha}$ is the order on X_α. Now for every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace X of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ is said to be the *lexicographic product of GO-spaces* X_α's, for more details see [8]. $\prod_{i\in\omega} X_i$ ($\prod_{i\leq n} X_i$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$. When X_α's are GO-spaces, $\prod_{\alpha<\gamma} X_\alpha$ usually means the lexicographic product otherwise stated.

2. The weight of GO-spaces

Recall that the weight $w(X)$ and the density $d(X)$ of a topological space X are defined as follows:

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\},$$

$$d(X) = \min\{|D| : D \text{ is dense in } X\}.$$

Because X is regular T_1, note that if X is infinite, then $\aleph_0 \leq d(X) \leq w(X)$ and $d(X) \leq |X|$ hold, and also that if X is finite, then $d(X) = w(X) = |X|$. For a GO-space X, let

$$N^+_X = \{x \in X : \text{ there is } y \in X \text{ with } x < y \text{ and } (x,y) = \emptyset\},$$

$$N^-_X = \{x \in X : \text{ there is } y \in X \text{ with } y < x \text{ and } (y,x) = \emptyset\}.$$

In other words, an element of N^+_X is called a left neighbor of neighbors in the sense of [2, p.6]. For every $x \in N^+_X$, assign $y \in X$ with $x < y$ and $(x,y) = \emptyset$. Then this assignment defines a 1-1 map on N^+_X onto N^-_X, so we have $|N^+_X| = |N^-_X|$. Obviously we have :

- if $x \in N^+_X$, then $(\leftarrow, x] \in \lambda_X \subset \tau_X$,
- $x \in X^+ \cup N^+_X$ iff $(x, \rightarrow)_X \neq \emptyset$ and $(\leftarrow, x]_X \in \tau_X$.

Lemma 2.1. *Let X be a GO-space and $x \in X$. Then $x \notin X^+ \cup N_X^+$ holds if and only if $(x, y)_{X^*} \neq \emptyset$ for every $y \in X^*$ with $x <_{X^*} y$, in other words, x has no immediate successor in X^*. Therefore, if $x \notin X^+ \cup N_X^+$, $(x, \to)_X \neq \emptyset$, D is dense in X and U is a neighborhood of x in X, then there is $d \in D$ with $x < d$ and $[x, d]_X \subset U$.*

Proof. The necessity is obvious. To see the the sufficiency, let $x \notin X^+ \cup N_X^+$, $(x, \to) \neq \emptyset$. If $y_0 \in X^*$ were an immediate successor of x, then y_0 would not belong to X otherwise $x \in N_X^+$ and $(\leftarrow, x] = (\leftarrow y_0)_{X^*} \cap X$ would be open in X. Thus $x \in X^+$, a contradiction.

To see the latter half, let $x \notin X^+ \cup N_X^+$, $(x, \to)_X \neq \emptyset$, D be dense in X and U a neighborhood of x. Take $y \in X^*$ with $x <_{X^*} y$ and $[x, y)_{X^*} \cap X \subset U$. Then y is not the immediate successor of x in X^*, so $(x, y)_{X^*} \neq \emptyset$ and $(x, y)_{X^*} \cap X$ is a non-empty open set in X. Therefore there exists a $d \in D$ such that $d \in (x, y)_{X^*} \cap X$, then $[x, d]_X \subset U$. ☐

We can also verify an analogous Lemma above for $x \notin X^- \cup N_X^-$. The weight $w(X)$ of a GO-space X is decided from $d(X), |N_X^+|, |X^+|$ and $|X^-|$.

Lemma 2.2. *Let X be a GO-space. Then*

$$w(X) = \max\{d(X), |N_X^+|, |X^+|, |X^-|\}.$$

Proof. If X is finite, then we have $|N_X^+| \leq |X| = w(X) = d(X)$ and $X^+ = X^- = \emptyset$. So we assume that X is infinite.

To see the inequality "$\geq$", let $\kappa = w(X)$ and $\mathcal{B}$ be a base for X with $|\mathcal{B}| = \kappa$. For each $x \in N_X^+$, assign $B_x \in \mathcal{B}$ with $x \in B_x \subset (\leftarrow, x]_X$. Then this assignment defines an injective function on N_X^+ to $\mathcal{B}$, so we have $|N_X^+| \leq \kappa$. Similarly we can see $|X^+| \leq \kappa$ and $|X^-| \leq \kappa$.

To see the other inequality, let $\kappa = \max\{d(X), |N_X^+|, |X^+|, |X^-|\}$ and fix a dense set D in X with $|D| = d(X)$. Since $|N_X^+| = |N_X^-|$ holds, it suffices to see the following claim.

Claim 1. The collection

$$\mathcal{S} := \{(\leftarrow, x) : x \in D\} \cup \{(x, \to) : x \in D\}$$

$$\cup \{(\leftarrow, x] : x \in N_X^+ \cup X^+\} \cup \{[x, \to) : x \in N_X^- \cup X^-\}$$

is a subbase for X, in fact, the collection of all non-empty intersections of at most two members of $\mathcal{S}$ is a base for X.

Proof. Let U be a non-empty open set with $x \in U$. We consider several cases.

Case 1. $(\leftarrow, x) = \emptyset$, that is, $x = \min X$.

Note $(x,\to) \neq \emptyset$. Whenever $x \in N_X^+ \cup X^+$, we have $x \in (\leftarrow, x] = \{x\} \subset U$. So let $x \notin N_X^+ \cup X^+$. ¿From Lemma 2.1, we can take $d \in D$ with $x < d$ and $[x,d] \subset U$. Then $x \in (\leftarrow, d)_X \subset U$ with $(\leftarrow, d)_X \in \mathcal{S}$.

Similarly we see:

Case 2. $(x,\to) = \emptyset$, that is, $x = \max X$.

Case 3. $(\leftarrow, x) \neq \emptyset$ and $(x,\to) \neq \emptyset$.

Whenever $x \notin N_X^+ \cup X^+$ and $x \notin N_X^- \cup X^-$, taking $d', d \in D$ with $d' < x < d$ and $[d', d] \subset U$ from Lemma 2.1, we have $x \in (\leftarrow, d) \cap (d', \to) \subset U$ with $(\leftarrow, d), (d', \to) \in \mathcal{S}$. Whenever $x \notin N_X^+ \cup X^+$ and $x \in N_X^- \cup X^-$, taking $d \in D$ with $x < d$ and $[x,d] \subset U$ from Lemma 2.1, we have $x \in (\leftarrow, d) \cap [x, \to) \subset U$ with $(\leftarrow, d), [x, \to) \in \mathcal{S}$. The case $x \in N_X^+ \cup X^+$ and $x \notin N_X^- \cup X^-$ is similar. Whenever $x \in N_X^+ \cup X^+$ and $x \in N_X^- \cup X^-$, we have $x \in (\leftarrow, x] \cap [x, \to) = \{x\} \subset U$ with $(\leftarrow, x]$, $[x, \to) \in \mathcal{S}$. □

□

Remark that this lemma also shows the well-known fact $w(X) \leq |X|$ about a GO-space X.

3. The weight of lexicographic products

In this section, we calculate the weight of the lexicographic products.

Lemma 3.1. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces with $|X| \geq \omega$ and κ an infinite cardinal. Then $w(X) \leq \kappa$ holds if and only if $|X_0| \leq \kappa$ and $w(X_1) \leq \kappa$.*

Proof. Let $\hat{X} = X_0^* \times X_1^*$.

To see "only if" part, let $w(X) \leq \kappa$ and $\mathcal{B}$ be a base for X with $|\mathcal{B}| = w(X)$. Take $v', v \in X_1$ with $v' < v$. Then for every $u \in X_0$, $(\langle u, v' \rangle, \to)_X$ is a neighborhood of $\langle u, v \rangle$. So for every $u \in X_0$ assign $B_u \in \mathcal{B}$ with $\langle u, v \rangle \in B_u \subset (\langle u, v' \rangle, \to)_X$. Then this assignment witnesses $|X_0| \leq |\mathcal{B}| \leq \kappa$. Now fix $u_0 \in X_0$, then obviously X_1 can be identified with the subspace $\{u_0\} \times X_1$, see also [9, Lemma 3.4]. Then we have $w(X_1) = w(\{u_0\} \times X_1) \leq w(X) \leq \kappa$.

To see "if" part, let $|X_0| \leq \kappa$ and $w(X_1) \leq \kappa$. Then by Lemma 2.2, we see $d(X_1) \leq \kappa$, $|N_{X_1}^+| \leq \kappa$, $|X_1^+| \leq \kappa$ and $|X_1^-| \leq \kappa$. So we can fix a dense set D_1 in X_1 with $|D_1| = d(X_1)$. Let

$$M = \{v \in X_1 : (\leftarrow, v) = \emptyset \text{ or } (v, \to) = \emptyset\},$$

that is, M is the set of a maximal element and a minimal element if exists, so $|M| \leq 2$. ¿From Lemma 2.2, it suffices to see the following claims.

Claim 1. $d(X) \leq \kappa$.

Proof. Let $D = X_0 \times (D_1 \cup M)$. The assumption ensures $|D| \leq \kappa$, so it suffices to see that D is dense in X. Let $x \in X$ and U be a neighborhood of x in X, say $x = \langle u, v \rangle$. When $v \in M$, obviously $U \cap D$ is non-empty. So assume $v \notin M$, then we can take $v_0^*, v_1^* \in X_1^*$ with $v_0^* < v < v_1^*$ and $(\langle u, v_0^* \rangle, \langle u, v_1^* \rangle)_{\hat{X}} \cap X \subset U$. Since $(v_0^*, v_1^*)_{X_1^*} \cap X_1$ is non-empty open set in X_1, we can find $d \in D_1 \cap ((v_0^*, v_1^*)_{X_1^*} \cap X_1)$. Now we have $\langle u, d \rangle \in U \cap D$. □

Claim 2. $|N_X^+| \leq \kappa$.

Proof. It suffices to see $N_X^+ \subset X_0 \times (N_{X_1}^+ \cup M)$. Let $x \in N_X^+$, say $x = \langle u, v \rangle$. As above, we may assume $v \notin M$. ¿From $x \in N_X^+$, we can find $y \in X$ with $x <_X y$ and $(x, y)_X = \emptyset$. By $(v, \rightarrow)_{X_1} \neq \emptyset$, y has to be $\langle u, v' \rangle$ for some $v' \in X_1$ with $v <_{X_1} v'$. Then we have $(v, v')_{X_1} = \emptyset$, therefore $v \in N_{X_1}^+$, so $x \in X_0 \times (N_{X_1}^+ \cup M)$. □

Claim 3. $|X^+| \leq \kappa$.

Proof. It suffices to see $X^+ \subset X_0 \times (X_1^+ \cup M)$. Let $x \in X^+$, say $x = \langle u, v \rangle$. As above, we may assume $v \notin M$. ¿From $x \in X^+$, note $(\leftarrow, x]_X \in \tau_X \setminus \lambda_X$. If $(\leftarrow, v]_{X_1} \in \lambda_{X_1}$ were true, then there is $v' \in X_1$ with $v <_{X_1} v'$ and $(v, v')_{X_1} = \emptyset$. Then we have $(\leftarrow, x]_X = (\leftarrow, \langle u, v' \rangle)_X \in \lambda_X$, a contradiction. So we have $(\leftarrow, v]_{X_1} \notin \lambda_{X_1}$. By $(\leftarrow, x]_X \in \tau_X$, we can find $y \in \hat{X}$ with $x <_{\hat{X}} y$ and $(x, y)_{\hat{X}} \cap X = \emptyset$. Since $(v, \rightarrow)_{X_1} \neq \emptyset$ holds, y can be represented as $\langle u, v^* \rangle$ with $v <_{X_1^*} v^* \in X_1^*$. Then we have $(v, v^*)_{X_1^*} = \emptyset$, otherwise $(x, y)_{\hat{X}} \cap X \neq \emptyset$. Now we have $(\leftarrow, v]_{X_1} = (\leftarrow, v^*)_{X_1^*} \cap X_1 \in \tau_{X_1}$, which implies, by $(\leftarrow, v]_{X_1} \notin \lambda_{X_1}$, $v \in X_1^+$ thus $x \in X_0 \times (X_1^+ \cup M)$. □

Similarly we see the following.

Claim 4. $|X^-| \leq \kappa$. □

Lemma 3.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and κ an infinite cardinal. Assume that γ is a limit ordinal. Then $w(X) \leq \kappa$ holds if and only if $\gamma \leq \kappa$ holds and for every $\beta < \gamma$, $|\prod_{\alpha \leq \beta} X_\alpha| \leq \kappa$ holds.*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X_\alpha^*$.

Assume $w(X) \leq \kappa$ and $\beta < \gamma$. It follows from $X = (\prod_{\alpha \leq \beta} X_\alpha) \times (\prod_{\beta<\alpha} X_\alpha)$, see [8, Lemma 1.5], that $|\prod_{\alpha \leq \beta} X_\alpha| \leq \kappa$ has to be true from the lemma above. If $\gamma > \kappa$ were true, then by $X = (\prod_{\alpha<\kappa} X_\alpha) \times$

$(\prod_{\kappa\leq\alpha} X_\alpha)$, applying the lemma above, we see $\kappa < 2^\kappa \leq |\prod_{\alpha<\kappa} X_\alpha| \leq \kappa$, a contradiction.

To see the other direction, let $\gamma \leq \kappa$ and for every $\beta < \gamma$, $|\prod_{\alpha\leq\beta} X_\alpha| \leq \kappa$. Define

$$J^+ = \{\alpha < \gamma :\ X_\alpha \text{ has no maximal element.}\},$$

$$J^- = \{\alpha < \gamma :\ X_\alpha \text{ has no minimal element.}\}.$$

By Lemma 2.2, it suffices to see the following claims.

Claim 1. $d(X) \leq \kappa$.

Proof. Fix $x_0 \in X$ and let

$$D = \bigcup_{\beta<\gamma} \{y^\wedge(x_0 \restriction (\beta,\gamma)) : y \in \prod_{\alpha\leq\beta} X_\alpha\}.$$

The assumption ensures $|D| \leq \kappa$, so it suffices to see that D is dense in X. Let $x \in X$ and U be a neighborhood of x in X. We consider some cases.

Case 1. $(\leftarrow, x) = \emptyset$.

Because of $x = \min X$ and $(x, \rightarrow) \neq \emptyset$, we can find $b \in \hat{X}$ with $x <_{\hat{X}} b$ and $[x, b)_{\hat{X}} \cap X \subset U$. Set $\alpha_0 = \min\{\alpha < \gamma : x(\alpha) \neq b(\alpha)\}$. Then $(x \restriction (\alpha_0 + 1))^\wedge(x_0 \restriction (\alpha_0, \gamma)) \in D \cap U$.

Similarly we see the following case.

Case 2. $(x, \rightarrow) = \emptyset$.

Case 3. $(\leftarrow, x) \neq \emptyset$ and $(x, \rightarrow) \neq \emptyset$.

Take $a, b \in \hat{X}$ with $a <_{\hat{X}} x <_{\hat{X}} b$ and $(a, b)_{\hat{X}} \cap X \subset U$. Set $\alpha_0 = \min\{\alpha < \gamma : a(\alpha) \neq b(\alpha)\}$. By $a < x < b$, we have $a \restriction \alpha_0 = x \restriction \alpha_0 = b \restriction \alpha_0$. We consider further 3 cases.

Case 3-1. $x \restriction (\alpha_0 + 1) = a \restriction (\alpha_0 + 1)$.

Let $\alpha_1 = \min\{\alpha < \gamma : a(\alpha) \neq x(\alpha)\}$. Then noting $\alpha_0 < \alpha_1$, we see $(x \restriction (\alpha_1 + 1))^\wedge(x_0 \restriction (\alpha_1, \gamma)) \in D \cap U$.

Similarly we see the following case.

Case 3-2. $x \restriction (\alpha_0 + 1) = b \restriction (\alpha_0 + 1)$.

Case 3-3. $x \restriction (\alpha_0 + 1) \neq a \restriction (\alpha_0 + 1)$ and $x \restriction (\alpha_0 + 1) \neq b \restriction (\alpha_0 + 1)$.

In this case, we have $a \restriction \alpha_0 = x \restriction \alpha_0 = b \restriction \alpha_0$ and $a(\alpha_0) < x(\alpha_0) < b(\alpha_0)$. Therefore we have $(x \restriction (\alpha_0 + 1))^\wedge(x_0 \restriction (\alpha_0, \gamma)) \in D \cap U$. □

Claim 2. $|N_X^+| \leq \kappa$.

Proof. We consider 2 cases.

Case 1. $\sup J^- = \gamma$ or $\sup J^+ = \gamma$.

We will see $N_X^+ = \emptyset$. Assuming $N_X^+ \neq \emptyset$, take $x \in N_X^+$. Then we can take $y \in X$ with $x < y$ and $(x, y) = \emptyset$. Let $\alpha_0 = \min\{\alpha < \gamma : x(\alpha) \neq y(\alpha)\}$. We consider further 2 subcases.

Case 1-1. $\sup J^+ = \gamma$.

Let $\alpha_1 = \min(J^+ \cap (\alpha_0, \gamma))$ and take $u \in X_{\alpha_1}$ with $x(\alpha_1) < u$. Then we have $(x \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha_1, \gamma)) \in (x, y)$, a contradiction.

Case 1-2. $\sup J^- = \gamma$.

Let $\alpha_1 = \min(J^- \cap (\alpha_0, \gamma))$ and take $u \in X_{\alpha_1}$ with $u < y(\alpha_1)$. Then we have $(y \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(y \restriction (\alpha_1, \gamma)) \in (x, y)$, a contradiction.

Case 2. $\sup J^- < \gamma$ and $\sup J^+ < \gamma$.

Let $\alpha_0 = \max\{\sup J^-, \sup J^+\}$. We consider 2 subcases.

Case 2-1. $\sup J^+ = \alpha_0$.

It suffices to see

$$N_X^+ \subset \bigcup_{\alpha_0 \leq \beta < \gamma} (\prod_{\alpha \leq \beta} X_\alpha) \times \{\langle \max X_\alpha : \alpha > \beta \rangle\},$$

because the cardinality of the right hand set is of $\leq \kappa$. To see this, let $x \in N_X^+$. Then there is $y \in X$ with $x < y$ and $(x, y) = \emptyset$. Let $\alpha_1 = \min\{\alpha < \gamma : x(\alpha) \neq y(\alpha)\}$. Then for every $\alpha < \gamma$ with $\alpha_1 < \alpha$, X_α has a maximal element and $x(\alpha) = \max X_\alpha$, otherwise, taking some $\alpha > \alpha_1$ and $u \in X_\alpha$ with $x(\alpha) < u$, we see $(x \restriction \alpha)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha, \gamma)) \in (x, y)$, a contradiction. Therefore $\alpha_0 \leq \alpha_1$ and $x \in (\prod_{\alpha \leq \alpha_1} X_\alpha) \times \{\langle \max X_\alpha : \alpha > \alpha_1 \rangle\}$.

Case 2-2. $\sup J^- = \alpha_0$.

In this case, as above, we can see

$$N_X^- \subset \bigcup_{\alpha_0 \leq \beta < \gamma} (\prod_{\alpha \leq \beta} X_\alpha) \times \{\langle \min X_\alpha : \alpha > \beta \rangle\}.$$

Then we see $|N_X^+| = |N_X^-| \leq \kappa$. □

Claim 3. $|X^+| \leq \kappa$.

Proof. We consider 2 cases.

Case 1. $\sup J^+ = \gamma$.

In this case, we prove $X^+ = \emptyset$. Assume $x \in X^+$, that is, $(\leftarrow, x]_X \in \tau_X \setminus \lambda_X$. We will get a contradiction. Note $(x, \rightarrow)_X \neq \emptyset$ because of $(\leftarrow, x]_X \notin \lambda_X$. Since $(\leftarrow, x]_X$ is open in X and X is a subspace of $\hat{X}$, we can find $b \in \hat{X}$ with $x <_{\hat{X}} b$ and $(x, b)_{\hat{X}} \cap X = \emptyset$. Let $\alpha_0 = \min\{\alpha < \gamma : x(\alpha) \neq b(\alpha)\}$. Then we have $x \restriction \alpha_0 = b \restriction \alpha_0$ and $X_{\alpha_0} \ni x(\alpha_0) <_{X^*_{\alpha_0}} b(\alpha_0) \in X^*_{\alpha_0}$. Further let $\alpha_1 = \min(J^+ \cap (\alpha_0, \gamma))$ and take $u \in X_{\alpha_1}$ with $x(\alpha_1) < u$. Then $(x \restriction \alpha_1)^\frown\langle u\rangle^\frown(x \restriction (\alpha_1.\gamma)) \in (x, b)_{\hat{X}} \cap X$, a contradiction. Thus we have $X^+ = \emptyset$.

Case 2. $\sup J^+ < \gamma$.

Let $\alpha_0 = \sup J^+$. As in Case 2-1 of Claim 2, it suffices to see

$$X^+ \subset \bigcup_{\alpha_0 \leq \beta < \gamma} (\prod_{\alpha \leq \beta} X_\alpha) \times \{\langle \max X_\alpha : \alpha > \beta \rangle\}.$$

Let $x \in X^+$. As in Case 1 above, take $b \in \hat{X}$ with $x <_{\hat{X}} b$ and $(x, b)_{\hat{X}} \cap X = \emptyset$. Let $\alpha_1 = \min\{\alpha < \gamma : x(\alpha) \neq b(\alpha)\}$. Then for every $\alpha > \alpha_1$, a maximal element of X_α exists and $x(\alpha) = \max X_\alpha$, otherwise for some $\alpha > \alpha_1$ and $u \in X_\alpha$, $x(\alpha) < u$ holds, now $(x \restriction \alpha)^\frown\langle u\rangle^\frown(x \restriction (\alpha, \gamma)) \in (x, b)_{\hat{X}} \cap X$, a contradiction. Thus we have $\alpha_0 \leq \alpha_1$ and $x \in (\prod_{\alpha \leq \alpha_1} X_\alpha) \times \{\langle \max X_\alpha : \alpha > \alpha_1 \rangle\}$. □

Similarly we see the following and the proof is complete.

Claim 4. $|X^-| \leq \kappa$. □

Theorem 3.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces with $|X| \geq \omega$. Then*

$$w(X) = \begin{cases} \sup\{|\prod_{\alpha \leq \beta} X_\alpha| : \beta < \gamma\} & \text{if } \gamma \text{ is limit,} \\ \max\{|\prod_{\alpha < \gamma - 1} X_\alpha|, w(X_{\gamma-1})\} & \text{if } \gamma \text{ is successor.} \end{cases}$$

Proof. First assume that γ is limit. The inequality "$\geq$" is obvious from Lemma 3.2. To see the inequality "$\leq$", let $\kappa = \sup\{|\prod_{\alpha\leq\beta} X_\alpha| : \beta < \gamma\}$. If $\gamma > \kappa$ were true, then we have $\kappa < 2^\kappa \leq |\prod_{\alpha\leq\kappa} X_\alpha| \leq \kappa$, a contradiction. So we have $\gamma \leq \kappa$. Now Lemma 3.2 shows $w(X) \leq \kappa$.

Next let γ be a successor. Because of $X = (\prod_{\alpha<\gamma-1} X_\alpha) \times X_{\gamma-1}$, Lemma 3.1 directly shows $w(X) = \max\{|\prod_{\alpha<\gamma-1} X_\alpha|, w(X_{\gamma-1})\}$. □

Example 3.4. Applying $\gamma = 2$ in the theorem above, we see $w(\mathbb{Q} \times \mathbb{R}) = \aleph_0$ but $w(\mathbb{R} \times \mathbb{Q}) = 2^{\aleph_0}$. This fact is also directly checked by the fact that $\mathbb{Q} \times \mathbb{R}$ is the topological sum of $|\mathbb{Q}|$-many $\mathbb{R}$'s but $\mathbb{R} \times \mathbb{Q}$ is the topological sum of $|\mathbb{R}|$-many $\mathbb{Q}$'s. Also note $w(\omega \times [0, 1)_\mathbb{R}) = \aleph_0$ but $w([0, 1)_\mathbb{R} \times \omega) = 2^{\aleph_0}$, where $[0, 1)_\mathbb{R}$ denotes the interval $[0, 1)$ in $\mathbb{R}$.

The theorem above extends Theorem 4.3.1 in [2] for lexicographic products of GO-spaces.

Corollary 3.5. [2, Theorem 4.3.1] *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be an infinite lexicographic product of GO-spaces. Then* X *is second countable if and only if the following clauses hold.*

(1) $\gamma \leq \omega$,
(2) *if* $\gamma = \omega$, *then for every* $\alpha < \gamma$, X_α *is countable,*
(3) *if* $\gamma < \omega$, *then the GO-space* $X_{\gamma-1}$ *is second countable and for every* $\alpha < \gamma - 1$, X_α *is countable.*

4. Applications

For a cardinal μ, μ^+ denotes the the smallest cardinal greater than μ. An uncountable cardinal λ with $\lambda = \mu^+$ for some cardinal μ is said to be a successor cardinal. A limit cardinal is an uncountable cardinal which is not a successor cardinal. For a cardinal κ and a limit cardinal λ, the cardinal function $\kappa^{<\lambda}$ is defined as follows:

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \ \mu \text{ is a cardinal and } \mu < \lambda \ \},$$

see [4, p.52, (5.10)]. However this cardinal function can be further extended as follows, for a cardinal κ and an ordinal γ,

$$\kappa^{<\gamma} = \sup\{\kappa^\mu : \ \mu \text{ is a cardinal and } \mu < \gamma \ \}, \text{ equivalently,}$$

$$\kappa^{<\gamma} = \sup\{\kappa^{|\alpha|} : \ \alpha \text{ is an ordinal and } \alpha < \gamma \ \}.$$

Note that under this definition, whenever μ^+ is a successor cardinal, we have $\kappa^{<\mu^+} = \kappa^\mu$. Obviously, whenever $\omega \leq \kappa < \gamma$, $\kappa^{<\gamma} = 2^{<\gamma}$ holds because of $\kappa^\mu = 2^\mu$ for $\omega \leq \kappa \leq \mu$. For every infinite cardinal κ, also note that $\kappa \leq 2^{<\kappa} \leq \kappa^{<\kappa}$ and $\kappa < \kappa^{\mathrm{cf}\kappa}$ hold, and that whenever the Generalized Continuum Hypothesis (GCH) is assumed, κ is a regular cardinal (that is, $\mathrm{cf}\kappa = \kappa$) if and only if $2^{<\kappa} = \kappa^{<\kappa}$, see [4, Theorem 5.15]. Moreover for example, we see $2^{<\omega} = \aleph_0^{<\omega} = \aleph_0$, $2^{<\omega+1} = \aleph_0^{<\omega+1} = 2^{\aleph_0}$, $\aleph_1^{<\omega} = \aleph_1$, $\aleph_1^{<\omega+1} = \aleph_1^{<\omega_1} = 2^{\aleph_0}$, $\aleph_1^{<\omega_1+1} = 2^{\aleph_1}, \cdots$, etc.

In this section, using this cardinal function, we will calculate the weight of special types of lexicographic products.

Corollary 4.1. *Let* γ *be an infinite ordinal, then the weight of the lexicographic product* 2^γ *is the cardinality* $2^{<\gamma}$, *that is,* $w(2^\gamma) = 2^{<\gamma}$.

Proof. When γ is limit, from Theorem 3.3, we see $w(2^\gamma) = \sup\{|2^{\beta+1}| : \beta < \gamma\} = \sup\{2^{|\beta|} : \beta < \gamma\} = 2^{<\gamma}$. When γ is successor, from Theorem 3.3, we see $w(2^\gamma) = \max\{|2^{\gamma-1}|, w(2)\} = 2^{|\gamma-1|} = 2^{<\gamma}$. □

Example 4.2. Applying the corollary above, we see $w(2^\omega) = \aleph_0$, $w(2^{\omega+1}) = w(2^{\omega_1}) = 2^{\aleph_0}$, $w(2^{\omega_1+1}) = w(2^{\omega_2}) = 2^{\aleph_1}$, $w(2^{\omega_\omega}) = 2^{<\aleph_\omega} \geq \aleph_\omega$, more generally for infinite cardinal κ, $w(2^\kappa) = 2^{<\kappa} \geq \kappa$ and $w(2^\gamma) = 2^\kappa$ whenever $\kappa < \gamma \leq \kappa^+$.

So we have:

Corollary 4.3. *The following hold.*

(1) *the assertion* $w(2^{\omega_1}) = \aleph_1$ *is equivalent to the Continuum Hypothesis (CH), that is,* $2^{\aleph_0} = \aleph_1$,
(2) *the assertion* $w(2^{\omega_1}) = w(2^{\omega_1+1})$ *is equivalent to the assertion* $2^{\aleph_0} = 2^{\aleph_1}$,
(3) $w(2^{\omega_\omega}) > \aleph_\omega$ *is equivalent to the assertion that* $\aleph_\omega < 2^{\aleph_n}$ *holds for some* $n \in \omega$.

Corollary 4.3 (1) shows that if the negation of CH is assumed, then the lexicographic product 2^{ω_1} and the usual Tychonoff product 2^{ω_1} are not homeomorphic. However, we will see in the next section that they are not homeomorphic without additional set theoretical assumptions.

Next we calculate the weight of lexicographic product $\prod_{\alpha<\gamma} X_\alpha$, where all X_α's have the same infinite cardinality κ.

Corollary 4.4. *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be a lexicographic product of GO-spaces and* κ *an infinite cardinal. If for every* $\alpha < \gamma$*, the cardinality of* X_α *is* κ*, then the weight of* X *is the cardinality* $\kappa^{<\gamma}$.

Proof. Noting $w(X_{\gamma-1}) \leq |X_{\gamma-1}| = \kappa \leq \kappa^{<\gamma}$, the proof is similar to Corollary 4.1. □

Example 4.5. Note that the weight of the real line $\mathbb{R}$ and the Sorgenfrey line $\mathbb{S}$ are $\aleph_0$ and $2^{\aleph_0}$ respectively. Applying the corollary above, we see $w(\mathbb{R}^2) = w(\mathbb{S}^2) = (2^{\aleph_0})^{<2} = 2^{\aleph_0}$, $w(\mathbb{R}^\omega) = w(\mathbb{S}^\omega) = (2^{\aleph_0})^{<\omega} = 2^{\aleph_0}$, $w(\mathbb{R}^{\omega+1}) = w(\mathbb{S}^{\omega+1}) = (2^{\aleph_0})^{<\omega+1} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $w(\mathbb{R}^{\omega_1}) = w(\mathbb{S}^{\omega_1}) = (2^{\aleph_0})^{<\omega_1} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $w(\mathbb{R}^{\omega_1+1}) = w(\mathbb{S}^{\omega_1+1}) = (2^{\aleph_0})^{<\omega_1+1} = (2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1}, \cdots$, etc., whereas $w(\mathbb{Q}^2) = w(\mathbb{Q}^\omega) = \aleph_0$, $w(\mathbb{Q}^{\omega+1}) = w(\mathbb{Q}^{\omega_1}) = 2^{\aleph_0}$ and $w(\mathbb{Q}^{\omega_1+1}) = 2^{\aleph_1}$.

For ordinal spaces, we see $w(\omega^2) = w(\omega^\omega) = \aleph_0$, $w(\omega^{\omega+1}) = w(\omega^{\omega_1}) = 2^{\aleph_0}$, $w(\omega^{\omega_1+1}) = w(\omega^{\omega_2}) = 2^{\aleph_1}$, $w(\omega_1^2) = w(\omega_1^\omega) = \aleph_1$, $w(\omega_1^{\omega+1}) = w(\omega_1^{\omega_1}) = 2^{\aleph_0}$, $w(\omega_1^{\omega_1+1}) = w(\omega_1^{\omega_2}) = 2^{\aleph_1}, \cdots$, etc. Also note $w((\omega_\omega)^{\omega_\omega}) = (\aleph_\omega)^{<\omega_\omega} \geq (\aleph_\omega)^{\aleph_0} > \aleph_\omega$, but note that if GCH is assumed, then $w(2^{\omega_\omega}) = 2^{<\omega_\omega} = \aleph_\omega$.

Similar to Corollary 4.3, we see:

Corollary 4.6. *The following hold.*

(1) *the assertions* $w(\mathbb{S}^2) = \aleph_1$, $w(\mathbb{S}^{\omega_1}) = \aleph_1$, $w(\omega^{\omega+1}) = \aleph_1$ *and* $w(\omega^{\omega_1}) = \aleph_1$ *are equivalent to CH,*
(2) *the assertions* $w(\mathbb{R}^{\omega_1}) = w(\mathbb{R}^{\omega_1+1})$, $w(\mathbb{S}^{\omega_1}) = w(\mathbb{S}^{\omega_1+1})$, $w(\omega^{\omega_1}) = w(\omega^{\omega_1+1})$ *and* $w(\omega_1^{\omega_1}) = w(\omega_1^{\omega_1+1})$ *are equivalent to the assertion* $2^{\aleph_0} = 2^{\aleph_1}$.

Finally, we calculate the weight of other types of lexicographic products.

Corollary 4.7. *Let γ be an infinite ordinal. Then the weight of the lexicographic product $\prod_{2\le\alpha<\gamma}\alpha$ is the cardinality $2^{<\gamma}$.*

Proof. First let γ be limit. For every $\beta<\gamma$ with $2\le\beta$, note $2^{|[2,\beta]|}=|\prod_{2\le\alpha\le\beta}\alpha|\le|\prod_{2\le\alpha\le\beta}\beta|\le|\beta|^{|[2,\beta]|}$. Therefore moreover if we assume $\omega\le\beta$, then we have $|\prod_{2\le\alpha\le\beta}\alpha|=2^{|\beta|}$. So, whenever $\gamma=\omega$, we have $w(\prod_{2\le\alpha<\gamma}\alpha)=\sup\{|\prod_{2\le\alpha\le\beta}\alpha|:2\le\beta<\gamma\}=\omega=2^{<\gamma}$. Whenever $\gamma>\omega$, we have $w(\prod_{2\le\alpha<\gamma}\alpha)=\sup\{|\prod_{2\le\alpha\le\beta}\alpha|:2\le\beta<\gamma\}=\sup\{2^{|\beta|}:\omega\le\beta<\gamma\}=2^{<\gamma}$.

Next let γ be successor. ¿From $\gamma>\omega$, we have

$$2^{|\gamma|}=|\prod_{2\le\alpha<\gamma-1}2|\le|\prod_{2\le\alpha<\gamma-1}\alpha|\le|\prod_{2\le\alpha<\gamma-1}(\gamma-1)|=2^{|\gamma|},$$

thus $|\prod_{2\le\alpha<\gamma-1}\alpha|=2^{|\gamma|}=2^{<\gamma}$. Moreover by $w(\gamma-1)\le|\gamma-1|\le|\gamma|<2^{|\gamma|}=2^{<\gamma}$, we also have $w(\prod_{2\le\alpha<\gamma}\alpha)=\max\{|\prod_{2\le\alpha\le\gamma-1}\alpha|,w(\gamma-1)\}=2^{<\gamma}$. □

Example 4.8. Using the corollary above, we see $w(\prod_{2\le\alpha<\omega}\alpha)=\aleph_0$, $w(\prod_{2\le\alpha<\omega+1}\alpha)=w(\prod_{2\le\alpha<\omega_1}\alpha)=2^{\aleph_0}$, $w(\prod_{2\le\alpha<\omega_1+1}\alpha)=2^{\aleph_1},\cdots$, etc. Also we remark $w(\prod_{\alpha<\omega}\omega_\alpha)=\sup\{|\prod_{\alpha\le\beta}\omega_\alpha|:\beta<\omega\}=\sup\{\aleph_\beta:\beta<\omega\}=\aleph_\omega$ and $w(\prod_{\alpha<\omega+1}\omega_\alpha)=\max\{|\prod_{\alpha<\omega}\omega_\alpha|,w(\omega_\omega)\}=|\prod_{\alpha<\omega}\omega_\alpha|=(\sup\{\aleph_\alpha:\alpha<\omega\})^{\aleph_0}=\aleph_\omega^{\aleph_0}>\aleph_\omega$, where for $|\prod_{\alpha<\omega}\omega_\alpha|=(\sup\{\aleph_\alpha:\alpha<\omega\})^{\aleph_0}$, use [4, Lemma 5.9].

5. THE LEXICOGRAPHIC PRODUCTS VERSUS THE TYCHONOFF PRODUCTS

In this section, we compare the lexicographic product 2^γ with the usual Tychonoff product 2^γ.

First recall that a topological space X is said to be *homogeneous* if for every $x,y\in X$, there is a homeomorphism h from X onto X with $h(x)=y$. Obviously:

- if topological spaces X_α's ($\alpha\in\Lambda$) are homogeneous, then the usual Tychonoff product $\prod_{\alpha\in\Lambda}X_\alpha$ is also homogeneous,
- if a topological space X is homogeneous, then there is a unique cardinal number κ such that $\chi(x,X)=\kappa$ for every $x\in X$, where $\chi(x,X)=\min\{|\mathcal{U}|:\mathcal{U}\text{ is a neighborhood base at }x\}$, which is called the *character* at x, see [1],

- if a topological space X is homogeneous with an isolated point, then it is discrete, thus whenever Λ is infinite, the usual Tychonoff product 2^Λ is homogeneous without isolated points.

Next we remember the cofinality of a compact LOTS discussed in [5]. Let L be a compact LOTS and $x \in L$. Note that every subset A of L has a least upper bound $\sup_L A$ (and greatest lower bound $\inf_L A$), see [1, 3.12.3 (a)]. A subset A of $(\leftarrow, x)_L$ is said to be 0-*unbounded* for x in L if for every $y < x$, there is $a \in A$ with $y \leq a$. Let

$$0\text{-}\operatorname{cf}_L x = \min\{|A| : A \text{ is 0-unbounded for } x \}.$$

Obviously $0\text{-}\operatorname{cf}_L x$ can be 0, 1 or an infinite regular cardinal, also $0\text{-}\operatorname{cf}_L x = 0$ ($0\text{-}\operatorname{cf}_L x = 1$) means that x is the minimal element of L (x has an immediate predecessor in L, respectively). Usually $0\text{-}\operatorname{cf}_L x$ is denoted by $0\text{-}\operatorname{cf} x$. Since L is a compact LOTS, for every $x \in L$, there is a sequence $\{x_\alpha : \alpha < 0\text{-}\operatorname{cf} x\}$, which is called a 0-*normal sequence* for x, such that:

- if $\beta < \alpha < 0\text{-}\operatorname{cf} x$, then $x_\beta <_L x_\alpha$,
- if $\alpha < 0\text{-}\operatorname{cf} x$ and α is limit, then $x_\alpha = \sup_L\{x_\beta : \beta < \alpha\}$,
- the set $\{x_\alpha : \alpha < 0\text{-}\operatorname{cf} x\}$ is 0-unbounded for x.

Analogous notions "$1\text{-}\operatorname{cf} x$", "1-*normal sequence* for x"..., etc can be defined, see [5, section 3]. Note $\chi(x, L) = \max\{0\text{-}\operatorname{cf} x, 1\text{-}\operatorname{cf} x\}$ for every $x \in L$ and also note that the lexicographic product 2^γ is a compact LOTS.

Lemma 5.1. *Let 2^γ be a lexicographic product and $x \in 2^\gamma$. Then the following hold:*

(1) *if $x^{-1}[\{1\}]$ has no maximal element, say $\delta = \sup x^{-1}[\{1\}]$, then $0\text{-}\operatorname{cf} x = \operatorname{cf} \delta$, where we consider as $\sup \emptyset = 0$ and $\operatorname{cf} 0 = 0$,*
(2) *if $x^{-1}[\{1\}]$ has a maximal element, say $\delta = \max x^{-1}[\{1\}]$, then $0\text{-}\operatorname{cf} x = 1$.*

Proof. (1) Assume that $x^{-1}[\{1\}]$ has no maximal element and let $\delta = \sup x^{-1}[\{1\}]$. Fix a strictly increasing sequence $\{\delta_\xi : \xi < \operatorname{cf} \delta\}$ in δ such that

- $(\delta_\xi, \delta_{\xi+1}) \cap x^{-1}[\{1\}] \neq \emptyset$ for every $\xi < \operatorname{cf} \delta$,
- $\delta_\xi = \sup\{\delta_\zeta : \zeta < \xi\}$ if ξ is limit,
- $\{\delta_\xi : \xi < \operatorname{cf} \delta\}$ is (0-)unbounded in δ.

Now for every $\xi < \operatorname{cf} \delta$, let $x_\xi = (x \restriction \delta_\xi)^\wedge\langle 0 : \delta_\xi \leq \alpha < \gamma\rangle$. Then obviously $\{x_\xi : \xi < \operatorname{cf} \delta\}$ is a 0-normal sequence for x in 2^γ, therefore $0\text{-}\operatorname{cf} x = \operatorname{cf} \delta$.

(2) Assume that $x^{-1}[\{1\}]$ has a maximal element δ. Let $y = (x \upharpoonright \delta)^\frown\langle 0\rangle^\frown\langle 1 : \delta < \alpha < \gamma\rangle$, then $y < x$ and $(y, x) = \emptyset$, which shows $0\text{-}\operatorname{cf} x = 1$. □

Changing 0 and 1 by 1 and 0, respectively, in the lemma above, we can get an analogous result for $1\text{-}\operatorname{cf} x$. For example, if x is an element of 2^{ω_1} so that both $x^{-1}[\{1\}]$ and $x^{-1}[\{0\}]$ are unbounded in ω_1, then we have $0\text{-}\operatorname{cf} x = 1\text{-}\operatorname{cf} x = \omega_1$, thus $\chi(x, 2^{\omega_1}) = \aleph_1$.

Definition 5.2. Let L be a compact LOTS. A point x in L is said to have *type I* if $\min\{0\text{-}\operatorname{cf} x, 1\text{-}\operatorname{cf} x\} \leq 1$. Otherwise, we say that x has *type II*, that is, $\omega \leq 0\text{-}\operatorname{cf} x$ and $\omega \leq 1\text{-}\operatorname{cf} x$.

Lemma 5.3. *Let L be a compact LOTS. If there are a type I point x with $\omega_1 \leq \max\{0\text{-}\operatorname{cf} x, 1\text{-}\operatorname{cf} x\}$ and a type II point y in L, then L is not homogeneous. .*

Proof. Assume that there is a homeomorphism $h : X \to X$ with $h(y) = x$, we may assume $\omega_1 \leq 0\text{-}\operatorname{cf} x$ and $1\text{-}\operatorname{cf} x \leq 1$. For each $i \in 2$, fix an i-normal sequence $A_i := \{y_i(\alpha) : \alpha < i\text{-}\operatorname{cf} y\}$ for y. Since y has type II, A_0 and A_1 are infinite and $\{y\} = \operatorname{Cl}_L A_0 \cap \operatorname{Cl}_L A_1$. Since h is a homeomorphism, we have $\{x\} = \operatorname{Cl}_L h[A_0] \cap \operatorname{Cl}_L h[A_1]$. It follows from $1\text{-}\operatorname{cf} x \leq 1$ that $\{x\} = \operatorname{Cl}_L B_0 \cap \operatorname{Cl}_L B_1$, where $B_i = h[A_i] \cap (\leftarrow, x)$. Fix a 0-normal sequence $\{x(\alpha) : \alpha < 0\text{-}\operatorname{cf} x\}$ for x. Since B_i's are 0-unbounded for x, by induction, for every $i \in 2$ and $n \in \omega$, we can fix $b_{in} \in B_i$ and $\alpha_n < 0\text{-}\operatorname{cf} x$ with $b_{0n} < b_{1n} < x(\alpha_n) < b_{0n+1}$. Then by letting $\alpha = \sup\{\alpha_n : n \in \omega\}$, we see $x(\alpha) \in \operatorname{Cl}_L B_0 \cap \operatorname{Cl}_L B_1$ and $x(\alpha) < x$, which contradicts $\{x\} = \operatorname{Cl}_L B_0 \cap \operatorname{Cl}_L B_1$. □

Lemma 5.4. *The following hold:*

(1) *if γ is a successor ordinal with $\gamma > \omega$, then the lexicographic product 2^γ is not homogeneous,*
(2) *if γ is a limit ordinal with $\gamma \geq \omega_1$, then the lexicographic product 2^γ is not homogeneous.*

Proof. (1) Let γ be a successor ordinal with $\gamma > \omega$. Then the maximal element $x = \langle 1 : \alpha < \gamma\rangle$ is isolated, because of $0\text{-}\operatorname{cf} x = 1$ and $1\text{-}\operatorname{cf} x = 0$, see Lemma 5.1. On the other hand, the element $y = \langle 1 : \alpha < \omega\rangle^\frown\langle 0 : \omega \leq \alpha < \gamma\rangle$ is not isolated, in fact, $0\text{-}\operatorname{cf} y = \omega$. Since y is not isolated, 2^γ is not homogeneous.

(2) Let γ be a limit ordinal with $\gamma \geq \omega_1$. First assume $\operatorname{cf}\gamma > \omega$. Let y be an element of 2^γ such that both $y^{-1}[\{1\}]$ and $y^{-1}[\{0\}]$ are unbounded in γ. Moreover let $x = \langle 0\rangle^\frown\langle 1 : 0 < \alpha < \gamma\rangle$. Then Lemma 5.1 shows $0\text{-}\operatorname{cf} y = 1\text{-}\operatorname{cf} y = \operatorname{cf}\gamma > \omega$, $0\text{-}\operatorname{cf} x = \operatorname{cf}\gamma > \omega$ and $1\text{-}\operatorname{cf} x = 1$.

Thus x has type I with $0\text{-}\operatorname{cf} x \geq \omega_1$ and y has type II, now Lemma 5.3 shows that 2^γ is not homogeneous.

Next assume $\operatorname{cf}\gamma = \omega$ and $\gamma \geq \omega_1$. Note $\gamma > \omega_1$. Let $x = \langle 1 : \alpha < \omega_1\rangle^\frown\langle 0 : \omega_1 \leq \alpha < \gamma\rangle$ and $y = \langle 0 : \alpha \leq \omega_1\rangle^\frown\langle 1 : \omega_1 < \alpha < \gamma\rangle$. Then Lemma 5.1 shows $0\text{-}\operatorname{cf} x = \omega_1$, $1\text{-}\operatorname{cf} x = \omega$, $0\text{-}\operatorname{cf} y = \omega$ and $1\text{-}\operatorname{cf} y = 1$, thus we we have $\chi(x, 2^\gamma) = \aleph_1$ and $\chi(y, 2^\gamma) = \aleph_0$. So 2^γ is not homogeneous. □

Theorem 5.5. *Let γ be an ordinal, then the following are equivalent:*

1. *the lexicographic product 2^γ and the usual Tychonoff product 2^γ are homeomorphic,*
2. *the identity map from the lexicographic product 2^γ onto the usual Tychonoff product 2^γ is a homeomorphism,*
3. *the lexicographic product 2^γ is homeomorphic to the usual Tychonoff product 2^Λ for some Λ,*
4. *$\gamma \leq \omega$.*

Proof. The implication (2) ⇒ (1) ⇒ (3) is obvious.

(3) ⇒ (4) Assume $\gamma > \omega$ and that the lexicographic product 2^γ is homeomorphic to the usual Tychonoff product 2^Λ for some Λ. Note $2^{|\Lambda|} = 2^{|\gamma|}$. Since the usual Tychonoff product 2^Λ is homogeneous, from Lemma 5.4, we see that γ is limit with $\omega < \gamma < \omega_1$. It follows from Corollary 4.1 that the weight of the the lexicographic product 2^γ is $2^{<\gamma} = 2^{|\gamma|}$. On the other hand, the weight of the product 2^Λ is at most $|\Lambda|$, which contradicts $|\Lambda| < 2^{|\Lambda|} = 2^{|\gamma|}$.

(4) ⇒ (2) Assume $\gamma \leq \omega$. Since the case "$\gamma < \omega$" is obvious, we may assume $\gamma = \omega$. Let L and T be the lexicographic product and the usual Tychonoff product 2^ω respectively, and $id : L \to T$ be the identity map. The following claims complete the proof. Also note that in [2, p78, Example 4], the fact that L and T are homeomorphic is proved by using a characterization theorem of the Cantor set.

Claim 1. *id* is continuous.

Proof. For every $n \in \omega$ and $i \in 2$, let $U_{ni} := \{x \in T : x(n) = i\}$. Since $\{U_{ni} : n \in \omega, i \in 2\}$ is a subbase for T, it suffices to see that each U_{ni} is open in L. So let $x \in U_{ni}$. We may assume $i = 0$, then note $(x, \to) \neq \emptyset$.

Fact 1. If $(\leftarrow, x) \neq \emptyset$, then there is $a \in L$ with $a < x$ and $(a, x] \subset U_{n0}$.

Proof. Let $(\leftarrow, x) \neq \emptyset$, then note $x^{-1}[\{1\}] \neq \emptyset$. Whenever $x^{-1}[\{1\}]$ has a maximal element m_0, let $a = (x \restriction m_0)^\frown\langle 0\rangle^\frown\langle 1 : m_0 < m < \omega\rangle$. Whenever $x^{-1}[\{1\}]$ has no maximal element, putting $m_0 = \min(x^{-1}[\{1\}] \cap (n, \omega))$, let $a = (x \restriction m_0)^\frown\langle 0\rangle^\frown\langle 1 : m_0 < m < \omega\rangle$. Then a is the required, see Lemma 5.1 □

Similarly wee see:

Fact 2. There is $b \in L$ with $x < b$ and $[x, b) \subset U_{n0}$.

Now let

$$V = \begin{cases} [x, b) & \text{if } (\leftarrow, x) = \emptyset, \\ (a, b) & \text{if } (\leftarrow, x) \neq \emptyset. \end{cases}$$

Then V is a neighborhood of x in L contained in U_{n0}, so U_{n0} is open in L. □

Claim 2. id^{-1} is continuous.

Proof. Since $\{(a, \rightarrow) : a \in 2^\omega\} \cup \{(\leftarrow, a) : a \in 2^\omega\}$ is a subbase for L, it suffices to see that $(a, \rightarrow)$ and $(\leftarrow, a)$ are open in T for every $a \in 2^\omega$. We check the former, because the latter is similar. Let $a \in 2^\omega$ and $x \in (a, \rightarrow)$. Putting $m_0 = \min\{m \in \omega : x(m) \neq a(m)\}$, let $V = \{y \in 2^\omega : y \restriction (m_0 + 1) = x \restriction (m_0 + 1)\}$. Then V is a neighborhood of x in T contained in $(a, \rightarrow)$, so $(a, \rightarrow)$ is open in T. □

□

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