HEREDITARY PARACOMPACTNESS OF LEXICOGRAPHIC PRODUCTS

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Abstract. Paracompactness and hereditary paracompactness of lexicographic products of LOTS’s are discussed in [2]. For instance, it is known in [2]:

- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS’s is paracompact whenever all $X_\alpha$’s are paracompact [2, Theorem 4.2.2],
- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS’s is hereditarily paracompact whenever $\gamma < \omega_1$ and all $X_\alpha$’s are hereditarily paracompact [2, Theorem 4.2.3],
- the lexicographic product $[0, 1]_\mathbb{R}^\omega$ is not hereditarily paracompact, where $[0, 1]_\mathbb{R}$ denotes the unit interval in the real line $\mathbb{R}$ [2, page 73].

Recently the author defined the notion of lexicographic products of GO-spaces and extended the first result above in [2] for lexicographic products of GO-spaces [4]. In this paper, we characterize the hereditary paracompactness of lexicographic products of GO-spaces and get some applications. For example, we see:

- the lexicographic products $\mathbb{S}^\gamma$, $\mathbb{M}^\gamma$, $\mathbb{R}^\gamma$ and $(0, 1)_\mathbb{R}^\gamma$ are hereditarily paracompact for every ordinal $\gamma$, where $\mathbb{S}$ and $\mathbb{M}$ denote the Sorgenfrey line and Michael line respectively,
- the lexicographic product $[0, 1]_\mathbb{R}^\omega$ is hereditarily paracompact, but the lexicographic product $[0, 1]_\mathbb{R}^{\omega_1}$ is not hereditarily paracompact,
- the lexicographic product $\omega_1 \times [0, 1]_\mathbb{R}$ is hereditarily paracompact but the lexicographic product $\omega_1 \times [0, 1]_\mathbb{R}$ is not paracompact,
- the lexicographic product $(\omega^2_1 \times (-\omega_1)^3)^{\omega_1}$ is hereditarily paracompact, but the lexicographic products $\omega^2_1$ and $\omega^{\omega_1}_1$ are not paracompact, where for a GO-space $X = (X, <_X, \tau_X)$, $\neg X$ denotes the GO-space $(X, >_X, \tau_X)$.

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1. Introduction

All spaces are assumed to be regular $T_1$ and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all $X_\alpha$’s are assumed to have cardinality at least 2 with $\gamma \geq 2$. Moreover, in this paper, $\prod_{\alpha<\gamma} X_\alpha$ usually means a lexicographic product defined below. Set theoretical and topological terminology follow [7] and [1]. The following are known:

- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS’s is paracompact whenever all $X_\alpha$’s are paracompact [2, Theorem 4.2.2],
- a lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of LOTS’s is hereditarily paracompact whenever $\gamma < \omega_1$ and all $X_\alpha$’s are hereditarily paracompact [2, Theorem 4.2.3],
- the lexicographic product $[0, 1]_R^\gamma$ is not hereditarily paracompact, where $[0, 1]_R$ denotes the unit interval in the real line $\mathbb{R}$ [2, page 73].

Recently the author defined the notion of lexicographic product of GO-spaces and extended the first result above for lexicographic products of GO-spaces [4]. Therefore we see:

- lexicographic products $S^\gamma$, $M^\gamma$, $R^\gamma$, $(0, 1)_R^\gamma$ and $[0, 1]_R^\gamma$ are paracompact for every ordinal $\gamma$, where $S$ and $M$ denote the Sorgenfrey line and Michael line respectively.

Since $\mathbb{R}$, $S$ and $M$ are hereditarily paracompact, it is natural to ask whether $S^\gamma$, $M^\gamma$, $\mathbb{R}^\gamma$, $(0, 1)_R^\gamma$ and $[0, 1]_R^\gamma$ are hereditarily paracompact even if $\gamma \geq \omega_1$. In this paper, we characterize the hereditary paracompactness of lexicographic products of GO-spaces. Applying this characterization, we see:

- lexicographic products $S^\gamma$, $M^\gamma$, $\mathbb{R}^\gamma$ and $(0, 1)_R^\gamma$ are hereditarily paracompact for every ordinal $\gamma$,
- the lexicographic product $[0, 1]_R^\gamma$ is hereditarily paracompact, but the lexicographic product $[0, 1]_R^{\omega_1}$ is paracompact but not hereditarily paracompact,
- the lexicographic product $\omega_1 \times (0, 1]_R$ is hereditarily paracompact but the lexicographic product $\omega_1 \times [0, 1]_R$ is not paracompact,
- the lexicographic product $(\omega_1^2 \times (-\omega_1)^3)\omega_1$ is hereditarily paracompact, but the lexicographic products $\omega_1^n$ and $\omega_1^\omega$ are not paracompact, where for a GO-space $X = \langle X, <_X, \tau_X \rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X \rangle$.

A linearly ordered set $\langle L, <_L \rangle$ has a natural topology $\lambda_L$, which is called an interval topology, generated by $\{(x, y)_L : x \in L \} \cup \{(x, \rightarrow)_L : x \in L \}$ as a subbase, where $(x, \rightarrow)_L = \{ z \in L : x <_L z \}$, $(x, y)_L = \{ z \in L : x <_L z <_L y \}$,
\{z \in L : x <_L z <_L y\}, (x, y)_L = \{z \in L : x <_L z \leq_L y\} and so on. The triple \(\langle L, <_L, \lambda_L \rangle\), which is simply denoted by \(L\), is called a LOTS.

A triple \(\langle X, <_X, \tau_X \rangle\) is said to be a GO-space, which is also simply denoted by \(X\), if \(\langle X, <_X \rangle\) is a linearly ordered set and \(\tau_X\) is a \(T_2\)-
topology on \(X\) having a base consisting of convex sets, where a subset \(C\) of \(X\) is convex if for every \(x, y \in C\) with \(x <_X y\), \([x, y]_X \subseteq C\) holds. For more information on LOTS’s or GO-spaces, see [8]. Usually \(<_L, (x, y)_L, \lambda_L\) or \(\tau_X\) are written simply \(<\), \((x, y)\), \(\lambda\) or \(\tau\) if contexts are clear.

The symbols \(\omega\) and \(\omega_1\) denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters \(\alpha, \beta, \gamma, \cdots\), are considered to be LOTS’s with the usual interval topology. For a subset \(A\) of an ordinal \(\alpha\), \(\text{Lim}(A)\) denotes the set \(\{\beta < \alpha : \beta = \sup(A \cap \beta)\}\), that is, the set of all cluster points of \(A\) in the topological space \(\alpha\). The cofinality of \(\alpha\) is denoted by \(\text{cf} \alpha\).

For GO-spaces \(X = \langle X, <_X, \tau_X \rangle\) and \(Y = \langle Y, <_Y, \tau_Y \rangle\), \(X\) is said to be a subspace of \(Y\) if \(X \subseteq Y\), the linear order \(<_X\) is the restriction \(<_Y\rvert X\) of the order \(<_Y\) and the topology \(\tau_X\) is the subspace topology \(\tau_Y \rvert X = \{U \cap X : U \in \tau_Y\}\) on \(X\) of the topology \(\tau_Y\). So a subset of a GO-space is naturally considered as a GO-space. For every GO-space \(X\), there is a LOTS \(X^*\) such that \(X\) is a dense subspace of \(X^*\) and \(X^*\) has the property that if \(L\) is a LOTS containing \(X\) as a dense subspace, then \(L\) also contains the LOTS \(X^*\) as a subspace, see [9]. Such a \(X^*\) is called the minimal \(d\)-extension of a GO-space \(X\). Indeed, \(X^*\) is constructed as follows, also see [4]. Let \(X^+ = \{x \in X : (\leftarrow, x) \in \tau_X \setminus \lambda_X\}\) and \(X^- = \{x \in X : (x, \rightarrow) \in \tau_X \setminus \lambda_X\}\). Then \(X^*\) is the LOTS \(X^- \times \{\{-1\}\} \cup X \times \{0\} \cup X \times \{1\}\), where the order \(<_{X^*}\) is the restriction of the usual lexicographic order on \(X \times \{-1, 0, 1\}\). Also we identify as \(X = X \times \{0\}\) in the obvious way.

Then, we can see:

- if \(X\) is a LOTS, then \(X^* = X\),
- \(X\) has a maximal element \(\max X\) if and only if \(X^*\) has a maximal element \(\max X^*\), in this case, \(\max X = \max X^*\) (similarly for minimal elements).

For every \(\alpha < \gamma\), let \(X_\alpha\) be a LOTS and \(X = \prod_{\alpha < \gamma} X_\alpha\). Every element \(x \in X\) is identified with the sequence \(\langle x(\alpha) : \alpha < \gamma \rangle\). The lexicographic order \(<_X\) on \(X\) is defined as follows: for every \(x, x' \in X\),

\(x <_X x'\) iff for some \(\alpha < \gamma\), \(x \upharpoonright \alpha = x' \upharpoonright \alpha\) and \(x(\alpha) <_{X_\alpha} x'(\alpha)\),

where \(x \upharpoonright \alpha = \langle x(\beta) : \beta < \alpha \rangle\) and \(<_{X_\alpha}\) is the order on \(X_\alpha\). Now for every \(\alpha < \gamma\), let \(X_\alpha\) be a GO-space and \(X = \prod_{\alpha < \gamma} X_\alpha\). The
subspace $X$ of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ is said to be
the lexicographic product of GO-spaces $X_\alpha$’s, for more details see [4].
$\prod_{i\in\omega} X_i \ (\prod_{i\leq n} X_i$ where $n \in \omega)$ is denoted by $X_0 \times X_1 \times X_2 \times \cdots$
$(X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by
$X^\gamma$ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let $X$ and $Y$ be LOTS’s. A map $f : X \to Y$ is said to be order
preserving or 0-order preserving if $f(x) <_Y f(x')$ whenever $x <_X x'$.
Similarly a map $f : X \to Y$ is said to be order reversing or 1-order
preserving if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order
preserving map (also 1-order preserving map) $f : X \to Y$ between
LOTS’s $X$ and $Y$, which is onto, is a homeomorphism, i.e., both $f$
and $f^{-1}$ are continuous. Now let $X$ and $Y$ be GO-spaces. A 0-order
preserving map $f : X \to Y$ is said to be 0-order preserving embedding
if $f$ is a homeomorphism between $X$ and $f[X]$, where $f[X]$ is the
subspace of the GO-space $Y$. In this case, we identify $X$ with $f[X]$ as
a GO-space and write $X = f[X]$ and $X \subseteq Y$.

Recall that a subset of a regular uncountable cardinal $\kappa$ is called
stationary if it intersects with all closed unbounded (= club) sets in $\kappa$.

Let $X$ be a GO-space. A subset $A$ of $X$ is called a 0-segment of $X$
if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. Similarly the
notion of 1-segment can be defined. Both $\emptyset$ and $X$ are 0-segments and
1-segments. Obviously, if $A$ is a 0-segment, then $X \setminus A$ is a 1-segment.

Let $A$ be a 0-segment of a GO-space $X$. A subset $U$ of $A$ is unbounded
in $A$ if for every $x \in A$, there is $x' \in U$ such that $x \leq x'$. Let

$$0\text{-}\text{cf}_X A = \min\{|U| : U \text{ is unbounded in } A\}.$$  

$0\text{-}\text{cf}_X A$ can be 0, 1 or a regular infinite cardinal, see also [3, 5, 6]. If
contexts are clear, $0\text{-}\text{cf}_X A$ is denoted by $0\text{-}\text{cf} A$. A 0-segment $A$ of a
GO-space $X$ is said to be stationary if $\kappa := 0\text{-}\text{cf} A \geq \omega_1$ and there are
a stationary set $S$ of $\kappa$ and a continuous map $\pi : S \to A$ such that $\pi[S]
$ is unbounded in $A$ (we say such a $\pi$ “an unbounded continuous map”).

Note that for a subspace $S$ of a regular uncountable cardinal $\kappa$, $S$
is stationary in $\kappa$ in the usual sense if and only if the 0-segment $S$ in the
GO-space $S$ is stationary in the sense above (e.g., use [5, Lemma 2.7]).
So this new term “stationarity of 0-segments” extends the usual term
“stationarity of subsets of a regular uncountable cardinal”.

A GO-space $X$ is said to be 0-paracompact if every closed 0-segment
is not stationary. Similarly the notions of 1-cf $A$, stationarity of a
1-segment and 1-paracompactness are defined. Remember that a GO-
space is paracompact if and only if it is both 0-paracompact and 1-
paracompact, see [4], where a topological space is paracompact if every
open cover has a locally finite open refinement [1]. It is well-known
that stationary sets of some regular uncountable cardinal are not paracompact. We frequently use the following basic lemmas from [5].

**Lemma 1.1.** [5, Lemma 2.7] Let $A$ be a 0-segment of a GO-space $X$ with $\kappa := 0-\text{cf } A \geq \omega_1$. If there are a stationary set $S$ of $\kappa$ and an unbounded continuous map $\pi : S \to A$, then there is a club set $C$ in $\kappa$ such that $\pi | (S \cap C) : S \cap C \to A$ is 0-order preserving embedding.

**Lemma 1.2.** [5, Lemma 3.4] Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and $u \in X_0$. Then the map $k_u : X_1 \to \{u\} \times X_1$ by $k_u(v) = \langle u, v \rangle$ is a 0-order preserving homeomorphism.

**Lemma 1.3.** [5, Lemma 3.6] Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and $A_0$ a 0-segment of $X_0$. Put $A = A_0 \times X_1$. Then the following hold:

1. $A$ is a 0-segment of $X$,
2. If $0-\text{cf}_{X_0} A_0 = 1$, then
   a) $0-\text{cf}_X A = 0-\text{cf}_{X_1} X_1$,
   b) $A$ is stationary if and only if the 0-segment $X_1$ is stationary,
3. If $0-\text{cf}_{X_0} A_0 \geq \omega$, then
   a) $0-\text{cf}_X A = 0-\text{cf}_{X_0} A_0$,
   b) $A$ is stationary if and only if $X_1$ has a minimal element and $A_0$ is stationary.

A GO-space $X$ is said to be hereditarily 0-paracompact if every 0-segment $A$ of $X$ is not stationary, similarly the notion of hereditary 1-paracompactness is defined. We can see the naming of these definitions are reasonable from the lemma below, where a topological space is hereditarily paracompact if all subspaces are paracompact.

**Lemma 1.4.** Let $X$ be a GO-space. Then $X$ is hereditarily paracompact if and only if it is both hereditarily 0-paracompact and hereditarily 1-paracompact.

**Proof.** First assume that $X$ is hereditarily paracompact and that $X$ is not hereditarily 0-paracompact, then there is a stationary 0-segment $A$ of $X$. Lemma 1.1 shows that $A$ has a copy of a stationary set of some regular uncountable cardinal, a contradiction. So $X$ is hereditarily 0-paracompact. Similarly $X$ is hereditarily 1-paracompact.

Next assume that there is a non-paracompact subspace $Y$ of $X$. We may assume that $Y$ is not 0-paracompact. So there is a closed stationary 0-segment $A$ of $Y$. Set $A' = \{ x \in X : \exists y \in A(x \leq y) \}$. Then it is easy to verify that $A'$ is also a stationary (need not be closed) 0-segment of $X$, which means that $X$ is not hereditarily 0-paracompact. \[\square\]
2. Products of two GO-spaces

In this section, we characterize the hereditary paracompactness of a lexicographic product \( X = X_0 \times X_1 \) of two GO-spaces.

**Lemma 2.1.** Let \( X = X_0 \times X_1 \) be a lexicographic product of GO-spaces. Then the following are equivalent:

1. \( X \) is hereditarily 0-paracompact,
2. the following clauses hold:
   
   a. \( X_1 \) is hereditarily 0-paracompact,
   b. if \( X_1 \) has a minimal element, then \( X_0 \) is hereditarily 0-paracompact.

**Proof.** (1) \( \Rightarrow \) (2) Assume that \( X \) is hereditarily 0-paracompact.

a. Assuming that \( X_1 \) is not hereditarily 0-paracompact, take a stationary 0-segment \( A_1 \) of \( X_1 \). Fixing \( u \in X_0 \), let \( A = \{ x \in X : \exists v \in A_1 : x \leq \langle u, v \rangle \} \). Obviously \( A \) is a 0-segment of \( X \). Since \( \{ u \} \times A_1 \) is a 1-segment (i.e., final segment) of \( A \), Lemma 1.2 shows that the 0-segment \( A \) is also stationary, a contradiction.

b. Assume that \( X_1 \) has a minimal element but \( X_0 \) is not hereditarily 0-paracompact. Taking a stationary 0-segment \( A_0 \) of \( X_0 \), let \( A = A_0 \times X_1 \). Then Lemma 1.3 (3b) shows that \( A \) is a stationary 0-segment of \( X \), a contradiction.

(2) \( \Rightarrow \) (1) Assuming (2) and the negation of (1), take a stationary 0-segment \( A \) of \( X \). Let \( A_0 = \{ u \in X_0 : \exists v \in X_1 : \langle u, v \rangle \in A \} \). Obviously \( A_0 \) is a non-empty 0-segment of \( X_0 \) with \( A \subseteq A_0 \times X_1 \). Assume that \( A_0 \) has a maximal element \( \max A_0 \) and let \( A_1 = \{ v \in X_1 : \langle \max A_0, v \rangle \in A \} \). Since \( \{ \max A_0 \} \times A_1 \) is a 1-segment of \( A \), Lemma 1.2 shows that \( A_1 \) is a stationary 0-segment of \( X_1 \), which contradicts the condition (2a).

Thus we see that \( A_0 \) has no maximal element, that is \( 0-\text{cf}_{X_0} A_0 \geq \omega \).

**Claim.** \( A = A_0 \times X_1 \).

**Proof.** The inclusion \( \subseteq \) is obvious. To see the inclusion \( \supseteq \), let \( x \in A_0 \times X_1 \). Since \( A_0 \) has no maximal element, we can take \( u \in A_0 \) with \( x(0) < u \). By \( u \in A_0 \), we can find \( v \in X_1 \) with \( \langle u, v \rangle \in A \). Then we have \( x < \langle u, v \rangle \). Now since \( A \) is a 0-segment, we see \( x \in A \). \( \square \)

Now Lemma 1.3 (3b) shows that \( X_1 \) has a minimal element and the 0-segment \( A_0 \) is stationary, which contradicts the condition (2b). \( \square \)

Analogously we see:

**Lemma 2.2.** Let \( X = X_0 \times X_1 \) be a lexicographic product of GO-spaces. Then the following are equivalent:
(1) $X$ is hereditarily 1-paracompact,
(2) the following clauses hold:
   (a) $X_1$ is hereditarily 1-paracompact,
   (b) if $X_1$ has a maximal element, then $X_0$ is hereditarily 1-paracompact.

The lemmas above show:

**Lemma 2.3.** Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then the following are equivalent:

(1) $X$ is hereditarily paracompact,
(2) the following clauses hold:
   (a) $X_1$ is hereditarily paracompact,
   (b) if $X_1$ has a minimal element, then $X_0$ is hereditarily 0-paracompact.
   (c) if $X_1$ has a maximal element, then $X_0$ is hereditarily 1-paracompact.

**Example 2.4.** The lemma above shows that $\omega_1 \times R$, $\omega_1 \times S$ and $\omega_1 \times M$ are hereditarily paracompact. But $\omega_1 \times [0,1]_R$ is not paracompact [5]. On the other hand, $\omega_1 \times (0,1]_R$ is hereditarily paracompact, indeed $\omega_1$ is hereditarily 1-paracompact because it is well-ordered.

### 3. Products of any length of GO-spaces

In this section, we characterize the hereditarily paracompactness of lexicographic products of any length of GO-spaces. The following notations are introduced in [4, Theorem 2.5]

**Definition 3.1.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. We use the following notations.

\[ J^+ = \{ \alpha < \gamma : X_\alpha \text{ has no maximal element.} \}, \]
\[ J^- = \{ \alpha < \gamma : X_\alpha \text{ has no minimal element.} \}. \]

Note $\sup J^+ \leq \gamma$ and $\sup J^- \leq \gamma$.

**Theorem 3.2.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:

(1) $X$ is hereditarily 0-paracompact,
(2) the following clauses hold:
   (a) $\gamma < \sup J^- + \omega_1$, where $\sup J^- + \omega_1$ is the usual ordinal sum,
   (b) for every $\alpha < \gamma$ with $\sup J^- \leq \alpha$, $X_\alpha$ is hereditarily 0-paracompact,
Proof. Let $\tilde{X} = \prod_{\alpha < \gamma} X_\alpha^*$ be the lexicographic product of LOTS’s $X_\alpha^*$’s.

(1) $\Rightarrow$ (2) Assume that $X$ is hereditarily 0-paracompact.

(a) Assume $\sup J^\gamma + \omega_1 \leq \gamma$. Letting $\alpha_0 = \sup J^\gamma$, fix $z \in \prod_{\alpha \leq \alpha_0} X_\alpha$. For every $\alpha < \gamma$ with $\alpha_0 < \alpha$, noting that $\min X_\alpha$ exists, fix $u(\alpha) \in X_\alpha$ with $\min X_\alpha < u(\alpha)$. First let $x = z^\gamma(\alpha), \alpha_0 < \alpha < \alpha_0 + \omega_1) = (\min X_\alpha : x_0 + \omega_1 < \alpha < \gamma)$, that is, $x$ is an element in $X$ such that $x(\alpha) = z(\alpha)$ when $\alpha \leq \alpha_0, x(\alpha) = u(\alpha)$ when $\alpha_0 < \alpha < \alpha_0 + \omega_1$, and $x(\alpha) = \min X_\alpha$ when $\alpha_0 + \omega_1 \leq \alpha < \gamma$. Next for $\beta < \omega_1$ with $1 < \beta$, let $x^\beta = z^\beta(\alpha), \alpha_0 < \alpha < \alpha_0 + \beta) = (\min X_\alpha : (\alpha_0 + \beta) < \alpha < \gamma)$. Set $A = (\langle x, x \rangle_X$ and $S = (1, \omega_1)$, and define $\pi : S \to A$ by $\pi(\beta) = x^\beta$. Obviously $\pi$ is 0-order preserving and unbounded (i.e., $\pi(\beta) \to \pi(\beta') < \pi(\beta)$) and $\pi[S]$ is unbounded in the 0-segment $A$.

Claim 1. $\pi$ is continuous.

Proof. Let $\beta \in S$ and $U$ be an open neighborhood of $\pi(\beta)$. We may assume $\beta \in \text{Lim}(S)$. Note $(\langle x, x \rangle_X) \neq \emptyset$. Then there is $y^* \in \tilde{X}$ with $y^* < \pi(\beta)$ and $(y^*, \pi(\beta)) \subseteq U$. Let $\beta_0 = \min\{\alpha < \gamma : y^*(\alpha) \not> \pi(\beta)(\alpha)\}$. The definition of $x^\beta = \pi(\beta)$ shows $x^\beta < \alpha_0 + \beta$. When $\beta_0 \leq \alpha_0$, obviously $\pi[S \cap (\beta + 1)] \subseteq U$ holds. So assuming $\alpha_0 < \beta_0 < \alpha_0 + \beta$, $\beta_0$ can be represented as $\beta_0 = \alpha_0 + \beta_1$ for some $\beta_1 < \beta$ with $0 < \beta_1$. Then for each $\beta' \in (\beta_1, \beta)$, we have $y^* < x^{\beta'} \leq x_\beta$. Therefore we see $\pi[S \cap (\beta_1, \beta)] \subseteq U$, so we have seen that $\pi$ is continuous.

Now since $S$ is stationary in $\omega_1$, the 0-segment $A$ is stationary, which contradicts the hereditary 0-paracompactness of $X$.

(b) Let sup $J^\gamma < \alpha_0 < \gamma$ and let $Y_0 = \prod_{\alpha \leq \alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha < \alpha_0} X_\alpha$ be lexicographic products. Then $X$ is identified with the lexicographic product $Y_0 \times Y_1$ [4, Lemma 1.5], where $X$ is identified with $Y_0$ whenever $\alpha_0 + 1 = \gamma$. Since $X = Y_0 \times Y_1$ is hereditarily 0-paracompact and $Y_1$ has the minimal element (min $X_\alpha : \alpha_0 < \alpha$), Lemma 2.1 (2b) shows that $Y_0$ is hereditarily 0-paracompact. Here note that $Y_0$ is itself hereditarily 0-paracompact whenever $X = Y_0$, so we will not mention such special cases. Now $Y_0 = \prod_{\alpha < \alpha_0} X_\alpha \times X_\alpha$ and Lemma 2.1 (2a) shows that $X_\alpha$ is hereditarily 0-paracompact.

(2) $\Rightarrow$ (1) Assume (2) and the negation of (1), then one can take a stationary 0-segment $A$ of $X$. We consider three cases and their subcases and in all cases, we will get contradictions. This argument is shown in [5, Theorem 4.8].

Case 1. $A = X$.

Since $A = X$ has no maximal element, $X_\alpha$ has no maximal element for some $\alpha < \gamma$. Let $\alpha_0 = \min\{\alpha < \gamma : X_\alpha$ has no maximal element.$\}$. 

Since $A = X = \prod_{\alpha \leq \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$, the 0-segment $A$ is stationary and $\prod_{\alpha \leq \alpha_0} X_\alpha$ has no maximal element, Lemma 1.3 (3b) shows that the 0-segment $\prod_{\alpha \leq \alpha_0} X_\alpha$ is stationary and $\prod_{\alpha_0 < \alpha} X_\alpha$ has a minimal element. Therefore $X_\alpha$ has a minimal element for every $\alpha > \alpha_0$, which means $\sup J^{-} \leq \alpha_0$. By the minimality of $\alpha_0$, $X_\alpha$ has a maximal element for every $\alpha < \alpha_0$. Then $\{\max X_\alpha : \alpha < \alpha_0\} \times X_{\alpha_0}$ is a 1-segment of $\prod_{\alpha_0 < \alpha} X_\alpha \times X_{\alpha_0}$. Now since the 0-segment $\prod_{\alpha_0 < \alpha} X_\alpha \times X_{\alpha_0}$ is stationary, Lemma 1.2 shows that the 0-segment $X_{\alpha_0}$ is also stationary, this contradicts the condition (2b).

Case 2. $A \neq X$ and $X \setminus A$ has a minimal element.

Let $B = X \setminus A$ and $b = \min B$, then note $A = (\leftarrow, b)_X$. Set $I = \{\alpha < \gamma : \exists a \in A(a \uparrow (\alpha + 1) = b \uparrow (\alpha + 1))\}$. Since $I$ is obviously a 0-segment of $\gamma$, for some $\alpha_0 \leq \gamma$, $I = \alpha_0$ holds. Now for every $\alpha < \alpha_0$, fix $a_\alpha \in A$ with $a_\alpha \uparrow (\alpha + 1) = b \uparrow (\alpha + 1)$.

Claim 2. For every $\alpha \in (\alpha_0, \gamma)$, $X_\alpha$ has a minimal element and $b(\alpha) = \min X_\alpha$, thus $\sup J^{-} \leq \alpha_0$.

Proof. Note that still we do not know whether $\alpha_0 < \gamma$ or not. Assume that for some $\alpha \in (\alpha_0, \gamma)$, there is $u \in X_\alpha$ with $u < b(\alpha)$. Let $\alpha_1 = \min\{\alpha > \alpha_0 : \exists u \in X_\alpha(u < b(\alpha))\}$ and take $u \in X_{\alpha_1}$ with $u < b(\alpha_1)$.

Let $a = b \uparrow \alpha_1 \uparrow \uparrow (u \uparrow b(\alpha_1))$. Then by $a < b$, we have $a \in A$ and $a \uparrow \alpha_1 = b \uparrow \alpha_1$. Now $\alpha_0 < \alpha_1$ shows $a \uparrow (\alpha_0 + 1) = b \uparrow (\alpha_0 + 1)$, which means $\alpha_0 \in I = \alpha_0$, a contradiction. □

We divide Case 2 into further two subcases.

Case 2-1. $\alpha_0$ is a successor ordinal.

Say $\alpha_0 = \beta_0 + 1$.

Claim 3. $\alpha_0 < \gamma$.

Proof. If $\alpha_0 = \gamma$ were true, then by $\beta_0 \in \alpha_0 = I$, we have $B \ni b = b \uparrow \alpha_0 = b \uparrow (\beta_0 + 1) = a_{\beta_0} \uparrow (\beta_0 + 1) = a_{\beta_0} \uparrow \alpha_0 = a_{\beta_0} \in A$, a contradiction. □

Claim 4. $b(\alpha_0)$ is not a minimal element of $X_{\alpha_0}$.

Proof. If $b(\alpha_0)$ were a minimal element of $X_{\alpha_0}$, then we have $A \ni a_{\beta_0} \geq b \in B$ because of $b(\alpha) = \min X_\alpha$ for every $\alpha \geq \alpha_0$, a contradiction. □

Let $Y_0 = \prod_{\alpha \leq \alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0 < \alpha} X_\alpha$.

Claim 5. $A = (\leftarrow, b \uparrow (\alpha_0 + 1))_{Y_0} \times Y_1$. 
Proof. To see the inclusion $\subseteq$, let $a \in (\leftarrow, b \restriction (\alpha_0 + 1)) \times Y_1$. Then $a \restriction (\alpha_0 + 1) < b \restriction (\alpha_0 + 1)$ shows $a < b = \min B$. So we have $a \in A$.

To see the inclusion $\supseteq$, let $a \in A$. Since $a < b$ and $b(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$, we have $a \restriction (\alpha_0 + 1) < b \restriction (\alpha_0 + 1)$, thus $a \in (\leftarrow, b \restriction (\alpha_0 + 1)) \times Y_1$.

We further divide Case 2-1 into two subcases.

Case 2-1-1. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ has no maximal element.

In this case, $(\leftarrow, b \restriction (\alpha_0 + 1))_{Y_0}$ has no maximal element, so Claim 5 and Lemma 1.3 (3b) show that the 0-segment $(\leftarrow, b \restriction (\alpha_0 + 1))$ in $Y_0$ is stationary. Then it is easy to see:

Claim 6. $(\leftarrow, b \restriction (\alpha_0 + 1))_{Y_0} = (\leftarrow, b \restriction \alpha_0) \times X_{\alpha_0} \cup \{b \restriction \alpha_0\} \times (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$.

Now Lemma 1.2 show that the 0-segment $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is stationary, because $\{b \restriction \alpha_0\} \times (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is a 1-segment of $(\leftarrow, b \restriction (\alpha_0 + 1))_{Y_0}$ by Claim 6. This contradicts the condition (2b) because of $\sup J^- \leq \alpha_0$.

Case 2-1-2. $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ has a maximal element.

Say $u_0 = \max (\leftarrow, b(\alpha_0))$, then note that $b \restriction \alpha_0 \times u_0$ is the immediate predecessor of $b \restriction (\alpha_0 + 1)$ in $Y_0$, so we see $(\leftarrow, b \restriction (\alpha_0 + 1)) = (\leftarrow, b \restriction \alpha_0)^\times \times X_{\alpha_0} \cup \{b \restriction \alpha_0\} \times (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$. Since $A$ has no maximal element and $A = (\leftarrow, b \restriction \alpha_0)^\times \times X_{\alpha_0}$ (Claim 5), $Y_1$ has no maximal element. So let $\alpha_1 = \min \{\alpha > \alpha_0 : X_\alpha$ has no maximal element\}. Now since $A = (\leftarrow, b \restriction (\alpha_0 + 1)) \times Y_1 = (\leftarrow, b \restriction \alpha_0)^\times \times X_{\alpha_0} \times \times X_{\alpha_0} = (\leftarrow, b \restriction \alpha_0)^\times \times X_{\alpha_0} \times \times X_{\alpha_0}$, $(\leftarrow, b \restriction \alpha_0)^\times \times X_{\alpha_0}$ has no maximal element and the 0-segment $A$ is stationary. Lemma 1.3 (3b) shows that the 0-segment $(\leftarrow, b \restriction \alpha_0)^\times \times \times X_{\alpha_0} \times \times X_{\alpha_0}$ is also stationary. Now since $\{b \restriction \alpha_0\} \times \times X_{\alpha_0}$ has a maximal element $X_{\alpha_1}$ is stationary. Since $\sup J^- \leq \alpha_0 < \alpha_1$, $X_{\alpha_1}$ has to be hereditarily 0-paracompact (condition (2b)), a contradiction.

Case 2-2. $\alpha_0$ is limit.

Claim 2 and the condition (2a) show $\sup J^- \leq \alpha_0 \leq \gamma < \sup J^- + \omega_1$, therefore we have $\cf \alpha_0 = \omega$.

Claim 7. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then note $\cf \gamma = \cf \alpha_0 = \omega$, so fix a 0-order preserving unbounded (i.e., strictly increasing cofinal) sequence $\{\gamma_n : n \in \omega\}$ in $\gamma$. Then $\{a_{\gamma_n} : n \in \omega\}$ is unbounded in the 0-segment $(\leftarrow, b)$.
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We divide Case 2-2 into three subcases.

Case 2-2-1. \((\leftarrow, b(\alpha_0))_{X_{\alpha_0}}\) is non-empty and has no maximal element.
In this case, using a similar argument to Case 2-1-1, we can get a contradiction.

Case 2-2-2. \((\leftarrow, b(\alpha_0))_{X_{\alpha_0}}\) is non-empty and has a maximal element.
In this case, using a similar argument to Case 2-1-2, we can get a contradiction.

Case 2-2-3. \((\leftarrow, b(\alpha_0))_{X_{\alpha_0}}\) is empty, that is, \(b(\alpha_0) = \min_{X_{\alpha_0}}\).
In this case, a 0-order preserving unbounded sequence \(f_n : n \in \omega\) in \(X\), for every \(\alpha < \omega\), we see that \(f_n(a) = b(a)\), so we have 0- \(\text{cf}_X A = \omega\), which contradicts the stationarity of the 0-segment \(A\).

Case 3. \(A \neq X\) and \(X \setminus A\) has no minimal element.
Let \(B = X \setminus A\) and
\[
I = \{\alpha < \gamma : \exists a \in A \exists b \in B \ (a \upharpoonright (\alpha + 1) = b \upharpoonright (\alpha + 1))\}.
\]
Since \(I\) is a 0-segment in \(\gamma\), for some \(\alpha_0 \leq \gamma\), \(I = \alpha_0\) holds. For every \(\alpha < \alpha_0\), fix \(a_\alpha \in A\) and \(b_\alpha \in B\) with \(a_\alpha \upharpoonright (\alpha + 1) = b_\alpha \upharpoonright (\alpha + 1)\) and consider the lexicographic products \(Y_0 = \prod_{\alpha < \alpha_0} X_{\alpha}\) and \(Y_1 = \prod_{\alpha_0 \leq \alpha} X_{\alpha}\). Define \(y_0 \in Y_0\) by \(y_0(\alpha) = a_\alpha(\alpha)\) for every \(\alpha < \alpha_0\).

Claim 8. For every \(\alpha < \alpha_0\), \(y_0 \upharpoonright (\alpha + 1) = a_\alpha \upharpoonright (\alpha + 1) = b_\alpha \upharpoonright (\alpha + 1)\) holds.

Proof. It suffices to see the first equality. Assuming \(y_0 \upharpoonright (\alpha + 1) \neq a_\alpha \upharpoonright (\alpha + 1)\) for some \(\alpha < \alpha_0\), let \(\alpha_1 = \min\{\alpha < \alpha_0 : y_0 \upharpoonright (\alpha + 1) \neq a_\alpha \upharpoonright (\alpha + 1)\}\) and \(\alpha_2 = \min(\alpha \leq \alpha_1 : y_0(\alpha) \neq a_\alpha(\alpha))\). Then \(y_0(\alpha_1) = a_{\alpha_1}(\alpha_1)\) shows \(\alpha_2 < \alpha_1\). Also the minimality of \(\alpha_1\) shows \(y_0 \upharpoonright (\alpha_2 + 1) = a_{\alpha_2} \upharpoonright (\alpha_2 + 1) = \beta(\alpha_2 + 1)\). When \(y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)\), we see \(B \ni b_\alpha < a_{\alpha_1} \in A\), a contradiction. When \(y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)\), we also see \(B \ni b_\alpha < a_{\alpha_2} \in A\), a contradiction.

Claim 9. \(\alpha_0 < \gamma\).

Proof. Assume \(\alpha_0 = \gamma\), then \(y_0 \in Y_0 = X = A \cup B\). Assume \(y_0 \in A\) and take \(a \in A\) with \(y_0 < b_\alpha < a\). Let \(\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq a(\beta)\}\). Then we have \(B \ni b_{\beta_0} < a \in A\), a contradiction. When \(y_0 \in B\), similarly we also get a contradiction.
Let \( A_0 = \{ a(\alpha_0) : a \in A, a \upharpoonright \alpha_0 = y_0 \} \) and \( B_0 = \{ b(\alpha_0) : b \in B, b \upharpoonright \alpha_0 = y_0 \} \).

**Claim 10.** The following hold:

1. for every \( a \in A, a \upharpoonright \alpha_0 \leq_{Y_0} y_0 \) holds,
2. for every \( x \in X, x \upharpoonright \alpha_0 <_{Y_0} y_0 \), then \( x \in A \).

**Proof.**

1. Assume \( a \upharpoonright \alpha_0 > y_0 \) for some \( a \in A \) and let \( \beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\} \). Now we have \( B \ni b_{\beta_0} < a \in A \), a contradiction.

2. Assume \( x \upharpoonright \alpha_0 < y_0 \) and let \( \beta_0 = \min\{\beta < \alpha_0 : x(\beta) \neq y_0(\beta)\} \). Then we have \( x < a_{\beta_0} \in A \), so we see \( x \in A \) because \( A \) is a 0-segment. \( \square \)

We similarly see:

**Claim 11.** The following hold:

1. for every \( b \in B, b \upharpoonright \alpha_0 \geq_{Y_0} y_0 \) holds,
2. for every \( x \in X, x \upharpoonright \alpha_0 >_{Y_0} y_0 \), then \( x \in B \).

**Claim 12.** \( A_0 \) is a 0-segment of \( X_{\alpha_0} \) and \( B_0 = X_{\alpha_0} \setminus A_0 \).

**Proof.** Let \( u' < u \in A_0 \) and take \( a \in A \) with \( a \uparrow (\alpha_0 + 1) = y_0 \uparrow \langle u \rangle \). Let \( a' = (a \uparrow \alpha_0) \uparrow \langle u' \rangle \uparrow (a \uparrow (\alpha_0, \gamma)) \). Since \( A \) is a 0-segment with \( a' < a \in A \), we have \( a' \in A \), thus \( u' \in A_0 \). So we have seen that \( A_0 \) is a 0-segment.

To see \( B_0 \subseteq X_{\alpha_0} \setminus A_0 \), let \( u \in B_0 \). Take \( b \in B \) with \( b \uparrow (\alpha_0 + 1) = y_0 \uparrow \langle u \rangle \). If \( u \in A_0 \) were true, then by taking \( a \in A \) with \( a \uparrow (\alpha_0 + 1) = y_0 \uparrow \langle u \rangle \), we see \( a \uparrow (\alpha_0 + 1) = b \uparrow (\alpha_0 + 1) \) thus \( \alpha_0 \in I = \alpha_0 \), a contradiction. So we have \( u \in X_{\alpha_0} \setminus A_0 \).

To see \( B_0 \supseteq X_{\alpha_0} \setminus A_0 \), let \( u \in X_{\alpha_0} \setminus A_0 \). Take \( x \in X \) with \( x \uparrow (\alpha_0 + 1) = y_0 \uparrow \langle u \rangle \). Then obviously we have \( x \in B \), thus \( u \in B_0 \). \( \square \)

**Claim 13.** \( A_0 \neq \emptyset \).

**Proof.** Assume \( A_0 = \emptyset \). We prove the following facts.

**Fact 1.** \( (\leftarrow, y_0)_{Y_0} \times Y_1 = A \).

**Proof.** Claim 10 (2) shows the inclusion \( \subseteq \). To see the other inclusion, let \( a \in A \). Then Claim 10 (1) shows \( a \uparrow \alpha_0 \leq y_0 \). If \( a \uparrow \alpha_0 = y_0 \) were true, then we have \( a(\alpha_0) \in A_0 \), which contradicts \( A_0 = \emptyset \). \( \square \)

**Fact 2.** \( \alpha_0 > 0 \) and \( \alpha_0 \) is limit.
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Proof. If \( \alpha_0 = 0 \) were true, then taking \( a \in A \), we see \( a(\alpha_0) \in A_0 \), a contradiction. If for some ordinal \( \beta_0, \alpha_0 = \beta_0 + 1 \) were true, then by \( \beta_0 \in I = \alpha_0 \) and \( a_{\beta_0} \upharpoonright \alpha_0 = a_{\beta_0} \upharpoonright (\beta_0 + 1) = y_0 \upharpoonright (\beta_0 + 1) = y_0 \upharpoonright \alpha_0 \), we see \( a_{\beta_0}(\alpha_0) \in A_0 \), a contradiction. \( \square \)

Fact 3. \( 0 \)-\( \text{cf} \) \( (\longleftarrow, y_0) \) \( Y_0 \geq \omega \).

Proof. Fact 1 with \( A \neq \emptyset \) shows \( (\longleftarrow, y_0) \neq \emptyset \), that is, \( 0 \)-\( \text{cf} \) \( (\longleftarrow, y_0) \geq 1 \). If \( 0 \)-\( \text{cf} \) \( (\longleftarrow, y_0) = 1 \) were true, then letting \( y_1 = \max(\longleftarrow, y_0) \) and \( \beta_0 = \min \{ \beta < \alpha_0 : y_1(\beta) \neq y_0(\beta) \} \), we see \( y_1 < a_{\beta_0} \upharpoonright \alpha_0 < y_0 \), a contradiction. \( \square \)

Since the 0-segment \( A \) is stationary, Lemma 1.3 (3) with Fact 1 and 3 shows that \( Y_1 \) has a minimal element. Now Claim 11 (1) shows that \( (\longleftarrow, y_0) \) is the minimal element of \( B \) in \( X \), which contradicts our case (=Case 3).

Now let \( Z_0 = \prod_{\alpha \leq \alpha_0} X_{\alpha}, Z_1 = \prod_{\alpha_0 < \alpha} X_{\alpha} \) and

\[
A^* = \{ z \in Z_0 : z \upharpoonright \alpha_0 < y_0 \} \text{ or } (z \upharpoonright \alpha_0 = y_0 \text{ and } z(\alpha_0) \in A_0). \}
\]

Observe that \( A^* \) is a 0-segment of \( Z_0 \) and \( A^* = (\longleftarrow, y_0) Y_0 \times X_{\alpha_0} \cup \{ y_0 \} \times A_0 \). Since \( \{ y_0 \} \times A_0 \) is a 1-segment of \( A^* \) because of \( A_0 \neq \emptyset \), Lemma 1.2 shows that \( 0 \)-\( \text{cf} \) \( A^* \) is equal to \( 0 \)-\( \text{cf} \) \( A_0 \) and that the stationarity of \( A^* \) is equivalent to the stationarity of \( A_0 \).

Claim 14. \( A = A^* \times Z_1 \).

Proof. The inclusion \( \subseteq \) follows from Claim 10 (1) and the definition of \( A_0 \). The inclusion \( \supseteq \) follows from Claim 10 (2) and the definition of \( A_0 \). \( \square \)

We divide Case 3 into two subcases.

Case 3-1. \( 0 \)-\( \text{cf} \) \( Z_0 \) \( A^* \geq \omega \).

In this case, since \( A \) is stationary, Lemma 1.3 (3b) with Claim 14 shows that \( Z_1 \) has a minimal element (so \( \sup J^- \leq \alpha_0 \)) and the 0-segment \( A^* \) is stationary (so the 0-segment \( A_0 \) is stationary), which contradicts our condition (2b).

Case 3-2. \( 0 \)-\( \text{cf} \) \( Z_0 \) \( A^* = 1 \), that is, \( \max A^* \) exists.

In this case, note \( \max A^* = y_0 ^\langle \max A_0 \rangle \). Since \( A = A^* \times Z_1 \), \( A \) has no maximal element but \( A^* \) has a maximal element, we see \( Z_1 \) has no maximal element. So let \( \alpha_1 = \min \{ \alpha_0 < \alpha : X_{\alpha} \text{ has no maximal element} \} \).

Note that \( X_{\alpha} \) has a maximal element for each \( \alpha \in (\alpha_0, \alpha_1) \). Since \( A = A^* \times Z_1 = (A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_{\alpha}) \times \prod_{\alpha_1 < \alpha} X_{\alpha} \) and \( A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_{\alpha} \)}
is a 0-segment in $\prod_{\alpha \leq \alpha_1} X_\alpha$ with no maximal element, Lemma 1.3 (3b) shows that the 0-segment $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$ is stationary and $\prod_{\alpha_1 < \alpha} X_\alpha$ has a minimal element (so $\sup J^- \leq \alpha_1$). Moreover since $\{y_0 \wedge (\max A_0) \wedge (\max X_\alpha : \alpha_0 < \alpha < \alpha_1)\} \times X_{\alpha_1}$ is a 1-segment in the stationary 0-segment $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$, Lemma 1.2 shows that the 0-segment $X_{\alpha_1}$ is also stationary, which contradicts our condition (2b).

Analogously we see the following.

**Theorem 3.3.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:

1. $X$ is hereditarily 1-paracompact,
2. the following clauses hold:
   1. $\gamma < \sup J^+ + \omega_1$,
   2. for every $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, $X_\alpha$ is hereditarily 1-paracompact,

4. Some Applications

In this section, we apply the theorems in the previous section to some special cases.

**Corollary 4.1.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $X_\alpha$ has both a minimal and a maximal element for every $\alpha < \gamma$, then the following are equivalent:

1. $X$ is hereditarily paracompact,
2. the following clauses hold:
   1. $\gamma < \omega_1$,
   2. for every $\alpha < \gamma$, $X_\alpha$ is hereditarily paracompact,

Proof. By the assumption, we have $J^- = J^+ = \emptyset$, then apply Theorems 3.2 and 3.3.

**Corollary 4.2.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $X_\alpha$ has neither a minimal nor a maximal element for every $\alpha < \gamma$, then the following are equivalent:

1. $X$ is hereditarily paracompact,
2. if $\gamma$ is successor, then $X_{\gamma - 1}$ is hereditarily paracompact, where $\gamma - 1$ is the immediate predecessor of $\gamma$.

Thus note that if $\gamma$ is limit, then $X$ is hereditarily paracompact.

Proof. By the assumption, we have $J^- = J^+ = \gamma$. So note that $\sup J^- = \sup J^+ = \gamma$ whenever $\gamma$ is limit and that $\sup J^- = \sup J^+ = \gamma - 1$ whenever $\gamma$ is successor. Then apply Theorems 3.2 and 3.3.
Example 4.3. The corollary above shows that the lexicographic products $S^\gamma$, $M^\gamma$, $R^\gamma$ and $(0,1)_{R}^\gamma$ are hereditarily paracompact for every ordinal $\gamma$.

Applying the theorems directly we can also see the following.

Corollary 4.4. Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $\sup J^- = \sup J^+ = \gamma$, then $X$ is hereditarily paracompact,

Here remark that $\sup J^- = \gamma$ implies that $\gamma$ is limit.

Example 4.5. The corollary above shows that $(\omega_1^2 \times (-\omega_1)^3)^{\omega_1}$ is hereditarily paracompact, where for a GO-space $X = \langle X, <, \tau_X \rangle$, $-X$ denotes the GO-space $\langle X, >, \tau_X \rangle$ which is called the reverse of $X$, see [5]. Note that $-X$ is topologically homeomorphic to $X$, because the identity map on $X$ to $-X$ ($= X$) is 1-order preserving and homeomorphism. Also note that the lexicographic products $\omega_1^{\omega_1}$ and $\omega_1^{\omega_1}$ are not paracompact [5].

Next we consider the case that all $X_\alpha$'s have minimal elements. Theorems 3.2 and 3.3 yield the following.

Corollary 4.6. Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $X_\alpha$ has a minimal element for every $\alpha < \gamma$, then the following are equivalent:

1) $X$ is hereditarily paracompact,

2) the following clauses hold:

(a) $\gamma < \omega_1$,

(b) for every $\alpha < \gamma$, $X_\alpha$ is hereditarily 0-paracompact,

(c) for every $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, $X_\alpha$ is hereditarily 1-paracompact.

Therefore we have the following.

Corollary 4.7. Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $X_\alpha$ has a minimal element but has no maximal element for every $\alpha < \gamma$, then the following are equivalent:

1) $X$ is hereditarily paracompact,

2) the following clauses hold:

(a) $\gamma < \omega_1$,

(b) for every $\alpha < \gamma$, $X_\alpha$ is hereditarily 0-paracompact,

(c) if $\gamma$ is successor, then $X_{\gamma - 1}$ is hereditarily 1-paracompact.

Now we consider hereditary paracompactness of $X^\gamma$.

Corollary 4.8. Let $X$ be a GO-space. Then the following hold:
(1) when $X$ has both a minimal and a maximal element, the lexicographic product $X^\gamma$ is hereditarily paracompact iff $\gamma < \omega_1$ and $X$ is hereditarily paracompact,

(2) when $X$ has neither a minimal nor a maximal element, the lexicographic product $X^\gamma$ is hereditarily paracompact iff $X$ is hereditarily paracompact whenever $\gamma$ is successor,

(3) when $X$ has a minimal element but has no maximal element, the lexicographic product $X^\gamma$ is hereditarily paracompact iff $\gamma < \omega_1$, $X$ is hereditarily $0$-paracompact and “if $\gamma$ is successor, then $X$ is hereditarily $1$-paracompact”.

Example 4.9. The corollary above shows the following:

(1) the lexicographic product $[0, 1]^\gamma_{\mathbb{R}}$ is hereditarily paracompact iff $\gamma < \omega_1$, see [2, page 73],

(2) the lexicographic product $2^\gamma$ is hereditarily paracompact iff $\gamma < \omega_1$, where $2 = \{0, 1\}$ with $0 < 1$,

(3) the lexicographic product $[0, 1)^\gamma_{\mathbb{R}}$ is hereditarily paracompact iff $\gamma < \omega_1$.

Example 4.10. Applying Theorems 3.2 and 3.3 directly, we see:

(1) the lexicographic product $[0, 1]^{\omega_1}_{\mathbb{R}} \times S^{\omega_1}$ is hereditarily paracompact,

(2) the lexicographic product $S^{\omega_1} \times [0, 1]^{\omega_1}_{\mathbb{R}}$ is not hereditarily paracompact,

(3) the lexicographic product $S^{\omega_1} \times [0, 1]^{\omega_1}_{\mathbb{R}}$ is hereditarily paracompact,

(4) the lexicographic product $(\omega_1 + 1)^\omega \times S^{\omega_1}$ is hereditarily paracompact,

(5) the lexicographic product $S^{\omega_1} \times (\omega_1 + 1)^\omega$ is not hereditarily paracompact,

(6) the lexicographic product $S^{\omega_1} \times [0, 1]^{\omega_1}_{\mathbb{R}}$ is hereditarily paracompact,

(7) the lexicographic product $S^{\omega_1} \times [0, 1]^{\omega_1}_{\mathbb{R}}$ is not hereditarily paracompact,

(8) the lexicographic product $[0, 1]^{\omega_1}_{\mathbb{R}} \times S^{\omega_1}$ is hereditarily paracompact,

Note that all spaces in Examples 4.9 and 4.10 are paracompact.

Finally we discuss on hereditarily paracompactness of lexicographic products of ordinal subspaces. Note that whenever $X$ is a subspace of an ordinal, then $X$ has a minimal element, more generally, all non-empty 1-segment of $X$ has a minimal element. Therefore when $X = \prod_{\alpha < \gamma} X_\alpha$ is a lexicographic product of subspaces of ordinals, we see:
\[ J^- = \emptyset, \]
\[ \text{is hereditarily } 1\text{-paracompact for every } \alpha < \gamma. \]

So Corollary 4.6 yields the following.

**Corollary 4.11.** Let \( X = \prod_{\alpha < \gamma} X_\alpha \) be a lexicographic product of subspaces of ordinals. Then the following are equivalent:

1. \( X \) is hereditarily paracompact,
2. the following clauses hold:
   a. \( \gamma < \omega_1 \),
   b. for every \( \alpha < \gamma \), \( X_\alpha \) is hereditarily \((0-)\)paracompact,

In particular, when \( X \) is an ordinal, \( X \) is hereditarily paracompact if and only if it is a countable ordinal. So we have the following.

**Corollary 4.12.** Let \( X = \prod_{\alpha < \gamma} X_\alpha \) be a lexicographic product of ordinals. Then the following are equivalent:

1. \( X \) is hereditarily paracompact,
2. \( \gamma < \omega_1 \) and for every \( \alpha < \gamma \), \( X_\alpha \) is a countable ordinal.

**Example 4.13.** The corollary above shows the following, where \( \mathbb{Z} \) denotes the GO-space of all integers with the usual order:

1. the lexicographic product \( (\omega + \omega)^{\omega+\omega} \) is hereditarily paracompact,
2. the lexicographic product \( (\omega + \omega)^{\omega_1} \) is paracompact but not hereditarily paracompact, on the other hand, the lexicographic product \( \mathbb{Z}^{\omega_1} \) is hereditarily paracompact

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**References**


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