COMPLETENESS OF LEXICOGRAPHIC PRODUCTS OF GO-SPACES

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Abstract. Variations of Dedekind completeness of lexicographic products of GO-spaces are studied. As a corollary, we see that whenever γ is limit and all GO-spaces X_α's have minimal elements but have no maximal elements, connectedness of a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ implies *non*-connectedness of all X_α's.

1. Introduction

All spaces are assumed to be regular T_1 and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α are assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminology follow [9] and [1]. The following are well known:

- a LOTS X is compact iff every subset A of X has a least upper bound $\sup_X A$, where $\sup_X \emptyset$ is defined to be the minimal element $\min X$ of X, see [1, Problem 3.12.3(a)],
- a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of LOTS's is compact iff all X_α's are compact, see [2, Theorem 4.2.1].

Obviously a LOTS X is compact iff both of the following properties hold:

(a) every *non-empty* subset A of X has a least upper bound $\sup_X A$,
(b) every *non-empty* subset A of X has a greatest lower bound $\inf_X A$.

One might conjecture that if a LOTS X_α satisfies the property (a) for every $\alpha < \gamma$, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ also satisfies the property (a). But immediately we see that this conjecture is false, for instance, the lexicographic product $(0,1]_{\mathbb{R}} \times (0,1]_{\mathbb{R}}$ with $A = (0, \frac{1}{2})_{\mathbb{R}} \times (0,1]_{\mathbb{R}}$, where $(0,1]_{\mathbb{R}}$ denotes the half open interval $(0,1]$

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in the real line $\mathbb{R}$, is a counter example. Recently the notion of lexicographic products of GO-spaces is defined and discussed in [5, 6]. In this paper, we will define properties, so called 0-completeness and 1-completeness, on GO-spaces which are related to (a) and (b) respectively, and characterize when a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of GO-spaces has such properties. As corollaries, we see:

- whenever all X_α's have minimal and maximal elements, a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is 0-complete iff all X_α's are 0-complete,
- whenever γ is limit and all X_α's have minimal elements but have no maximal elements, a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is 0-complete iff $\gamma = \omega$ and all X_α's are ordinals, this yields:
- whenever γ is limit and all X_α's have minimal elements but have no maximal elements, connectedness of a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ implies *non*-connectedness of all X_α's.

A linearly ordered set $\langle L, <_L \rangle$ has a natural topology λ_L, which is called an interval topology, generated by $\{(\leftarrow, x)_L : x \in L\} \cup \{(x, \rightarrow)_L : x \in L\}$ as a subbase, where $(x, \rightarrow)_L = \{z \in L : x <_L z\}$, $(x, y)_L = \{z \in L : x <_L z <_L y\}$, $(x, y]_L = \{z \in L : x <_L z \leq_L y\}$ and so on. The triple $\langle L, <_L, \lambda_L \rangle$, which is simply denoted by L, is called a *LOTS*.

A triple $\langle X, <_X, \tau_X \rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X \rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subset C$ holds. The linearly ordered set $\langle X, <_X \rangle$ is called the underlying linearly ordered set of the GO-space X. Usually $<_L$, $(x, y)_L$, λ_L or τ_X are written simply $<$, (x, y), λ or τ if contexts are clear. For a GO-space X, X^+ (X^-) denotes the set $\{x \in X : (\leftarrow, x] \in \tau_X \setminus \lambda_X\}$ ($\{x \in X : [x, \rightarrow) \in \tau_X \setminus \lambda_X\}$). Obviously if $x \in X^+$, then $(x, \rightarrow)$ is non-empty and has no minimal element. Note that a GO-space X is a LOTS iff $X^+ \cup X^- = \emptyset$. The Sorgenfrey line $\mathbb{S}$ is known to be a GO-space but not a LOTS, where the underlying linearly ordered set of $\mathbb{S}$ is $\mathbb{R}$ and sets of type $[a, b)$ are declared to be open in $\mathbb{S}$. For more information on LOTS's or GO-spaces, see [10].

ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \cdots$, are considered to be LOTS's with the usual intereval topology.

For GO-spaces $X = \langle X, <_X, \tau_X \rangle$ and $Y = \langle Y, <_Y, \tau_Y \rangle$, X is said to be a *subspace* of Y if $X \subset Y$, the linear order $<_X$ is the restriction $<_Y \restriction X$ of the order $<_Y$ and the topology τ_X is the subspace topology

$\tau_Y \restriction X$ $(= \{U \cap X : U \in \tau_Y\})$ on X of the topology τ_Y. So a subset of a GO-space is naturally considered as a GO-space. For every GO-space X, there is a LOTS X^* such that X is a dense subspace of X^* and X^* has the property that if L is a LOTS containing X as a dense subspace, then L also contains the LOTS X^* as a subspace, see [11]. Such a X^* is called the *minimal d-extension of a GO-space* X. The construction of X^* is also shown in [5]. Obviously, we can see:

- if X is a LOTS, then $X^* = X$,
- X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

For every $\alpha < \gamma$, let X_α be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with $\langle x(\alpha) : \alpha < \gamma \rangle$. The lexicographic order $<_X$ on X is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha \rangle$ and $<_{X_\alpha}$ is the order on X_α. Now for every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace X of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X_\alpha^*$ is said to be the *lexicographic product of GO-spaces* X_α's, for more details see [5]. $\prod_{i\in\omega} X_i$ ($\prod_{i\leq n} X_i$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let X and Y be LOTS's. A map $f : X \to Y$ is said to be *order preserving* or 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be *order reversing* or 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map (also 1-order preserving map) $f : X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. Now let X and Y be GO-spaces. A 0-order preserving map $f : X \to Y$ is said to be 0-*order preserving embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we identify X with $f[X]$ as a GO-space and write $X = f[X]$.

Let X be a GO-space. A subset A of X is called a 0-*segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. A 0-*segment* A is said to be *bounded* if $X \setminus A$ is non-empty. Similarly the notion of (bounded) 1-segment can be defined. Both $\emptyset$ and X are 0-segments and 1-segments.

In our discussion, we mainly consider the following properties.

Definition 1.1. A GO-space X is said to be 0-*complete* (1-*complete*) if every non-empty bounded closed 0-segment (1-segment) has a maximal (minimal) element. When a GO-space X is 0-complete and 1-complete, it is simply called *complete*. Similarly a GO-space X is said to be 0-*compact* (1-*compact*) if every non-empty closed 0-segment (1-segment) has a maximal (minimal) element.

A linearly ordered set (or LOTS) X is called *Dedekind* 0-*complete* (*Dedekind* 1-*complete*) if every non-empty subset A having an upper bound (lower bound) has a least upper bound $\sup_X A$ (a greatest lower bound $\inf_X A$). A LOTS is *Dedekind complete* if it is both Dedekind 0-complete and Dedekind 1-complete.

Obviously a GO-space X is 0-compact iff it is 0-complete and has a maximal element $\max X$. We emphasize that Dedekind completeness is a property on LOTS's (or underlying linearly ordered sets of GO-spaces), on the other hand, completeness is a property on GO-spaces.

Remark 1.2. These notions above can be represented by using classical terms cut, gap, ... etc., as follows. An ordered pair $\langle A, B\rangle$ of open sets (equivalently, closed sets) in a GO-space X is said to be a *cut* if it satisfies

(i) $X = A \cup B$ and

(ii) $\forall a \in A\ \forall b \in B(a < b)$.

A cut $\langle A, B\rangle$ is a *gap* if it satisfies

(iii) A has no maximal element, and B has no minimal element.

Furthermore a gap $\langle A, B\rangle$ is an *internal gap* if it satisfies

(iv) $A \neq \emptyset$ and $B \neq \emptyset$.

A cut $\langle A, B\rangle$ with (iv) is a *pseudogap from left* if it also satisfies

(v) A has no maximal element, but B has a minimal element.

A pseudogap from right can be similarly defined. Now it is easy to see that a GO-space X is 0-complete in our sense if and only if there exist neither internal gaps nor pseudogaps from left.

Recently using our notation, (hereditary) paracompactness, countable compactness, the weight ... and so on of lexicographic products have been discussed, see [3, 4, 5, 6, 7, 8].

First we clarify the relationship between 0-completeness and Dedekind 0-completeness.

Lemma 1.3. *Let X be a GO-space. Then the following are equivalent*

(1) X *is* 0-*complete,*
(2) *the following properties hold:*
    (a) $X^- = \emptyset$,

(b) *the underlying linearly ordered set of* X *is Dedekind* 0-*complete.*

Proof. (1) $\Rightarrow$ (2) Assume (1). (a) is obvious, because if $x \in X^-$ then $(\leftarrow, x)$ is a non-empty bounded closed 0-segment of X with no maximal element.
(b) If there were a non-empty subset C having an upper bound such that C does not have a least upper bound, then $A := \{x \in X : \exists c \in C(x \leq c)\}$ is a non-empty closed 0-segment with no maximal element.

(2) $\Rightarrow$ (1) Assuming that X is not 0-complete, take a non-empty bounded closed 0-segment A with no maximal element. It follows from (2b) that $\sup_X A$ exists and obviously $\sup_X A \notin A$. Since A is closed, take $x \in X^*$ with $x <_{X^*} \sup_X A$ and $A \cap ((x, \rightarrow)_{X^*} \cap X) = \emptyset$. By (2a), we have $\sup_X A \notin X^-$, so we may assume $x \in X$. Then we have $x \in A$ by $x < \sup_X A$. Take $a \in A$ with $x < a$, then we see $a \in A \cap ((x, \rightarrow)_{X^*} \cap X)$, a contradiction. □

This lemma shows that whenever X is a LOTS, 0-completeness of X is equivalent to Dedekind 0-completeness. So with the analogous result of Lemma 1.3, we see:

Corollary 1.4. *The following hold:*

(1) *a GO-space* X *is complete iff it is a Dedekind complete LOTS,*
(2) *a GO-space* X *is* 0-*compact and* 1-*compact iff it is a compact LOTS.*

Since the underlying linearly ordered set of a subspace of an ordinal is Dedekind complete, the corollary above yields:

Corollary 1.5. *A subspace of an ordinal is complete iff it is a LOTS. In particular, ordinals are complete.*

In our discussion, we will consider 0-completeness and 1-completeness separately.

Example 1.6. The following hold:

(1) the LOTS $(0, 1]_{\mathbb{R}}$ is a 0-compact and 1-complete, but not 1-compact,
(2) the Sorgenfrey line (where intervals of type $[a, b)_{\mathbb{R}}$ are declared to be open) is 1-complete, but neither 0-complete nor 1-compact. Note that $\mathbb{S}$ is a GO-space but not a LOTS.
(3) ω_1 is 0-complete and 1-compact, but $-\omega_1$ is 1-complete and 0-compact, where $-X$ denotes the GO-space $\langle X, >_X, \tau_X \rangle$ for a GO-space $X = \langle X, <_X, \tau_X \rangle$, see [6],

(4) the subspace $\omega_1 \setminus \{\omega\}$ of ω_1 is 1-compact but not 0-complete, thus it is not a LOTS. Note that the underlying linearly ordered set of $\omega_1 \setminus \{\omega\}$, which can be identified with ω_1, is Dedekind complete.

2. Characterizations

In this section, we characterize 0-(1-)completeness of lexicographic products of GO-spaces. First we consider a special case.

Lemma 2.1. *Let $X = X_0 \times X_1$ be a lexicographic product of two GO-spaces. Then X is 0-complete iff the following clauses hold:*

(1) *both X_0 and X_1 are 0-complete,*
(2) *X_1 has a minimal element or a maximal element,*
(3) *if X_1 has no minimal element, then for every $u \in X_0$ with $(\leftarrow, u) \neq \emptyset$, $(\leftarrow, u)$ has a maximal element,*
(4) *if X_1 has no maximal element, then for every $u \in X_0$ with $(u, \rightarrow) \neq \emptyset$, $(u, \rightarrow)$ has a minimal element.*

Proof. Set $\hat{X} = X_0^* \times X_1^*$. To see one direction, let X be 0-complete.
(1) Assuming that X_0 is not 0-complete, let A_0 be a non-empty bounded closed 0-segment of X_0 with no maximal element. Let $A = A_0 \times X_1$, obviously it is a non-empty bounded 0-segment in X with no maximal element.

Claim 1. A is closed.

Proof. Let $x \in X \setminus A$. Since $x(0) \notin A_0$, $(\leftarrow, x(0)) \neq \emptyset$ and A_0 is closed, there is $u^* \in X_0^*$ with $u^* < x(0)$ such that $A_0 \cap ((u^*, \rightarrow)_{X_0^*} \cap X_0) = \emptyset$. Fixing $v \in X_1$, let $U = (\langle u^*, v\rangle, \rightarrow)_{\hat{X}} \cap X$. Then U is a neighborhood of x in X. To see $U \cap A = \emptyset$, assume $a \in U \cap A$ for some a. Since $u^* \leq a(0) \in A_0$ and A_0 has no maximal element, take $u \in A_0$ with $a(0) < u$. Then we have $u \in A_0 \cap ((u^*, \rightarrow)_{X_0^*} \cap X_0)$, a contradiction. So U is a neighborhood of x disjoint from A. □

Next assuming that X_1 is not 0-complete, let A_1 be a non-empty bounded closed 0-segment of X_1 with no maximal element. Fixing $u \in X_0$, let $A = \{x \in X : \exists v \in A_1 (x \leq \langle u, v\rangle\}$. Obviously it is a non-empty bounded 0-segment with no maximal element. Now the following claim contradicts 0-completeness of X.

Claim 2. A is closed.

Proof. Let $x \in X \setminus A$. Since the 0-segment A_1 is bounded, fix $v \in X_1 \setminus A_1$. Whenever $\langle u, v\rangle < x$, $(\langle u, v\rangle, \rightarrow)_X$ is a neighborhood of x disjoint from A. So let $x \leq \langle u, v\rangle$, then by $x \notin A$, we have $x(0) = u$

and $x(1) \notin A_1$. Since A_1 is closed, there is $v^* \in X_1^*$ with $v^* <_{X_1^*} x(1)$ such that $A_1 \cap ((v^*, \rightarrow)_{X_1^*} \cap X_1) = \emptyset$. Then $(\langle u, v^* \rangle, \rightarrow)_{\hat{X}} \cap X$ is a neighborhood of x disjoint from A. □

(2) Assume that X_1 has neither a minimal nor a maximal element. Fixing $u \in X_0$ with $(\leftarrow, u)_{X_0} \neq \emptyset$, let $A = (\leftarrow, u) \times X_1$. Then A is a non-empty bounded closed 0-segment of X with no maximal element, a contradiction.
(3) Assume that X_1 has no minimal element but there is $u \in X_0$ with $(\leftarrow, u) \neq \emptyset$ such that $(\leftarrow, u)$ has no maximal element. Let $A = (\leftarrow, u) \times X_1$. Then A is a non-empty bounded closed 0-segment of X with no maximal element, a contradiction.
(4) Assume that X_1 has no maximal element but there is $u \in X_0$ with $(u, \rightarrow) \neq \emptyset$ such that $(u, \rightarrow)$ has no minimal element. Let $A = (\leftarrow, u] \times X_1$. Then A is a non-empty bounded closed 0-segment of X with no maximal element, a contradiction.

To see the other direction, assuming the clauses (1)-(4) and that X is not 0-complete, take a non-empty bounded closed 0-segment A of X with no maximal element. Let $A_0 = \{u \in X_0 : \exists v \in X_1(\langle u, v \rangle \in A)\}$. Obviously A_0 is a non-empty 0-segment of X_0. We consider 2 cases, and in both cases we will get contradictions.

Case 1. A_0 has a maximal element.

In this case, let $A_1 = \{v \in X_1 : \langle \max A_0, v \rangle \in A\}$. Then A_1 is a non-empty 0-segment of X_1. If $A_1 \neq X_1$ were true, then A_1 is a non-empty bounded in X_1 and $\{\max A_0\} \times A_1$ is a 1-segment (i.e., final segment) of the 0-segment A. So A_1 is closed in X_1 and has no maximal element, which contradicts (1). So we have $A_1 = X_1$, therefore $A = (\leftarrow, \max A_0] \times X_1$. Since the 0-segment A is bounded and has no maximal element, the 0-segment A_0 is bounded in X_0 and X_1 has no maximal element. So by $(\max A_0, \rightarrow) \neq \emptyset$ with the condition (4), $\min(\max A_0, \rightarrow)$ exists. Moreover the condition (2) shows that X_1 has a minimal element. Now we have $\min(X \setminus A) = \langle \min(\max A_0, \rightarrow), \min X_1 \rangle \notin A$. Since A is closed in X, there is $\langle u^*, v^* \rangle \in \hat{X}$ with $\langle u^*, v^* \rangle < \langle \min(\max A_0, \rightarrow), \min X_1 \rangle$ such that $A \cap (((\langle u^*, v^* \rangle, \rightarrow)_{\hat{X}} \cap X) = \emptyset$. Since $\min X_1 \leq v^*$, $u^* < \min(\max A_0, \rightarrow)$ has to be true. Moreover since X_0 is dense in X_0^*, we have $u^* \leq \max A_0$. If $u^* < \max A_0$ were true, then $\langle \max A_0, \min X_1 \rangle \in A \cap (\langle u^*, v^* \rangle, \rightarrow)_{\hat{X}} \cap X$, a contradiction. So we have $u^* = \max A_0$. Since X_1 has no maximal element, take $v \in X_1$ with $v^* < v$. Then $\langle \max A_0, v \rangle \in A \cap (((\langle u^*, v^* \rangle, \rightarrow)_{\hat{X}} \cap X)$, a contradiction.

Case 2. A_0 has no maximal element.

In this case, we have:

Claim 3. $A = A_0 \times X_1$.

Proof. $A \subset A_0 \times X_1$ is obvious. To see $A \supset A_0 \times X_1$, let $x \in A_0 \times X_1$. Since A_0 has no maximal element, one can take $u \in A_0$ with $x(0) < u$. Then for some $v \in X_1$, we have $\langle u, v \rangle \in A$. Since A is a 0-segment and $x < \langle u, v \rangle \in A$, we have $x \in A$. □

Since A is bounded, the 0-segment A_0 is also bounded in X_0. If $X_0 \setminus A_0$ have no minimal element, then the 0-segment A_0 is closed, which contradicts (1). So $X_0 \setminus A_0$ has a minimal element. The following claim yields a contradiction to (3) with $u = \min(X_0 \setminus A_0)$.

Claim 4. X_1 has no minimal element.

Proof. Assume that X_1 has a minimal element. Since A is closed and $\langle \min(X_0 \setminus A_0), \min X_1 \rangle \notin A$, there is $\langle u^*, v^* \rangle \in \hat{X}$ with $\langle u^*, v^* \rangle < \langle \min(X_0 \setminus A_0), \min X_1 \rangle$ such that $A \cap ((\langle u^*, v^* \rangle, \rightarrow)_{\hat{X}} \cap X) = \emptyset$. Then $(u^*, \rightarrow)_{X_0^*} \cap X_0$ is a neighborhood of $\min(X_0 \setminus A_0)$ disjoint from A_0, which shows that A_0 is closed in X_0. So A_0 witnesses non-0-completeness of X_0, which contradicts (1). □

□

Remark that 0-compactness is equivalent to 0-completeness + the existence of a maximal element and also that $X_0 \times X_1$ has a maximal element iff both X_0 and X_1 have maximal elements. So we have:

Lemma 2.2. *Let $X = X_0 \times X_1$ be a lexicographic product of two GO-spaces. Then X is 0-compact iff the following clauses hold:*

1. *both X_0 and X_1 are 0-compact,*
2. *if X_1 has no minimal element, then for every $u \in X_0$ with $(\leftarrow, u) \neq \emptyset$, $(\leftarrow, u)$ has a maximal element.*

Replacing 0-, minimal, maximal, $(\leftarrow, u)$ and $(u, \rightarrow)$ by 1-, maximal, minimal, $(u, \rightarrow)$ and $(\leftarrow, u)$ respectively, we have analogous results of Lemma 2.1 and 2.2.

Example 2.3. Applying the lemmas above, we see:

1. $\mathbb{R} \times \mathbb{I}$ is complete, but $\mathbb{I} \times \mathbb{R}$ is neither 0-complete nor 1-complete, where $\mathbb{I} = [0, 1]_{\mathbb{R}}$,
2. $\mathbb{S} \times \mathbb{I}$ and $\mathbb{S} \times [0, 1]_{\mathbb{S}}$ are 1-complete, where $[0, 1]_{\mathbb{S}}$ denotes the subspace $[0, 1]$ of $\mathbb{S}$,
3. $(0, 1]_{\mathbb{R}} \times \mathbb{I}$ and $(-\omega_1) \times (0, 1]_{\mathbb{R}}$ are 0-compact, but $(0, 1)^2_{\mathbb{R}}$ and $\omega_1 \times (0, 1]_{\mathbb{R}}$ are not 0-complete,
4. $\omega_1 \times \mathbb{I}$, $\omega_1 \times [0, 1)_{\mathbb{R}}$ and $(\omega_1 + 1) \times [0, 1)_{\mathbb{R}}$ are complete,

(5) $\mathbb{I} \times \omega_1$ and $[0,1)_{\mathbb{R}} \times \omega_1$ are neither 0-complete nor 1-complete, but $[0,1)_{\mathbb{R}} \times (\omega_1 + 1)$ is complete.

Next, we consider general cases.

Definition 2.4. Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. We use the following notation in [5].

$$J^+ = \{\alpha < \gamma : \ X_\alpha \text{ has no maximal element } \},$$
$$J^- = \{\alpha < \gamma : \ X_\alpha \text{ has no minimal element } \},$$

Theorem 2.5. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is 0-complete iff the following clauses hold:*

(1) *for every $\alpha < \gamma$, X_α is 0-complete,*
(2) *$J^- \subset \{0\}$ or $J^+ \subset \{0\}$,*
(3) *$J^- \cup J^+ \subset \omega$,*
(4) *for every $\alpha < \sup J^-$ and $u \in X_\alpha$, if $(\leftarrow, u) \neq \emptyset$, then $(\leftarrow, u)$ has a maximal element,*
(5) *for every $\alpha < \sup J^+$ and $u \in X_\alpha$, if $(u, \rightarrow) \neq \emptyset$, then $(u, \rightarrow)$ has a minimal element,*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$. First assuming that X is 0-complete, we will see the clauses (1) - (5).
(1) Assuming that (1) is false, fix $\alpha_0 < \gamma$ such that X_{α_0} is not 0-complete, moreover let $Y_0 = \prod_{\alpha\leq\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0<\alpha} X_\alpha$. Since $X = Y_0 \times Y_1$ (see [5, Lemma 1.5]) and X is 0-complete, from Lemma 2.1, we see that Y_0 is 0-complete. Now since $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$, again by Lemma 2.1, we see that X_{α_0} is 0-complete.
(2) Assume $J^- \setminus \{0\} \neq \emptyset$ and $J^+ \setminus \{0\} \neq \emptyset$. Then $\prod_{0<\alpha} X_\alpha$ has neither a minimal nor a maximal elements. Since X $(= X_0 \times \prod_{0<\alpha} X_\alpha)$ is 0-complete, we get a contradiction from Lemma 2.1 (2).
(3) Assume $J^- \cup J^+ \not\subset \omega$. We may assume $J^- \not\subset \omega$. Let $\alpha_0 = \min(J^- \setminus \omega)$ and fix $u \in \prod_{\alpha<\omega} X_\alpha$ with $(\leftarrow, u(\alpha))_{X_\alpha} \neq \emptyset$ for every $\alpha < \omega$. Since X $(= \prod_{\alpha\leq\alpha_0} X_\alpha \times \prod_{\alpha_0<\alpha} X_\alpha)$ is 0-complete, by Lemma 2.1 (1), $\prod_{\alpha\leq\alpha_0} X_\alpha$ $(= \prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0})$ is 0-complete. Let $z = u \,^\frown\langle \min X_\alpha : \omega \leq \alpha < \alpha_0\rangle$, that is, $z \in \prod_{\alpha<\alpha_0} X_\alpha$ with $z(\alpha) = u(\alpha)$ in case $\alpha < \omega$ and $z(\alpha) = \min X_\alpha$ in case $\omega \leq \alpha < \alpha_0$. Then $(\leftarrow, z) \neq \emptyset$ and it has no maximal element. Since X_{α_0} has no minimal element, it contradicts Lemma 2.1 (3).
(4) Let $\alpha_0 < \sup J^-$ and

$$\alpha_1 = \min\{\alpha > \alpha_0 : \ X_\alpha \text{ has no minimal element.}\}.$$

Then for each $\alpha \in (\alpha_0, \alpha_1)$, X_α has a minimal element. Assume that for some $u_0 \in X_{\alpha_0}$ with $(\leftarrow, u_0) \neq \emptyset$, $(\leftarrow, u_0)$ has no maximal element.

By Lemma 2.1 (1), $\prod_{\alpha\le\alpha_1} X_\alpha$ is 0-complete. Fixing $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$, let $z_0 = y_0 {}^\wedge\langle u_0\rangle^\wedge\langle \min X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle$. Then obviously $z_0 \in \prod_{\alpha<\alpha_1} X_\alpha$ and $(\leftarrow, z_0)$ is non-empty having no maximal element, which contradicts Lemma 2.1 (3). (5) is similar to (4).

To see the other direction. Assume that the clauses (1) - (5) holds, but X is not 0-complete. Then we can fix a non-empty bounded closed 0-segment A in X with no maximal element. Let $B = X \setminus A$. Note $B \neq \emptyset$.

Claim 1. B has no minimal element.

Proof. Assume that B has a minimal element, say $b = \min B$. Since A is closed and $b \notin A$, there is $x^* \in \hat{X}$ with $x^* <_{\hat{X}} b$ and $(x^*, b)_{\hat{X}} = \emptyset$. Let $\alpha_0 = \min\{\alpha < \gamma : x^*(\alpha) \neq b(\alpha)\}$. Then $(x^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} = \emptyset$, otherwise, taking $u \in (x^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}}$, $(b \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(b \restriction (\alpha_0, \gamma)) \in (x^*, b)_{\hat{X}}$, a contradiction. Therefore we have $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is equal to $(\leftarrow, x^*(\alpha_0)]_{X^*_{\alpha_0}} \cap X_{\alpha_0}$, so it is a non-empty bounded closed 0-segment in X_{α_0}. Since X_{α_0} is 0-complete, from (1), $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ has a maximal element $u \in X_{\alpha_0}$. Also by $(x^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} = \emptyset$, we see $u = x^*(\alpha_0)$.

Fact. For every $\alpha > \alpha_0$, X_α has a maximal element and $x^*(\alpha) = \max X_\alpha$.

Proof. Otherwise, set $\alpha_1 = \min\{\alpha > \alpha_0 : \exists u \in X_\alpha(x^*(\alpha) < u)\}$ and take $u_1 \in X_{\alpha_1}$ with $x^*(\alpha_1) < u_1$. Then we have $(x^* \restriction \alpha_1)^\wedge\langle u_1\rangle^\wedge(x^* \restriction (\alpha_1, \gamma)) \in (x^*, b)_{\hat{X}}$, a contradiction. □

This fact with $x^*(\alpha_0) = u \in X_{\alpha_0}$ shows $x^* \in X$, so $A = (\leftarrow, b)_X$ has a maximal element x^*, a contradiction. This completes the proof of Claim 1. □

Now let $I = \{\alpha < \gamma : \exists a \in A \exists b \in B(a \restriction (\alpha+1) = b \restriction (\alpha+1))\}$. Since I is a 0-segment (= initial segment) of γ, I is equal to α_0 for some $\alpha_0 \le \gamma$. For every $\alpha < \alpha_0$, fix $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$. Letting $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0\le\alpha} X_\alpha$, define $y_0 \in Y_0$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$. These arguments below are somewhat similar to the arguments in [6, Theorem 4.8]. But since there are some technical differences, we will give their details.

Claim 2. For every $\alpha < \alpha_0$, $y_0 \restriction (\alpha+1) = a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$.

Proof. Assuming $y_0 \restriction (\alpha+1) \neq a_\alpha \restriction (\alpha+1)$ for some $\alpha < \alpha_0$, put $\alpha_1 = \min\{\alpha < \alpha_0 : y_0 \restriction (\alpha+1) \neq a_\alpha \restriction (\alpha+1)\}$ and $\alpha_2 = \min\{\alpha \le \alpha_1 : y_0(\alpha) \neq a_{\alpha_1}(\alpha)\}$. Then $\alpha_2 < \alpha_1$ and $y_0 \restriction (\alpha_2+1) = a_{\alpha_2} \restriction (\alpha_2+1) = b_{\alpha_2} \restriction (\alpha_2+1)$ holds. When $y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)$, we see $B \ni b_{\alpha_2} < a_{\alpha_1} \in A$, a contradiction. When $y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)$, we see $B \ni b_{\alpha_1} < a_{\alpha_2} \in A$, a contradiction. The second equality follows from the definitions of a_α and b_α. $\square$

Claim 3. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then we may assume $y_0 \in A$. Since A has no maximal element, fix $a \in A$ with $y_0 < a$. Letting $\beta_0 = \min\{\alpha < \gamma : y_0(\alpha) \neq a(\alpha)\}$, we see $B \ni b_{\beta_0} < a \in A$, a contradiction. $\square$

Claim 4. The following hold:

(1) for every $a \in A$, $a \restriction \alpha_0 \leq y_0$ holds,
(2) for every $b \in B$, $b \restriction \alpha_0 \geq y_0$ holds,
(3) for every $x \in X$, if $x \restriction \alpha_0 < y_0$ holds, then $x \in A$,
(4) for every $x \in X$, if $x \restriction \alpha_0 > y_0$ holds, then $x \in B$.

Proof. (1) Assuming $a \restriction \alpha_0 > y_0$ for some $a \in A$, let $\beta_0 = \min\{\alpha < \alpha_0 : a(\alpha) \neq y_0(\alpha)\}$. Then we see $B \ni b_{\beta_0} < a \in A$, a contradiction. (2) is similar.
(3) Assuming $x \restriction \alpha_0 < y_0$, let $\beta_0 = \min\{\alpha < \alpha_0 : x(\alpha) \neq y_0(\alpha)\}$. Then we see $x < a_{\beta_0} \in A$. Since A is a 0-segment, we have $x \in A$. (4) is similar. $\square$

Let $A_0 = \{a(\alpha_0) : a \in A, a \restriction \alpha_0 = y_0\}$ and $B_0 = \{b(\alpha_0) : b \in B, b \restriction \alpha_0 = y_0\}$.

Claim 5. A_0 is a 0-segment of X_{α_0} and $B_0 = X_{\alpha_0} \setminus A_0$.

Proof. Letting $u' < u \in A_0$, take $a \in A$ with $a \restriction (\alpha_0 + 1) = y_0 {}^\frown\langle u\rangle$. Set $a' = (a \restriction \alpha_0)^\frown\langle u'\rangle^\frown(a \restriction (\alpha_0, \gamma))$, then $a' < a \in A$. Since A is a 0-segment, we see $a' \in A$ thus $u' \in A_0$. So A_0 is a 0-segment of X_{α_0}.

To see $B_0 \subset X_{\alpha_0} \setminus A_0$, let $u \in B_0$. Take $b \in B$ with $b \restriction (\alpha_0 + 1) = y_0 {}^\frown\langle u\rangle$. If $u \in A_0$ were true, then one can take $a \in A$ with $a \restriction (\alpha_0 + 1) = y_0 {}^\frown\langle u\rangle$ $(= b \restriction (\alpha_0 + 1))$, thus we have $\alpha_0 \in I = \alpha_0$, a contradiction. So we have $u \in X_{\alpha_0} \setminus A_0$.

To see $B_0 \supset X_{\alpha_0} \setminus A_0$, let $u \in X_{\alpha_0} \setminus A_0$. Take $x \in X$ with $x \restriction (\alpha_0 + 1) = y_0 {}^\frown\langle u\rangle$. Then obviously $x \in B$ has to be true, so we have $u \in B_0$. $\square$

Claim 6. $A_0 \neq \emptyset$.

Proof. Assume $A_0 = \emptyset$. Then from Claim 4, we see $(\leftarrow, y_0) \times Y_1 = A$, so we have $(\leftarrow, y_0) \neq \emptyset$. If $\alpha_0 = 0$ were true, then by taking $a \in A$, we see $a(\alpha_0) \in A_0$, a contradiction. So we have $\alpha_0 > 0$. If $\alpha_0 = \beta_0 + 1$ were true for some ordinal β_0, then by $\beta_0 \in I = \alpha_0$ and $a_{\beta_0} \restriction \alpha_0 = y_0$, we have $a_{\beta_0}(\alpha_0) \in A_0$, a contradiction. So α_0 is limit. If $(\leftarrow, y_0)$ has a maximal element, say $y_1 = \max(\leftarrow, y_0)$, then by letting $\beta_0 = \min\{\alpha < \alpha_0 : y_1(\alpha) \neq y_0(\alpha)\}$, we see $y_1 < a_{\beta_0} \restriction \alpha_0 < y_0$, a contradiction. So $(\leftarrow, y_0)$ has no maximal element. From our condition (3), we have $J^- \subset \omega \leq \alpha_0$, so Y_1 $(= \prod_{\alpha_0 \leq \alpha} X_\alpha)$ has a minimal element. Then $\langle y_0, \min Y_1 \rangle$ is the minimal element of $B = X \setminus A$, which contradicts Claim 1. □

Claim 7. $A_0 \neq X_{\alpha_0}$.

Proof. Assume $A_0 = X_{\alpha_0}$. It follows from Claim 4 that $A = (\leftarrow, y_0] \times Y_1$. If $\alpha_0 = 0$ were true, then by taking $b \in B$, we see $b(\alpha_0) \in B_0$, a contradiction. So we have $\alpha_0 > 0$. Now if $\alpha_0 = \beta_0 + 1$ were true for some ordinal β_0, then by $\beta_0 \in I = \alpha_0$ and $b_{\beta_0} \restriction \alpha_0 = y_0$, we see $b_{\beta_0}(\alpha_0) \in B_0$, a contradiction. So α_0 is limit. Since A has no maximal element, it follows from $A = (\leftarrow, y_0] \times Y_1$ that Y_1 has no maximal element, so there is $\alpha \geq \alpha_0$ such that X_α has no maximal element, but this contradicts $J^+ \subset \omega \leq \alpha_0$. □

Now let $Z_0 = \prod_{\alpha \leq \alpha_0} X_\alpha$, $Z_1 = \prod_{\alpha_0 < \alpha} X_\alpha$ and $A^* = \{z \in Z_0 : z \restriction \alpha_0 < y_0 \text{ or } (z \restriction \alpha_0 = y_0, z(\alpha_0) \in A_0)\}$. Note $A^* = (\leftarrow, y_0) \times X_{\alpha_0} \cup \{y_0\} \times A_0$, therefore $\{y_0\} \times A_0$ is a 1-segment of the 0-segment A^* in Z_0. Using Claim 4, we can easily check $A = A^* \times Z_1$, so A^* is a non-empty bounded 0-segment of Z_0.

Assume that A^* has a maximal element, that is, A_0 has a maximal element. It follows from $A = A^* \times Z_1$ that Z_1 has no maximal element, which means $\alpha_0 < \sup J^+$. The clause (5) shows that $(\max A_0, \rightarrow)$ $(= B_0)$ has a minimal element. On the other hand, since B has no minimal element, there is $\alpha > \alpha_0$ such that X_α has no minimal element, which contradicts the clause (2). So we see that A^* has no maximal element, i.e., A_0 has no maximal element.

Claim 8. Z_1 has a minimal element.

Proof. Assuming that Z_1 has no minimal element, let $\alpha_1 = \min\{\alpha > \alpha_0 : X_\alpha \text{ has no minimal element }\}$. Since $\alpha_0 < \alpha_1 \in J^- \subset \omega$ holds by the clause (3), we have $\alpha_0 < \sup J^-$. If $\min B_0$ exists, then applying clause (4) with $u = \min B_0$, we see that A_0 $(= (\leftarrow, \min B_0))$ has a maximal element, a contradiction. So B_0 has no minimal element, thus A_0 is a non-empty bounded closed 0-segment of X_{α_0} with no maximal element, and this contradicts the clause (1). □

Claim 9. A_0 is closed in X_{α_0}.

Proof. Let $u \in X_{\alpha_0} \setminus A_0$ and $x = y_0 {}^\frown \langle u \rangle {}^\frown \langle \min X_\alpha : \alpha_0 < \alpha \rangle$. Then $x \in X \setminus A$. Since A is closed, there is $x^* \in \hat{X}$ such that $x^* < x$ and

$A \cap (x^*, x)_{\hat{X}} = \emptyset$. Let $\beta_0 = \min\{\alpha < \gamma : x^*(\alpha) \neq x(\alpha)\}$. Note $\beta_0 \leq \alpha_0$. If $\beta_0 < \alpha_0$ were true, then we have $a_{\beta_0} \in (x^*, x)_{\hat{X}}$, a contradiction. So we have $\beta_0 = \alpha_0$ and $A_0 \cap (x^*(\alpha_0), u)_{X^*_{\alpha_0}} = \emptyset$, which means that $(x^*(\alpha_0), \rightarrow)_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ is a neighborhood of u disjoint from A_0. □

We have seen that A_0 is a non-empty bounded closed 0-segment of X_{α_0} with no maximal element, which contradicts the clause (1). □

With the analogous result of Theorem 2.5, we have:

Corollary 2.6. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is complete iff the following clauses hold:*

1. *for every $\alpha < \gamma$, X_α is complete,*
2. *$J^- \subset \{0\}$ or $J^+ \subset \{0\}$,*
3. *$J^- \cup J^+ \subset \omega$,*
4. *for every $\alpha < \sup J^-$ and $u \in X_\alpha$, if $(\leftarrow, u) \neq \emptyset$, then $(\leftarrow, u)$ has a maximal element,*
5. *for every $\alpha < \sup J^+$ and $u \in X_\alpha$, if $(u, \rightarrow) \neq \emptyset$, then $(u, \rightarrow)$ has a minimal element,*

3. Applications

In this section, we apply the results of previous section. With the fact "0-compactness = 0-completeness with a maximal element", the theorem above shows:

Corollary 3.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is 0-compact iff the following clauses hold:*

1. *for every $\alpha < \gamma$, X_α is 0-compact,*
2. *$J^- \subset \omega$,*
3. *for every $\alpha < \sup J^-$ and $u \in X_\alpha$, if $(\leftarrow, u) \neq \emptyset$, then $(\leftarrow, u)$ has a maximal element,*

With the analogous result of the corollary above, we see:

Corollary 3.2. [2, Theorem 4.2.1] *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is compact iff for every $\alpha < \gamma$, X_α is compact.*

Example 3.3. Note that $[0,1]_{\mathbb{S}}^\gamma$ is 1-compact for every ordinal γ, but $[0,1]_{\mathbb{S}}$ is not 0-complete. Moreover note that $[0,1)_{\mathbb{S}}$ is 1-compact but $[0,1)_{\mathbb{S}}^2$ is not 1-complete. $[0,1)_{\mathbb{S}} \times [0,1]_{\mathbb{S}}^\gamma$ is 1-compact for every γ.

For later use, we discuss GO-spaces which are similar to ordinals.

Definition 3.4. A GO-space X is called an *almost ordinal* if for every $x \in X$, the subspace $[x, \rightarrow)_X$ of X is identified with some ordinal α (we say "$[x, \rightarrow)_X$ is an ordinal"), that is, there is a 0-order preserving onto map $f : \alpha \to [x, \rightarrow)_X$ such that both f and f^{-1} are continuous. A GO-space X is called a *almost reverse ordinal* if $-X$ is an almost ordinal. When α is ordinal, $-\alpha$ is called a reverse ordinal.

We can easily check:

- almost (reverse) ordinals are complete LOTS's,
- an almost ordinal with a minimal element is (identified with) an ordinal,
- if α is an ordinal, then the lexicographic product $(-\omega) \times \alpha$ is an almost ordinal,
- the lexicographic product $\omega \times (-\omega)$ is complete but not an almost ordinal,
- $-\omega$ is an almost ordinal, but $-\omega_1$ is not an almost ordinal.

The property "almost ordinal" is just equivalent to both properties in (1) and (5) in Theorem 2.5.

Lemma 3.5. *A GO-space X is almost ordinal iff the following properties hold:*

(1) *X is 0-complete,*
(2) *for every $u \in X$, if $(u, \rightarrow)_X \neq \emptyset$, then $(u, \rightarrow)_X$ has a minimal element.*

Proof. One direction is obvious. To see the other direction, let $x_0 \in X$ and $f(0) = x_0$. Inductively we define a 0-order preserving onto map $f : \alpha_0 \to [x_0, \rightarrow)$ for some ordinal α_0 such that f and f^{-1} are continuous. Assume that $\alpha > 0$ and a 0-order preserving map $f \restriction \alpha : \alpha \to [x_0, \rightarrow)$ is defined so that for every $\beta < \alpha$ with $\beta + 1 < \alpha$, $(f(\beta), f(\beta + 1))_X = \emptyset$ holds. Let $A(f \restriction \alpha) = \{x \in X : \exists \beta < \alpha (x \leq_X f(\beta))\}$. Obviously $A(f \restriction \alpha)$ is a non-empty 0-segment of X. Whenever $A(f \restriction \alpha) = X$, stop the induction and let $\alpha_0 = \alpha$. So let $A(f \restriction \alpha) \neq X$. If $X \setminus A(f \restriction \alpha)$ has no minimal element, then $A(f \restriction \alpha)$ is a non-empty bounded closed 0-segment, so by (1), $A(f \restriction \alpha)$ has a maximal element. Then by (2), $X \setminus A(f \restriction \alpha)$ has a minimal element, a contradiction. Thus $X \setminus A(f \restriction \alpha)$ has a minimal element, so let $f(\alpha) = \min(X \setminus A(f \restriction \alpha))$. After finishing the induction, we have got a 0-order preserving onto map $f : \alpha_0 \to [x_0, \rightarrow)$. It suffices to see that both f and f^{-1} are continuous.

To see that f is continuous at $\alpha < \alpha_0$, let U be a convex open set in X with $f(\alpha) \in U$. We may assume that α is limit. Then since $(\leftarrow, f(\alpha))$ is a non-empty bounded 0-segment with no maximal element, $(\leftarrow, f(\alpha))$

is not closed in X, so $f(\alpha) \in \mathrm{Cl}_X(\leftarrow, f(\alpha))$, where Cl_X denotes the closure in X. Then one can pick $\beta < \alpha$ with $f(\beta) \in U$. Now by convexity of U, we see $f[(\beta, \alpha]] \subset U$. This shows the continuity of f.

To see that f^{-1} is continuous at $f(\alpha)$ with $\alpha < \alpha_0$, let $\beta < \alpha$. Put

$$U = \begin{cases} (f(\beta), \rightarrow)_X & \text{if } \alpha + 1 = \alpha_0, \\ (f(\beta), f(\alpha+1))_X & \text{otherwise.} \end{cases}$$

Then $(\beta, \alpha] = f^{-1}[U]$. This shows the continuity of f^{-1}. □

Theorem 2.5 and the lemma above yields:

Corollary 3.6. *If a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is 0-complete (1-complete), then for every $\alpha < \sup J^+$ ($\alpha < \sup J^-$), X_α is an almost ordinal (almost reverse ordinal, respectively). In particular, X_α is an ordinal (reverse ordinal) for every $\alpha < \sup J^+$ ($\alpha < \sup J^-$) with $0 < \alpha$.*

Corollary 3.7. *If a GO-space Y has no maximal element, then the lexicographic products $\mathbb{R} \times Y$, $\mathbb{I} \times Y$, $(-\mathbb{S}) \times Y$, $[0, 1)_\mathbb{R} \times Y$, $(-\omega_1) \times Y$ and $\omega \times (-\omega) \times Y$ are not 0-complete.*

Applying Theorem 2.5, we consider completeness of a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ on the following typical cases:

(a) all X_α's have neither minimal nor maximal elements, that is, $J^- = J^+ = \gamma$,
(b) all X_α's have a minimal and a maximal elements, that is, $J^- = J^+ = \emptyset$,
(c) all X_α's have a minimal element, that is, $J^- = \emptyset$.

The case (a) is not interesting, e.g., consider the lexicographic product $\mathbb{R} \times \mathbb{R}$. More generally we have:

Corollary 3.8. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $J^- \setminus \{0\} \neq \emptyset$ and $J^+ \setminus \{0\} \neq \emptyset$, then X is neither 0-complete nor 1-complete.*

About (b), we have:

Corollary 3.9. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $J^- = J^+ = \emptyset$, then X is 0-complete (1-complete, complete) iff for every $\alpha < \gamma$, X_α is 0-complete (1-complete, complete, respectively).*

(c) is the most interesting case.

Corollary 3.10. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $J^- = \emptyset$, then X is 0-complete (1-complete) iff the following clauses hold:*

(1) *X_α is 0-complete (1-complete) for every $\alpha < \gamma$,*
(2) *$J^+ \subset \omega$,*
(3) *for every $\alpha < \sup J^+$ and $u \in X_\alpha$, if $(u, \rightarrow) \neq \emptyset$, then $(u, \rightarrow)$ has a minimal element,*

Example 3.11. Applying the corollary above, we see:

(1) $[0,1)^\omega_{\mathbb{R}}$ is neither 0-complete nor 1-complete,
(2) $\omega_1^3 \times [0,1)_{\mathbb{S}}$ is 1-complete but not 0-complete, but $[0,1)_{\mathbb{S}} \times \omega_1^3$ is neither 0-complete nor 1-complete,
(3) $\omega_1^3 \times [1,0)_{-\mathbb{S}}$ is 0-complete but not 1-complete, but $[1,0)_{-\mathbb{S}} \times \omega_1^3$ is neither 0-complete nor 1-complete,
(4) ω_1^ω is complete but $\omega_1^{\omega+1}$ is neither 0-complete nor 1-complete, or the lexicographic product of a ω-sequence of ordinals is complete, e.g., $\prod_{n\in\omega} \omega_n$,
(5) $\omega_1^\omega \times (\omega_1 + 1)^\omega$ and $\omega_1^\omega \times \mathbb{I}^\omega$ are complete,
(6) $(\omega_1 \times (\omega_1 + 1))^\omega$ and $\prod_{n\in\omega}(\omega_n \times (\omega_n + 1))$ are complete, but $(\omega_1 \times \mathbb{I})^\omega$ and $\prod_{n\in\omega}(\omega_n \times \mathbb{I})$ are neither 0-complete nor 1-complete.

Now we consider further special case:

(c*) γ is limit, all X_α's have a minimal element and X_α has no maximal element for cofinally many $\alpha < \gamma$, that is, $J^- = \emptyset$ and $\sup J^+ = \gamma$.

Corollary 3.12. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $J^- = \emptyset$ and $\sup J^+ = \gamma$, then the following hold:*

(1) *X is 0-complete iff $\gamma = \omega$ and all X_α's are ordinals,*
(2) *if X is 0-complete, then it is 1-complete thus complete.*

Proof. (1) follows from Corollary 3.10 and Lemma 3.5.

(2) If X is 0-complete, then by (1), $\gamma = \omega$ and all X_α's are ordinal so 1-complete. Again applying Corollary 3.10, we see that X is 1-complete. □

Example 3.13. The reverse implication of (2) in the corollary above is not true. Note that the subspace $\omega_1 \setminus \{\omega\}$ of ω_1 is 1-complete but not 0-complete. So from Corollary 3.10, we see that $(\omega_1 \setminus \{\omega\})^\omega$ is 1-complete but not 0-complete. Also note that $(\omega_1 \setminus \{\omega\})^\omega$ is not a LOTS, because of $\langle \omega + 1, 0, 0, 0, \dots \rangle \in X^-$.

4. CONNECTEDNESS

It is well known that the usual Tychonoff product $\prod_{\alpha<\gamma} X_\alpha$ is connected iff all X_α's are connected [1, Theorem 6.1.15] and that the lexicographic product ω_1^ω is connected [2, p.68, Remark]. Obviously connected GO-spaces are a complete LOTS. In this section, we will clarify the relationship between connectedness and completeness, also we will prove that in some situations, connectedness of a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ implies non-connectedness of all X_α's.

Lemma 4.1. *A GO-space X is connected iff the following clauses hold:*

(1) *X is complete (hence a LOTS),*
(2) *for every $u \in X$, if $(u, \rightarrow) \neq \emptyset$, then $(u \rightarrow)$ has no minimal element, that is, X has no jumps in the sense of* [2].

Proof. One direction is obvious. Assume (1) and (2) and that X is not connected. Take non-empty disjoint open sets U and V with $X = U \cup V$, moreover fix $u \in U$ and $v \in V$. We may assume $u < v$. Let $A = \{x \in X : \exists w \in X(u \leq w, x \leq w, [u, w] \subset U)\}$. Then obviously $(\leftarrow, u] \subset A$ and A is a 0-segment of X with $v \notin A$. Let $B = X \setminus A$.

Claim 1. A is open in X.

Proof. Let $x \in A$. When $x < u$, $(\leftarrow, u)$ is a neighborhood of x contained in A. So assume $u \leq x$. Then there is $w \in X$ such that $x \leq w$ and $[u, w] \subset U$. Since $w \in U$ and U is open in X, there is $w' \in X$ with $w < w'$ and $[w, w') \subset U$. Then it is easy to see that $(\leftarrow, w')$ is a neighborhood of x contained in A. □

This claim shows that B is a non-empty bounded closed 1-segment of X. So by 1-completeness of X, B has a minimal element, say $b = \min B$.

Claim 2. $b \in V$.

Proof. Assume $b \in U$. To see $[u, b) \subset U$, let $x \in [u, b)$. Then by $x \in A$, we have $x \in [u, w] \subset U$ for some $w \in X$ with $x \leq w$, so we have $[x, b) \subset U$. Now we see $[u, b] = [u, b) \cup \{b\} \subset U$, thus $b \in A$, a contradiction. □

Since V is open, there is $b' \in X$ with $b' < b$ such that $(b', b] \cap X \subset V$. Then we have $(b', b] \subset B$, which shows that B is open in X. Thus A is a non-empty bounded closed 0-segment of X. By the 0-completeness of X, A has a maximal element, say $u = \max A$. Now we have $b = \min(u, \rightarrow)$, which contradicts our assumption (2). □

Now we also discuss connectedness of lexicographic products in some cases. Note from Corollary 3.8 that a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is not connected whenever $J^- \setminus \{0\} \neq \emptyset$ and $J^+ \setminus \{0\} \neq \emptyset$, e.g., for every GO-space X, lexicographic products $X \times \mathbb{R}$, $X \times [0,1)_{\mathbb{S}} \times (0,1]_{\mathbb{R}}$, ... etc. are not connected.

When all X_α's have a minimal and a maximal elements, we have an expected result:

Corollary 4.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $J^- = J^+ = \emptyset$, then X is connected iff for every $\alpha < \gamma$, X_α is connected.*

Proof. Assume that X is connected. Since X is complete, by Corollary 3.9, all X_α's are complete. It suffices to see that all X_α's have the property in (2) in Lemma 4.1. So assume that for some $\alpha_0 < \gamma$ and $u_0, u_1 \in X_{\alpha_0}$, $u_0 < u_1$ and $(u_0, u_1)_{X_{\alpha_0}} = \emptyset$ are true. Fixing $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$, let $x_0 = y_0 {}^\wedge\langle u_0\rangle^\wedge\langle \max X_\alpha : \alpha_0 < \alpha\rangle$ and $x_1 = y_0 {}^\wedge\langle u_1\rangle^\wedge\langle \min X_\alpha : \alpha_0 < \alpha\rangle$. Then we have $x_0 < x_1$ and $(x_0, x_1)_X = \emptyset$ which contradicts connectedness of X.

To see the other direction, assume that X_α's are connected. From Corollary 3.9, X is complete. It suffices to see (2) in the lemma above. If there were $x_0, x_1 \in X$ with $x_0 < x_1$ and $(x_0, x_1)_X = \emptyset$, then letting $\alpha_0 = \min\{\alpha < \gamma : x_0(\alpha) \neq x_1(\alpha)\}$, we see $x_0(\alpha_0) < x_1(\alpha_0)$ and $(x_0(\alpha_0), x_1(\alpha_0))_{X_{\alpha_0}} = \emptyset$, a contradiction. □

Finally we discuss connectedness in the special case (c*) of the previous section, that is, $J^- = \emptyset$ and $\sup J^+ = \gamma$.

Corollary 4.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Assume $J^- = \emptyset$ and $\sup J^+ = \gamma$. Then X is 0-complete iff X is connected.*

Proof. One direction is obvious. Assume that X is 0-complete. From Corollary 3.12 (2), it suffices to see the property (2) in Lemma 4.1. So let $u, u' \in X$ with $u < u'$ and fix $v_\alpha \in X_\alpha$ with $u(\alpha) < v_\alpha$ for every $\alpha \in J^+$. Let $\alpha_0 = \min\{\alpha < \gamma : u(\alpha) \neq u'(\alpha)\}$. Noting $\sup J^+ = \gamma$, take $\alpha_1 \in J^+$ with $\alpha_0 < \alpha_1$. Then we have $u < (u \restriction \alpha_1)^\wedge\langle v_{\alpha_1}\rangle^\wedge(u \restriction (\alpha_1, \gamma)) < u'$. □

From Corollary 3.12 (1), we get a strange result:

Corollary 4.4. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Assume $J^- = \emptyset$ and $\sup J^+ = \gamma$. If X is 0-complete, then $\gamma = \omega$ and all X_α's are ordinals. In particular, if X is connected, then all X_α's are not connected.*

Comparing with the above two corollaries, remark that $(\omega_1 \setminus \{\omega\})^\omega$ is a 1-complete non-connected GO-space which is not a LOTS, see Example 3.13.

Example 4.5. Using the corollaries above, we see:

(1) the lexicographic product $\mathbb{I}^\gamma$ is connected for every ordinal γ,
(2) the lexicographic product $(\omega_1 + 1)^\omega$ is not connected,
(3) the lexicographic products $(\omega_1 \times (\omega_1 + 1))^\omega$ and $\prod_{n\in\omega}((\omega_n \times (\omega_n + 1))$ are connected, but $(\omega_1 \times \mathbb{I})^\omega$ is not connected (in fact, not complete),
(4) the lexicographic products $\omega_1^\omega \times (\omega_1+1)^\omega$, $\prod_{n\in\omega} \omega_n \times \prod_{n\in\omega}(\omega_n + 1)$ and $\omega_1^\omega \times \mathbb{I}^\omega$ are complete, moreover we see that $\omega_1^\omega \times (\omega_1+1)^\omega$ and $\prod_{n\in\omega} \omega_n \times \prod_{n\in\omega}(\omega_n + 1)$ are not connected, but $\omega_1^\omega \times \mathbb{I}^\omega$ is connected.

Question 4.6. Find a topological property P so that, in some situations such as in Corollary 4.4, if a lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ of GO-spaces satisfies the property P, then for every $\alpha < \gamma$, X_α does not satisfy P.

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