HEREDITARY PARACOMPACTNESS OF LEXICOGRAPHIC PRODUCTS

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Abstract. Paracompactness and hereditary paracompactness of lexicographic products of LOTS's are discussed in [2]. For instance, it is known in [2]:

- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS's is paracompact whenever all X_α's are paracompact [2, Theorem 4.2.2],
- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS's is hereditarily paracompact whenever $\gamma < \omega_1$ and all X_α's are hereditarily paracompact [2, Theorem 4.2.3],
- the lexicographic product $[0,1]_{\mathbb{R}}^{\omega_1}$ is not hereditarily paracompact, where $[0,1]_{\mathbb{R}}$ denotes the unit interval in the real line $\mathbb{R}$ [2, page 73].

Recently the author defined the notion of lexicographic products of GO-spaces and extended the first result above in [2] for lexicographic products of GO-spaces [4]. In this paper, we characterize the hereditary paracompactness of lexicographic products of GO-spaces and get some applications. For example, we see:

- the lexicographic products $\mathbb{S}^\gamma$, $\mathbb{M}^\gamma$, $\mathbb{R}^\gamma$ and $(0,1)_{\mathbb{R}}^\gamma$ are hereditarily paracompact for every ordinal γ, where $\mathbb{S}$ and $\mathbb{M}$ denote the Sorgenfrey line and Michael line respectively,
- the lexicographic product $[0,1)_{\mathbb{R}}^\omega$ is hereditarily paracompact, but the lexicographic product $[0,1)_{\mathbb{R}}^{\omega_1}$ is not hereditarily paracompact,
- the lexicographic product $\omega_1 \times (0,1]_{\mathbb{R}}$ is hereditarily paracompact but the lexicographic product $\omega_1 \times [0,1)_{\mathbb{R}}$ is not paracompact,
- the lexicographic product $(\omega_1^2 \times (-\omega_1)^3)^{\omega_1}$ is hereditarily paracompact, but the lexicographic products ω_1^ω and $\omega_1^{\omega_1}$ are not paracompact, where for a GO-space $X = \langle X, <_X, \tau_X\rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X\rangle$.

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1. Introduction

All spaces are assumed to be regular T_1 and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α's are assumed to have cardinality at least 2 with $\gamma \geq 2$. Moreover, in this paper, $\prod_{\alpha<\gamma} X_\alpha$ usually means a lexicographic product defined below. Set theoretical and topological terminology follow [7] and [1]. The following are known:

- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS's is paracompact whenever all X_α's are paracompact [2, Theorem 4.2.2],
- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS's is hereditarily paracompact whenever $\gamma < \omega_1$ and all X_α's are hereditarily paracompact [2, Theorem 4.2.3],
- the lexicographic product $[0,1]_{\mathbb{R}}^{\omega_1}$ is not hereditarily paracompact, where $[0,1]_{\mathbb{R}}$ denotes the unit interval in the real line $\mathbb{R}$ [2, page 73].

Recently the author defined the notion of lexicographic product of GO-spaces and extended the first result above for lexicographic products of GO-spaces [4]. Therefore we see:

- lexicographic products $\mathbb{S}^\gamma$, $\mathbb{M}^\gamma$, $\mathbb{R}^\gamma$, $(0,1)_{\mathbb{R}}^\gamma$ and $[0,1)_{\mathbb{R}}^\gamma$ are paracompact for every ordinal γ, where $\mathbb{S}$ and $\mathbb{M}$ denote the Sorgenfrey line and Michael line respectively.

Since $\mathbb{R}$, $\mathbb{S}$ and $\mathbb{M}$ are hereditarily paracompact, it is natural to ask whether $\mathbb{S}^\gamma$, $\mathbb{M}^\gamma$, $\mathbb{R}^\gamma$, $(0,1)_{\mathbb{R}}^\gamma$ and $[0,1)_{\mathbb{R}}^\gamma$ are hereditarily paracompact even if $\gamma \geq \omega_1$. In this paper, we characterize the hereditary paracompactness of lexicographic products of GO-spaces. Applying this characterization, we see:

- lexicographic products $\mathbb{S}^\gamma$, $\mathbb{M}^\gamma$, $\mathbb{R}^\gamma$ and $(0,1)_{\mathbb{R}}^\gamma$ are hereditarily paracompact for every ordinal γ,
- the lexicographic product $[0,1)_{\mathbb{R}}^\omega$ is hereditarily paracompact, but the lexicographic product $[0,1)_{\mathbb{R}}^{\omega_1}$ is paracompact but not hereditarily paracompact,
- the lexicographic product $\omega_1 \times (0,1]_{\mathbb{R}}$ is hereditarily paracompact but the lexicographic product $\omega_1 \times [0,1)_{\mathbb{R}}$ is not paracompact,
- the lexicographic product $(\omega_1^2 \times (-\omega_1)^3)^{\omega_1}$ is hereditarily paracompact, but the lexicographic products ω_1^ω and $\omega_1^{\omega_1}$ are not paracompact, where for a GO-space $X = \langle X, <_X, \tau_X\rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X\rangle$.

A linearly ordered set $\langle L, <_L\rangle$ has a natural topology λ_L, which is called an interval topology, generated by $\{(\leftarrow, x)_L : x \in L\} \cup \{(x, \rightarrow)_L : x \in L\}$ as a subbase, where $(x, \rightarrow)_L = \{z \in L : x <_L z\}$, $(x, y)_L =$

$\{z \in L : x <_L z <_L y\}$, $(x, y]_L = \{z \in L : x <_L z \leq_L y\}$ and so on. The triple $\langle L, <_L, \lambda_L\rangle$, which is simply denoted by L, is called a *LOTS*.

A triple $\langle X, <_X, \tau_X\rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X\rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subseteq C$ holds. For more information on LOTS's or GO-spaces, see [8]. Usually $<_L$, $(x, y)_L$, λ_L or τ_X are written simply $<$, (x, y), λ or τ if contexts are clear.

The symbols ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \cdots$, are considered to be LOTS's with the usual intereval topology. For a subset A of an ordinal α, $\mathrm{Lim}(A)$ denotes the set $\{\beta < \alpha : \beta = \sup(A \cap \beta)\}$, that is, the set of all cluster points of A in the topological space α. The cofinality of α is denoted by $\mathrm{cf}\,\alpha$..

For GO-spaces $X = \langle X, <_X, \tau_X\rangle$ and $Y = \langle Y, <_Y, \tau_Y\rangle$, X is said to be a *subspace* of Y if $X \subseteq Y$, the linear order $<_X$ is the restriction $<_Y \restriction X$ of the order $<_Y$ and the topology τ_X is the subspace topology $\tau_Y \restriction X$ $(= \{U \cap X : U \in \tau_Y\})$ on X of the topology τ_Y. So a subset of a GO-space is naturally considered as a GO-space. For every GO-space X, there is a LOTS X^* such that X is a dense subspace of X^* and X^* has the property that if L is a LOTS containing X as a dense subspace, then L also contains the LOTS X^* as a subspace, see [9]. Such a X^* is called the *minimal d-extension of a GO-space* X. Indeed, X^* is constructed as follows, also see [4]. Let $X^+ = \{x \in X : (\leftarrow, x] \in \tau_X \setminus \lambda_X\}$ and $X^- = \{x \in X : [x, \rightarrow) \in \tau_X \setminus \lambda_X\}$. Then X^* is the LOTS $X^- \times \{-1\} \cup X \times \{0\} \cup X \times \{1\}$, where the order $<_{X^*}$ is the restriction of the usual lexicographic order on $X \times \{-1, 0, 1\}$. Also we identify as $X = X \times \{0\}$ in the obvious way.

Then, we can see:

- if X is a LOTS, then $X^* = X$,
- X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

For every $\alpha < \gamma$, let X_α be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma\rangle$. The lexicographic order $<_X$ on X is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha\rangle$ and $<_{X_\alpha}$ is the order on X_α. Now for every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace X of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ is said to be the *lexicographic product of GO-spaces* X_α's, for more details see [4]. $\prod_{i\in\omega} X_i$ ($\prod_{i\le n} X_i$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let X and Y be LOTS's. A map $f : X \to Y$ is said to be *order preserving* or 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be *order reversing* or 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map (also 1-order preserving map) $f : X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. Now let X and Y be GO-spaces. A 0-order preserving map $f : X \to Y$ is said to be 0-*order preserving embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we identify X with $f[X]$ as a GO-space and write $X = f[X]$ and $X \subseteq Y$.

Recall that a subset of a regular uncountable cardinal κ is called *stationary* if it intersects with all closed unbounded (= club) sets in κ.

Let X be a GO-space. A subset A of X is called a 0-*segment* of X if for every $x, x' \in X$ with $x \le x'$, if $x' \in A$, then $x \in A$. Similarly the notion of 1-segment can be defined. Both $\emptyset$ and X are 0-segments and 1-segments. Obviously, if A is a 0-segment, then $X \setminus A$ is a 1-segment.

Let A be a 0-segment of a GO-space X. A subset U of A is *unbounded in* A if for every $x \in A$, there is $x' \in U$ such that $x \le x'$. Let

$$0\text{-}\operatorname{cf}_X A = \min\{|U| : U \text{ is unbounded in } A.\}.$$

$0\text{-}\operatorname{cf}_X A$ can be $0, 1$ or a regular infinite cardinal, see also [3, 5, 6]. If contexts are clear, $0\text{-}\operatorname{cf}_X A$ is denoted by $0\text{-}\operatorname{cf} A$. A 0-segment A of a GO-space X is said to be *stationary* if $\kappa := 0\text{-}\operatorname{cf} A \ge \omega_1$ and there are a stationary set S of κ and a continuous map $\pi : S \to A$ such that $\pi[S]$ is unbounded in A (we say such a π "an unbounded continuous map").

Note that for a subspace S of a regular uncountable cardinal κ, S is stationary in κ in the usual sense if and only if the 0-segment S in the GO-space S is stationary in the sense above (e.g., use [5, Lemma 2.7]). So this new term "stationarity of 0-segments" extends the usual term "stationarity of subsets of a regular uncountable cardinal".

A GO-space X is said to be 0-*paracompact* if every closed 0-segment is not stationary. Similarly the notions of $1\text{-}\operatorname{cf} A$, stationarity of a 1-segment and 1-paracompactness are defined. Remember that a GO-space is paracompact if and only if it is both 0-paracompact and 1-paracompact, see [4], where a topological space is *paracompact* if every open cover has a locally finite open refinement [1]. It is well-known that stationary sets of some regular uncountable cardinal are not paracompact. We frequently use the following basic lemmas from [5].

Lemma 1.1. [5, Lemma 2.7] *Let A be a 0-segment of a GO-space X with $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$. If there are a stationary set S of κ and an unbounded continuous map $\pi : S \to A$, then there is a club set C in κ such that $\pi \restriction (S \cap C) : S \cap C \to A$ is 0-order preserving embedding.*

Lemma 1.2. [5, Lemma 3.4] *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and $u \in X_0$. Then the map $k_u : X_1 \to \{u\} \times X_1$ by $k_u(v) = \langle u, v \rangle$ is a 0-order preserving homeomorphism.*

Lemma 1.3. [5, Lemma 3.6] *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and A_0 a 0-segment of X_0. Put $A = A_0 \times X_1$. Then the following hold:*

1. (1) *A is a 0-segment of X,*
2. (2) *if $0\text{-}\operatorname{cf}_{X_0} A_0 = 1$, then*
   - (a) $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{X_1} X_1$,
   - (b) *A is stationary if and only if the 0-segment X_1 is stationary,*
3. (3) *if $0\text{-}\operatorname{cf}_{X_0} A_0 \geq \omega$, then*
   - (a) $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{X_0} A_0$,
   - (b) *A is stationary if and only if X_1 has a minimal element and A_0 is stationary,*

A GO-space X is said to be *hereditarily 0-paracompact* if every 0-segment A of X is not stationary, similarly the notion of hereditary 1-paracompactness is defined. We can see the naming of these definitions are reasonable from the lemma below, where a topological space is *hereditarily paracompact* if all subspaces are paracompact.

Lemma 1.4. *Let X be a GO-space. Then X is hereditarily paracompact if and only if it is both hereditarily 0-paracompact and hereditarily 1-paracompact.*

Proof. First assume that X is hereditarily paracompact and that X is not hereditarily 0-paracompact, then there is a stationary 0-segment A of X. Lemma 1.1 shows that A has a copy of a stationary set of some regular uncountable cardinal, a contradiction. So X is hereditarily 0-paracompact. Similarly X is hereditarily 1-paracompact.

Next assume that there is a non-paracompact subspace Y of X. We may assume that Y is not 0-paracompact. So there is a closed stationary 0-segment A of Y. Set $A' = \{x \in X : \exists y \in A(x \leq y)\}$. Then it is easy to verify that A' is also a stationary (need not be closed) 0-segment of X, which means that X is not hereditarily 0-paracompact. □

2. Products of two GO-spaces

In this section, we characterize the hereditary paracompactness of a lexicographic product $X = X_0 \times X_1$ of two GO-spaces.

Lemma 2.1. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) *X is hereditarily 0-paracompact,*
(2) *the following clauses hold:*
   (a) *X_1 is hereditarily 0-paracompact,*
   (b) *if X_1 has a minimal element, then X_0 is hereditarily 0-paracompact.*

Proof. (1) $\Rightarrow$ (2) Assume that X is hereditarily 0-paracompact.

(a) Assuming that X_1 is not hereditarily 0-paracompact, take a stationary 0-segment A_1 of X_1. Fixing $u \in X_0$, let $A = \{x \in X : \exists v \in A_1(x \leq \langle u, v\rangle)\}$. Obviously A is a 0-segment of X. Since $\{u\} \times A_1$ is a 1-segment (i.e., final segment) of A, Lemma 1.2 shows that the 0-segment A is also stationary, a contradiction.

(b) Assume that X_1 has a minimal element but X_0 is not hereditarily 0-paracompact. Taking a stationary 0-segment A_0 of X_0, let $A = A_0 \times X_1$. Then Lemma 1.3 (3b) shows that A is a stationary 0-segment of X, a contradiction.

(2) $\Rightarrow$ (1) Assumimg (2) and the negation of (1), take a staionary 0-segment A of X. Let $A_0 = \{u \in X_0 : \exists v \in X_1(\langle u, v\rangle \in A)\}$. Obviously A_0 is a non-empty 0-segment of X_0 with $A \subseteq A_0 \times X_1$. Assume that A_0 has a maximal element $\max A_0$ and let $A_1 = \{v \in X_1 : \langle \max A_0, v\rangle \in A\}$. Since $\{\max A_0\} \times A_1$ is a 1-segment of A, Lemma 1.2 shows that A_1 is a stationary 0-segment of X_1, which contradicts the condition (2a). Thus we see that A_0 has no maximal element, that is $0\text{-}\operatorname{cf}_{X_0} A_0 \geq \omega$.

Claim. $A = A_0 \times X_1$.

Proof. The inclusion $\subseteq$ is obvious. To see the inclusion $\supset$, let $x \in A_0 \times X_1$. Since A_0 has no maximal element, we can take $u \in A_0$ with $x(0) < u$. By $u \in A_0$, we can find $v \in X_1$ with $\langle u, v\rangle \in A$. Then we have $x < \langle u, v\rangle$. Now since A is a 0-segment, we see $x \in A$. □

Now Lemma 1.3 (3b) shows that X_1 has a minimal element and the 0-segment A_0 is stationary, which contradicts the condition (2b). □

Analogously we see:

Lemma 2.2. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) *X is hereditarily* 1*-paracompact,*
(2) *the following clauses hold:*
  (a) *X_1 is hereditarily* 1*-paracompact,*
  (b) *if X_1 has a maximal element, then X_0 is hereditarily* 1*-paracompact.*

The lemmas above show:

Lemma 2.3. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then the following are equivalent:*
(1) *X is hereditarily paracompact,*
(2) *the following clauses hold:*
  (a) *X_1 is hereditarily paracompact,*
  (b) *if X_1 has a minimal element, then X_0 is hereditarily* 0*-paracompact.*
  (c) *if X_1 has a maximal element, then X_0 is hereditarily* 1*-paracompact.*

Example 2.4. The lemma above shows that $\omega_1 \times \mathbb{R}$, $\omega_1 \times \mathbb{S}$ and $\omega_1 \times \mathbb{M}$ are hereditarily paracompact. But $\omega_1 \times [0,1)_{\mathbb{R}}$ is not paracompact [5]. On the other hand, $\omega_1 \times (0,1]_{\mathbb{R}}$ is hereditarily paracompact, indeed ω_1 is hereditarily 1-paracompact because it is well-ordered.

3. Products of any length of GO-spaces

In this section, we characterize the hereditarily paracompactness of lexicographic products of any length of GO-spaces. The following notations are introduced in [4, Theorem 2.5]

Definition 3.1. Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. We use the following notations.

$$J^+ = \{\alpha < \gamma : \ X_\alpha \text{ has no maximal element.}\},$$
$$J^- = \{\alpha < \gamma : \ X_\alpha \text{ has no minimal element.}\}.$$

Note $\sup J^+ \leq \gamma$ and $\sup J^- \leq \gamma$.

Theorem 3.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:*
(1) *X is hereditarily* 0*-paracompact,*
(2) *the following clauses hold:*
  (a) *$\gamma < \sup J^- + \omega_1$, where $\sup J^- + \omega_1$ is the usual ordinal sum,*
  (b) *for every $\alpha < \gamma$ with $\sup J^- \leq \alpha$, X_α is hereditarily* 0*-paracompact,*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ be the lexicographic product of LOTS's X^*_α's.

(1) $\Rightarrow$ (2) Assume that X is hereditarily 0-paracompact.

(a) Assume $\sup J^- + \omega_1 \leq \gamma$. Letting $\alpha_0 = \sup J^-$, fix $z \in \prod_{\alpha\leq\alpha_0} X_\alpha$. For every $\alpha < \gamma$ with $\alpha_0 < \alpha$, noting that $\min X_\alpha$ exists, fix $u(\alpha) \in X_\alpha$ with $\min X_\alpha < u(\alpha)$. First let $x = z^\wedge\langle u(\alpha) : \alpha_0 < \alpha < \alpha_0 + \omega_1\rangle^\wedge\langle \min X_\alpha : \alpha_0 + \omega_1 \leq \alpha < \gamma\rangle$, that is, x is an element in X such that $x(\alpha) = z(\alpha)$ when $\alpha \leq \alpha_0$, $x(\alpha) = u(\alpha)$ when $\alpha_0 < \alpha < \alpha_0 + \omega_1$ and $x(\alpha) = \min X_\alpha$ when $\alpha_0 + \omega_1 \leq \alpha < \gamma$. Next for $\beta < \omega_1$ with $1 < \beta$, let $x_\beta = z^\wedge\langle u(\alpha) : \alpha_0 < \alpha < \alpha_0 + \beta\rangle^\wedge\langle \min X_\alpha : \alpha_0 + \beta \leq \alpha < \gamma\rangle$. Set $A = (\leftarrow, x)_X$ and $S = (1, \omega_1)$, and define $\pi : S \to A$ by $\pi(\beta) = x_\beta$. Obviously π is 0-order preserving and unbounded (i.e., "$\beta' < \beta \to \pi(\beta') < \pi(\beta)$" and $\pi[S]$ is unbounded in the 0-segment A).

Claim 1. π is continuous.

Proof. Let $\beta \in S$ and U be an open neighborhood of $\pi(\beta)$. We may assume $\beta \in \mathrm{Lim}(S)$. Note $(\leftarrow, \pi(\beta))_X \neq \emptyset$. Then there is $y^* \in \hat{X}$ with $y^* < \pi(\beta)$ and $(y^*, \pi(\beta)]_{\hat{X}} \cap X \subseteq U$. Let $\beta_0 = \min\{\alpha < \gamma : y^*(\alpha) \neq \pi(\beta)(\alpha)\}$. The definition of x_β $(= \pi(\beta))$ shows $\beta_0 < \alpha_0 + \beta$. When $\beta_0 \leq \alpha_0$, obviously $\pi[S \cap (\beta + 1)] \subseteq U$ holds. So assumeing $\alpha_0 < \beta_0 < \alpha_0 + \beta$, β_0 can be represented as $\beta_0 = \alpha_0 + \beta_1$ for some $\beta_1 < \beta$ with $0 < \beta_1$. Then for each $\beta' \in (\beta_1, \beta]$, we have $y^* < x_{\beta'} \leq x_\beta$. Therefore we see $\pi[S \cap (\beta_1, \beta]] \subseteq U$, so we have seen that π is continuous. □

Now since S is stationary in ω_1, the 0-segment A is stationary, which contradicts the hereditary 0-paracompactness of X.

(b) Let $\sup J^- \leq \alpha_0 < \gamma$ and let $Y_0 = \prod_{\alpha\leq\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0<\alpha} X_\alpha$ be lexicographic products. Then X is identified with the lexicographic product $Y_0 \times Y_1$ [4, Lemma 1.5], where X is identified with Y_0 whenever $\alpha_0 + 1 = \gamma$. Since X $(= Y_0 \times Y_1)$ is hereditarily 0-paracompact and Y_1 has the minimal element $\langle \min X_\alpha : \alpha_0 < \alpha\rangle$, Lemma 2.1 (2b) shows that Y_0 is hereditarily 0-paracompact. Here note that Y_0 is itself hereditarily 0-paracompact whenever $X = Y_0$, so we will not mention such special cases. Now $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$ and Lemma 2.1 (2a) shows that X_{α_0} is hereditarily 0-paracompact.

(2) $\Rightarrow$ (1) Assume (2) and the negation of (1), then one can take a stationary 0-segment A of X. We consider three cases and their subcases and in all cases, we will get contradictions. This argument is shown in [5, Theorem 4.8].

Case 1. $A = X$.

Since A $(= X)$ has no maximal element, X_α has no maximal element for some $\alpha < \gamma$. Let $\alpha_0 = \min\{\alpha < \gamma : X_\alpha \text{ has no maximal element.}\}$.

Since $A = X = \prod_{\alpha\le\alpha_0} X_\alpha \times \prod_{\alpha_0<\alpha} X_\alpha$, the 0-segment A is stationary and $\prod_{\alpha\le\alpha_0} X_\alpha$ has no maximal element, Lemma 1.3 (3b) shows that the 0-segment $\prod_{\alpha\le\alpha_0} X_\alpha$ is stationary and $\prod_{\alpha_0<\alpha} X_\alpha$ has a minimal element. Therefore X_α has a minimal element for every $\alpha > \alpha_0$, which means $\sup J^- \le \alpha_0$. By the minimality of α_0, X_α has a maximal element for every $\alpha < \alpha_0$. Then $\{\langle \max X_\alpha : \alpha < \alpha_0\rangle\} \times X_{\alpha_0}$ is a 1-segment of $\prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$. Now since the 0-segment $\prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$ is stationary, Lemma 1.2 shows that the 0-segment X_{α_0} is also stationary, this contradicts the condition (2b).

Case 2. $A \ne X$ and $X \setminus A$ has a minimal element.

Let $B = X \setminus A$ and $b = \min B$, then note $A = (\leftarrow, b)_X$. Set $I = \{\alpha < \gamma : \exists a \in A(a \restriction (\alpha+1) = b \restriction (\alpha+1))\}$. Since I is obviously a 0-segment of γ, for some $\alpha_0 \le \gamma$, $I = \alpha_0$ holds. Now for every $\alpha < \alpha_0$, fix $a_\alpha \in A$ with $a_\alpha \restriction (\alpha+1) = b \restriction (\alpha+1)$.

Claim 2. For every $\alpha \in (\alpha_0, \gamma)$, X_α has a minimal element and $b(\alpha) = \min X_\alpha$, thus $\sup J^- \le \alpha_0$.

Proof. Note that still we do not know whether $\alpha_0 < \gamma$ or not. Assume that for some $\alpha \in (\alpha_0, \gamma)$, there is $u \in X_\alpha$ with $u < b(\alpha)$. Let $\alpha_1 = \min\{\alpha > \alpha_0 : \exists u \in X_\alpha(u < b(\alpha))\}$ and take $u \in X_{\alpha_1}$ with $u < b(\alpha_1)$. Let $a = b \restriction \alpha_1 {}^\wedge\langle u\rangle^\wedge b \restriction (\alpha_1, \gamma)$. Then by $a < b$, we have $a \in A$ and $a \restriction \alpha_1 = b \restriction \alpha_1$. Now $\alpha_0 < \alpha_1$ shows $a \restriction (\alpha_0+1) = b \restriction (\alpha_0+1)$, which means $\alpha_0 \in I = \alpha_0$, a contradiction. □

We divide Case 2 into further two subcases.

Case 2-1. α_0 is a successor ordinal.

Say $\alpha_0 = \beta_0 + 1$.

Claim 3. $\alpha_0 < \gamma$.

Proof. If $\alpha_0 = \gamma$ were true, then by $\beta_0 \in \alpha_0 = I$, we have $B \ni b = b \restriction \alpha_0 = b \restriction (\beta_0+1) = a_{\beta_0} \restriction (\beta_0+1) = a_{\beta_0} \restriction \alpha_0 = a_{\beta_0} \in A$, a contradiction. □

Claim 4. $b(\alpha_0)$ is not a minimal element of X_{α_0}.

Proof. If $b(\alpha_0)$ were a minimal element of X_{α_0}, then we have $A \ni a_{\beta_0} \ge b \in B$ because of $b(\alpha) = \min X_\alpha$ for every $\alpha \ge \alpha_0$, a contradiction. □

Let $Y_0 = \prod_{\alpha\le\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0<\alpha} X_\alpha$.

Claim 5. $A = (\leftarrow, b \restriction (\alpha_0+1))_{Y_0} \times Y_1$.

Proof. To see the inclusion $\supset$, let $a \in (\leftarrow, b \restriction (\alpha_0+1)) \times Y_1$. Then $a \restriction (\alpha_0+1) < b \restriction (\alpha_0+1)$ shows $a < b = \min B$. So we have $a \in A$.

To see the inclusion $\subseteq$, let $a \in A$. Since $a < b$ and $b(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$, we have $a \restriction (\alpha_0+1) < b \restriction (\alpha_0+1)$, thus $a \in (\leftarrow, b \restriction (\alpha_0+1)) \times Y_1$. □

We further divide Case 2-1 into two subcases.

Case 2-1-1. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ has no maximal element.

In this case, $(\leftarrow, b \restriction (\alpha_0+1))_{Y_0}$ has no maximal element, so Claim 5 and Lemma 1.3 (3b) show that the 0-segment $(\leftarrow, b \restriction (\alpha_0+1))$ in Y_0 is stationary. Then it is easy to see:

Claim 6. $(\leftarrow, b \restriction (\alpha_0+1))_{Y_0} = (\leftarrow, b \restriction \alpha_0) \times X_{\alpha_0} \cup \{b \restriction \alpha_0\} \times (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$.

Now Lemma 1.2 show that the 0-segment $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is stationary, because $\{b \restriction \alpha_0\} \times (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is a 1-segment of $(\leftarrow, b \restriction (\alpha_0+1))_{Y_0}$ by Claim 6. This contradicts the condition (2b) because of $\sup J^- \leq \alpha_0$.

Case 2-1-2. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ has a maximal element.

Say $u_0 = \max(\leftarrow, b(\alpha_0))$, then note that $(b \restriction \alpha_0)^\wedge\langle u_0\rangle$ is the immediate predecessor of $b \restriction (\alpha_0+1)$ in Y_0, so we see $(\leftarrow, b \restriction (\alpha_0+1)) = (\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle]$. Since A has no maximal element and $A = (\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times Y_1$ (Claim 5), Y_1 has no maximal element. So let $\alpha_1 = \min\{\alpha > \alpha_0 : X_\alpha \text{ has no maximal element.}\}$. Now since $A = (\leftarrow, b \restriction (\alpha_0+1)) \times Y_1 = (\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times \prod_{\alpha_0<\alpha} X_\alpha = (\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times (\prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha \times \prod_{\alpha_1<\alpha} X_\alpha) = ((\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times \prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha) \times \prod_{\alpha_1<\alpha} X_\alpha$, $(\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times \prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha$ has no maximal element and the 0-segment A is stationary, Lemma 1.3 (3b) shows that the 0-segment $(\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times \prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha$ in $\prod_{\alpha\leq\alpha_1} X_\alpha$ is also stationary. Now since $\{(b \restriction \alpha_0)^\wedge\langle u_0\rangle^\wedge\langle \max X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle\} \times X_{\alpha_1}$ is a 1-segment of $(\leftarrow, (b \restriction \alpha_0)^\wedge\langle u_0\rangle] \times \prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha$, Lemma 1.2 shows that X_{α_1} is stationary. Since $\sup J^- \leq \alpha_0 < \alpha_1$, X_{α_1} has to be hereditarily 0-paracompact (condition (2b)), a contradiction.

Case 2-2. α_0 is limit.

Claim 2 and the condition (2a) show $\sup J^- \leq \alpha_0 \leq \gamma < \sup J^- + \omega_1$, therefore we have $\operatorname{cf} \alpha_0 = \omega$.

Claim 7. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then note $\operatorname{cf} \gamma = \operatorname{cf} \alpha_0 = \omega$, so fix a 0-order preserving unbounded (i.e., strictly increasing cofinal) sequence $\{\gamma_n : n \in \omega\}$ in γ. Then $\{a_{\gamma_n} : n \in \omega\}$ is unbounded in the 0-segment $(\leftarrow, b)$

$(= A)$, so we have $0\text{-}\operatorname{cf}_X A = \omega$, which contradicts the stationarity of the 0-segment A. □

We divide Case 2-2 into three subcases.

Case 2-2-1. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is non-empty and has no maximal element.

In this case, using a similar argument to Case 2-1-1, we can get a contradiction.

Case 2-2-2. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is non-empty and has a maximal element.

In this case, using a similar argument to Case 2-1-2, we can get a contradiction.

Case 2-2-3. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is empty, that is, $b(\alpha_0) = \min X_{\alpha_0}$.

In this case, fix a 0-order preserving unbounded sequence $\{\gamma_n : n \in \omega\}$ in α_0. Since $b(\alpha) = \min X_\alpha$ for every $\alpha \geq \alpha_0$, we see that $\{a_{\gamma_n} : n \in \omega\}$ is unbounded in the 0-segment $(\leftarrow, b)$ $(= A)$, so we have $0\text{-}\operatorname{cf}_X A = \omega$, which contradicts the stationarity of the 0-segment A.

Case 3. $A \neq X$ and $X \setminus A$ has no minimal element.

Let $B = X \setminus A$ and

$$I = \{\alpha < \gamma : \exists a \in A\ \exists b \in B\ (a \restriction (\alpha+1) = b \restriction (\alpha+1))\}.$$

Since I is a 0-segment in γ, for some $\alpha_0 \leq \gamma$, $I = \alpha_0$ holds. For every $\alpha < \alpha_0$, fix $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$ and consider the lexicographic products $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0 \leq \alpha} X_\alpha$. Define $y_0 \in Y_0$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$.

Claim 8. For every $\alpha < \alpha_0$, $y_0 \restriction (\alpha+1) = a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$ holds.

Proof. It suffices to see the first equality. Assuming $y_0 \restriction (\alpha+1) \neq a_\alpha \restriction (\alpha+1)$ for some $\alpha < \alpha_0$, let $\alpha_1 = \min\{\alpha < \alpha_0 : y_0 \restriction (\alpha+1) \neq a_\alpha \restriction (\alpha+1)\}$ and $\alpha_2 = \min\{\alpha \leq \alpha_1 : y_0(\alpha) \neq a_{\alpha_1}(\alpha)\}$. Then $y_0(\alpha_1) = a_{\alpha_1}(\alpha_1)$ shows $\alpha_2 < \alpha_1$. Also the minimality of α_1 shows $y_0 \restriction (\alpha_2+1) = a_{\alpha_2} \restriction (\alpha_2+1)$ $(= b_{\alpha_2} \restriction (\alpha_2+1))$. When $y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)$, we see $B \ni b_{\alpha_2} < a_{\alpha_1} \in A$, a contradiction. When $y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)$, we also see $B \ni b_{\alpha_1} < a_{\alpha_2} \in A$, a contradiction. □

Claim 9. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then $y_0 \in Y_0 = X = A \cup B$. Assume $y_0 \in A$ and take $a \in A$ with $y_0 < a$. Let $\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq a(\beta)\}$. Then we have $B \ni b_{\beta_0} < a \in A$, a contradiction. When $y_0 \in B$, similarly we also get a contradiction. □

Let $A_0 = \{a(\alpha_0) : a \in A, a \restriction \alpha_0 = y_0\}$ and $B_0 = \{b(\alpha_0) : b \in B, b \restriction \alpha_0 = y_0\}$.

Claim 10. The following hold:

(1) for every $a \in A$, $a \restriction \alpha_0 \leq_{Y_0} y_0$ holds,
(2) for every $x \in X$, if $x \restriction \alpha_0 <_{Y_0} y_0$, then $x \in A$.

Proof. (1) Assume $a \restriction \alpha_0 > y_0$ for some $a \in A$ and let $\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}$. Now we have $B \ni b_{\beta_0} < a \in A$, a contradiction.

(2) Assume $x \restriction \alpha_0 < y_0$ and let $\beta_0 = \min\{\beta < \alpha_0 : x(\beta) \neq y_0(\beta)\}$. Then we have $x < a_{\beta_0} \in A$, so we see $x \in A$ because A is a 0-segment. □

We similarly see:

Claim 11. The following hold:

(1) for every $b \in B$, $b \restriction \alpha_0 \geq_{Y_0} y_0$ holds,
(2) for every $x \in X$, if $x \restriction \alpha_0 >_{Y_0} y_0$, then $x \in B$.

Claim 12. A_0 is a 0-segment of X_{α_0} and $B_0 = X_{\alpha_0} \setminus A_0$.

Proof. Let $u' < u \in A_0$ and take $a \in A$ with $a \restriction (\alpha_0 + 1) = y_0 {}^\wedge\langle u\rangle$. Let $a' = (a \restriction \alpha_0)^\wedge\langle u'\rangle^\wedge(a \restriction (\alpha_0, \gamma))$. Since A is a 0-segment with $a' < a \in A$, we have $a' \in A$, thus $u' \in A_0$. So we have seen that A_0 is a 0-segment.

To see $B_0 \subseteq X_{\alpha_0} \setminus A_0$, let $u \in B_0$. Take $b \in B$ with $b \restriction (\alpha_0 + 1) = y_0 {}^\wedge\langle u\rangle$. If $u \in A_0$ were true, then by taking $a \in A$ with $a \restriction (\alpha_0 + 1) = y_0 {}^\wedge\langle u\rangle$, we see $a \restriction (\alpha_0 + 1) = b \restriction (\alpha_0 + 1)$ thus $\alpha_0 \in I = \alpha_0$, a contradiction. So we have $u \in X_{\alpha_0} \setminus A_0$.

To see $B_0 \supset X_{\alpha_0} \setminus A_0$, let $u \in X_{\alpha_0} \setminus A_0$. Take $x \in X$ with $x \restriction (\alpha_0 + 1) = y_0 {}^\wedge\langle u\rangle$. Then obviously we have $x \in B$, thus $u \in B_0$. □

Claim 13. $A_0 \neq \emptyset$.

Proof. Assume $A_0 = \emptyset$. We prove the following facts.

Fact 1. $(\leftarrow, y_0)_{Y_0} \times Y_1 = A$.

Proof. Claim 10 (2) shows the inclusion $\subseteq$. To see the other inclusion, let $a \in A$. Then Claim 10 (1) shows $a \restriction \alpha_0 \leq y_0$. If $a \restriction \alpha_0 = y_0$ were true, then we have $a(\alpha_0) \in A_0$, which contradicts $A_0 = \emptyset$. □

Fact 2. $\alpha_0 > 0$ and α_0 is limit.

Proof. If $\alpha_0 = 0$ were true, then taking $a \in A$, we see $a(\alpha_0) \in A_0$, a contradiction. If for some ordinal β_0, $\alpha_0 = \beta_0 + 1$ were true, then by $\beta_0 \in I = \alpha_0$ and $a_{\beta_0} \restriction \alpha_0 = a_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$, we see $a_{\beta_0}(\alpha_0) \in A_0$, a contradiction. □

Fact 3. $0\text{-}\mathrm{cf}_{Y_0}(\leftarrow, y_0)_{Y_0} \geq \omega$.

Proof. Fact 1 with $A \neq \emptyset$ shows $(\leftarrow, y_0) \neq \emptyset$, that is, $0\text{-}\mathrm{cf}_{Y_0}(\leftarrow, y_0) \geq 1$. If $0\text{-}\mathrm{cf}_{Y_0}(\leftarrow, y_0) = 1$ were true, then letting $y_1 = \max(\leftarrow, y_0)$ and $\beta_0 = \min\{\beta < \alpha_0 : y_1(\beta) \neq y_0(\beta)\}$, we see $y_1 < a_{\beta_0} \restriction \alpha_0 < y_0$, a contradiction. □

Since the 0-segment A is stationary, Lemma 1.3 (3) with Fact 1 and 3 shows that Y_1 has a minimal element. Now Claim 11 (1) shows that $y_0 {}^\wedge \langle \min X_\alpha : \alpha_0 \leq \alpha \rangle$ is the minimal element of B in X, which contradicts our case (=Case 3). □

Now let $Z_0 = \prod_{\alpha \leq \alpha_0} X_\alpha$, $Z_1 = \prod_{\alpha_0 < \alpha} X_\alpha$ and

$$A^* = \{z \in Z_0 : z \restriction \alpha_0 <_{Y_0} y_0 \text{ or } (z \restriction \alpha_0 = y_0 \text{ and } z(\alpha_0) \in A_0).\}.$$

Observe that A^* is a 0-segment of Z_0 and $A^* = (\leftarrow, y_0)_{Y_0} \times X_{\alpha_0} \cup \{y_0\} \times A_0$. Since $\{y_0\} \times A_0$ is a 1-segment of A^* because of $A_0 \neq \emptyset$, Lemma 1.2 shows that $0\text{-}\mathrm{cf}_{Z_0} A^*$ is equal to $0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0$ and that the stationarity of A^* is equivalent to the stationarity of A_0.

Claim 14. $A = A^* \times Z_1$.

Proof. The inclusion $\subseteq$ follows from Claim 10 (1) and the definition of A_0. The inclusion $\supseteq$ follows from Claim 10 (2) and the definition of A_0. □

We divide Case 3 into two subcases.

Case 3-1. $0\text{-}\mathrm{cf}_{Z_0} A^* \geq \omega$.

In this case, since A is stationary, Lemma 1.3 (3b) with Claim 14 shows that Z_1 has a minimal element (so $\sup J^- \leq \alpha_0$) and the 0-segment A^* is stationary (so the 0-segment A_0 is stationary), which contradicts our condition (2b).

Case 3-2. $0\text{-}\mathrm{cf}_{Z_0} A^* = 1$, that is, $\max A^*$ exists.

In this case, note $\max A^* = y_0 {}^\wedge \langle \max A_0 \rangle$. Since $A = A^* \times Z_1$, A has no maximal element but A^* has a maximal element, we see Z_1 has no maximal element. So let $\alpha_1 = \min\{\alpha_0 < \alpha : X_\alpha \text{ has no maximal element.}\}$. Note that X_α has a maximal element for each $\alpha \in (\alpha_0, \alpha_1)$. Since $A = A^* \times Z_1 = (A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha) \times \prod_{\alpha_1 < \alpha} X_\alpha$ and $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$

is a 0-segment in $\prod_{\alpha\leq\alpha_1} X_\alpha$ with no maximal element, Lemma 1.3 (3b) shows that the 0-segment $A^* \times \prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha$ is stationary and $\prod_{\alpha_1<\alpha} X_\alpha$ has a minimal element (so $\sup J^- \leq \alpha_1$). Moreover since $\{y_0 {}^\wedge\langle\max A_0\rangle^\wedge\langle\max X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle\} \times X_{\alpha_1}$ is a 1-segment in the stationary 0-segment $A^* \times \prod_{\alpha_0<\alpha\leq\alpha_1} X_\alpha$, Lemma 1.2 shows that the 0-segment X_{α_1} is also stationary, which contradicts our condition (2b). □

Analogously we see the following.

Theorem 3.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) *X is hereditarily 1-paracompact,*
(2) *the following clauses hold:*
  (a) *$\gamma < \sup J^+ + \omega_1$,*
  (b) *for every $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, X_α is hereditarily 1-paracompact,*

4. Some applications

In this section, we apply the theorems in the previous section to some special cases.

Corollary 4.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If X_α has both a minimal and a maximal element for every $\alpha < \gamma$, then the following are equivalent:*

(1) *X is hereditarily paracompact,*
(2) *the following clauses hold:*
  (a) *$\gamma < \omega_1$,*
  (b) *for every $\alpha < \gamma$, X_α is hereditarily paracompact,*

Proof. By the assumption, we have $J^- = J^+ = \emptyset$, then apply Theorems 3.2 and 3.3. □

Corollary 4.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If X_α has neither a minimal nor a maximal element for every $\alpha < \gamma$, then the following are equivalent:*

(1) *X is hereditarily paracompact,*
(2) *if γ is successor, then $X_{\gamma-1}$ is hereditarily paracompact, where $\gamma - 1$ is the immediate predecessor of γ,*

thus note that if γ is limit, then X is hereditarily paracompact.

Proof. By the assumption, we have $J^- = J^+ = \gamma$. So note that $\sup J^- = \sup J^+ = \gamma$ whenever γ is limit and that $\sup J^- = \sup J^+ = \gamma - 1$ whenever γ is successor. Then apply Theorems 3.2 and 3.3. □

Example 4.3. The corollary above shows that the lexicographic products $\mathbb{S}^\gamma$, $\mathbb{M}^\gamma$, $\mathbb{R}^\gamma$ and $(0,1)_{\mathbb{R}}^\gamma$ are hereditarily paracompact for every ordinal γ.

Applying the theorems directly we can also see the following.

Corollary 4.4. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $\sup J^- = \sup J^+ = \gamma$, then X is hereditarily paracompact,*

Here remark that $\sup J^- = \gamma$ implies that γ is limit.

Example 4.5. The corollary above shows that $(\omega_1^2 \times (-\omega_1)^3)^{\omega_1}$ is hereditarily paracompact, where for a GO-space $X = \langle X, <_X, \tau_X \rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X \rangle$ which is called the reverse of X, see [5]. Note that $-X$ is topologically homeomorphic to X, because the identity map on X to $-X$ $(= X)$ is 1-order preserving and homeomorphism. Also note that the lexicographic products ω_1^ω and $\omega_1^{\omega_1}$ are not paracompact [5].

Next we consider the case that all X_α's have minimal elements. Theorems 3.2 and 3.3 yield the following.

Corollary 4.6. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If X_α has a minimal element for every $\alpha < \gamma$, then the following are equivalent:*

(1) *X is hereditarily paracompact,*
(2) *the following clauses hold:*
    (a) $\gamma < \omega_1$,
    (b) *for every $\alpha < \gamma$, X_α is hereditarily 0-paracompact,*
    (c) *for every $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, X_α is hereditarily 1-paracompact.*

Therefore we have the following.

Corollary 4.7. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If X_α has a minimal element but has no maximal element for every $\alpha < \gamma$, then the following are equivalent:*

(1) *X is hereditarily paracompact,*
(2) *the following clauses hold:*
    (a) $\gamma < \omega_1$,
    (b) *for every $\alpha < \gamma$, X_α is hereditarily 0-paracompact,*
    (c) *if γ is successor, then $X_{\gamma-1}$ is hereditarily 1-paracompact.*

Now we consider hereditary paracompactness of X^γ.

Corollary 4.8. *Let X be a GO-space. Then the following hold:*

(1) *when X has both a minimal and a maximal element, the lexicographic product X^γ is hereditarily paracompact iff $\gamma < \omega_1$ and X is hereditarily paracompact,*
(2) *when X has neither a minimal nor a maximal element, the lexicographic product X^γ is hereditarily paracompact iff X is hereditarily paracompact whenever γ is successor,*
(3) *when X has a minimal element but has no maximal element, the lexicographic product X^γ is hereditarily paracompact iff $\gamma < \omega_1$, X is hereditarily 0-paracompact and "if γ is successor, then X is hereditarily 1-paracompact".*

Example 4.9. The corollary above shows the following:
(1) the lexicographic product $[0,1]_{\mathbb{R}}^\gamma$ is hereditarily paracompact iff $\gamma < \omega_1$, see [2, page 73],
(2) the lexicographic product 2^γ is hereditarily paracompact iff $\gamma < \omega_1$, where $2 = \{0,1\}$ with $0 < 1$,
(3) the lexicographic product $[0,1)_{\mathbb{R}}^\gamma$ is hereditarily paracompact iff $\gamma < \omega_1$.

Example 4.10. Applying Theorems 3.2 and 3.3 directly, we see:
(1) the lexicographic product $[0,1]_{\mathbb{R}}^{\omega_1} \times \mathbb{S}^{\omega_1}$ is hereditarily paracompact,
(2) the lexicographic product $\mathbb{S}^{\omega_1} \times [0,1]_{\mathbb{R}}^{\omega_1}$ is not hereditarily paracompact,
(3) the lexicographic product $\mathbb{S}^{\omega_1} \times [0,1]_{\mathbb{R}}^{\omega}$ is hereditarily paracompact,
(4) the lexicographic product $(\omega_1 + 1)^\omega \times \mathbb{S}^{\omega_1}$ is hereditarily paracompact,
(5) the lexicographic product $\mathbb{S}^{\omega_1} \times (\omega_1 + 1)^\omega$ is not hereditarily paracompact,
(6) the lexicographic product $\mathbb{S}^{\omega_1} \times [0,1)_{\mathbb{R}}^{\omega}$ is hereditarily paracompact,
(7) the lexicographic product $\mathbb{S}^{\omega_1} \times [0,1)_{\mathbb{R}}^{\omega_1}$ is not hereditarily paracompact,
(8) the lexicographic product $[0,1)_{\mathbb{R}}^{\omega} \times \mathbb{S}^{\omega_1}$ is hereditarily paracompact,

Note that all spaces in Examples 4.9 and 4.10 are paracompact.

Finally we discuss on hereditarily paracompactness of lexicographic products of ordinal subspaces. Note that whenever X is a subspace of an ordinal, then X has a minimal element, more generally, all nonempty 1-segment of X has a minimal element. Therefore when $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic product of subspaces of ordinals, we see:

- $J^- = \emptyset$,
- X_α is hereditarily 1-paracompact for every $\alpha < \gamma$.

So Corollary 4.6 yields the following.

Corollary 4.11. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of subspaces of ordinals. Then the following are equivalent:*

(1) *X is hereditarily paracompact,*
(2) *the following clauses hold:*
  (a) $\gamma < \omega_1$,
  (b) *for every $\alpha < \gamma$, X_α is hereditarily* (0-)*paracompact,*

In particular, when X is an ordinal, X is hereditarily paracompact iff it is a countable ordinal. So we have the following.

Corollary 4.12. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of ordinals. Then the following are equivalent:*

(1) *X is hereditarily paracompact,*
(2) *$\gamma < \omega_1$ and for every $\alpha < \gamma$, X_α is a countable ordinal.*

Example 4.13. The corollary above shows the following, where $\mathbb{Z}$ denotes the GO-space of all integers with the usual order:

(1) the lexicographic product $(\omega + \omega)^{\omega+\omega}$ is hereditarily paracompact,
(2) the lexicographic product $(\omega + \omega)^{\omega_1}$ is paracompact but not hereditarily paracompact, on the other hand, the lexicographic product $\mathbb{Z}^{\omega_1}$ is hereditarily paracompact

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