THE LINDELÖF PROPERTY OF LEXICOGRAPHIC PRODUCTS

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Abstract. The Lindelöf property of lexicographic products of GO-spaces is characterized.

1. Introduction

All spaces are assumed to be regular T_1 and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α's are assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminology follow [9] and [1].

Recently the notion of lexicographic products of GO-spaces is defined and discussed in [3, 4, 5, 6, 7]. The following are known:

(1) if for every $\alpha < \gamma$, X_α is a Lindelöf GO-space with both a minimal element and a maximal element, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is Lindelöf, see [11, Theorem 2.10],

(2) if $\gamma \leq \omega_1$ and for every $\alpha < \gamma$, X_α is a Lindelöf subspace of an ordinal, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is Lindelöf, see [11, Theorem 3.2 and 3.3],

(3) if for every $\alpha < \gamma$, X_α is a paracompact GO-space, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact, see [5, Corollary 4.7] and [2, Theorem 4.2.2].

Question 1.1. Related to the results above, it is natural to ask:

(Q1) if the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of GO-spaces is Lindelöf, where for every $\alpha < \gamma$, X_α has both a minimal element and a maximal element, then are all X_α's Lindelöf ?,

(Q2) if $\gamma \leq \omega_1$ and the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of subspaces of ordinals is Lindelöf, then are all X_α's Lindelöf ?,

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Date: February 10, 2025.

2020 *Mathematics Subject Classification.* Primary 54F05, 54B10, 54B05 . Secondary 54C05.

Key words and phrases. lexicographic product, GO-space, LOTS, Lindelöf, ω_1-compact .

This research was supported by Grant-in-Aid for Scientific Research (C) 21K03339, 23K03206.

(Q3) if for every $\alpha < \gamma$, X_α is a Lindelöf GO-space, then is the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ Lindelöf?

However immediately we get negative answers for (Q2) and (Q3). For (Q2), consider the lexicographic product $\omega \times \omega_1$. For (Q3), consider the lexicographic product $\mathbb{R}^2$, where $\mathbb{R}$ denotes the usual real line.

In this paper, we will characterize the Lindelöf property of lexicographic products of GO-spaces. As corollaries we see:

- (Q1) in Question above is true,
- the lexicographic product $[0,1)_{\mathbb{R}}^\gamma$ is Lindelöf if and only if $\gamma \leq \omega_1$, where $[0,1)_{\mathbb{R}}$ denotes the interval $[0,1)$ in the real line $\mathbb{R}$,
- the lexicographic product $[0,1)_{\mathbb{S}}^2$ is not Lindelöf but the lexicographic product $(0,1]_{\mathbb{S}}^2$ is Lindelöf, where $[0,1)_{\mathbb{S}}$ denote the interval $[0,1)$ in the Sorgenfrey line $\mathbb{S}$ (that is, $\mathbb{S} = \mathbb{R}$ and sets of type $[a,b)$ are declared to be open),
- the lexicographic product $(\omega \times \omega_1)^\gamma$ is Lindelöf if and only if $\gamma \leq \omega_1$, whereas the lexicographic product $\omega_1 \times \omega$ is not Lindelöf,
- the lexicographic product $(\omega \times \omega_1 \times (\omega_1+1))^{\omega_1}$ is Lindelöf, but the lexicographic product $((\omega_1+1) \times \omega_1 \times \omega)^{\omega_1}$ is not Lindelöf,
- the lexicographic product $\omega \times \omega_1 \times (\omega_1+1) \times \omega_1$ is Lindelöf, but the lexicographic product $\omega \times \omega_1 \times [0,1]_{\mathbb{R}} \times \omega_1$ is not Lindelöf,
- the lexicographic product $\prod_{\alpha<\omega_1} \omega_\alpha$ is Lindelöf, moreover the lexicographic product $\prod_{\alpha<\omega_1} \omega_\alpha \times \prod_{\omega_1 \leq \alpha < \omega_2} (\omega_\alpha + 1)$ is also Lindelöf, but the lexicographic products $\prod_{\alpha<\omega_1} \omega_{\alpha+1}$ and $\prod_{\alpha\leq\omega_1} \omega_\alpha$ are not Lindelöf.

A linearly ordered set $\langle X, <_X \rangle$ has a natural topology λ_X, which is called an *interval topology*, generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\}$ as a subbase, where $(x, \rightarrow)_X = \{z \in X : x <_X z\}$, $(x,y)_X = \{z \in X : x <_X z <_X y\}$, $(x,y]_X = \{z \in X : x <_X z \leq_X y\}$ and so on. The triple $\langle X, <_X, \lambda_X \rangle$, which is simply denoted by X, is called a *linearly ordered topological space* (LOTS).

A triple $\langle X, <_X, \tau_X \rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X \rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x,y]_X \subset C$ holds. In this case, the linearly ordered set $\langle X, <_X \rangle$ is said to be an *underlying linearly ordered set of the GO-space X*. Note $\lambda_X \subset \tau_X$. For more information on LOTS's or GO-spaces, see [10]. Usually $<_X$, $(x,y)_X$, λ_X or τ_X are written simply $<$, (x,y), λ or τ if contexts are clear.

ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \cdots$, are considered to be LOTS's with the usual intereval topology. cfα denotes the cofinality of the ordinal α. For a subset A of an ordinal α, $\mathrm{Lim}_\alpha(A)$ (or simply $\mathrm{Lim}(A)$) denotes the set $\{\beta < \alpha : \beta = \sup(A \cap \beta)\}$, that is, the set of all cluster points of A in the topological space α.

For GO-spaces $X = \langle X, <_X, \tau_X \rangle$ and $Y = \langle Y, <_Y, \tau_Y \rangle$, X is said to be a *subspace* of Y if $X \subset Y$, the linear order $<_X$ is the restriction $<_Y \restriction X$ on X of the order $<_Y$ and the topology τ_X is the subspace topology $\tau_Y \restriction X$ $(= \{U \cap X : U \in \tau_Y\})$ on X of the topology τ_Y. So a subset Y of a GO-space X is naturally considered as a GO-space, that is, $\langle Y, <_X \restriction Y, \tau_X \restriction Y \rangle$. Note that a GO-space is characterized as a subspace of some LOTS' and generally a GO-space can be a subspace of many LOTS'. However a GO-space X is a subspace of the following nice LOTS X^*.

For a GO-space $X = \langle X, <_X, \tau_X \rangle$, let

$$X^* = (X^- \times \{-1\}) \cup (X \times \{0\}) \cup (X^+ \times \{1\}),$$

where λ_X is the interval topology of $\langle X, <_X \rangle$,

$$X^+ = \{x \in X : (\leftarrow, x] \in \tau_X \setminus \lambda_X\},$$

$$X^- = \{x \in X : [x, \rightarrow) \in \tau_X \setminus \lambda_X\},$$

and the order $<_{X^*}$ on X^* is the restriction of the usual lexicographic order on $X \times \{-1, 0, 1\}$ with $-1 < 0 < 1$, also we identify $X \times \{0\}$ with X by $\langle x, 0 \rangle = x$. Then X is a dense subspace of the LOTS X^*, obviously we can see:

- if X is a LOTS, then $X^* = X$,
- X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

Also X^* has the following nice property [12].

- if L is a LOTS containing X as a dense subspace, then L contains X^* as a subspace.

We call X^* as the *minimal d-extension* of X.

For every $\alpha < \gamma$, let X_α be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma \rangle$. In the present paper, a sequence means a function whose domain is an ordinal. For notational convenience, $\prod_{\alpha<\gamma} X_\alpha$ is considered as $\{\emptyset\}$ whenever $\gamma = 0$, where $\emptyset$ is considered to be a function whose domain is 0. When $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$ and $y_1 \in \prod_{\beta\leq\alpha} X_\alpha$, $y_0 {}^\wedge y_1$

denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ y_1(\alpha) & \text{if } \beta \leq \alpha. \end{cases}$$

In this case, whenever $\beta = 0$, $\emptyset\ {}^\wedge y_1$ is considered as y_1. In case $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$, $u \in X_\beta$ and $y_1 \in \prod_{\beta<\alpha} X_\alpha$, $y_0\ {}^\wedge\langle u\rangle{}^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ u & \text{if } \alpha = \beta, \\ y_1(\alpha) & \text{if } \beta < \alpha. \end{cases}$$

More general cases are similarly defined.

The lexicographic order $<_X$ on X is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha\rangle$ and $<_{X_\alpha}$ is the order on X_α. Now for every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace X of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X_\alpha^*$ is said to be the *lexicographic product of GO-spaces* X_α's, for more details see [5]. $\prod_{i\in\omega} X_i$ ($\prod_{i\leq n} X_i$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let X and Y be LOTS's. A map $f : X \to Y$ is said to be *order preserving* or 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be *order reversing* or 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map (also 1-order preserving map) $f : X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. Now let X and Y be GO-spaces. A 0-order preserving map $f : X \to Y$ is said to be 0-*order preserving embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we identify X with $f[X]$ as a GO-space and write $X = f[X]$.

Recall that a subset of a regular uncountable cardinal κ is called *stationary* if it intersects with all closed unbounded (= club) sets in κ.

Let X be a GO-space and $Y \subset X$. A subset Z of Y is 0-*unbounded in* Y if for every $x \in Y$, there is $x' \in Z$ such that $x \leq x'$. A subset Z of Y is 0-*bounded in* Y if it is not 0-unbounded in Y. A subset A of X is called a 0-*segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. "1-(un)boundedness" and ''1-segment" are similarly defined. Both $\emptyset$ and X are 0-segments and 1-segments.

For a 0-segment A of a GO-space X, let

$$0\text{-}\operatorname{cf}_X A = \min\{|U| : U \text{ is 0-unbounded in } A.\}.$$

$0\text{-}\operatorname{cf}_X A$ can be $0, 1$ or regular infinite cardinals. If contexts are clear, $0\text{-}\operatorname{cf}_X A$ is denoted by $0\text{-}\operatorname{cf} A$. Note that for a given 0-order preserving sequence $\{x_\beta : \beta < \delta\}$ in X indexed by an ordinal δ (i.e., if $\beta < \beta'$, then $x_\beta <_X x_{\beta'}$), the set $\{x \in X : \exists \beta < \delta (x \leq x_\beta)\}$, which is denoted by $A(\{x_\beta : \beta < \delta\})$, is a 0-segment of X with $0\text{-}\operatorname{cf}_X A(\{x_\beta : \beta < \delta\}) = \operatorname{cf}\delta$. Conversely, note that for a given 0-segment A of X, there is a 0-order preserving and 0-unbounded sequence $\{x_\beta : \beta < 0\text{-}\operatorname{cf}_X A\}$ in A, in this case, $A = A(\{x_\beta : \beta < 0\text{-}\operatorname{cf}_X A\})$ holds.

Definition 1.2. A GO-space X is said to have *countable* (*closed*) 0-*cofinality* if for every (closed, respectively) 0-segment A of X, $0\text{-}\operatorname{cf}_X A \leq \omega$ holds. Also a GO-space X is said to have *countable* 0-*bounded closed* 0-*cofinality* if for every 0-bounded closed 0-segment A of X, $0\text{-}\operatorname{cf}_X A \leq \omega$ hold. Obviously if X has countable 0-bounded closed 0-cofinality and $0\text{-}\operatorname{cf}_X X \leq \omega$, then X has countable closed 0-cofinality. Analogous notions (0 is replaced by 1) are defined.

Note that $\omega_1 + \omega$ has countable closed 0-cofinality but does not have countable 0-cofinality. Also note that $(\omega_1 + \omega) \setminus \{\omega_1\}$ does not have countable closed 0-cofinality but its underlying linearly ordered set has countable closed 0-cofinality, because it is identified with $\omega_1 + \omega$.

Recall that a topological space X is ω_1-*compact* if every uncountable subset H of X has a cluster point x, that is, for every neighborhood U of x, $(U \setminus \{x\}) \cap H$ is non-empty (equivalently, $U \cap H$ is infinite).

Definition 1.3. A GO-space X is said to be *boundedly* ω_1-*compact* if every uncountable subset H of X, which is both 0-bounded and 1-bounded, has a cluster point.

Obviously Lindelöf topological spaces are ω_1-compact and ω_1-compact GO-spaces are boundedly ω_1-compact, also boundedly ω_1-compact GO-spaces with both a maximal element and a minimal element are ω_1-compact. Using the notions defined above, we characterize the Lindelöfness of GO-spaces.

Lemma 1.4. *Let X be a GO-space. Then X is Lindelöf if and only if the following clauses hold:*

(1) *X has countable closed* 0-*cofinality,*
(2) *X has countable closed* 1-*cofinality,*

(3) *X is (boundedly) ω_1-compact.*

Proof. One direction is obvious. To see another direction, assume (1), (2) and that X is boundedly ω_1-compact but not Lindelöf. By (1) and (2), we can find $x_0, x_1 \in X$ with $x_0 < x_1$ such that $Y := [x_0, x_1]_X$ is not Lindelöf (use $0\text{-}\operatorname{cf}_X X \leq \omega$ and $1\text{-}\operatorname{cf}_X X \leq \omega$). So one can find a collection $\mathcal{U}$ of open sets with $Y \subset \bigcup \mathcal{U}$ such that $\mathcal{U}$ has no countable subcollection which covers Y. Since open sets in a GO-space can be decomposed into convex open sets, we may assume that every $U \in \mathcal{U}$ is a convex open set meeting Y. For every $U, U' \in \mathcal{U}$, define $U \sim U'$ when there are $n \in \omega$ and a sequence $\{U_i : i \leq n\} \subset \mathcal{U}$ such that $U = U_0$, $U' = U_n$ and for each $i < n$, $U_i \cap U_{i+1}$ is non-empty. Obviously the relation $\sim$ is an equivalence relation on $\mathcal{U}$, so $\{\bigcup \mathcal{V} : \mathcal{V} \in \mathcal{U}/_\sim\}$ is a decomposition of $\bigcup \mathcal{U}$ by non-empty open sets, where $\mathcal{U}/_\sim$ denotes the collection of all equivalence classes of $\mathcal{U}$ by $\sim$. Fixing a point $x(\mathcal{V}) \in Y \cap (\bigcup \mathcal{V})$ for every $\mathcal{V} \in \mathcal{U}/_\sim$, let $H = \{x(\mathcal{V}) : \mathcal{V} \in \mathcal{U}/_\sim\}$. Obviously H has no cluster points in X. It follows from $H \subset Y$ that $H \setminus \{x_0, x_1\}$ is 0-bounded and 1-bounded in X. By bounded ω_1-compactness, we see that H is countable, so $\mathcal{U}/_\sim$ is countable. Therefore for some $\mathcal{V}_0 \in \mathcal{U}/_\sim$, any countable subcollection of $\mathcal{V}_0$ cannot cover $Y \cap (\bigcup \mathcal{V}_0)$. Fix $x_2 \in Y \cap (\bigcup \mathcal{V}_0)$. We may assume that any countable subcollection of $\mathcal{V}_0$ cannot cover $[x_2, x_1] \cap (\bigcup \mathcal{V}_0)$, otherwise any countable subcollection of $\mathcal{V}_0$ cannot cover $[x_0, x_2] \cap (\bigcup \mathcal{V}_0)$. Put $A = \{x \in X : \exists y \in [x_2, x_1] \cap (\bigcup \mathcal{V}_0)(x \leq y)\}$. Obviously A is a 0-segment of X with $A \cap [x_2, x_1] = [x_2, x_1] \cap (\bigcup \mathcal{V}_0)$. To see that A is closed in X, let $x \in X \setminus A$. Then obviously we have $x > x_2$. When $x > x_1$, $(x_1, \rightarrow)$ is a neighborhood of x disjoint from A. So let $x \leq x_1$. Since $x \in Y \subset \bigcup \mathcal{U}$, there is $\mathcal{V}_1 \in \mathcal{U}/_\sim$ with $x \in \bigcup \mathcal{V}_1$. Then it is straightforward to see that $\bigcup \mathcal{V}_1$ is a neighborhood of x disjoint from A. Now by (1), we have $0\text{-}\operatorname{cf}_X A \leq \omega$. First assume $0\text{-}\operatorname{cf}_X A = 1$, that is, A has a maximal element $\max A$. "$x_2, \max A \in \bigcup \mathcal{V}_0$" and the definition $\sim$ witness that some finite subcollection $\mathcal{V}'$ of $\mathcal{V}_0$ covers $[x_2, \max A]$ $(= [x_2, x_1] \cap (\bigcup \mathcal{V}_0))$, a contradiction. Next assume $0\text{-}\operatorname{cf}_X A = \omega$. Fix a 0-order preserving and 0-unbounded sequence $\{a_n : n \in \omega\}$ in A with $x_2 < a_0$. As above "$x_2, a_n \in \bigcup \mathcal{V}_0$" witnesses that some finite subcollection $\mathcal{V}'_n$ of $\mathcal{V}_0$ covers $[x_2, a_n]$ for every $n \in \omega$. Now $\bigcup_{n \in \omega} \mathcal{V}'_n$ covers $[x_2, x_1] \cap (\bigcup \mathcal{V}_0)$, a contradiction. $\square$

In later sections, we will separately characterize countable closed 0-cofinality, countable closed 1-cofinality and ω_1-compactness of lexicographic products.

Example 1.5. In the lemma above, the clause (3) cannot be omitted, e.g., the lexicographic product $X = \mathbb{R}^2$ satisfies the clauses (1) and (2) but is not Lindelöf.

But in some special cases, the clause (3) is unnecessary.

Corollary 1.6. *Whenever X is a subspace of an ordinal, X is Lindelöf if and only if it has countable closed 0-cofinality.*

Proof. One direction follows from the lemma above. Let X be a subspace of an ordinal which has countable closed 0-cofinality. It suffices to see that X satisfies (2) and (3) in the lemma above. (2) is obvious, since X is well-order. To see (3), assume that there is an uncountable subset H of X having no cluster points. Enumerate $H = \{x_\alpha : \alpha < \delta\}$ as 0-order preserving, that is, $x_\alpha <_X x_{\alpha'}$ whenever $\alpha < \alpha' < \delta$. Since H is uncountable, we have $\omega_1 \le \delta$. Let $A = \{x \in X : \exists \alpha < \omega_1 (x \le_X x_\alpha)\}$. Obviously A is a 0-segment of X with $0\text{-}\operatorname{cf}_X A = \omega_1$. It suffices to see that A is closed in X. Let $x \in X \setminus A$. If there is $y \in X$ with $y < x$ such that $x_\alpha \le y$ holds for every $\alpha < \omega_1$, then obviously $(y, \rightarrow)$ is a neighborhood of x disjoint from A. Assume that for every $y \in X$ with $y < x$, there is $\alpha < \omega_1$ such that $y < x_\alpha$ holds. Since x is not a cluster point of H, there is $x^* \in X^*$ such that $H \cap (x^*, x) = \emptyset$. Now $(x^*, \rightarrow)_{X^*} \cap X$ is a neighborhood of x disjoint from A. $\square$

2. Countable closed 0-cofinality of lexicographic products

In this section, we characterize countable closed 0-cofinality of lexicographic products. We need the following terminologies shown in [6].

Definition 2.1. Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. We use the following special notation.

$$J^+ = \{\alpha < \gamma : \ X_\alpha \text{ has no maximal elements.}\},$$

$$J^- = \{\alpha < \gamma : \ X_\alpha \text{ has no minimal elements.}\},$$

$$K^+ = \{\alpha < \gamma : \text{ there is } x \in X_\alpha \text{ such that } (x, \rightarrow)_{X_\alpha} \text{ is non-empty and has no minimal element.}\},$$

$$K^- = \{\alpha < \gamma : \text{ there is } x \in X_\alpha \text{ such that } (\leftarrow, x)_{X_\alpha} \text{ is non-empty and has no maximal elements.}\}$$

Let α be an ordinal and let

$$l(\alpha) = \begin{cases} 0 & \text{if } \alpha < \omega, \\ \sup\{\beta \le \alpha : \beta \text{ is limit.}\} & \text{if } \alpha \ge \omega. \end{cases}$$

Note that $l(\alpha)$ is the largest limit ordinal less than or equal to α whenever $\alpha \geq \omega$. So the interval $[l(\alpha), \alpha)$ of ordinals is finite, thus every ordinal α can be uniquely represented as $l(\alpha) + n(\alpha)$ for some $n(\alpha) \in \omega$. When $l(\alpha) = \alpha$, we let $[l(\alpha), \alpha) = \emptyset$.

Lemma 2.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X has countable closed 0-cofinality if and only if the following clauses hold:*

(1) $J^- \subset \omega_1$
(2) *for every ordinal $\alpha < \gamma$ with $\sup J^- \leq \alpha$, the following hold:*
  (a) *X_α has countable 0-bounded closed 0-cofinality,*
  (b) *in each of the following cases, $0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha \leq \omega$ holds,*
    (i) $J^+ \cap [l(\alpha), \alpha) = \emptyset$,
    (ii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $J^- \cap (\alpha', \alpha] \neq \emptyset$,
    (iii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^+ \cap [\alpha', \alpha) \neq \emptyset$,
    where $\alpha' = \max(J^+ \cap [l(\alpha), \alpha))$ *in case* $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$.
(3) *for every $\alpha < \sup J^-$, X_α has countable 0-cofinality.*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ and fix $u_0, u_1 \in X$ with $u_0(\alpha) < u_1(\alpha)$ for every $\alpha < \gamma$.

To see one direction, assume that X has countable closed 0-cofinality. We will see (1), (2) and (3).

(1) Assume $J^- \setminus \omega_1 \neq \emptyset$ and let $\alpha_0 = \min(J^- \setminus \omega_1)$. Noting $\omega_1 \leq \alpha_0$, let $y = \langle u_1(\alpha) : \alpha < \omega_1\rangle^\wedge\langle \min X_\alpha : \omega_1 \leq \alpha < \alpha_0\rangle$ and $A = (\leftarrow, y)_{\prod_{\alpha<\alpha_0} X_\alpha} \times \prod_{\alpha_0 \leq \alpha} X_\alpha$. Then A is a 0-segment of X. To see that A is closed in X, let $x \in X \setminus A$. Note $y \leq x \restriction \alpha_0$. Since X_{α_0} has no minimal elements, we can take $u \in X_{\alpha_0}$ with $u < x(\alpha_0)$. Letting $x' = (x \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha_0, \gamma))$, $(x', \rightarrow)_X$ is a neighborhood of x disjoint from A. We have seen that A is closed in X. Fixing $z \in \prod_{\alpha_0 \leq \alpha} X_\alpha$, for each $\beta < \omega_1$, let $y_\beta = \langle u_1(\alpha) : \alpha < \beta\rangle^\wedge\langle u_0(\alpha) : \beta \leq \alpha < \omega_1\rangle^\wedge\langle \min X_\alpha : \omega_1 \leq \alpha < \alpha_0\rangle^\wedge z$. Then the sequence $\{y_\beta : \beta < \omega_1\}$ is 0-order preserving and 0-unbounded in A, therefore $0\text{-}\mathrm{cf}_X A = \omega_1$, which contradicts that X has countable closed 0-cofinality. Thus we see $J^- \subset \omega_1$.

(2) Let $\sup J^- \leq \alpha_0 < \gamma$. To see (a), assume that there is a 0-bounded closed 0-segment A_0 of X_{α_0} with $\lambda := 0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0 \geq \omega_1$ and fix $u \in X_{\alpha_0} \setminus A_0$. Then there is a 0-order preserving sequence $\{u_\beta : \beta < \lambda\}$ which is 0-unbounded in A_0 and $(\leftarrow, u_0)_{X_{\alpha_0}} \neq \emptyset$. Fixing $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$, for every $\beta < \lambda$, let $x_\beta = y_0{}^\wedge\langle u_\beta\rangle^\wedge\langle \min X_\alpha : \alpha_0 < \alpha\rangle$. Noting that the sequence $\{x_\beta : \beta < \lambda\}$ is 0-order preserving, put $A = A(\{x_\beta : \beta < \lambda\})$. To see that A is closed in X, let $x \in X \setminus A$. When $x \restriction (\alpha_0+1) > y_0{}^\wedge\langle u\rangle$,

by letting $x' = y_0{}^\frown\langle u\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$, $(x', \rightarrow)$ is a neighborhood of x disjoint from A. So assume $x \restriction (\alpha_0+1) \leq y_0{}^\frown\langle u\rangle$, then $x \restriction \alpha_0 = y_0$ has to be true (otherwise, $x \in A$). Therefore we have $x(\alpha_0) \notin A_0$ and $x(\alpha_0) \leq u$. Since A_0 is closed in X_{α_0}, one can take $u^* \in X^*_{\alpha_0}$ with $u^* < x(\alpha_0)$ and $A_0 \cap (u^*, \rightarrow)_{X^*_{\alpha_0}} = \emptyset$. Letting $x^* = (x \restriction \alpha_0)^\frown\langle u^*\rangle^\frown(x \restriction (\alpha_0, \gamma))$, we see that $(x^*, \rightarrow)_{\hat{X}} \cap X$ is a neighborhood of x disjoint from A. Thus A is closed in X. Now by $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, we have a contradiction to countable closed 0-cofinality of X.

To see (b), assume $\lambda := 0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \geq \omega_1$ and fix a 0-order preserving sequence $\{u_\beta : \beta < \lambda\}$ which is 0-unbounded in X_{α_0} with $(\leftarrow, u_0) \neq \emptyset$. In each case of (i), (ii) and (iii), we will get a contradiction.

(i) $J^+ \cap [l(\alpha_0), \alpha_0) = \emptyset$.

Note that X_α has a maximal element for every $\alpha \in [l(\alpha_0), \alpha_0)$, and that X_α has a minimal element for every $\alpha > \alpha_0$. We consider 2 cases.

Case 1. $l(\alpha_0) = 0$.

Letting $x_\beta = \langle \max X_\alpha : \alpha < \alpha_0\rangle^\frown\langle u_\beta\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$ for every $\beta < \lambda$, consider $A := A(\{x_\beta : \beta < \lambda\})$. Then A is a 0-segment in X with $0\text{-}\operatorname{cf}_X A = \lambda$. To see $A = X$, let $x \in X$. When $x \restriction \alpha_0 < \langle \max X_\alpha : \alpha < \alpha_0\rangle$, we have $x < x_0$, so $x \in A$. When $x \restriction \alpha_0 = \langle \max X_\alpha : \alpha < \alpha_0\rangle$, for some $\beta < \lambda$, $x(\alpha_0) < u_\beta$ holds which implies $x < x_\beta$ thus $x \in A$. We have seen $A = X$. Now A $(= X)$ is a closed 0-segment with $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, a contradiction.

Case 2. $l(\alpha_0) \geq \omega$.

Letting $x_\beta = \langle u_0(\alpha) : \alpha < l(\alpha_0)\rangle^\frown\langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0\rangle^\frown\langle u_\beta\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$ for every $\beta < \lambda$, consider the 0-segment $A := A(\{x_\beta : \beta < \lambda\})$ in X with $0\text{-}\operatorname{cf}_X A = \lambda$. To see that A is closed, let $x \in X \setminus A$. If $x \restriction \alpha_0 < \langle u_0(\alpha) : \alpha < l(\alpha_0)\rangle^\frown\langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0\rangle$ were true, then we have $x < x_0 \in A$ so $x \in A$, a contradiction. If $x \restriction \alpha_0 = \langle u_0(\alpha) : \alpha < l(\alpha_0)\rangle^\frown\langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0\rangle$ were true, then by taking $\beta < \lambda$ with $x(\alpha_0) < u_\beta$, we see $x < x_\beta \in A$ so $x \in A$, a contradiction. We have seen $x \restriction \alpha_0 > \langle u_0(\alpha) : \alpha < l(\alpha_0)\rangle^\frown\langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0\rangle$. Now letting $\alpha_1 = \min\{\alpha < l(\alpha_0) : x(\alpha) \neq u_0(\alpha)\}$ and $x' = \langle u_0(\alpha) : \alpha \leq \alpha_1\rangle^\frown\langle u_1(\alpha) : \alpha_1 < \alpha < l(\alpha_0)\rangle^\frown(x \restriction [l(\alpha_0), \gamma))$, $(x', \rightarrow)$ is a neighborhood of x disjoint from A. So A is a closed 0-segment with $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, a contradiction.

(ii) $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $J^- \cap (\alpha_1, \alpha_0] \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Let $\alpha_2 = \max(J^- \cap (\alpha_1, \alpha_0])$. Note that α_2 exists, because $[l(\alpha_0), \alpha_0)$ is finite. Also note $l(\alpha_0) \leq \alpha_1 < \alpha_2 \leq \alpha_0$. We consider 2 cases.

Case 1. $\alpha_2 = \alpha_0$.

Fixing $y_0 \in \prod_{\alpha \leq \alpha_1} X_\alpha$, let $x_\beta = y_0 {}^\wedge\langle \max X_\alpha : \alpha_1 < \alpha < \alpha_0 \rangle^\wedge \langle u_\beta \rangle^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$ for every $\beta < \lambda$. Consider the 0-segment $A := A(\{x_\beta : \beta < \lambda\})$ in X with $0\text{-}\operatorname{cf}_X A = \lambda$. To see that A is closed, let $x \in X \setminus A$. If $x \restriction \alpha_0 < y_0 {}^\wedge\langle \max X_\alpha : \alpha_1 < \alpha < \alpha_0 \rangle$ were true, then we have $x < x_0 \in A$ so $x \in A$, a contradiction. If $x \restriction \alpha_0 = y_0 {}^\wedge\langle \max X_\alpha : \alpha_1 < \alpha < \alpha_0 \rangle$ were true, then by taking $\beta < \lambda$ with $x(\alpha_0) < u_\beta$, we see $x < x_\beta \in A$ so $x \in A$, a contradiction. We have seen $x \restriction \alpha_0 > y_0 {}^\wedge\langle \max X_\alpha : \alpha_1 < \alpha < \alpha_0 \rangle$ thus $x \restriction (\alpha_1 + 1) > y_0$. Taking $u \in X_{\alpha_2}$ $(= X_{\alpha_0})$ with $u < x(\alpha_0)$, let $x' = (x \restriction \alpha_0)^\wedge \langle u \rangle^\wedge (x \restriction (\alpha_0, \gamma))$. Then $(x', \rightarrow)$ is a neighborhood of x disjoint from A. So A is a closed 0-segment with $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, a contradiction.

Case 2. $\alpha_2 < \alpha_0$.

Fixing $y_0 \in \prod_{\alpha < \alpha_2} X_\alpha$, let $x_\beta = y_0 {}^\wedge\langle \max X_\alpha : \alpha_2 \leq \alpha < \alpha_0 \rangle^\wedge \langle u_\beta \rangle^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$ for every $\beta < \lambda$. Consider the 0-segment $A := A(\{x_\beta : \beta < \lambda\})$ in X with $0\text{-}\operatorname{cf}_X A = \lambda$. To see that A is closed, let $x \in X \setminus A$. By a similar argument of Case 1 above, we have $x \restriction \alpha_0 > y_0 {}^\wedge\langle \max X_\alpha : \alpha_2 \leq \alpha < \alpha_0 \rangle$ thus $x \restriction \alpha_2 > y_0 \restriction \alpha_2$. Taking $u \in X_{\alpha_2}$ with $u < x(\alpha_2)$, let $x' = (x \restriction \alpha_2)^\wedge \langle u \rangle^\wedge (x \restriction (\alpha_2, \gamma))$. Then $(x', \rightarrow)$ is a neighborhood of x disjoint from A. So A is a closed 0-segment with $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, a contradiction.

(iii) $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $K^+ \cap [\alpha_1, \alpha_0) \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Let $\alpha_2 = \max(K^+ \cap [\alpha_1, \alpha_0))$, then note $l(\alpha_0) \leq \alpha_1 \leq \alpha_2 < \alpha_0$. From $\alpha_2 \in K^+$, we can take $u \in X_{\alpha_2}$ such that $(u, \rightarrow)_{X_{\alpha_2}}$ is non-empty and has no minimal elements. Fixing $y_0 \in \prod_{\alpha < \alpha_2} X_\alpha$, let $x_\beta = y_0 {}^\wedge\langle u \rangle^\wedge \langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0 \rangle^\wedge \langle u_\beta \rangle^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$ for every $\beta < \lambda$. Consider the 0-segment $A := A(\{x_\beta : \beta < \lambda\})$ in X with $0\text{-}\operatorname{cf}_X A = \lambda$. To see that A is closed, let $x \in X \setminus A$. By $x \restriction \alpha_0 > y_0 {}^\wedge\langle u \rangle^\wedge \langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0 \rangle$ (otherwise, $x \in A$), we have $x \restriction (\alpha_2 + 1) > y_0 {}^\wedge\langle u \rangle$. Since $(y_0 {}^\wedge\langle u \rangle, \rightarrow)$ is non-empty and has no minimal elements, taking $z \in \prod_{\alpha \leq \alpha_2} X_\alpha$ with $y_0 {}^\wedge\langle u \rangle < z < x \restriction (\alpha_2 + 1)$, let $x' = z {}^\wedge (x \restriction (\alpha_2, \gamma))$. Then $(x', \rightarrow)$ is a neighborhood of x disjoint from A. So A is a closed 0-segment with $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, a contradiction.

(3) Assume $\alpha_0 < \sup J^-$ and that there exists a 0-segment A_0 in X_{α_0} with $\lambda := 0\text{-}\operatorname{cf}_{X_{\alpha_1}} A_0 \geq \omega_1$. Let $\alpha_1 = \min(J^- \setminus (\alpha_0 + 1))$ and $\{u_\beta : \beta < \lambda\}$ be a 0-order preserving and 0-unbounded sequence in A_0 with

$(\leftarrow, u_0) \neq \emptyset$. Fixing $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$ and $y_1 \in \prod_{\alpha_0<\alpha} X_\alpha$, let $x_\beta = y_0{}^\wedge\langle u_\beta\rangle^\wedge y_1$ for every $\beta < \lambda$. Consider the 0-segment $A := A(\{x_\beta : \beta < \lambda\})$ in X with $0\text{-}\operatorname{cf}_X A = \lambda$. To see that A is closed, let $x \in X \setminus A$. Then we have $x \restriction \alpha_0 \geq y_0$, otherwise $x < x_0 \in A$. Fixing $u \in X_{\alpha_1}$ with $u < x(\alpha_1)$ and letting $x' = (x \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha_1, \gamma))$, $(x', \rightarrow)$ is a neighborhood of x disjoint from A. So A is a closed 0-segment with $0\text{-}\operatorname{cf}_X A = \lambda \geq \omega_1$, a contradiction.

To see the other direction, assume (1)-(3) and that there is a closed 0-segment A of X with $0\text{-}\operatorname{cf}_X A \geq \omega_1$. Let $B = X \setminus A$.

Claim 1. $B \neq \emptyset$.

Proof. Assume $B = \emptyset$. Since A $(=X)$ has no maximal elements, J^+ is non-empty. Let $\alpha_0 = \min J^+$, then the clause (2bi) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} = \omega$. Fix a 0-order preserving sequence $\{u_n : n \in \omega\}$ with $(\leftarrow, u_0) \neq \emptyset$ which is 0-unbounded in X_{α_0}. Fixing $y_1 \in \prod_{\alpha_0<\alpha} X_\alpha$, let $x_n = \langle \max X_\alpha : \alpha < \alpha_0\rangle^\wedge\langle u_n\rangle^\wedge y_1$ for every $n \in \omega$. Then $\{x_n : n \in \omega\}$ is 0-order preserving 0-unbounded in A $(= X)$, which contradicts $0\text{-}\operatorname{cf}_X A \geq \omega_1$. This completes the proof of Claim 1.

We consider 2 cases.

Case 1. B has a minimal element.

Let $b = \min B$. Since A is closed in X with $b \notin A$, we can find $b^* \in \hat{X}$ such that $b^* < b$ and $(b^*, b)_{\hat{X}} \cap A = \emptyset$. Since A has no maximal elements, we have $b^* \notin X$. Let $\alpha_0 = \min\{\alpha < \gamma : b^*(\alpha) \neq b(\alpha)\}$.

Claim 2. For every $\alpha > \alpha_0$, X_α has a minimal element and $b(\alpha) = \min X_\alpha$.

Proof. If there were $\alpha > \alpha_0$ and $u \in X_\alpha$ with $u < b(\alpha)$, then we have $b^* < (b \restriction \alpha)^\wedge\langle u\rangle^\wedge(b \restriction (\alpha, \gamma)) < b$, a contradiction. This completes the proof of Claim 2.

Claim 3. $(b^*(\alpha_0), b(\alpha_0))_{X_{\alpha_0}} = \emptyset$.

Proof. If there were $u \in (b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}}$, then we have $b^* < (b \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(b \restriction (\alpha_0, \gamma)) < b$, a contradiction. This completes the proof of Claim 3.

Claim 4. $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$, where $\lambda_{X_{\alpha_0}}$ denotes the interval topology of the linearly ordered set X_{α_0}.

Proof. It follows from $b^*(\alpha_0) < b(\alpha_0)$ that $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is not empty. Assume $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, then there is $u \in X_{\alpha_0}$ with $u < b(\alpha_0)$ and $(u, b(\alpha_0))_{X_{\alpha_0}} = \emptyset$. By Claim 3, $u = b^*(\alpha_0)$ holds. Similarly to

Claim 2, we see that for every $\alpha > \alpha_0$, X_α has a maximal element and $b^*(\alpha) = \max X_\alpha$, which means $b^* \in X$ from $b \restriction \alpha_0 = b^* \restriction \alpha_0$ and $b^*(\alpha_0) = u \in X_{\alpha_0}$, a contradiction. This completes the proof of Claim 4.

Now let $A_0 = (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$. Claim 4 says that A_0 is 0-bounded closed 0-segment of X_{α_0} with no maximal elements. From Claim 2, we see $\sup J^- \leq \alpha_0$, therefore by the clause (2a), we have $0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0 = \omega$. Fix a 0-order preserving 0-unbounded sequence $\{u_n : n \in \omega\}$ in A_0. Letting $x_n = (b \restriction \alpha_0)^\wedge \langle u_n \rangle^\wedge (b \restriction (\alpha_0, \gamma))$ for every $n \in \omega$, we see $A = A(\{x_n : n \in \omega\})$ thus $0\text{-}\mathrm{cf}_X A = \omega$, a contradiction.

Case 2. B has no minimal elements.

Set $I = \{\alpha < \gamma : \exists a \in A \exists b \in B (a \restriction (\alpha+1) = b \restriction (\alpha+1))\}$. Since I is a 0-segment in γ, we have $I = \alpha_0$ for some ordinal $\alpha_0 \leq \gamma$. For every $\alpha < \alpha_0$, fix $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$. Define $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$.

Claim 5. For every $\alpha < \alpha_0$, $y_0 \restriction (\alpha+1) = a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$.

Proof. It suffices to see the first equality. Assuming $y_0 \restriction (\alpha+1) \neq a_\alpha \restriction (\alpha+1)$ for some $\alpha < \alpha_0$, let $\alpha_1 = \min\{\alpha < \alpha_0 : y_0 \restriction (\alpha + 1) \neq a_\alpha \restriction (\alpha+1)\}$ and $\alpha_2 = \min\{\alpha \leq \alpha_1 : y_0(\alpha) \neq a_{\alpha_1}(\alpha)\}$. Then obviously $\alpha_2 < \alpha_1 < \alpha_0$ holds because of $y_0(\alpha_1) = a_{\alpha_1}(\alpha_1)$. When $y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_2} < a_{\alpha_1} \in A$, a contradiction. When $y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_1} < a_{\alpha_2} \in A$, a contradiction. This completes the proof of Claim 5.

Claim 6. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then $y_0 \in X = A \cup B$. Let $y_0 \in A$. Since A has no maximal elements, fixing $a \in A$ with $y_0 < a$, let $\beta_0 = \min\{\alpha < \gamma : y_0(\alpha) \neq a(\alpha)\}$. Then we have $B \ni b_{\beta_0} < a \in A$, a contradiction. The case "$y_0 \in B$" is similar, since we are in the Case 2. This completes the proof of Claim 6.

Claim 7. The following hold:

(1) if $a \in A$, then $a \restriction \alpha_0 \leq y_0$,
(2) if $b \in B$, then $b \restriction \alpha_0 \geq y_0$,
(3) if $x \in X$ and $x \restriction \alpha_0 < y_0$, then $x \in A$,
(4) if $x \in X$ and $x \restriction \alpha_0 > y_0$, then $x \in B$.

Proof. (1) If $a \in A$ and $a \restriction \alpha_0 > y_0$ were true, then letting $\beta_0 = \min\{\alpha < \alpha_0 : a(\alpha) \neq y_0(\alpha)\}$, we have $B \ni b_{\beta_0} < a \in A$, a contradiction. (2) is similar.

(3) Let $x \in X$ and $x \restriction \alpha_0 < y_0$. By letting $\beta_0 = \min\{\alpha < \alpha_0 : x(\alpha) \neq y_0(\alpha)\}$, we have $x < a_{\beta_0} \in A$, thus $x \in A$. (4) is similar. This completes the proof of Claim 7.

Put $A_0 = \{a(\alpha_0) : a \in A, a \restriction \alpha_0 = y_0\}$ and $B_0 = \{b(\alpha_0) : b \in B, b \restriction \alpha_0 = y_0\}$.

Claim 8. A_0 is a 0-segment of X_{α_0} and $B_0 = X_{\alpha_0} \setminus A_0$.

Proof. To see that A_0 is a 0-segment of X_{α_0}, let $u' < u \in A_0$. Take $a \in A$ with $a \restriction \alpha_0 = y_0$ and $a(\alpha_0) = u$. Let $a' = (a \restriction \alpha_0)^\wedge\langle u'\rangle^\wedge(a \restriction (\alpha_0, \gamma))$. Since A is a 0-segment with $a' < a \in A$, we have $a' \in A$. Then $u' = a'(\alpha_0)$ and $a' \restriction \alpha_0 = y_0$ show $u' \in A_0$.

To see $B_0 \subset X_{\alpha_0} \setminus A_0$, let $u \in B_0$. Take $b \in B$ with $b \restriction \alpha_0 = y_0$ and $b(\alpha_0) = u$. If $u \in A_0$ were true, then by taking $a \in A$ with $a \restriction \alpha_0 = y_0$ and $a(\alpha_0) = u$, we see $a \restriction (\alpha_0 + 1) = b \restriction (\alpha_0 + 1)$, thus $\alpha_0 \in I = \alpha_0$, a contradiction.

To see $B_0 \supset X_{\alpha_0} \setminus A_0$, let $u \in X_{\alpha_0} \setminus A_0$. Take $x \in X$ with $x \restriction \alpha_0 = y_0$ and $x(\alpha_0) = u$. If $x \in A$ were true, then by the definition of A_0, we see $u = x(\alpha_0) \in A_0$, a contradiction. Now we have $x \in B$ thus $u \in B_0$ This completes the proof of Claim 8.

Claim 9. $A_0 \neq \emptyset$.

Proof. Assume $A_0 = \emptyset$. We will check the following Facts.

Fact 1. $(\leftarrow, y_0)_{\prod_{\alpha<\alpha_0} X_\alpha} \times \prod_{\alpha_0 \leq \alpha} X_\alpha = A$.

Proof. The inclusion "$\subset$" follows from Claim 7 (3). To see the other inclusion, let $a \in A$. By Claim 7 (1), we have $a \restriction \alpha_0 \leq y_0$. If $a \restriction \alpha_0 = y_0$ were true, then we have $a(\alpha_0) \in A_0$, which contradicts $A_0 = \emptyset$. So we have $a \restriction \alpha_0 < y_0$, that is, $a \in (\leftarrow, y_0) \times \prod_{\alpha_0 \leq \alpha} X_\alpha$. This completes the proof of Fact 1.

This fact shows $(\leftarrow, y_0) \neq \emptyset$ because of $A \neq \emptyset$.

Fact 2. $\alpha_0 > 0$ and α_0 is limit.

Proof. If $\alpha_0 = 0$ were true, then by taking $a \in A$, we see $a(\alpha_0) \in A_0$, a contradiction. If $\alpha_0 = \beta_0 + 1$ were true for some ordinal β_0, then by $\beta_0 \in I$ $(= \alpha_0)$, we see $a_{\beta_0} \restriction \alpha_0 = a_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$ which shows $a_{\beta_0}(\alpha_0) \in A_0$, a contradiction. This completes the proof of Fact 2.

Fact 3. $0\text{-}\operatorname{cf}_{\prod_{\alpha<\alpha_0} X_\alpha}(\leftarrow, y_0) = \operatorname{cf}\alpha_0$.

Proof. Let $\lambda = \operatorname{cf}\alpha_0$. It follows from $A_0 = \emptyset$ that for every $\alpha < \alpha_0$, there is $\alpha' < \alpha_0$ with $\alpha < \alpha'$ such that $a_\alpha \restriction \alpha_0 < a_{\alpha'} \restriction \alpha_0$. So using the induction on λ, we can find a 0-order preserving and 0-unbounded sequence $\{\delta_\beta : \beta < \lambda\}$ in α_0 such that $\{a_{\delta_\beta} : \beta < \lambda\}$ is also 0-order preserving and 0-unbounded in $(\leftarrow, y_0)$. Therefore $0\text{-}\operatorname{cf}(\leftarrow, y_0) = \lambda$. This completes the proof of Fact 3.

Fact 4. There is $\alpha < \gamma$ with $\alpha_0 \le \alpha$ such that X_α has no minimal elements, that is, $J^- \cap [\alpha_0, \gamma) \neq \emptyset$.

Proof. Otherwise, $y_0 {}^\wedge\langle \min X_\alpha : \alpha_0 \le \alpha\rangle$ is the minimal element of B, a contradiction to Case 2. This completes the proof of Fact 4.

It follows from $J^- \subset \omega_1$ and Fact 4 that $\alpha_0 < \omega_1$, therefore by Fact 2, we see $\operatorname{cf}\alpha_0 = \omega$. Now Fact 1 and 3 show $0\text{-}\operatorname{cf} A = 0\text{-}\operatorname{cf}(\leftarrow, y_0) = \operatorname{cf}\alpha_0 = \omega$, a contradiction. We have seen $A_0 \neq \emptyset$. This completes the proof of Claim 9.

Put $A^* = (\leftarrow, y_0) \times X_{\alpha_0} \cup \{y_0\} \times A_0$. Since A_0 is a 0-segment of X_{α_0}, A^* is also a 0-segment of $\prod_{\alpha \le \alpha_0} X_\alpha$ and $\{y_0\} \times A_0$ is a 1-segment (i.e., final segment) of A^*, therefore by Claim 9, we have $0\text{-}\operatorname{cf}_{\prod_{\alpha \le \alpha_0} X_\alpha} A^* = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0$.

Claim 10. $A = A^* \times \prod_{\alpha_0 < \alpha} X_\alpha$.

Proof. To see the inclusion "$\subset$", let $a \in A$. By Claim 7 (1), we have $a \restriction \alpha_0 \le y_0$. When $a \restriction \alpha_0 < y_0$, we can see $a \restriction (\alpha_0 + 1) \in A^*$ thus $a \in A^* \times \prod_{\alpha_0 < \alpha} X_\alpha$. When $a \restriction \alpha_0 = y_0$, by $a(\alpha_0) \in A_0$, we can see $a \restriction (\alpha_0 + 1) \in A^*$ thus $a \in A^* \times \prod_{\alpha_0 < \alpha} X_\alpha$.

To see the inclusion "$\supset$", let $a \in A^* \times \prod_{\alpha_0 < \alpha} X_\alpha$. Note $a \restriction (\alpha_0 + 1) \in A^*$. When $a \restriction (\alpha_0 + 1) \in (\leftarrow, y_0) \times X_{\alpha_0}$, from Claim 7(3), we see $a \in A$. When $a \restriction (\alpha_0 + 1) \in \{y_0\} \times A_0$, from $a \restriction \alpha_0 = y_0$ and $a(\alpha_0) \in A_0$, we see $a \in A$ from Claim 8. This completes the proof of Claim 10.

Claim 10 shows $A^* \neq \emptyset$. We consider further 2 subcases of Case 2.

Case 2-1. $0\text{-}\operatorname{cf}_{\prod_{\alpha \le \alpha_0} X_\alpha} A^* \ge \omega$.

In this case, by Claim 10, we easily see $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf} A^*$ $(= 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0)$, so we have $0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 \ge \omega_1$. Then the clause (3) shows $\sup J^- \le \alpha_0$.

Claim 11. $A_0 \neq X_{\alpha_0}$.

Proof. Assume $A_0 = X_{\alpha_0}$. Since A_0 has no maximal elements, we have $\alpha_0 \in J^+$. If $\alpha_0 = \beta_0 + 1$ were true for some ordinal β_0, then by $b_{\beta_0} \in B$ and $b_{\beta_0} \restriction \alpha_0 = b_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$, we have $b_{\beta_0}(\alpha_0) \in B_0$, a contradiction. So we see that α_0 is 0 or limit, thus $[l(\alpha_0), \alpha_0) = \emptyset$. It follows from $\sup J^- \le \alpha_0$ and the clause (2bi) that $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} = \omega$, which contradicts $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 \ge \omega_1$. This completes the proof of Claim 11.

Claim 12. A_0 is closed in X_{α_0}.

Proof. Let $u \in X_{\alpha_0} \setminus A_0$ $(= B_0)$. When there is $u' \in B_0$ with $u' < u$, $(u', \rightarrow)_{X_{\alpha_0}}$ is a neighborhood of u disjoint from A_0. When there is no such u' (that is, $u = \min B_0$), letting $b = y_0{}^\frown\langle u\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$, we see $b = \min B$, which contradicts Case 2. This completes the proof of Claim 12.

Now we have seen that $\sup J^- \leq \alpha_0$ and A_0 is a 0-bounded closed 0-segment with $\omega_1 \leq 0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0$, a contradiction to (2a).

Case 2-2. $0\text{-}\mathrm{cf}_{\prod_{\alpha \leq \alpha_0} X_\alpha} A^* = 1$, that is, A^* (hence A_0) has a maximal element.

In this case, since A has no maximal elements but A^* has a maximal element, by Claim 10, for some ordinal $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has no maximal elements. So let

$$\alpha_1 = \min\{\alpha > \alpha_0 : \ X_\alpha \text{ has no maximal elements. }\}$$

and $\lambda = 0\text{-}\mathrm{cf}_{X_{\alpha_1}} X_{\alpha_1}$. Take a 0-order preserving and 0-unbounded sequence $\{u_\beta : \beta < \lambda\}$ in X_{α_1} with $(\leftarrow, u_0) \neq \emptyset$. Fixing $y_1 \in \prod_{\alpha_1 < \alpha} X_\alpha$, let $x_\beta = y_0{}^\frown\langle \max A_0\rangle^\frown\langle \max X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle^\frown\langle u_\beta\rangle^\frown y_1$ for every $\beta < \lambda$. Then the sequence $\{x_\beta : \beta < \lambda\}$ is 0-order preserving and 0-unbounded in A, so $\lambda = 0\text{-}\mathrm{cf}_X A \geq \omega_1$ thus $0\text{-}\mathrm{cf}_{X_{\alpha_1}} X_{\alpha_1} \geq \omega_1$. Now the clause (3) shows $\sup J^- \leq \alpha_1$. Also note $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ from the definition of α_1.

Claim 13. $l(\alpha_1) \leq \alpha_0$ and $J^+ \cap [l(\alpha_1), \alpha_0] \neq \emptyset$ hold, therefore $J^+ \cap [l(\alpha_1), \alpha_1) \neq \emptyset$.

Proof. If $\alpha_0 < l(\alpha_1)$ were true, then by $J^+ \cap [l(\alpha_1), \alpha_1) \subset J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ and the clause (2bi), we have $0\text{-}\mathrm{cf}_{X_{\alpha_1}} X_{\alpha_1} \leq \omega$, a contradiction. Thus we have $l(\alpha_1) \leq \alpha_0$. If $J^+ \cap [l(\alpha_1), \alpha_0] = \emptyset$ were true, then by $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$, we have $J^+ \cap [l(\alpha_1), \alpha_1) = \emptyset$, thus $0\text{-}\mathrm{cf}_{X_{\alpha_1}} X_{\alpha_1} \leq \omega$ by the clause (2bi), a contradiction. This completes the proof of Claim 13.

So let $\alpha_2 = \max(J^+ \cap [l(\alpha_1), \alpha_1))$. From $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$, note $\alpha_2 \leq \alpha_0$.

Claim 14. B_0 has a minimal element.

Proof. First we will check $B_0 \neq \emptyset$. When $\alpha_0 = \alpha_2$, we see $A_0 \neq X_{\alpha_0}$ (thus $B_0 \neq \emptyset$), because A_0 has a maximal element but X_{α_0} $(= X_{\alpha_2})$ has no maximal elements. So let $\alpha_2 < \alpha_0$. It follows from $l(\alpha_1) \leq \alpha_2 < \alpha_0 < \alpha_1$ that α_0 is a successor ordinal, say $\alpha_0 = \beta_0 + 1$. Because of $b_{\beta_0} \in B$ and $b \restriction \alpha_0 = y_0 \restriction \alpha_0$, we have $b_{\beta_0}(\alpha_0) \in B_0$, thus $B_0 \neq \emptyset$.

Next assume that B_0 has no minimal elements, then $(\max A_0, \to)$ has no minimal elements therefore $\alpha_0 \in K^+ \cap [\alpha_2, \alpha_1)$. Now by clause (2biii), we have $0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} \leq \omega$, a contradiction. Thus B_0 has a minimal element. This completes the proof of Claim 14.

Since B has no minimal elements but B_0 has a minimal element, there is $\alpha < \gamma$ with $\alpha_0 < \alpha$ such that X_α has no minimal elements (otherwise, B has the minimal element $y_0 {}^\wedge\langle \min B_0\rangle^\wedge\langle \min X_\alpha : \alpha_0 < \alpha\rangle$). So let $\alpha_3 = \min\{\alpha > \alpha_0 : X_\alpha \text{ has no minimal elements.}\}$, then $\alpha_3 \in J^- \cap (\alpha_2, \alpha_1]$ because of $\sup J^- \leq \alpha_1$. Now the assumption (2bii) with $\sup J^- \leq \alpha_1$ shows $0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} \leq \omega$, a contradiction. □

Analogously we have:

Lemma 2.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X has countable closed 1-cofinality if and only if the following clauses hold:*

(1) $J^+ \subset \omega_1$,
(2) *for every ordinal $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, the following hold:*
  (a) *X_α has countable 1-bounded closed 1-cofinality,*
  (b) *in each of the following cases, $1\text{-}\operatorname{cf}_{X_\alpha} X_\alpha \leq \omega$ holds,*
    (i) $J^- \cap [l(\alpha), \alpha) = \emptyset$,
    (ii) $J^- \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $J^+ \cap (\alpha', \alpha] \neq \emptyset$,
    (iii) $J^- \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^- \cap [\alpha', \alpha) \neq \emptyset$,
    where $\alpha' = \max(J^- \cap [l(\alpha), \alpha))$ in case $J^- \cap [l(\alpha), \alpha) \neq \emptyset$.
(3) *for every $\alpha < \sup J^+$, X_α has countable 1-cofinality.*

3. ω_1-COMPACTNESS OF LEXICOGRAPHIC PRODUCTS

In this section, we characterize ω_1-compactness of lexicographic products. To do this, we need further observation of ω_1-compactness. When X is a GO-space, we can consider some variations of notions of cluster points and ω_1-compactness.

Definition 3.1. Let X be a GO-space, $x \in X$ and $H \subset X$. A point x is said to be a 0-*cluster point* of H in X if for every neighborhood U of x in X, $(U \cap (\leftarrow, x)) \cap H$ is non-empty (equivalently, infinite), in other words, for every $x^* \in X^*$ with $x^* <_{X^*} x$, $(x^*, x)_{X^*} \cap H$ is non-empty (equivalently, infinite). The notion of 1-*cluster points* of H in X is similarly defined. Moreover x is said to be a 2-*cluster point* of H in X if it is a both 0-cluster and 1-cluster point of H in X.

A GO-space X is said to be 0-(1-, 2-)ω_1-*compact* if every uncountable subset of X has 0-(1-, 2-, respectively)cluster points in X.

When H is represented as $H = \{x_\beta : \beta < \delta\}$, where δ is an ordinal and "$\beta' < \beta \to x_{\beta'} < x_\beta$" (i.e., H is a 0-order preserving sequence), obviously H has no 1-cluster points. By the definitions, we easily see:

- countable GO-spaces are 2-ω_1-compact,
- Lindelöf GO-spaces are ω_1-compact,
- 2-ω_1-compact GO-spaces are 0-ω_1-compact and 1-ω_1-compact,
- 0-ω_1-compact (1-ω_1-compact) GO-spaces are ω_1-compact,
- all subspaces of $\mathbb{R}$ are ω_1-compact, because they are Lindelöf, see Lemma 1.4.

In fact, we first show:

Proposition 3.2. *If H is an uncountable subset of $\mathbb{R}$, then $\{x \in H :$ x is not a 2-cluster point of H in $H.\}$ is countable. Therefore, all subspaces of $\mathbb{R}$ are 2-ω_1-compact.*

Proof. Let H be an arbitrary uncountable subset of $\mathbb{R}$. Let

$$\mathcal{C} = \{C \subset \mathbb{R} : C \text{ is a convex set in } \mathbb{R},\ |C \cap H| \le \omega\},$$

$$\mathcal{C}_M = \{C \in \mathcal{C} : \text{ there is no } C' \in \mathcal{C} \text{ with } C \subsetneq C'\},$$

$$\mathcal{C}_L = \{C \in \mathcal{C}_M : |C| \ge 2\}.$$

It is routine to check that

(1) for each $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| \le \omega$, if $C_0 \cap C_1 \ne \emptyset$ for every $C_0, C_1 \in \mathcal{C}'$, then $\bigcup \mathcal{C}' \in \mathcal{C}$.

(2) if $C_0, C_1 \in \mathcal{C}$ and $C_0 \cap C_1 \ne \emptyset$, then $C_0 \cup C_1 \in \mathcal{C}$,

(3) if $\xi < \omega_1$ and $\{C_\alpha : \alpha < \xi\} \subset \mathcal{C}$ is ascending (that is, if $\alpha < \alpha'$ then $C_\alpha \subset C_{\alpha'}$), then $\bigcup_{\alpha<\xi} C_\alpha \in \mathcal{C}$.

By (2),

(4) $\mathcal{C}_M$ is pairwise disjoint.

Notice that

(5) $\{\mathrm{Int}_{\mathbb{R}} C : C \in \mathcal{C}_L\}$ is a pairwise disjoint collection of non-empty open sets of $\mathbb{R}$, where $\mathrm{Int}_{\mathbb{R}} C$ denotes the interior of C in $\mathbb{R}$.

Since $\mathbb{R}$ has the c.c.c. (that is, the cardinality of every pairwise disjoint collection of non-empty open sets is countable), we see:

(6) $|\mathcal{C}_L| \le \omega$, in particular, $|\bigcup_{C \in \mathcal{C}_L} (C \cap H)| \le \omega$,

(7) there is not a strictly ascending sequence of uncountable length by convex sets (in particular, by members of $\mathcal{C}$).

By (3) and (7), we have:

(8) for each $C \in \mathcal{C}$, there is a $C_M \in \mathcal{C}_M$ with $C \subset C_M$.

It is trivial that

(9) $\{x\} \in \mathcal{C}$ for every $x \in \mathbb{R}$.

By (8) and (9), we have

(10) $\mathbb{R} = \bigcup \mathcal{C}_M$.

It suffices to see:

Claim. $\{x \in H : x \text{ is not a 2-cluster point of } H \text{ in } H.\} \subset \bigcup_{C \in \mathcal{C}_L}(C \cap H)$.

Proof. Let $x \in H \setminus \bigcup_{C \in \mathcal{C}_L}(C \cap H)$. By (10), we obtain a $C \in \mathcal{C}_M$ with $x \in C$, then $C \notin \mathcal{C}_L$ so $C = \{x\}$. Let U be an arbitrary neighborhood of x in H. Take $a, b \in \mathbb{R}$ with $a < x < b$ and $[a, b] \cap H \subset U$. By $C \in \mathcal{C}_M$ and $C = \{x\} \subsetneq [a, x]$, we have $[a, x] \notin \mathcal{C}$, i.e. $|[a, x] \cap H| > \omega$. Hence, $(U \cap (\leftarrow, x)) \cap H \neq \emptyset$. Similarly, we see $(U \cap (x, \rightarrow)) \cap H \neq \emptyset$. Therefore x is a 2-cluster point of H. □

□

To characterize ω_1-compactness of lexicographic product, we need further variations of it.

Definition 3.3. A GO-space X is said to be *0-boundedly 0-ω_1-compact* if every uncountable 0-bounded subset H of X has a 0-cluster point in X. 1-boundedly 1-ω_1-compactness is similarly defined.

Example 3.4.
(1) Let X be a subspace of $\mathbb{R}$. Obviously X has both countable 0-cofinality and countable 1-cofinality, moreover it is 2-ω_1-compact (Proposition 3.2).
(2) Let X be a subspace of $\mathbb{S}$, then X has both countable 0-cofinality and countable 1-cofinality, moreover it is 1-ω_1-compact. But X is not 0-ω_1-compact whenever X is uncountable.
(3) Let X be a subspace of an ordinal. Then X has countable 1-cofinality, but X is not 1-ω_1-compact whenever X is uncountable. The GO-space $X := \{\alpha < \omega_1 : \alpha \text{ is a successor ordinal.}\}$ is 0-boundedly 0-ω_1-compact, but not 0-ω_1-compact.
(4) All ordinals are 0-ω_1-compact and have countable 0-bounded closed 0-cofinality.
(5) Let $X = \langle X, <_X, \tau_X \rangle$ be a GO-space. The *reverse* of X, which is denoted by $-X$, is the GO-space $\langle X, >_X, \tau_X \rangle$, see [6, Definition 3.10]. Obviously X has countable (closed) 0-cofinality if and only if $-X$ has countable (closed) 1-cofinality, also X is (0-boundedly) 0-ω_1-compact if and only if $-X$ is (1-boundedly) 1-ω_1-compact. Therefore if α is an ordinal, then $-\alpha$ is 1-ω_1-compact, moreover has countable 0-cofinality and countable 1-bounded closed 1-cofinality.

(6) Regarding $X \cap Y = \emptyset$, let $X = \langle X, <_X \rangle$ and $Y = \langle Y, <_Y \rangle$ be LOTS's. $X + Y$ denotes the LOTS $\langle X \cup Y, <_{X+Y} \rangle$, where the linear order $<_{X+Y}$ extends both $<_X$ and $<_Y$, moreover it satisfies $x <_{X+Y} y$ for every $x \in X$ and $y \in Y$, that is, $X + Y$ is the resulting LOTS that Y is added after X. Then $(-\omega_1) + \omega_1$ and $\omega_1 + (-\omega_1)$ are topologically homeomorphic and ω_1-compact. $(-\omega_1) + \omega_1$ has countable 0-bounded closed 0-cofinality and countable 1-bounded closed 1-cofinality. On the other hand, $\omega_1 + (-\omega_1)$ has neither countable 0-bounded closed 0-cofinality nor countable 1-bounded closed 1-cofinality. $(-\omega_1) + \omega_1$ and $\omega_1 + (-\omega_1)$ are neither 0-ω_1-compact nor 1-ω_1-compact.

We use the following notation.

Definition 3.5. Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a product of linearly ordered sets (need not be a lexicographic product) and $H \subset X$. For every $\alpha < \gamma$ and $y \in \prod_{\beta<\alpha} X_\beta$ and $x \in X$, let

$$E(y, H) = \{z(\alpha) : z \in H, z \restriction \alpha = y\},$$

$$F^-(x, H) = \{\alpha < \gamma : E(x \restriction \alpha, H) \cap (\leftarrow, x(\alpha))_{X_\alpha} \neq \emptyset\},$$

$$F^+(x, H) = \{\alpha < \gamma : E(x \restriction \alpha, H) \cap (x(\alpha), \rightarrow)_{X_\alpha} \neq \emptyset\}.$$

Lemma 3.6. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a product of linearly ordered sets and $H \subset X$. If the following clauses hold, then H is countable.*

(1) *$E(y, H)$ is countable for every $\alpha < \gamma$ and $y \in \prod_{\beta<\alpha} X_\beta$,*
(2) *$F^-(x, H)$ is finite for every $x \in H$,*
(3) *$F^+(x, H)$ is countable for every $x \in X$.*

Proof. For every $n \in \omega$ and $y \in \prod_{\beta<\alpha} X_\beta$ with $\alpha \leq \gamma$, let

$$H(y, n) = \{x \in H : x \restriction \alpha = y, |F^-(x, H) \cap [\alpha, \gamma)| = n\}.$$

We will prove, by induction on $n \in \omega$,

$$(*)_n \text{ for every } y \in \prod_{\beta<\alpha} X_\beta \text{ with } \alpha \leq \gamma,\ H(y, n) \text{ is countable.}$$

If we have done this induction, then from the clause (2), we see $H = \bigcup_{n\in\omega} H(\emptyset, n)$. Therefore H is countable.

Claim 1. Let $x, x' \in H$ and $\alpha \leq \alpha' \leq \gamma$. If $x \restriction \alpha = x' \restriction \alpha$, $F^-(x, H) \cap [\alpha, \alpha') = \emptyset$ and $F^-(x', H) \cap [\alpha, \alpha') = \emptyset$ hold, then we have $x \restriction \alpha' = x' \restriction \alpha'$.

Proof. Assume that for some $x, x' \in H$ and α, α' with $\alpha \leq \alpha' \leq \gamma$, $x \restriction \alpha = x' \restriction \alpha$, $F^-(x, H) \cap [\alpha, \alpha') = \emptyset$ and $F^-(x', H) \cap [\alpha, \alpha') = \emptyset$

hold but $x \restriction \alpha' \neq x' \restriction \alpha'$. Let $\alpha_0 = \min\{\beta < \alpha' : x(\beta) \neq x'(\beta)\}$. Note $\alpha \leq \alpha_0 < \alpha'$, $x \restriction \alpha_0 = x' \restriction \alpha_0$ and $x(\alpha_0) \neq x'(\alpha_0)$. Since X_{α_0} is a linearly ordered set, we may assume $x(\alpha_0) < x'(\alpha_0)$, then $x(\alpha_0) \in E(x' \restriction \alpha_0, H) \cap (\leftarrow, x'(\alpha_0))$. It follows from $\alpha_0 \notin F^-(x', H)$ that $E(x' \restriction \alpha_0, H) \cap (\leftarrow, x'(\alpha_0)) = \emptyset$, a contradiction. This completes the proof of Claim 1.

Claim 2. $(*)_0$ holds.

Proof. Let $\alpha \leq \gamma$, $y \in \prod_{\beta<\alpha} X_\beta$ and $x, x' \in H(y, 0)$. Then $x \restriction \alpha = x' \restriction \alpha$, $F^-(x, H) \cap [\alpha, \gamma) = \emptyset$ and $F^-(x', H) \cap [\alpha, \gamma) = \emptyset$ hold, which imply $x = x'$ by Claim 1. Thus we have $|H(y, 0)| \leq 1$. $\square$

Let $n \in \omega$ with $n > 0$. Assuming $(*)_k$ for every $k < n$, we will see $(*)_n$. So let $\alpha \leq \gamma$ and $y \in \prod_{\beta<\alpha} X_\beta$. Let

$$N = \bigcup_{x \in H, x \restriction \alpha = y} \{\delta \in F^-(x, H) \cap [\alpha, \gamma) : F^-(x, H) \cap [\alpha, \delta) = \emptyset\}.$$

Notice $N \subset [\alpha, \gamma)$. For every $\delta \in N$, fix $x_\delta \in H$ with $x_\delta \restriction \alpha = y$ such that $\delta \in F^-(x_\delta, H) \cap [\alpha, \gamma)$ and $F^-(x_\delta, H) \cap [\alpha, \delta) = \emptyset$.

Claim 3. If $\delta, \delta' \in N$ with $\delta \leq \delta'$, then $x_\delta \restriction \delta = x_{\delta'} \restriction \delta$.

Proof. Let $\delta, \delta' \in N$ with $\delta \leq \delta'$. Since $F^-(x_\delta, H) \cap [\alpha, \delta) = \emptyset$ and $F^-(x_{\delta'}, H) \cap [\alpha, \delta) = \emptyset$, apply Claim 1 with $\alpha' = \delta$. $\square$

Claim 4. N is countable.

Proof. From Claim 3, we can take $x \in X$ with $x \restriction \delta = x_\delta \restriction \delta$ for every $\delta \in N$. By the clause (3), it suffices to see $\{\delta \in N : \delta \text{ is not maximal in } N.\} \subset F^+(x, H)$. Let δ be a member of N which is not maximal in N. Then we can take $\delta' \in N$ with $(\alpha \leq)\ \delta < \delta'$. It follows from $\delta \in F^-(x_\delta, H) \setminus F^-(x_{\delta'}, H)$ that $E(x_\delta \restriction \delta, H) \cap (\leftarrow, x_\delta(\delta)) \neq \emptyset$ and $E(x_{\delta'} \restriction \delta, H) \cap (\leftarrow, x_{\delta'}(\delta)) = \emptyset$. By $x_\delta \restriction \delta = x_{\delta'} \restriction \delta$, we see $x_{\delta'}(\delta) < x_\delta(\delta)$, so $\delta \in F^+(x_{\delta'}, H)$. Now by $x_{\delta'} \restriction (\delta + 1) = x \restriction (\delta + 1)$, we have $\delta \in F^+(x, H)$. This completes the proof of Claim 4.

For every $\delta \in N$, the clause (1) ensures that $E(x_\delta \restriction \delta, H)$ is countable, moreover by the inductive assumption $(*)_{n-1}$, $H((x_\delta \restriction \delta) {}^\frown\langle u \rangle, n-1)$ is countable for every $u \in E(x_\delta \restriction \delta, H)$. Now by Claim 4, the following claim completes the proof.

Claim 5. $H(y, n) \subset \bigcup_{\delta \in N} \bigcup_{u \in E(x_\delta \restriction \delta, H)} H((x_\delta \restriction \delta) {}^\frown\langle u \rangle, n - 1)$.

Proof. Let $x \in H(y, n)$ and $\delta = \min(F^-(x, H) \cap [\alpha, \gamma))$. Note that δ is defined because of $x \in H(y, n)$ (so $|F^-(x, H) \cap [\alpha, \gamma)| = n > 0$). By the minimality of δ, we see $\delta \in N$. Then by $|F^-(x, H) \cap [\alpha, \gamma)| = n$, we have $|F^-(x,H)\cap[\delta+1,\gamma)| = n-1$ and $x \in H$, so we see $x \in H(x \upharpoonright (\delta+1), n-1)$ and $x(\delta) \in E(x \upharpoonright \delta, H)$. Also note $x \upharpoonright \alpha = y = x_\delta \upharpoonright \alpha$, $F^-(x,H) \cap [\alpha,\delta) = \emptyset$ and $F^-(x_\delta,H)\cap[\alpha,\delta) = \emptyset$. So by Claim 1, we have $x \upharpoonright \delta = x_\delta \upharpoonright \delta$, so $x(\delta) \in E(x_\delta \upharpoonright \delta, H)$, which shows $x \in H((x_\delta \upharpoonright \delta)\,{}^\frown\langle x(\delta)\rangle, n-1)$. This completes the proof of Claim 5.

□

The example "$X = \omega^\omega$ with $H = X$" obviously shows that "finite" in Lemma 3.6 (2) cannot be replaced by "countable".

Corollary 3.7. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and H a subset of X with no cluster points. If the following clauses hold, then H is countable.*

1. *$E(y,H)$ is countable for every $\alpha < \gamma$ and $y \in \prod_{\beta<\alpha} X_\beta$,*
2. *$J^- \subset \omega$,*
3. *$J^+ \subset \omega_1$.*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, H be a subset of X with no cluster points. Assume the clauses (1) - (3). It suffices to see (2) and (3) in Lemma 3.6. To see (3) of Lemma 3.6, assume that $F^+(x,H)$ is uncountable for some $x \in X$. Then note that H is uncountable and $\omega_1 \leq \gamma$. Since $F^+(x,H)$ is an uncountable subset of γ, there is $\alpha_0 \leq \gamma$ with $\omega_1 \leq \alpha_0$ which is a (0-)cluster point of $F^+(x,H)$, that is, $\omega_1 \leq \alpha_0 \in \mathrm{Lim}_{\gamma+1} F^+(x,H)$. Let $z = (x \upharpoonright \alpha_0)^\frown\langle \max X_\alpha : \alpha_0 \leq \alpha\rangle$ and z^* be a point in $\hat{X}$ with $z <_{\hat{X}} z^*$, where z can be defined because of $J^+ \subset \omega_1$. Put $\alpha_1 = \min\{\alpha < \gamma : z(\alpha) \neq z^*(\alpha)\}$. Then by the definition of z, we have $\alpha_1 < \alpha_0$. Since α_0 is a (0-)cluster point of $F^+(x,H)$, take $\alpha_2 \in F^+(x,H)$ with $\alpha_1 < \alpha_2 < \alpha_0$. Then there is $z' \in H$ with $z' \upharpoonright \alpha_2 = x \upharpoonright \alpha_2$ $(= z \upharpoonright \alpha_2)$ and $z'(\alpha_2) > x(\alpha_2)$ $(= z(\alpha_2))$, thus $z < z'$. Also by $\alpha_1 < \alpha_2$, we see $z' <_{\hat{X}} z^*$, so $H \cap (z, z^*) \neq \emptyset$, which means that z is a 1-cluster point of H, a contradiction. (2) is similar. □

There are also an analogous result changed by $+$ and $-$ by $-$ and $+$ respectively of Lemma 3.6 and Corollary 3.7.

Definition 3.8. A 0-segment A of a GO-space X is said to be *stationary* if $\kappa := 0\text{-}\mathrm{cf}_X A \geq \omega_1$ and there are a stationary set $S \subset \kappa$ and a continuous map $\pi : S \to A$ such that $\pi[S]$ is 0-unbounded in A, see [6]. Similarly the stationarity of 1-segments can be defined.

Lemma 3.9. *Let A be a 0-segment of a GO-space X with $\kappa := 0\text{-}\mathrm{cf}_X A \geq \omega_1$. The following are equivalent:*

1. *A is stationary,*

(2) *there are a stationary set S in κ and a 0-order preserving embedding $\pi : S \to A$ such that $\pi[S]$ is closed and 0-unbounded in A,*
(3) *every closed 0-unbounded subset H of A has a cluster point,*
(4) *every 0-unbounded subset H of A has a (0-)cluster point.*

Proof. The equivalences between (1), (2) and (3) are shown in [6, Lemma 2.7]. (4) $\Rightarrow$ (3) is obvious.

(2) $\Rightarrow$ (4) Assuming (2), fix such a stationary set S and a 0-order preserving embedding π. Let H be a 0-unbounded subset of A. For each $\alpha \in S$, fix $x_\alpha \in H$ and $f(\alpha) \in S$ with $\pi(\alpha) <_X x_\alpha <_X \pi(f(\alpha))$. Then $C := \{\alpha < \kappa : \forall \alpha' \in S \cap \alpha (f(\alpha') < \alpha)\}$ is a club set in κ. Pick $\alpha_0 \in S \cap \mathrm{Lim}(S) \cap C$. Such an α_0 exists, because S is stationary in κ, and both $\mathrm{Lim}(S)$ and C are club. To see that $\pi(\alpha_0)$ is a 0-cluster point of H, let U be a convex neighborhood of $\pi(\alpha_0)$. Since π is continuous, we can find $\beta < \alpha_0$ with $\pi[S \cap (\beta, \alpha_0]] \subset U$. From $\alpha_0 \in \mathrm{Lim}(S)$, we can fix $\alpha' \in S$ with $\beta < \alpha' < \alpha_0$. Since $\pi(\alpha') < x_{\alpha'} < \pi((f(\alpha')) < \pi(\alpha_0)$ and U is convex with $\pi(\alpha'), \pi(\alpha_0) \in U$, we have $x_{\alpha'} \in H \cap (U \cap (\leftarrow, \pi(\alpha_0))$. So $\pi(\alpha_0)$ is a 0-cluster point of H. ☐

Also we have an analogous result of the lemma above. Now we have the following.

Lemma 3.10. *A GO-space X is ω_1-compact if and only if the following hold,*
(1) *X is boundedly ω_1-compact,*
(2) *if $0\text{-}\operatorname{cf}_X X = \omega_1$, then the 0-segment X is stationary,*
(3) *if $1\text{-}\operatorname{cf}_X X = \omega_1$, then the 1-segment X is stationary.*

Now we have prepared to characterize ω_1-compactness of lexicographic products.

Lemma 3.11. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is ω_1-compact if and only if the following clauses hold:*
(1) *$J^- \subset \omega$ or $J^+ \subset \omega$,*
(2) *$J^- \subset \omega_1$ and $J^+ \subset \omega_1$,*
(3) *for every ordinal $\alpha < \gamma$ with $\sup J^- \leq \alpha$ and $0\text{-}\operatorname{cf} X_\alpha = \omega_1$, in each of the following cases, the 0-segment X_α is stationary,*
  (i) *$J^+ \cap [l(\alpha), \alpha) = \emptyset$,*
  (ii) *$J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ and $J^- \cap (\alpha', \alpha] \neq \emptyset$,*
  (iii) *$J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ and $K^+ \cap [\alpha', \alpha) \neq \emptyset$,*
  where $\alpha' = \max(J^+ \cap [l(\alpha), \alpha))$ in case $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$.
(4) *for every ordinal $\alpha < \gamma$ with $\sup J^+ \leq \alpha$ and $1\text{-}\operatorname{cf} X_\alpha = \omega_1$, in each of the following cases, the 1-segment X_α is stationary,*

(i) $J^- \cap [l(\alpha), \alpha) = \emptyset$,
(ii) $J^- \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $J^+ \cap (\alpha', \alpha] \neq \emptyset$,
(iii) $J^- \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^- \cap [\alpha', \alpha) \neq \emptyset$,
where $\alpha' = \max(J^- \cap [l(\alpha), \alpha))$ *in case* $J^- \cap [l(\alpha), \alpha) \neq \emptyset$.

(5) *for every* $\alpha < \gamma$ *with* $\alpha < \min\{\sup J^-, \sup J^+\}$, X_α *is countable,*
(6) *for every* $\alpha < \gamma$ *with* $\sup J^- \leq \alpha < \sup J^+$, X_α *is* 0*-boundedly* 0*-*ω_1*-compact,*
(7) *for every* $\alpha < \gamma$ *with* $\sup J^+ \leq \alpha < \sup J^-$, X_α *is* 1*-boundedly* 1*-*ω_1*-compact,*
(8) *for every* $\alpha < \gamma$ *with* $\max\{\sup J^-, \sup J^+\} \leq \alpha$, X_α *is boundedly* ω_1*-compact.*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$.

"if" part: Assume all clauses (1) - (8) but that there is an uncountable subset H of X with no cluster points. We may assume $|H| = \omega_1$. From (1), we may also assume $J^- \subset \omega$. From (2), we have $J^+ \subset \omega_1$. By Corollary 3.7, the following claim completes the proof of this part.

Claim 1. For every $\alpha < \gamma$ and $y \in \prod_{\beta<\alpha} X_\beta$, $E(y, H)$ is countable.

Proof. First we prove the following fact.

Fact 1. For every $\alpha < \gamma$ with $\sup J^- \leq \alpha$, and for every $y \in \prod_{\beta<\alpha} X_\beta$ and $u \in X_\alpha$, $E(y, H) \cap (\leftarrow, u)_{X_\alpha}$ is countable

Proof. Assume that there are $\alpha < \gamma$ with $\sup J^- \leq \alpha$, $y \in \prod_{\beta<\alpha} X_\beta$ and $u \in X_\alpha$ such that $H_0 := E(y, H) \cap (\leftarrow, u)_{X_\alpha}$ is uncountable. We divide into 2 cases and their subcases. In each case, we we will get a contradiction.

Case 1. $\sup J^- \leq \alpha < \sup J^+$.

In this case, the clause (6) shows that H_0 has a 0-cluster point w in X_α. Then obviously $x := y\,^\frown\langle w\rangle^\frown\langle \min X_\beta : \alpha < \beta\rangle$ is a 0-cluster point of H, a contradiction.

Case 2. $\max\{\sup J^-, \sup J^+\} \leq \alpha$.

We further consider 2 subcases.

Case 2-1. $1\text{-}\mathrm{cf}_{X_\alpha} X_\alpha \neq \omega_1$ or $E(y, H)$ is 1-bounded in X_α.

In this case, we can find $u' \in X_\alpha$ with $H'_0 := H_0 \cap (u', \rightarrow)_{X_\alpha}$ is uncountable. It follows from $H'_0 \subset (u', u)$ and the clause (8) that H'_0 has a cluster point w. When w is a 0-cluster point of H'_0, $x := y\,^\frown\langle w\rangle^\frown\langle \min X_\beta : \alpha < \beta\rangle$ is a 0-cluster point of H, a contradiction. When w is a 1-cluster point of H'_0, $x := y\,^\frown\langle w\rangle^\frown\langle \max X_\beta : \alpha < \beta\rangle$ is a 1-cluster point of H, a contradiction.

Case 2-2. $1\text{-}\mathrm{cf}_{X_\alpha} X_\alpha = \omega_1$ and $E(y,H)$ is 1-unbounded in X_α.

We further divide into 2 subcases.

Case 2-2-1. Either one of cases (i), (ii) and (iii) of the clause (4) holds.

Since by the clause (4), the 1-segment X_α is stationary, the analogous result of Lemma 3.9 ensures the existence of a 1-cluster point w of $E(y,H)$ in X_α. Then $x := y\,{}^\frown\langle w\rangle{}^\frown\langle \max X_\beta : \alpha < \beta\rangle$ is a 1-cluster point of H, a contradiction.

Case 2-2-2. None of cases (i), (ii) and (iii) of the clause (4) holds.

From the negation of (i), $\alpha' = \max(J^- \cap [l(\alpha),\alpha))$ is defined. Let $J^-(y) = \{\beta < \alpha : \exists u \in X_\beta(u < y(\beta))\}$, then we have $\alpha' \in J^-(y)$. Since $[\alpha',\alpha)$ is a subset of the finite set $[l(\alpha),\alpha)$, we can let $\alpha_1 = \max J^-(y)$. Then $l(\alpha) \le \alpha' \le \alpha_1 < \alpha$. From the negation of (ii) and $\sup J^+ \le \alpha$, we have $J^+ \cap (\alpha',\gamma) = \emptyset$. Also from the negation of (iii), we have $\alpha_1 \notin K^-$, thus $w := \max(\leftarrow, y(\alpha_1))_{X_{\alpha_1}}$ exists, that is, w is the immediate predecessor of $y(\alpha_1)$. Let $x = (y \restriction \alpha_1){}^\frown\langle w\rangle{}^\frown\langle \max X_\beta : \alpha_1 < \beta\rangle$. To complete this case, it suffices to see that x is a 1-cluster point of H (then we have a contradiction). To see this, let $x^* \in \hat{X}$ with $x <_{\hat{X}} x^*$ and $\beta_0 = \min\{\alpha < \gamma : x(\alpha) \ne x^*(\alpha)\}$. Note $\beta_0 \le \alpha_1$. When $\beta_0 < \alpha_1$, we have $\emptyset \ne \{z \in H : z \restriction \alpha = y\} \subset (x,x^*) \cap H$. So assume $\beta_0 = \alpha_1$, then note $y \restriction \alpha_1 = x \restriction \alpha_1 = x^* \restriction \alpha_1$ and $y(\alpha_1) \le x^*(\alpha_1)$. When $y(\alpha_1) < x^*(\alpha_1)$, we have $\emptyset \ne \{z \in H : z \restriction \alpha = y\} \subset (x,x^*) \cap H$. When $y(\alpha_1) = x^*(\alpha_1)$, by the 1-unboundedness of $E(y,H)$, taking $u' \in E(y,H)$ with $u' < x^*(\alpha)$ and $z \in H$ with $z \restriction (\alpha+1) = y\,{}^\frown\langle u'\rangle$, we see $z \in (x,x^*) \cap H$.

This completes the proof of Fact 1.

In fact, we have a stronger result:

Fact 2. For every $\alpha < \gamma$ with $\sup J^- \le \alpha$ and $y \in \prod_{\beta<\alpha} X_\alpha$, $E(y,H)$ is countable.

Proof. Assume that for some $\alpha < \gamma$ with $\sup J^- \le \alpha$ and $y \in \prod_{\beta<\alpha} X_\alpha$, $E(y,H)$ is uncountable. If X_α has a maximal element, then by letting $u = \max X_\alpha$ in Fact 1, we get a contradiction. If $0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha = \omega$ were true, then taking a 0-order preserving and 0-unbounded sequence $\{u_n : n \in \omega\}$ in X_α, we see $E(y,H) = \bigcup_{n\in\omega}(E(y,H) \cap (\leftarrow, u_n])$ so $E(y,H)$ is countable by Fact 1, a contradiction. We have seen $0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha \ge \omega_1$.

Whenever $0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha > \omega_1$ or $E(y,H)$ is 0-bounded in X_α, then we have $E(y,H) = E(y,H) \cap (\leftarrow, u)$ for some $u \in X_\alpha$, a contradiction to Fact 1. Whenever $0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha = \omega_1$ and $E(y,H)$ is 0-unbounded in X_α, then as in Case 2-2 in Fact 1 (using the clause (3) instead of (4)), we get a contradiction. This completes the proof of Fact 2.

Now, the following fact completes the proof of Claim 1.

Fact 3. For every $\alpha < \sup J^-$ and $y \in \prod_{\beta<\alpha} X_\alpha$, $E(y, H)$ is countable.

Proof. Assume that for some $\alpha < \sup J^-$ and $y \in \prod_{\beta<\alpha} X_\alpha$, $E(y, H)$ is uncountable. From the clause (5), we see $\sup J^+ \leq \alpha < \sup J^-$. If for some $u \in X_\alpha$, $E(y, H) \cap (u, \rightarrow)$ were uncountable, then by the clause (7), $E(y, H)$ has a 1-cluster point w in X_α, therefore $x := y\,{}^\wedge\langle w\rangle^\wedge\langle \max X_\beta : \alpha < \beta\rangle$ is a 1-cluster point of H in X, a contradiction. Thus $E(y, H) \cap (u, \rightarrow)$ is countable for every $u \in X_\alpha$. Then using an analogous argument in the proof of Fact 2, we see that $E(y, H)$ is countable, a contradiction. This completes the proof of Fact 3.

This completes the proof of Claim.1

"only if" part: Assume that X is ω_1-compact and fix $u_0(\alpha), u_1(\alpha) \in X_\alpha$ with $u_0(\alpha) < u_1(\alpha)$ for every $\alpha < \gamma$. We prove (1) - (8).

(1) Assuming both $J^- \not\subset \omega$ and $J^+ \not\subset \omega$, fix $\alpha_0 \in J^- \setminus \omega$ and $\alpha_1 \in J^+ \setminus \omega$. Let $Y_0 = \prod_{\alpha<\omega} X_\alpha$ and $Y_1 = \prod_{\omega\leq\alpha} X_\alpha$. Since $X = Y_0 \times Y_1$ (see [5, Lemma 1.5]) and Y_1 has neither a minimal element nor a maximal element, fixing $y_1 \in Y_1$, we see that $H := \{\langle y, y_1\rangle : y \in Y_0\}$ has no cluster points and $|H| = |Y_0| \geq 2^\omega \geq \omega_1$, a contradiction to ω_1-compactness ot X.

(2) Assuming $J^- \not\subset \omega_1$, let $\alpha_0 = \min(J^- \setminus \omega_1)$, $Y_0 = \prod_{\alpha<\omega_1} X_\alpha$ and $Y_1 = \prod_{\omega_1\leq\alpha} X_\alpha$. From the clause (1), we see $J^+ \subset \omega$ therefore Y_1 has no minimal elements but has a maximal element. Letting $x_\beta = \langle u_1(\alpha) : \alpha < \beta\rangle^\wedge\langle u_0(\alpha) : \beta \leq \alpha < \omega_1\rangle^\wedge\langle \max X_\alpha : \omega_1 \leq \alpha\rangle$ for every $\beta < \omega_1$, put $H = \{x_\beta : \beta < \omega_1\}$.

Claim 2. H has no cluster points.

Proof. Since H is a 0-order preserving sequence indexed by ω_1, obviously it has no 1-cluster points. Let $x \in X$. Since X_{α_0} has no minimal elements, we can fix $u \in X_{\alpha_0}$ with $u < x(\alpha_0)$. Letting $y = (x \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha_0, \gamma)))$, we see $H \cap (y, x) = \emptyset$. Therefore H has no 0-cluster points. This completes the proof of Claim 2.

Since X is ω_1-compact, we have a contradiction. We have shown $J^- \subset \omega_1$. "$J^+ \subset \omega_1$" is similar.

(3) Assume that $\sup J^- \leq \alpha_0$ and $0\text{-}\operatorname{cf} X_{\alpha_0} = \omega_1$, but the 0-segment X_{α_0} is not stationary. It follows from Lemma 3.9 that there is a 0-unbounded subset K of X_{α_0} with no cluster points. Because of

$0\text{-}\operatorname{cf} X_{\alpha_0} = \omega_1$, we may assume that K is a 0-order preserving and 0-unbounded sequence $\{u_\beta : \beta < \omega_1\}$ in X_{α_0} with $(\leftarrow, u_0) \neq \emptyset$. In each cases (i), (ii) and (iii) of (3), we will get a contradiction.

(i) $J^+ \cap [l(\alpha_0), \alpha_0) = \emptyset$.

We consider 2 cases.

Case 1. $l(\alpha_0) = 0$.

In this case, letting $x_\beta = \langle \max X_\alpha : \alpha < \alpha_0 \rangle ^\wedge \langle u_\beta \rangle ^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$ for every $\beta < \omega_1$, put $H = \{x_\beta : \beta < \omega_1\}$. Since H is 0-order preserving, it has no 1-cluster points. To see that H has no 0-cluster points, let $x \in X$. Whenever $x \restriction \alpha_0 < \langle \max X_\alpha : \alpha < \alpha_0 \rangle$ or "$x \restriction \alpha_0 = \langle \max X_\alpha : \alpha < \alpha_0 \rangle$ and $x(\alpha_0) \leq u_0$", obviously we see $H \cap (\leftarrow, x) = \emptyset$. So let us consider the case "$x \restriction \alpha_0 = \langle \max X_\alpha : \alpha < \alpha_0 \rangle$ and $u_0 < x(\alpha_0)$". In this case, since K has no (0-)cluster points, we can take $u^* \in X^*_{\alpha_0}$ with $u^* < x(\alpha_0)$ and $K \cap (u^*, x(\alpha_0)) = \emptyset$. Then by letting $x^* = (x \restriction \alpha_0)^\wedge \langle u^* \rangle ^\wedge (x \restriction (\alpha_0, \gamma))$, we see $H \cap (x^*, x) = \emptyset$. So H is a uncountable subset of X with no cluster points, a contradiction to ω_1-compactness of X.

Case 2. $l(\alpha_0) > 0$.

In this case, letting $y = \langle u_0(\alpha) : \alpha < l(\alpha_0) \rangle ^\wedge \langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0 \rangle$ and $x_\beta = y\ ^\wedge \langle u_\beta \rangle ^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$ for every $\beta < \omega_1$, set $H = \{x_\beta : \beta < \omega_1\}$. Obviously H has no 1-cluster points. To see that H has no 0-cluster points, let $x \in X$. First consider the case "$x \restriction \alpha_0 > y$". In this case, let $\beta_0 = \min\{\alpha < \alpha_0 : x(\alpha) \neq y(\alpha)\}$. Noting $\beta_0 < l(\alpha_0)$, let $y' = \langle u_0(\alpha) : \alpha \leq \beta_0 \rangle ^\wedge \langle u_1(\alpha) : \beta_0 < \alpha < l(\alpha_0) \rangle ^\wedge \langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0 \rangle$. Then we have $y < y' < x \restriction \alpha_0$. Now by letting $z = y'\ ^\wedge (x \restriction [\alpha_0, \gamma))$, we see $z < x$ and $H \cap (z, x) = \emptyset$. Next in the case "$x \restriction \alpha_0 < y$", we obviously have $H \cap (\leftarrow, x) = \emptyset$. Finally consider the case "$x \restriction \alpha_0 = y$". Whenever $x(\alpha_0) \leq u_0$, we have $H \cap (\leftarrow, x) = \emptyset$. Whenever $x(\alpha_0) > u_0$, noting that K has no (0-)cluster points, by a similar argument in Case 1, we can find $x^* \in \hat{X}$ with $x^* < x$ and $H \cap (x^*, x) = \emptyset$. These arguments show that H is an uncountable subset of X with no cluster points, a contradiction.

(ii) $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $J^- \cap (\alpha_1, \alpha_0] \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

In this case, let $\alpha_2 = \max(J^- \cap (\alpha_1, \alpha_0])$, then note $l(\alpha_0) \leq \alpha_1 < \alpha_2 \leq \alpha_0$. We consider 2 cases.

Case 1. $\alpha_2 = \alpha_0$.

In this case, note that $\prod_{\alpha_0\le\alpha} X_\alpha$ has neither a minimal element nor a maximal element. Fixing $y \in \prod_{\alpha<\alpha_0} X_\alpha$, let $x_\beta = y\,{}^\frown\langle u_\beta\rangle{}^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$ for every $\beta < \omega_1$. Then $H := \{x_\beta : \beta < \omega_1\}$ has no 1-cluster points. To see that H has no 0-cluster points, let $x \in X$. When $x \restriction \alpha_0 < y$, obviously we have $H \cap (\leftarrow, x) = \emptyset$. When $x \restriction \alpha_0 > y$, taking $z \in \prod_{\alpha_0\le\alpha} X_\alpha$ with $z < x \restriction [\alpha_0, \gamma)$, we see $H \cap (x', x) = \emptyset$, where $x' = (x \restriction \alpha_0){}^\frown z$. When $x \restriction \alpha_0 = y$, as above in Case 1 or the final paragraph of Case 2 in (i), we see that x is not a 0-cluster point of H. So H is an uncountable subset of X with no cluster points, a contradiction.

Case 2. $\alpha_2 < \alpha_0$.

In this case, fixing $y \in \prod_{\alpha<\alpha_2} X_\alpha$, let $x_\beta = y\,{}^\frown\langle \max X_\alpha : \alpha_2 \le \alpha < \alpha_0\rangle{}^\frown\langle u_\beta\rangle{}^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$ for every $\beta < \omega_1$. Then by a similar argument as above, we see that $H := \{x_\beta : \beta < \omega_1\}$ has no cluster points, a contradiction.

(iii) $J^+ \cap [l(\alpha_0), \alpha_0) \ne \emptyset$ and $K^+ \cap [\alpha_1, \alpha_0) \ne \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

In this case, let $\alpha_2 = \max(K^+ \cap [\alpha_1, \alpha_0))$ and fix $u \in X_{\alpha_2}$ such that $(u, \rightarrow)$ is non-empty and has no minimal elements. Note $l(\alpha_0) \le \alpha_1 \le \alpha_2 < \alpha_0$. Fixing $y \in \prod_{\alpha<\alpha_2} X_\alpha$, let $x_\beta = y\,{}^\frown\langle u\rangle{}^\frown\langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0\rangle{}^\frown\langle u_\beta\rangle{}^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$ for every $\beta < \omega_1$. Obviously, $H := \{x_\beta : \beta < \omega_1\}$ has no 1-cluster points. To see that H has no 0-cluster points, let $x \in X$. When $x \restriction \alpha_0 < y\,{}^\frown\langle u\rangle{}^\frown\langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0\rangle$, we obviously see $H \cap (\leftarrow, x) = \emptyset$. When $x \restriction \alpha_0 = y\,{}^\frown\langle u\rangle{}^\frown\langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0\rangle$, as above in Case 1 or the final paragraph of Case 2 in (i), we see that x is not a 0-cluster point of H. Now we consider the remaining case "$x \restriction \alpha_0 > y\,{}^\frown\langle u\rangle{}^\frown\langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0\rangle$". In this case, since $x \restriction (\alpha_2+1) > y\,{}^\frown\langle u\rangle$ and $(y\,{}^\frown\langle u\rangle, \rightarrow)_{\prod_{\alpha\le\alpha_2} X_\alpha}$ has no minimal elements, taking $z \in \prod_{\alpha\le\alpha_2} X_\alpha$ with $x \restriction (\alpha_2+1) > z > y\,{}^\frown\langle u\rangle$ and $z' \in \prod_{\alpha_2<\alpha} X_\alpha$, we see $z\,{}^\frown z' < x$ and $H \cap (z\,{}^\frown z', x) = \emptyset$, a contradiction.
(4) is similar.

(5) Let $\alpha_0 < \min\{\sup J^-, \sup J^+\}$. Fix $\alpha_1 \in J^-$ and $\alpha_2 \in J^+$ with $\alpha_0 < \alpha_1$ and $\alpha_0 < \alpha_2$. Since X $(= \prod_{\alpha\le\alpha_0} X_\alpha \times \prod_{\alpha_0<\alpha} X_\alpha)$ is ω_1-compact and $\prod_{\alpha_0<\alpha} X_\alpha$ has neither minimal elements nor maximal elements, $\{\{x\} \times \prod_{\alpha_0<\alpha} X_\alpha : x \in \prod_{\alpha\le\alpha_0} X_\alpha\}$ is a discrete collection of non-empty open sets so $\prod_{\alpha\le\alpha_0} X_\alpha$ is countable. Therefore X_{α_0} is countable.

(6) Assume that $\sup J^- \le \alpha_0 < \sup J^+$ and X_{α_0} is not 0-boundedly 0-ω_1-compact. Take an uncountable 0-bounded subset K of X_{α_0} with no

0-cluster points. Fixing $y \in \prod_{\alpha<\alpha_0} X_\alpha$, consider $H := \{y \,^\frown\langle u\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle : u \in K\}$. First to see that H has no 0-cluster points, let $x \in X$ and take $u_0 \in X_{\alpha_0}$ with $u \leq u_0$ for every $u \in K$. When $x \upharpoonright \alpha_0 < y$ or "$x \upharpoonright \alpha_0 = y$ and $x(\alpha_0) = \min X_{\alpha_0}$", we have $H \cap (\leftarrow, x) = \emptyset$. When $x \upharpoonright \alpha_0 > y$, letting $x' = y \,^\frown\langle u_0\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$, we see $x' < x$ and $H \cap (x', x) = \emptyset$. When $x \upharpoonright \alpha_0 = y$ and $x(\alpha_0) \neq \min X_{\alpha_0}$, taking $u^* \in X^*_{\alpha_0}$ with $u^* <_{X^*_{\alpha_0}} x(\alpha_0)$ and $K \cap (u^*, x(\alpha_0))_{X^*_{\alpha_0}} = \emptyset$, we see $|H \cap ((x \upharpoonright \alpha_0)^\frown\langle u^*\rangle^\frown(x \upharpoonright (\alpha_0, \gamma)), x)_{\hat{X}}| \leq 1$. Therefore H has no 0-cluster points in X.

Next to see that H has no 1-cluster points, let $x \in X$ and fix $\alpha_1 \in J^+$ with $\alpha_0 < \alpha_1$. Also fix $u \in X_{\alpha_1}$ with $x(\alpha_1) < u$. Letting $x' = (x \upharpoonright \alpha_1) \,^\frown\langle u\rangle^\frown(x \upharpoonright (\alpha_1, \gamma))$, obviously we see $x < x'$ and $H \cap (x, x') = \emptyset$. Therefore H has no 1-cluster points in X. (7) is similar to (6).

(8) Assume that $\max\{\sup J^-, \sup J^+\} \leq \alpha_0$ and X_{α_0} is not boundedly ω_1-compact. Take an uncountable bounded subset K of X_{α_0} with no cluster points and fix $u_0, u_1 \in X_{\alpha_0}$ with $u_0 < u_1$ and $K \subset [u_0, u_1]_{X_{\alpha_0}}$. Fixing $y \in \prod_{\alpha<\alpha_0} X_\alpha$ and $z \in \prod_{\alpha>\alpha_0} X_\alpha$, consider $H := \{y \,^\frown\langle u\rangle^\frown z : u \in K\}$. To see that H has no 0-cluster points, let $x \in X$. When $x \upharpoonright (\alpha_0 + 1) \leq y \,^\frown\langle u_0\rangle$, obviously $|H \cap (\leftarrow, x)| \leq 1$. When $x \upharpoonright (\alpha_0 + 1) > y \,^\frown\langle u_1\rangle$, obviously $y \,^\frown\langle u_1\rangle^\frown\langle \max X_\alpha : \alpha_0 < \alpha\rangle < x$ and $H \cap (y \,^\frown\langle u_1\rangle^\frown\langle \max X_\alpha : \alpha_0 < \alpha\rangle, x) = \emptyset$. Let us consider the remaining case "$y \,^\frown\langle u_0\rangle < x \upharpoonright (\alpha_0+1) \leq y \,^\frown\langle u_1\rangle$", that is, $x \upharpoonright \alpha_0 = y$ and $u_0 < x(\alpha_0) \leq u_1$. Since K has no (0-)cluster points and $(\leftarrow, x(\alpha_0)) \neq \emptyset$, there is $u^* \in X^*_{\alpha_0}$ with $u^* < x(\alpha_0)$ such that $K \cap (u^*, x(\alpha_0)) = \emptyset$. Then we can check $y \,^\frown\langle u^*\rangle^\frown\langle \max X_\alpha : \alpha_0 < \alpha\rangle < x$ and $|H \cap (y \,^\frown\langle u^*\rangle^\frown\langle \max X_\alpha : \alpha_0 < \alpha\rangle, x)| \leq 1$. Similarly we can check that H has no 1-cluster points. $\square$

4. The Lindelöf property of lexicographic products

We have characterized "having countable closed 0(1)-cofinality" and ω_1-compactness of lexicographic products in the sections above. Also we have seen that there are some common or similar clauses in these characterizations. Using Lemma 1.4 and combining these characterizations, we can characterize the Lindelöfness of lexicographic products.

Theorem 4.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is Lindelöf if and only if the following clauses hold:*

(1) *$J^- \subset \omega$ or $J^+ \subset \omega$,*
(2) *$J^- \subset \omega_1$ and $J^+ \subset \omega_1$,*
(3) *for every ordinal $\alpha < \gamma$ with $\sup J^- \leq \alpha$, the following hold:*
  (a) *X_α has countable 0-bounded closed 0-cofinality,*

(b) *in each of the following cases,* $0\text{-}\operatorname{cf} X_\alpha \leq \omega$ *holds,*
   (i) $J^+ \cap [l(\alpha), \alpha) = \emptyset$,
   (ii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $J^- \cap (\alpha', \alpha] \neq \emptyset$,
   (iii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^+ \cap [\alpha', \alpha) \neq \emptyset$,
   where $\alpha' = \max(J^+ \cap [l(\alpha), \alpha))$ *in case* $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$.

(4) *for every ordinal* $\alpha < \gamma$ *with* $\sup J^+ \leq \alpha$, *the following hold:*
  (a) X_α *has countable* 1*-bounded closed* 1*-cofinality,*
  (b) *in each of the following cases,* $1\text{-}\operatorname{cf} X_\alpha \leq \omega$ *holds,*
   (i) $J^- \cap [l(\alpha), \alpha) = \emptyset$,
   (ii) $J^- \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $J^+ \cap (\alpha', \alpha] \neq \emptyset$,
   (iii) $J^- \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^- \cap [\alpha', \alpha) \neq \emptyset$,
   where $\alpha' = \max(J^- \cap [l(\alpha), \alpha))$ *in case* $J^- \cap [l(\alpha), \alpha) \neq \emptyset$.

(5) *for every* $\alpha < \gamma$ *with* $\alpha < \min\{\sup J^-, \sup J^+\}$, X_α *is countable,*
(6) *for every* $\alpha < \gamma$ *with* $\sup J^- \leq \alpha < \sup J^+$, X_α *is* 0*-boundedly* 0*-*ω_1*-compact,*
(7) *for every* $\alpha < \gamma$ *with* $\sup J^+ \leq \alpha < \sup J^-$, X_α *is* 1*-boundedly* 1*-*ω_1*-compact,*
(8) *for every* $\alpha < \gamma$ *with* $\max\{\sup J^-, \sup J^+\} \leq \alpha$, X_α *is boundedly* ω_1*-compact.*

Proof. "only if" part: Assume that X is Lindelöf, then by Lemma 1.4, all clauses in Lemma 2.2, 2.3 and 3.11 hold. So all clauses (1) - (8) of this theorem hold.

"if" part: Assume all clauses above. It suffices to see all clauses in Lemma 2.2, 2.3 and 3.11. Note that the clause (3b) of this theorem implies the clause (3) of Lemma 3.11. It suffices to see the following claim.

Claim. The following hold:

(1) the clauses (5) and (7) in this theorem imply the clause (3) of Lemma 2.2,
(2) the clauses (5) and (6) in this theorem imply the clause (3) of Lemma 2.3.

Proof. We only prove (1), since the other is similar. Let $\alpha_0 < \sup J^-$ and A_0 be a 0-segment of X_{α_0}. We will check $0\text{-}\operatorname{cf} A_0 \leq \omega$. Whenever $\alpha_0 < \sup J^+$, from (5), we obviously have $0\text{-}\operatorname{cf} A_0 \leq \omega$. Now let $\sup J^+ \leq \alpha_0 < \sup J^-$ and assume $\lambda := 0\text{-}\operatorname{cf} A_0 \geq \omega_1$. Take a 0-order preserving and 0-unbounded sequence $H := \{u_\beta : \beta < \lambda\}$ in A_0 with

$(\leftarrow, u_0) \neq \emptyset$. Then H has no 1-cluster points, a contradiction to (7) in this theorem. This completes the proof of Claim.

□

5. Applications

In this section, we apply the results in the previous sections. We first apply to the special case that all GO-spaces X_α's have both minimal and maximal elements.

Corollary 5.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and assume that all X_α's have both minimal and maximal elements, that is, $J^- = J^+ = \emptyset$.*

- (I) *X has countable closed 0-(1-)cofinality if and only if all X_α's have countable closed 0-(1-, respectively)cofinality,*
- (II) *X is ω_1-compact if and only if all X_α's are ω_1-compact,*
- (III) *X is Lindelöf if and only if all X_α's are Lindelöf.*

Proof. To (I), check all clauses in Lemma 2.2 (Lemma 2.3, respectively). Other clauses are similar (for (III), use Lemma 1.4). □

Note that one direction of (III) in the corollary above is Theorem 2.10 in [11], also the other direction gives an affirmative answer to (Q1) in the Question 1.1.

Example 5.2. From Corollary 5.1, we see the following.

1. The lexicographic products $([0,1]_{\mathbb{R}} \cup (2,3]_{\mathbb{R}})^\gamma$ and $[0,1]_{\mathbb{S}}^\gamma$ are Lindelöf for every ordinal γ.
2. The lexicographic product $(\omega_1 + (-\omega_1))^\gamma$ does not have countable closed 0-(1-)cofinality, but is ω_1-compact for every ordinal γ.

Next, we consider the case that all GO-spaces X_α's have neither minimal nor maximal elements. In the following corollary, note

$$\sup\gamma = \begin{cases} \gamma & \text{if } \gamma \text{ is limit,} \\ \gamma - 1 & \text{if } \gamma \text{ is successor,} \end{cases}$$

where $\gamma - 1$ denotes the immediate predecessor of a successor ordinal γ. In the following, remember that we are assuming $\gamma \geq 2$.

Corollary 5.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and assume that all X_α's have neither minimal nor maximal elements, that is, $J^- = J^+ = \gamma$.*

- (I) *X has countable closed 0-(1-)cofinality if and only if the following hold:*

(1) *$\gamma \leq \omega_1$,*
(2) *if γ is successor, then the following hold:*
    (a) *$X_{\gamma-1}$ has countable closed 0-(1-, respectively)cofinality,*
    (b) *for every $\alpha < \gamma - 1$, X_α has countable 0-(1-, respectively)cofinality,*
(3) *if γ is limit, then for every $\alpha < \gamma$, X_α has countable 0-(1-, respectively) cofinality,*

(II) *X is ω_1-compact if and only if the following hold:*
(1) *$\gamma \leq \omega$,*
(2) *if $\gamma < \omega$, then the following hold:*
    (a) *$X_{\gamma-1}$ is ω_1-compact,*
    (b) *for every $\alpha < \gamma - 1$, X_α is countable,*
(3) *if $\gamma = \omega$, then for every $\alpha < \gamma$, X_α is countable,*

(III) *X is Lindelöf if and only if the following hold:*
(1) *$\gamma \leq \omega$,*
(2) *if $\gamma < \omega$, then the following hold:*
    (a) *$X_{\gamma-1}$ is Lindelöf,*
    (b) *for every $\alpha < \gamma - 1$, X_α is countable,*
(3) *if $\gamma = \omega$, then for every $\alpha < \gamma$, X_α is countable.*

For the proof of (2a) in (II) above, use Lemma 3.10. Also for the proof of (2a) in (III) above, use Lemma 1.4. From this corollary, we see that whenever all X_α's have neither minimal nor maximal elements,

- if $\gamma = \omega$, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is ω_1-compact if and only if it is Lindelöf,
- if $\gamma > \omega$, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is not ω_1-compact,
- if $\prod_{\alpha<\gamma} X_\alpha$ is Lindelöf, then $\gamma \leq \omega$ and all X_α's are Lindelöf.

Example 5.4. From Corollary 5.3, we see the following.

(1) Each of the lexicographic products $(0,1)_{\mathbb{R}}^\gamma$, $((0,1]_{\mathbb{R}} \cup (2,3)_{\mathbb{R}})^\gamma$, $(0,1)_{\mathbb{S}}^\gamma$, $\mathbb{R}^\gamma$ and $\mathbb{S}^\gamma$ has countable closed 0-cofinality if and only if $\gamma \leq \omega_1$. Note that $(0,1)_{\mathbb{R}}$, $(0,1]_{\mathbb{R}} \cup (2,3)_{\mathbb{R}}$, $(0,1)_{\mathbb{S}}$, $\mathbb{R}$ and $\mathbb{S}$ are Lindelöf.
(2) Each of the lexicographic products $(0,1)_{\mathbb{R}}^\gamma$, $((0,1]_{\mathbb{R}} \cup (2,3)_{\mathbb{R}})^\gamma$, $(0,1)_{\mathbb{S}}^\gamma$, $\mathbb{R}^\gamma$ and $\mathbb{S}^\gamma$ is not ω_1-compact (if $\gamma \geq 2$).
(3) Each of the lexicographic products $\mathbb{Z}^\gamma$ and $\mathbb{Q}^\gamma$ has countable closed 0-cofinality if and only if $\gamma \leq \omega_1$, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the LOTS's of all integers and rationals respectively.
(4) Each of the lexicographic products $\mathbb{Z}^\gamma$ and $\mathbb{Q}^\gamma$ is ω_1-compact if and only if $\gamma \leq \omega$.

(5) The lexicographic product $((-\omega_1)+\omega_1)^2$ is not ω_1-compact, whereas $(-\omega_1)+\omega_1$ is ω_1-compact but does not have countable closed 0-cofinality.
(6) The lexicographic product $\mathbb{Z}\times((-\omega_1)+\omega_1)$ is ω_1-compact but not Lindelöf.

Finally, we consider the special case "$J^- = \emptyset$".

Corollary 5.5. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and assume that all X_α's have minimal elements, that is, $J^- = \emptyset$.*

(I)$_0$ *X has countable closed 0-cofinality if and only if the following hold:*
  (1) *for every $\alpha<\gamma$, X_α has countable 0-bounded closed 0-cofinality,*
  (2) *for every $\alpha<\gamma$, in each of the following cases, $0\text{-}\operatorname{cf} X_\alpha \le \omega$ holds,*
    (i) *$J^+\cap[l(\alpha),\alpha)=\emptyset$,*
    (ii) *$J^+\cap[l(\alpha),\alpha)\neq\emptyset$ and $K^+\cap[\alpha',\alpha)\neq\emptyset$,*
    where $\alpha'=\max(J^+\cap[l(\alpha),\alpha))$ in case $J^+\cap[l(\alpha),\alpha)\neq\emptyset$.

(I)$_1$ *X has countable closed 1-cofinality if and only if the following hold:*
  (1) *$J^+\subset\omega_1$,*
  (2) *for every $\alpha<\sup J^+$, X_α has countable 1-cofinality,*
  (3) *for every $\alpha<\gamma$ with $\sup J^+\le\alpha$, X_α has countable closed 1-cofinality,*

(II) *X is ω_1-compact if and only if the following hold:*
  (1) *$J^+\subset\omega_1$,*
  (2) *for every $\alpha<\gamma$ with $0\text{-}\operatorname{cf} X_\alpha=\omega_1$, in each of the following cases, the 0-segment X_α is stationary,*
    (i) *$J^+\cap[l(\alpha),\alpha)=\emptyset$,*
    (ii) *$J^+\cap[l(\alpha),\alpha)\neq\emptyset$ and $K^+\cap[\alpha',\alpha)\neq\emptyset$,*
    where $\alpha'=\max(J^+\cap[l(\alpha),\alpha))$ in case $J^+\cap[l(\alpha),\alpha)\neq\emptyset$.
  (3) *for every $\alpha<\sup J^+$, X_α is 0-boundedly 0-ω_1-compact,*
  (4) *for every $\alpha<\gamma$ with $\sup J^+\le\alpha$, X_α is boundedly ω_1-compact,*

(III) *X is Lindelöf if and only if the following hold:*
  (1) *$J^+\subset\omega_1$,*
  (2) *for every $\alpha<\gamma$,*
    (a) *X_α has countable 0-bounded closed 0-cofinality,*
    (b) *in each of the following cases, $0\text{-}\operatorname{cf} X_\alpha\le\omega$ holds,*

(i) $J^+ \cap [l(\alpha), \alpha) = \emptyset$,
(ii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^+ \cap [\alpha', \alpha) \neq \emptyset$, *where* $\alpha' = \max(J^+ \cap [l(\alpha), \alpha))$ *in case* $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$.

(3) *for every* $\alpha < \sup J^+$, X_α *is* 0*-boundedly* 0*-*ω_1*-compact,*
(4) *for every* $\alpha < \gamma$ *with* $\sup J^+ \leq \alpha$, X_α *has countable closed* 1*-cofinality and is boundedly* ω_1*-compact.*

The corollary above and Example 3.4 (3) & (4) give the following:

Corollary 5.6. *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be a lexicographic product of subspaces of ordinals.*

$(\mathrm{I})_0$ *X has countable closed* 0*-cofinality if and only if for every* $\alpha < \gamma$,
  (1) X_α *has countable* 0*-bounded closed* 0*-cofinality,*
  (2) *if* $J^+ \cap [l(\alpha), \alpha) = \emptyset$, *then* $0\text{-}\operatorname{cf} X_\alpha \leq \omega$ *holds,*

$(\mathrm{I})_1$ *X has countable closed* 1*-cofinality if and only if* $J^+ \subset \omega_1$,

(II) *X is* ω_1*-compact if and only if the following hold:*
  (1) $J^+ \subset \omega_1$,
  (2) *for every* $\alpha < \gamma$ *with* $0\text{-}\operatorname{cf} X_\alpha = \omega_1$, *if* $J^+ \cap [l(\alpha), \alpha) = \emptyset$, *then the* 0*-segment* X_α *is stationary,*
  (3) *for every* $\alpha < \gamma$, X_α *is boundedly* ω_1*-compact,*

(III) *X is Lindelöf if and only if the following hold:*
  (1) $J^+ \subset \omega_1$,
  (2) *for every* $\alpha < \gamma$,
    (a) X_α *has countable* 0*-bounded closed* 0*-cofinality,*
    (b) *if* $J^+ \cap [l(\alpha), \alpha) = \emptyset$, *then* $0\text{-}\operatorname{cf} X_\alpha \leq \omega$ *holds.*

Proof. $(\mathrm{I})_0$, $(\mathrm{I})_1$ and (II) follow from the corollary above. (III) follows from the fact that if a subspace of an ordinal has countable 0-bounded closed 0-cofinality, then it is boundedly ω_1-compact (modify the proof of Corollary 1.6). □

Corollary 5.6 (III) extends Theorem 3.2 and 3.3 in [11].

The following corollary generalizes Corollary 1.6.

Corollary 5.7. *A lexicographic product* $\prod_{\alpha<\gamma} X_\alpha$ *of subspaces of ordinals is Lindelöf if and only if it has both countable closed* 0*-cofinality and countable closed* 1*-cofinality. Moreover, whenever* $J^+ \subset \omega_1$, $\prod_{\alpha<\gamma} X_\alpha$ *is Lindelöf if and only if it has countable closed* 0*-cofinality.*

In particular, we see the following.

Corollary 5.8. *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be a lexicographic product of ordinals.*

(I)$_0$ *X has countable closed 0-cofinality if and only if for every $\alpha < \gamma$, if $J^+ \cap [l(\alpha), \alpha) = \emptyset$, then $\mathrm{cf} X_\alpha \leq \omega$ holds,*

(I)$_1$ *X has countable closed 1-cofinality if and only if $J^+ \subset \omega_1$,*

(II) *X is ω_1-compact if and only if $J^+ \subset \omega_1$,*

(III) *X is Lindelöf if and only if the following hold:*
  1. *$J^+ \subset \omega_1$,*
  2. *for every $\alpha < \gamma$, if $J^+ \cap [l(\alpha), \alpha) = \emptyset$, then $\mathrm{cf} X_\alpha \leq \omega$ holds,*

Thus we have:

Corollary 5.9. *A lexicographic product of ordinals has countable closed 1-cofinality if and only if it is ω_1-compact.*

When all X_α's are limit ordinals, we have:

Corollary 5.10. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of limit ordinals.*

(I)$_0$ *X has countable closed 0-cofinality if and only if for every $\alpha < \gamma$, if α is 0 or limit, then $\mathrm{cf} X_\alpha = \omega$ holds,*

(I)$_1$ *X has countable closed 1-cofinality if and only if $\gamma \leq \omega_1$,*

therefore X is Lindelöf if and only if $\gamma \leq \omega_1$ and for every $\alpha < \gamma$, if α is 0 or limit, then $\mathrm{cf} X_\alpha = \omega$ holds.

Example 5.11. From Corollaries above, we see the following.

1. The lexicographic product $[0,1)_{\mathbb{R}}^\gamma$ has countable closed 0-cofinality for every ordinal γ. But it has countable closed 1-cofinality (is ω_1-compact, is Lindelöf) if and only if $\gamma \leq \omega_1$. Also the lexicographic product $[0,1)_{\mathbb{R}}^{\omega_1} \times [0,1]_{\mathbb{R}}^{\omega_2}$ has countable closed 1-cofinality.
2. The lexicographic product $([0,1)_{\mathbb{R}} \times [0,1]_{\mathbb{R}})^\gamma$ has countable closed 0-cofinality for every ordinal γ. But it has countable closed 1-cofinality (is ω_1-compact, is Lindelöf) if and only if $\gamma \leq \omega_1$.
3. The lexicographic product $[0,1)_{\mathbb{R}}^{\omega_1} \times [0,1]_{\mathbb{R}}^{\omega_1}$ is Lindelöf. But the lexicographic product $[0,1]_{\mathbb{R}}^{\omega_1} \times [0,1)_{\mathbb{R}}^{\omega_1}$ is not Lindelöf.
4. The lexicographic product $[0,1)_{\mathbb{R}}^2$ is Lindelöf from (1). But the lexicographic product $[0,1)_{\mathbb{S}}^2$ is not ω_1-compact because $[0,1)_{\mathbb{S}}$ is not 0-boundedly 0-ω_1-compact (see (3) of Corollary 5.5 (II)).
5. The lexicographic product $[1,0)_{-\mathbb{S}}^2$ is Lindelöf (use Corollary 5.5 (III)), analogously $(0,1]_{\mathbb{S}}^2$ is Lindelöf.
6. The lexicographic product $(\omega \times \omega_1)^\gamma$ has countable closed 0-cofinality for every ordinal γ. But it has countable closed 1-cofinality if and only if $\gamma \leq \omega_1$. Thus it is Lindelöf (ω_1-compact) if and only if $\gamma \leq \omega_1$.

(7) The lexicographic product $(\omega_1 \times \omega)^\gamma$ does not have countable closed 0-cofinality for every ordinal $\gamma \geq 1$. But it has countable closed 1-cofinality (ω_1-compact) if and only if $\gamma \leq \omega_1$.

(8) The lexicographic products $(\omega \times \omega_1 \times (\omega_1 + 1))^{\omega_1}$ and $((\omega_1 + 1) \times \omega \times \omega_1)^{\omega_1}$ are also Lindelöf. But the lexicographic products $(\omega_1 \times \omega \times (\omega_1 + 1))^2$ and $((\omega_1 + 1) \times \omega_1 \times \omega)^2$ are not Lindelöf.

(9) The lexicographic product $\omega \times \omega_1 \times (\omega_1 + 1) \times \omega_1$ is Lindelöf. But the lexicographic product $\omega \times \omega_1 \times [0, 1]_{\mathbb{R}} \times \omega_1$ is not Lindelöf (use Corollary 5.5 (III) (2bii)).

(10) The lexicographic product $\prod_{\alpha<\omega_1} \omega_\alpha$ is Lindelöf, moreover the lexicographic product $\prod_{\alpha<\omega_1} \omega_\alpha \times \prod_{\omega_1 \leq \alpha < \omega_2} (\omega_\alpha + 1)$ is also Lindelöf. But the lexicographic products $\prod_{\alpha<\omega_1} \omega_{\alpha+1}$ and $\prod_{\alpha \leq \omega_1} \omega_\alpha$ are not Lindelöf (use Corollary 5.10).

(11) Let $\mathbb{L}(\omega_1)$ denote the lexicographic product $\omega_1 \times [0, 1)_{\mathbb{R}}$, which is called the Long line of the length ω_1, then obviously $\mathbb{L}(\omega_1)$ is ω_1-compact but not Lindelöf. Moreover the lexicographic product $\mathbb{L}(\omega_1)^\gamma$ is ω_1-compact if and only if $\gamma \leq \omega_1$ (apply Corollary 5.5(II) for $(\omega_1 \times [0, 1)_{\mathbb{R}})^\gamma$).

It is known in [8, Corollary 4.4] that if a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of GO-spaces is connected, where γ is a limit ordinal and for each $\alpha < \gamma$, X_α has a minimal element but does not have maximal elements, then all X_α's are not connected. In this connection, we would like to ask:

Question 5.12. In some special situations on a sequence $\{X_\alpha : \alpha < \gamma\}$ of GO-spaces, can the assumption that the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of GO-spaces is Lindelöf imply that all X_α's are not Lindelöf?

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