COUNTABLE COMPACTNESS OF LEXICOGRAPHIC PRODUCTS OF GO-SPACES

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Abstract. It is well-known:

• the usual Tychonoff product $X^2$ of a paracompact space $X$
  need not be paracompact, for instance, the Sorgenfrey line $S$
  is such an example.

On the other hand, the following is known:

• the lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ of paracompact
  LOTS's is also paracompact [2].

In [6], the notion of the lexicographic product of GO-spaces is
  defined and the result above in [2] is extended for GO-spaces [6, 7],
  so the lexicographic product $S^2$ is paracompact. It is also known
  that:

• the usual Tychonoff product of countably compact GO-spaces
  is also countably compact, therefore the usual Tychonoff prod-
  uct $\omega^\gamma_1$ is countably compact for every ordinal $\gamma$,

• the lexicographic product $\omega_1^\gamma$ is countably compact, but the
  lexicographic product $\omega_1^{\omega+1}$ is not countably compact [4].

In this paper, we will characterize the countable compactness of
  lexicographic products of GO-spaces. Applying this characteriza-
  tion, about lexicographic products, we see:

• the lexicographic product $X^2$ of a countably compact GO-
  space $X$ need not be countably compact,

• $\omega_1^2$, $\omega_1 \times \omega$, $(\omega+1) \times (\omega_1 + 1) \times \omega_1 \times \omega$, $\omega_1 \times \omega \times \omega_1$, $\omega_1 \times \omega \times \cdots$, $\omega_1 \times \omega_\omega^\omega$, $\omega_1 \times \omega_\omega \times (\omega+1)$, $\omega_1^\gamma$, $\omega_1^{\omega+1}$ and $\prod_{\alpha \in \omega} \omega_\alpha$ are countably compact,

• $\omega \times \omega_1$, $(\omega+1) \times (\omega_1 + 1) \times \omega \times \omega_1$, $\omega \times \omega_1 \times \omega \times \omega_1 \times \cdots$, $\omega \times \omega_\omega^\omega$, $\omega_1 \times \omega_\omega \times \omega_1$, $\omega_1^\gamma \times \omega$, $\prod_{n \in \omega} \omega_1$ and $\prod_{n \geq \omega} \omega_1$ are not countably compact,

• $[0,1)_R \times \omega_1$, where $[0,1)_R$ denotes the half open interval
  in the real line $\mathbb{R}$, is not countably compact,

• $\omega_1 \times (0,1)_R$ is countably compact,

• both $S \times \omega_1$ and $\omega_1 \times S$ are not countably compact,

• $\omega_1 \times (\omega_1)$ is not countably compact, where for a GO-space
  $X = \langle X, \leq_X, \tau_X \rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X \rangle$.

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1. Introduction

Lexicographic products of LOTS’s were studied in [2] and it was proved:

- a lexicographic product of LOTS’s is compact iff all factors are compact,
- a lexicographic products of paracompact LOTS’s is also paracompact,

Recently, the author defined the notion of the lexicographic product of GO-spaces and extended the results above for GO-spaces, see [6, 7]. It is also known:

- the usual Tychonoff product of GO-spaces is countably compact iff all factors are countably compact, therefore the usual Tychonoff product $\omega^\gamma$ is countably compact for every ordinal $\gamma$,
- the lexicographic product $\omega^\omega$ is countably compact, but the lexicographic product $\omega^{\omega+1}$ is not countably compact [4].

In this paper, we will characterize the countable compactness of lexicographic products of GO-spaces, further give some applications.

When we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all $X_\alpha$ are assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminology follow [9] and [1].

A linearly ordered set $\langle L; <_L \rangle$ has a natural topology $\tau_L$, which is called an interval topology, generated by $f((x, y)_L : x <_L y)$ as a subbase, where $(x, y)_L = \{z \in L : x <_L z <_L y\}$ and so on. The triple $\langle L; <_L; \lambda_L \rangle$, which is simply denoted by $L$, is called a LOTS.

A triple $\langle X; <_X; \tau_X \rangle$ is said to be a GO-space, which is also simply denoted by $X$, if $\langle X; <_X \rangle$ is a linearly ordered set and $\tau_X$ is a $T_2$-topology on $X$ having a base consisting of convex sets, where a subset $C$ of $X$ is convex if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subseteq C$ holds. For more information on LOTS’s or GO-spaces, see [10]. Usually $<_L$, $(x, y)_L$, $\lambda_L$ or $\tau_X$ are written simply $<$, $(x, y)$, $\lambda$ or $\tau$ if contexts are clear.

The symbols $\omega$ and $\omega_1$ denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \ldots$, are considered to be LOTS’s with the usual interval topologies. The cofinality of $\alpha$ is denoted by $\text{cf } \alpha$.

For GO-spaces $X = \langle X; <_X; \tau_X \rangle$ and $Y = \langle Y; <_Y, \tau_Y \rangle$, $X$ is said to be a subspace of $Y$ if $X \subseteq Y$, the linear order $<_X$ is the restriction $<_Y \upharpoonright X$ of the order $<_Y$ and the topology $\tau_X$ is the subspace topology $\tau_Y \upharpoonright X (= \{U \cap X : U \in \tau_Y\})$ on $X$ of the topology $\tau_Y$. So a subset of a
GO-space is naturally considered as a GO-space. For every GO-space $X$, there is a LOTS $X^*$ such that $X$ is a dense subspace of $X^*$ and $X^*$ has the property that if $L$ is a LOTS containing $X$ as a dense subspace, then $L$ also contains the LOTS $X^*$ as a subspace, see [11]. Such a $X^*$ is called the minimal $d$-extension of a GO-space $X$. The construction of $X^*$ is also shown in [6]. Obviously, we can see:

- if $X$ is a LOTS, then $X^* = X$,
- $X$ has a maximal element $\text{max } X$ if and only if $X^*$ has a maximal element $\text{max } X^*$, in this case, $\text{max } X = \text{max } X^*$ (similarly for minimal elements).

For every $\alpha < \gamma$, let $X_\alpha$ be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with the sequence $(x(\alpha) : \alpha < \gamma)$. For notational convenience, $\prod_{\alpha<\gamma} X_\alpha$ is considered as the trivial one point LOTS $\{\emptyset\}$ whenever $\gamma = 0$, where $\emptyset$ is considered to be a function whose domain is $0 (= \emptyset)$. When $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$ and $y_1 \in \prod_{\beta \leq \alpha} X_\alpha$, $y_0 \wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ y_1(\alpha) & \text{if } \beta \leq \alpha. \end{cases}$$

In this case, whenever $\beta = 0$, $\emptyset \wedge y_1$ is considered as $y_1$. In case $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$, $u \in X_\beta$ and $y_1 \in \prod_{\beta \leq \alpha} X_\alpha$, $y_0 \wedge (u) \wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ u & \text{if } \alpha = \beta, \\ y_1(\alpha) & \text{if } \beta < \alpha. \end{cases}$$

More general cases are similarly defined. The lexicographic order $<_X$ on $X$ is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma, x \upharpoonright \alpha = x' \upharpoonright \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \upharpoonright \alpha = (x(\beta) : \beta < \alpha)$ (in particular $x \upharpoonright 0 = \emptyset$) and $<_X$ is the order on $X_\alpha$. Now for every $\alpha < \gamma$, let $X_\alpha$ be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace $X$ of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X_\alpha^*$ is said to be the lexicographic product of GO-spaces $X_\alpha$’s, for more details see [6]. $\prod_{i \in \omega} X_i \left( \prod_{i \leq n} X_i \right.$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively).

$\prod_{\alpha<\gamma} X_\alpha$ is also denoted by $X^\gamma$ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let $X$ and $Y$ be LOTS’s. A map $f : X \to Y$ is said to be order preserving or 0-order preserving if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be order reversing or 1-order preserving.
preserving if \( f(x) >_Y f(x') \) whenever \( x <_X x' \). Obviously a 0-order preserving map (also 1-order preserving map) \( f : X \to Y \) between LOTS’s \( X \) and \( Y \), which is onto, is a homeomorphism, i.e., both \( f \) and \( f^{-1} \) are continuous. Now let \( X \) and \( Y \) be GO-spaces. A 0-order preserving map \( f : X \to Y \) is said to be a 0-order preserving embedding if \( f \) is a homeomorphism between \( X \) and \( f[X] \), where \( f[X] \) is the subspace of the GO-space \( Y \). In this case, we identify \( X \) with \( f[X] \) as a GO-space and write \( X = f[X] \) and \( X \subseteq Y \).

Let \( X \) be a GO-space. A subset \( A \) of \( X \) is called a 0-segment of \( X \) if for every \( x, x' \in X \) with \( x \leq x' \), if \( x' \in A \), then \( x \in A \). A 0-segment \( A \) is said to be bounded if \( X \setminus A \) is non-empty. Similarly the notion of (bounded) 1-segment can be defined. Both \( \emptyset \) and \( X \) are 0-segments and 1-segments. Obviously if \( A \) is a 0-segment of \( X \), then \( X \setminus A \) is a 1-segment of \( X \).

Let \( A \) be a 0-segment of a GO-space \( X \). A subset \( U \) of \( A \) is unbounded in \( A \) if for every \( x \in A \), there is \( x' \in U \) such that \( x \leq x' \). Let

\[
0\text{-}cf_{X \setminus A} = \min\{|U| : U \text{ is unbounded in } A\}.
\]

0-\( cf_{X \setminus A} \) can be 0, 1 or regular infinite cardinals. 0-\( cf_{X \setminus A} = 0 \) means \( A = \emptyset \) and 0-\( cf_{X \setminus A} = 1 \) means that \( A \) has a maximal element. If contexts are clear, 0-\( cf_{X \setminus A} \) is denoted by 0-\( cf_A \). For cofinality in compact LOTS and linearly ordered compactifications, see also [3, 8].

Remember that a topological space is said to be countably compact if every infinite subset has a cluster point.

**Definition 1.1.** A GO-space \( X \) is (boundedly) countably 0-compact if for every (bounded) closed 0-segment \( A \) of \( X \), 0-\( cf_A \neq \omega \) holds. The term “Boundedly countably 1-compact” is analogously defined.

Obviously a GO-space \( X \) is countably 0-compact iff it is boundedly countably 0-compact and 0-\( cf_X \neq \omega \). Note that subspaces of ordinals are always countably 1-compact because they are well-ordered. Also note that ordinals are boundedly countably 0-compact but in general not countably 0-compact, e.g., \( \omega \), \( \mathfrak{N}_\omega \) etc.

We first check:

**Lemma 1.2.** A GO-space \( X \) is countably 0-compact if and only if every 0-order preserving sequence \( \{x_n : n \in \omega\} \) (i.e., \( m < n \to x_m < x_n \)) has a cluster point.

**Proof.** Assuming the existence of a 0-order preserving sequence \( \{x_n : n \in \omega\} \) with no cluster points, set \( A = \{x \in X : \exists n \in \omega(x \leq x_n)\} \).

Then \( A \) is closed 0-segment with 0-\( cf_A = \omega \).
To see the other direction, assuming the existence a closed 0-segment $A$ with $0 \text{-cf } A = \omega$, by induction, we can construct a 0-order preserving sequence with no cluster points. □

Using the lemma, we can see that a GO-space is countably compact if and only if it is both countably 0-compact and countably 1-compact, see also [5].

2. A simple case

In this section, we characterize countable 0-compactness of lexicographic products of two GO-spaces. The following is easy to prove, see also [7, Lemma 3.6 (3a)].

**Lemma 2.1.** Let $X = X_0 \times X_1$ be a lexicographic product of two GO-spaces and $A_0$ a 0-segment of $X_0$ with $0 \text{-cf } X_0 A_0 \geq \omega$. Then $A = A_0 \times X_1$ is also a 0-segment of $X$ with $0 \text{-cf } X A = 0 \text{-cf } X_0 A_0$.

The following lemma will be a useful tool for handling general cases.

**Lemma 2.2.** Let $X = X_0 \times X_1$ be a lexicographic product of two GO-spaces. Then the following are equivalent.

1. $X$ is countably 0-compact,
2. the following clauses hold:
   a. $X_0$ is countably 0-compact,
   b. $X_1$ is boundedly countably 0-compact,
   c. if $X_1$ has no minimal element or $(u, \to)_{X_0}$ has no minimal element (that is, $1 \text{-cf } X_0 (u, \to) \neq 1$) for some $u \in X_0$, then $0 \text{-cf } X_1 X_1 \neq \omega$,
   d. if $X_1$ has no minimal element, then $0 \text{-cf } X_0 (\to, u) \neq \omega$ for every $u \in X_0$.

**Proof.** Set $\hat{X} = X_0^* \times X_1^*$.

1. $\Rightarrow$ 2. Let $X$ be countably 0-compact.

   a. Assuming that $X_0$ is not countably 0-compact, take a closed 0-segment $A_0$ of $X_0$ with $0 \text{-cf } X_0 A_0 = \omega$. By the lemma above, $A = A_0 \times X_1$ is a 0-segment of $X$ with $0 \text{-cf } X A = \omega$. It suffices to see that $A$ is closed, which contradicts countable 0-compactness of $X$. So let $x \notin A$, then $x(0) \notin A_0$. Since $A_0$ is closed in $X_0$, there is $u^* \in X_0^*$ such that $u^* <_{X_0^*} x(0)$ and $((u^*, \to)_{X_0^*} \cap X_0) \cap A_0 = \emptyset$ (this means $(u^*, x(0))_{X_0^*} = \emptyset$). Fix $w \in X_1$ and let $x^* = (w^*, w) \in \hat{X}$. Let $U = (x^*, \to)_{X} \cap X$, then $U$ is a neighborhood of $x$. To see $U \cap A = \emptyset$, assume $a \in U \cap A$ for some $a$. By $a(0) \in A_0$, we can take $u \in A_0$ with $a(0) < u$. Now $u^* \leq a(0) < u$ shows $u \in (x^*, \to)_{X_0} \cap A_0$, a contradiction.
(b) Assuming that $X_1$ is not boundedly countably 0-compact, take a bounded closed 0-segment $A_1$ of $X_1$ with $0\text{-}\text{cf}_{X_1} A_1 = \omega$. Fix $u \in X_0$ and let $A = \{ x \in X : \exists v \in A_1 (x \leq_X \langle u, v \rangle) \}$. Obviously $A$ is a 0-segment of $X$ and $\{ u \} \times A_1$ is unbounded in the 0-segment $A$, so we see $0\text{-}\text{cf}_X A = 0\text{-}\text{cf}_{X_1} A_1 = \omega$. It suffices to see that $A$ is closed, so let $x \in X \setminus A$. Note $u \leq x(0)$. Since $A_1$ is bounded, fix $v \in X_1 \setminus A_1$ and let $y = \langle u, v \rangle$. When $y < x$, $U = (y, \rightarrow)_X$ is a neighborhood of $x$ disjoint from $A$. So let $x \leq y$, then we have $x(0) = u$ and $x(1) \notin A_1$. Since $A_1$ is closed in $X_1$, take $v^* \in X_1^*$ such that $v^* < x(1)$ and $\langle (v^*, \rightarrow) \rangle_X \cap A_1 = \emptyset$. Then $U = (\langle u, v^* \rangle, \rightarrow)_X \cap X$ is a neighborhood of $x$ disjoint from $A$.

(c) First assume that $X_1$ has no minimal element. Fix $u \in X_0$. Then $A = (\leftarrow, u] \times X_1$ is a closed 0-segment of $X$ and $\{ u \} \times X_1$ is unbounded in the 0-segment $A$, therefore $0\text{-}\text{cf}_{X_1} X_1 = 0\text{-}\text{cf}_X A \neq \omega$.

Next assume that $\langle u, \rightarrow \rangle_{X_0}$ has no minimal element. Then putting $A = (\leftarrow, u] \times X_1$, similarly we see $0\text{-}\text{cf}_{X_1} X_1 \neq \omega$.

(d) Assuming that $X_1$ has no minimal element and $0\text{-}\text{cf}_{X_0}(\leftarrow, u) = \omega$ for some $u \in X_0$, let $A = (\leftarrow, u] \times X_1$. Then $A$ is a closed 0-segment of $X$ with $0\text{-}\text{cf}_X A = 0\text{-}\text{cf}_{X_0}(\leftarrow, u)$ by Lemma 2.1. This contradicts countable 0-compactness of $X$.

(2) $\Rightarrow$ (1) Assuming (2) and that $X$ is not countably 0-compact, take a closed 0-segment $A$ of $X$ with $0\text{-}\text{cf}_X A = \omega$. Let $A_0 = \{ u \in X_0 : \exists v \in X_1 (\langle u, v \rangle \in A) \}$. Since $A$ is a non-empty 0-segment of $X$, $A_0$ is also a non-empty 0-segment of $X_0$. We consider two cases, and in each case, we will derive a contradiction.

Case 1. $A_0$ has no maximal element, i.e., $0\text{-}\text{cf} A_0 \geq \omega$.

In this case, we have:

Claim 1. $A = A_0 \times X_1$.

Proof. The inclusion $\subseteq$ is obvious. Let $\langle u, v \rangle \in A_0 \times X_1$. Since $u \in A_0$ and $A_0$ has no maximal element, we can take $u' \in A_0$ with $u < u'$. By $u' \in A_0$, there is $v' \in X_1$ with $\langle u', v' \rangle \in A$. Then from $\langle u, v \rangle < \langle u', v' \rangle \in A$, we see $\langle u, v \rangle \in A$, because $A$ is a 0-segment. $\square$

Lemma 2.1 shows $0\text{-}\text{cf} A_0 = 0\text{-}\text{cf} A = \omega$. The following claim contradicts the condition (2a).

Claim 2. $A_0$ is closed in $X_0$.

Proof. Let $u \in X_0 \setminus A_0$. Whenever $u' < u$ for some $u' \in X_0 \setminus A_0$, $(u', \rightarrow)$ is a neighborhood of $u$ disjoint from $A_0$. So assume the other case, that is, $u = \min(X_0 \setminus A_0)$. Note $A_0 = (\leftarrow, u)$. If $X_1$ has no minimal element,
then by (2d), we have $0$-cf$(\leftarrow, u) \neq \omega$, a contradiction. Thus $X_1$ has a minimal element, therefore $\langle u, \min X_1 \rangle = \min(X \setminus A) \notin A$. Since $A$ is closed, there are $u^* \in X_0^*$ and $v^* \in X_1^*$ such that $\langle u^*, v^* \rangle < \langle u, \min X_1 \rangle$ and $\langle (\langle u^*, v^* \rangle, \rightarrow \rangle) \setminus X \rangle \cap A = \emptyset$. $\langle u^*, v^* \rangle < \langle u, \min X_1 \rangle$ shows $u^* < u$, so $(u^*, \rightarrow) \cap X_0$ is a neighborhood of $u$ disjoint from $A_0$. 

**Case 2.** $A_0$ has a maximal element.

In this case, let $A_1 = \{v \in X_1 : \langle \max A_0, v \rangle \in A\}$. Then $A_1$ is a non-empty 0-segment of $X_1$. Since $\{\max A_0\} \times A_1$ is unbounded in the 0-segment $A$, we see $0$-cf$_{X_1} A_1 = 0$-cf$_X A = \omega$.

**Claim 3.** $A_1$ is closed in $X_1$.

**Proof.** Let $v \in X_1 \setminus A_1$. Since $\langle \max A_0, v \rangle \notin A$ and $A$ is closed, there are $u^* \in X_0^*$ and $v^* \in X_1^*$ such that $\langle u^*, v^* \rangle < \langle \max A_0, v \rangle$ and $\langle (\langle u^*, v^* \rangle, \rightarrow) \setminus X \rangle \cap A = \emptyset$. It follows from $A_1 \neq \emptyset$ that $u^* = \max A_0$ and so $v^* < v$. Then we see that $\langle v^*, \rightarrow \rangle_{X_1} \cap X_1$ is a neighborhood of $v$ disjoint from $A_1$. 

This claim with the condition (2b) shows $A_1 = X_1$, which says $A = (\leftarrow, \max A_0) \times X_1$, in particular, we see that $X_1$ has no maximal element.

**Claim 4.** $\langle \max A_0, \rightarrow \rangle$ has no minimal element or $X_1$ has no minimal element.

**Proof.** Assume that $\langle \max A_0, \rightarrow \rangle$ has a minimal element $u_0$ and $X_1$ has a minimal element, then note $\langle u_0, \min X_1 \rangle = \min(X \setminus A)$. Since $A$ is closed in $X$, there are $u^* \in X_0^*$ and $v^* \in X_1^*$ such that $\langle u^*, v^* \rangle < \langle u_0, \min X_1 \rangle$ and $\langle (\langle u^*, v^* \rangle, \rightarrow) \setminus X \rangle \cap A = \emptyset$. Then we have $u^* = \max A_0$. Since $X_1$ has no maximal element, pick $v \in X_1$ with $v^* < v$. Then we see $\langle \max A_0, v \rangle \in \langle (\langle u^*, v^* \rangle, \rightarrow) \setminus X \rangle \cap A$, a contradiction. 

Now the condition (2c) shows $0$-cf$_{X_1} X_1 \neq \omega$, a contradiction. This completes the proof of the lemma. 

3. **A general case**

In this section, using the results in the previous section, we characterize the countable compactness of lexicographic products of any length of GO-spaces. We use the following notations.

**Definition 3.1.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. Define:

$$J^+ = \{\alpha < \gamma : X_\alpha \text{ has no maximal element}\}.$$
\[ J^- = \{ \alpha < \gamma : X_\alpha \text{ has no minimal element.} \}, \]

\[ K^+ = \{ \alpha < \gamma : \text{there is } x \in X_\alpha \text{ such that } (x, \to)_{X_\alpha} \text{ is non-empty and has no minimal element.} \}, \]

\[ K^- = \{ \alpha < \gamma : \text{there is } x \in X_\alpha \text{ such that } (\leftarrow, x)_{X_\alpha} \text{ is non-empty and has no maximal element.} \}, \]

\[ L^+ = \{ \alpha \leq \gamma : \text{there is } u \in \prod_{\beta < \alpha} X_\beta \text{ with } 0-\text{cf}_{\prod_{\beta < \alpha}} x_\beta (\leftarrow, u) = \omega \}, \]

\[ L^- = \{ \alpha \leq \gamma : \text{there is } u \in \prod_{\beta < \alpha} X_\beta \text{ with } 1-\text{cf}_{\prod_{\beta < \alpha}} x_\beta (u, \to) = \omega \}, \]

For an ordinal \( \alpha \), let

\[ l(\alpha) = \begin{cases} 0 & \text{if } \alpha < \omega, \\ \sup \{ \beta \leq \alpha : \beta \text{ is limit.} \} & \text{if } \alpha \geq \omega. \end{cases} \]

Some of the definitions above are introduced in [7]. Note that 0 \( \notin \) \( L^+ \cup L^- \) and for an ordinal \( \alpha \geq \omega \), \( l(\alpha) \) is the largest limit ordinal less than or equal to \( \alpha \), therefore the half open interval \([l(\alpha), \alpha)\) of ordinals is finite.

We also remark:

**Lemma 3.2.** Let \( X = \prod_{\alpha \leq \gamma} X_\alpha \) be a lexicographic product of GO-spaces. If \( \omega \leq \gamma \), then \( \omega \in L^+ \cap L^- \) holds.

**Proof.** Assume \( \omega \leq \gamma \). For each \( n \in \omega \), fix \( u_0(n), u_1(n) \in X_n \) with \( u_0(n) < u_1(n) \). Set \( y = \langle u_1(n) : n \in \omega \rangle \). Moreover for each \( n \in \omega \), set \( y_n = \langle u_1(i) : i < n \rangle \wedge \langle u_0(i) : n \leq i \rangle \). Then \( \{y_n : n \in \omega \} \) is a 0-order preserving unbounded sequence in \((\leftarrow, y)\) in \( \prod_{n \in \omega} X_n \), therefore \( \omega \in L^+ \). The statement \( \omega \in L^- \) is similar. \( \square \)

**Theorem 3.3.** Let \( X = \prod_{\alpha \leq \gamma} X_\alpha \) be a lexicographic product of GO-spaces. Then the following are equivalent:

1. \( X \) is countably 0-compact,
2. the following clauses hold:
   a. \( X_\alpha \) is boundedly countably 0-compact for every \( \alpha < \gamma \),
   b. if \( L^+ \neq \emptyset \), then \( J^- \subset \min L^+ \),
   c. for every \( \alpha < \gamma \), if any one of the following cases holds, then \( 0-\text{cf}_{X_\alpha} X_\alpha \neq \omega \) holds,
      i. \( J^+ \cap [l(\alpha), \alpha) = \emptyset \),
      ii. \( J^+ \cap [l(\alpha), \alpha) \neq \emptyset \) and \( (\alpha_0, \alpha] \cap J^- \neq \emptyset \), where \( \alpha_0 = \max(J^+ \cap [l(\alpha), \alpha)) \),
      iii. \( J^+ \cap [l(\alpha), \alpha) \neq \emptyset \) and \( [\alpha_0, \alpha) \cap K^+ \neq \emptyset \), where \( \alpha_0 = \max(J^+ \cap [l(\alpha), \alpha)) \).
Proof. Note that (2a)+(2ci) implies that $X_0$ is countably 0-compact. Let $X = \prod_{\alpha < \gamma} X^*_\alpha$.

(1) $\Rightarrow$ (2) Assume that $X$ is countably 0-compact.

(a) Let $\alpha_0 < \gamma$. Since $X = \prod_{\alpha \leq \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$, see [6, Lemma 1.5], and $X$ is countably 0-compact, Lemma 2.2 shows that $\prod_{\alpha \leq \alpha_0} X_\alpha$ is countably 0-compact. Now by $\prod_{\alpha < \alpha_0} X_\alpha = \prod_{\alpha < \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$ and Lemma 2.2 again, we see that $X_{\alpha_0}$ is boundedly countably 0-compact.

(b) Assume $L^+ \neq \emptyset$ and $\alpha_0 = \min L^+$. Then Lemma 3.2 shows $\alpha_0 \leq \omega$. From $\alpha_0 \in L^+$, one can take $u \in \prod_{\alpha < \alpha_0} X_\alpha$ such that $0$-$\text{cf}\prod_{\alpha < \alpha_0} X_\alpha(\leftarrow, u) = \omega$. Now since $X = \prod_{\alpha < \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$ is countably 0-compact, Lemma 2.2 (d) shows that $\prod_{\alpha_0 < \alpha} X_\alpha$ has a minimal element. Therefore $X_\alpha$ has a minimal element for every $\alpha \geq \alpha_0$, which shows $J^- \subset \alpha_0$.

(c) Let $\alpha_0 < \gamma$. We will see $0$-$\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$ in each case of (i), (ii) and (iii).

Case (i), i.e., $J^+ \cap [l(\alpha_0), \alpha_0) = \emptyset$.

Since $X$ is countably 0-compact and $X = \prod_{\alpha \leq \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$, Lemma 2.2 shows that $\prod_{\alpha \leq \alpha_0} X_\alpha$ is also countably 0-compact. When $\alpha_0 = 0$, by countable 0-compactness of $\prod_{\alpha \leq \alpha_0} X_\alpha = X_{\alpha_0}$, we see $0$-$\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$. So let $\alpha_0 > 0$. We divide into two cases.

Case (i)-1. $l(\alpha_0) = 0$, i.e., $\alpha_0 < \omega$.

In this case, since $\prod_{\alpha < \alpha_0} X_\alpha$ has a maximal element, which implies $(\max_{\alpha < \alpha_0} X_\alpha, \to)$ has no minimal element, and $\prod_{\alpha < \alpha_0} X_\alpha \times X_{\alpha_0}$ is countably 0-compact, Lemma 2.2 (2c) shows $0$-$\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (i)-2. $l(\alpha_0) \geq \omega$ i.e., $\alpha_0 \geq \omega$.

In this case, note that for every $\alpha \in [l(\alpha_0), \alpha_0)$, $X_\alpha$ has a maximal element. For every $\alpha < l(\alpha_0)$, fix $x_0(\alpha), x_1(\alpha) \in X_\alpha$ with $x_0(\alpha) < x_1(\alpha)$, and let $y = (x_0(\alpha) : \alpha < l(\alpha_0))^\land (\max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0)$. Moreover for every $\beta < l(\alpha_0)$, let $y_\beta = (x_0(\alpha) : \alpha < \beta)^\land (x_1(\alpha) : \beta \leq \alpha < l(\alpha_0))^\land (\max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0)$. Then $\{y_\beta : \beta < l(\alpha_0)\}$ is 1-order preserving and unbounded in $(y, \to)$, in particular, the interval $(y, \to)$ in $\prod_{\alpha < \alpha_0} X_\alpha$ has no minimal element. Now Lemma 2.2 (2c) shows $0$-$\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (ii), i.e., $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $(\alpha_1, \alpha_0) \cap J^- \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Note that $\alpha_1$ is well-defined because $[l(\alpha_0), \alpha_0)$ is finite. Also let $\alpha_2 = \max((\alpha_1, \alpha_0) \cap J^-)$, then note $0 \leq l(\alpha_0) \leq \alpha_1 < \alpha_2 \leq \alpha_0$, in particular $[0, \alpha_2) \neq \emptyset$. 

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Case (ii)-1. $\alpha_2 = \alpha_0$.

Since $\prod_{\alpha < \alpha_0} X_\alpha \times X_{\alpha_0}$ ($= \prod_{\alpha \leq \alpha_0} X_\alpha$) is countably 0-compact, Lemma 2.2 (2c) shows $0\text{-}\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (ii)-2. $\alpha_2 < \alpha_0$.

Note that by the definition of $\alpha_2$, $X_\alpha$ has a minimal element for every $\alpha \in (\alpha_2, \alpha_0]$. Fixing $z \in \prod_{\alpha < \alpha_2} X_\alpha$, let $y = z^\langle \max X_\alpha : \alpha_2 \leq \alpha < \alpha_0 \rangle$, then $y \in \prod_{\alpha < \alpha_0} X_\alpha$.

Claim 1. $(y, \to)_{\prod_{\alpha < \alpha_0} X_\alpha}$ is non-empty and has no minimal element.

Proof. Because $X_{\alpha_2}$ has no maximal element, fix $u \in X_{\alpha_2}$ with $y(\alpha_1) < u$. Then $(y \upharpoonright \alpha_2)^\langle u \rangle \langle (\alpha_1, \alpha_0) \rangle \in (y, \to)$, which shows $(y, \to) \neq \emptyset$. Next assume $y < y' \in \prod_{\alpha < \alpha_0} X_\alpha$. Since $y(\alpha) = \max X_\alpha$ for every $\alpha \in [\alpha_2, \alpha_0)$, we have $y \upharpoonright \alpha_2 < y' \upharpoonright \alpha_2$. Since $X_{\alpha_2}$ has no minimal element, fix $u \in X_{\alpha_2}$ with $u < y'(\alpha_2)$. Then we have $y < (y' \upharpoonright \alpha_2)^\langle u \rangle \langle (\alpha_2, \alpha_0) \rangle < y'$, which shows that $(y, \to)$ has no minimal element. \hfill \Box

Now because $\prod_{\alpha < \alpha_0} X_\alpha \times X_{\alpha_0}$ is countably 0-compact, Lemma 2.2 (2c) and the claim above show $0\text{-}\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (iii), i.e., $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $[\alpha_1, \alpha_0) \cap K^+ \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Let $\alpha_2 = \max([\alpha_1, \alpha_0) \cap K^+]$, then note $l(\alpha_0) \leq \alpha_1 \leq \alpha_2 < \alpha_0$. Fixing $z \in \prod_{\alpha < \alpha_2} X_\alpha$ and $u \in X_{\alpha_2}$ satisfying that $(u, \to)$ is non-empty and has no minimal element, let $y = z^\langle u \rangle \langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0 \rangle$. Then obviously $y \in \prod_{\alpha < \alpha_0} X_\alpha$ and $(y, \to)$ has no minimal element. Since $\prod_{\alpha < \alpha_0} X_\alpha \times X_{\alpha_0}$ is countable 0-compact, Lemma 2.2 (2c) shows $0\text{-}\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

$(2) \Rightarrow (1)$: Assuming $(2)$ and the negation of $(1)$, take a closed 0-segment $A$ of $X$ with $0\text{-}\text{cf}_X A = \omega$. Modifying the proof of Theorem 4.8 in [7], we consider 3 cases and their subcases. In each case, we will derive a contradiction.

Case 1. $A = X$.

In this case, since $X$ has no maximal element, we have $J^+ \neq \emptyset$, so let $\alpha_0 = \min J^+$. Then $J^+ \cap [l(\alpha_0), \alpha_0) \subset J^+ \cap [0, \alpha_0) = \emptyset$ and the condition (2ci) shows $0\text{-}\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \geq \omega_1$. Since $\{\max X_\alpha : \alpha < \alpha_0\} \times X_{\alpha_0}$ is unbounded in $\prod_{\alpha \leq \alpha_0} X_\alpha$, we have $0\text{-}\text{cf}_{\prod_{\alpha \leq \alpha_0} X_\alpha} \prod_{\alpha \leq \alpha_0} X_\alpha = 0\text{-}\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \geq \omega_1$. Now by $X = \prod_{\alpha \leq \alpha_0} X_\alpha \times \prod_{\alpha \leq \alpha_0} X_\alpha$, Lemma 2.1 shows $0\text{-}\text{cf}_X A = 0\text{-}\text{cf}_X X = 0\text{-}\text{cf}_{\prod_{\alpha \leq \alpha_0} X_\alpha} \prod_{\alpha \leq \alpha_0} X_\alpha = 0\text{-}\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \geq \omega_1$, a contradiction.
Case 2. $A \neq X$ and $X \setminus A$ has a minimal element.

Let $B = X \setminus A$ and $b = \min B$. Since $A$ is non-empty closed and $B = [b, \to)$, there is $b^* \in X$ with $b^* < b$ and $((b^*, \to) \cap X) \cap A = \emptyset$, equivalently $(b^*, b) \notin X$. Note $b^* \notin X$ because $A$ has no maximal element. Let $\alpha_0 = \min \{ \alpha < \gamma : b^*(\alpha) \neq b(\alpha) \}$.

Claim 2. For every $\alpha > \alpha_0$, $X_\alpha$ has a minimal element and $b(\alpha) = \min X_\alpha$.

Proof. Assuming $b(\alpha) > u$ for some $\alpha > \alpha_0$ and $u \in X_\alpha$, let $\alpha_1 = \min \{ \alpha > \alpha_0 : \exists u \in X_\alpha (b(\alpha) > u) \}$ and fix $u \in X_{\alpha_1}$ with $b(\alpha_1) > u$. Then we have $b^* < (b \upharpoonright \alpha_1)^{\langle u \rangle ^\wedge} (b \upharpoonright (\alpha_1, \gamma)) < b$, a contradiction. \hfill $\square$

Claim 3. $(b^*(\alpha_0), b(\alpha_0))_{X_{\alpha_0}} \cap X_{\alpha_0} = \emptyset$.

Proof. Assume $u \in (b^*(\alpha_0), b(\alpha_0))_{X_{\alpha_0}} \cap X_{\alpha_0}$ for some $u$. Then we have $b^* < (b \upharpoonright \alpha_0)^{\langle u \rangle ^\wedge} (b \upharpoonright (\alpha_0, \gamma)) < b$, a contradiction. \hfill $\square$

Claim 4. $[b(\alpha_0), \to)_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$, therefore $b^*(\alpha_0) \notin X_{\alpha_0}$.

Proof. It follows from $b^*(\alpha_0) \in (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ that $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}} \neq \emptyset$. Assume $(b(\alpha_0), \to)_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, then for some $u \in X_{\alpha_0}$ with $u < b(\alpha_0)$, $(u, b(\alpha_0)) = \emptyset$. Claim 3 shows $b^*(\alpha_0) = u \in X_{\alpha_0}$. If there were $\alpha > \alpha_0$ and $v \in X_\alpha$ with $v > b^*(\alpha)$, then by letting $\alpha_1 = \min \{ \alpha > \alpha_0 : \exists v \in X_\alpha (v > b^*(\alpha)) \}$ and taking $v \in X_{\alpha_1}$ with $v > b^*(\alpha_1)$, we have $b^* < (b^* \upharpoonright \alpha_1)^{\langle u \rangle ^\wedge} (b^* \upharpoonright (\alpha_1, \gamma)) < b$, a contradiction. Therefore for every $\alpha > \alpha_0$, max $X_\alpha$ exists and $b^*(\alpha) = \text{max } X_\alpha$. Thus we have $b^* = (b \upharpoonright \alpha_0)^{\langle u \rangle ^\wedge} (\text{max } X_\alpha : \alpha_0 < \alpha) \notin X$ a contradiction. \hfill $\square$

Claims 3 and 4 show that $A_0 := (\leftarrow, b(\alpha_0))$ is a bounded closed 0-segment of $X_{\alpha_0}$ without a maximal element. Now the condition (2a) shows $0$-cf $X_{\alpha_0} A_0 \geq \omega_1$. Since $\{ b \upharpoonright \alpha_0 \} \times A_0 \times \{ b \upharpoonright (\alpha_0, \gamma) \}$ is unbounded in the 0-segment in $A (= (\leftarrow, b)_X)$, we have $\omega = 0$-cf $X = 0$-cf $X_{\alpha_0} A_0 \geq \omega_1$, a contradiction. This completes Case 2.

Case 3. $A \neq X$ and $X \setminus A$ has no minimal element.

Let $B = X \setminus A$ and

$$I = \{ \alpha < \gamma : \exists a \in A \exists b \in B (a \upharpoonright (\alpha + 1) = b \upharpoonright (\alpha + 1)) \}.$$

Obviously $I$ is a 0-segment of $\gamma$, so $I = \alpha_0$ for some $\alpha_0 \leq \gamma$. For each $\alpha < \alpha_0$, fix $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \upharpoonright (\alpha + 1) = b_\alpha \upharpoonright (\alpha + 1)$. By letting $Y_0 = \prod_{\alpha < \alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0 \leq \alpha} X_\alpha$, define $y_0 \in Y_0$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$. The ordinal $\alpha_0$ can be 0, then in this case, $Y_0 = \{ \emptyset \}$ and $y_0 = \emptyset$. 
Claim 5. For every $\alpha < \alpha_0$, $y_0 \upharpoonright (\alpha + 1) = a_\alpha \upharpoonright (\alpha + 1) = b_\alpha \upharpoonright (\alpha + 1)$ holds.

Proof. The second equality is obvious. To see the first equality, assuming $y_0 \upharpoonright (\alpha + 1) \neq a_\alpha \upharpoonright (\alpha + 1)$ for some $\alpha < \alpha_0$, let $\alpha_1 = \min\{\alpha < \alpha_0 : y_0 \upharpoonright (\alpha + 1) \neq a_\alpha \upharpoonright (\alpha + 1)\}$. Moreover let $\alpha_2 = \min\{\alpha \leq \alpha_1 : y_0(\alpha) \neq a_{\alpha_1}(\alpha)\}$. It follows from $y_0(\alpha_1) = a_{\alpha_1}(\alpha_1)$ that $\alpha_2 < \alpha_1$. Since $y_0 \upharpoonright \alpha_2 = a_{\alpha_1} \upharpoonright \alpha_2$, and $y_0(\alpha_2) \neq a_{\alpha_1}(\alpha_2)$ holds, by the minimality of $\alpha_1$, we have $y_0 \upharpoonright (\alpha_2 + 1) = a_{\alpha_2} \upharpoonright (\alpha_2 + 1) = b_{\alpha_2} \upharpoonright (\alpha_2 + 1)$. When $y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_2} < a_{\alpha_1} \in A$, a contradiction. When $y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_2} < a_{\alpha_2} \in A$, a contradiction. □

Claim 5 remains true when $\alpha_0 = 0$, because there is no ordinal $\alpha$ with $\alpha < \alpha_0$.

Claim 6. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then note $y_0 \in Y_0 = X = A \cup B$. Assume $y_0 \in A$. Since $A$ has no maximal element, one can take $a \in A$ with $y_0 < a$. Letting $\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq a(\beta)\}$, we see $A \ni b_{\beta_0} < B$, a contradiction. The remaining case is similar. □

Let $A_0 = \{a(\alpha_0) : a \in A, a \upharpoonright \alpha_0 = y_0\}$ and $B_0 = \{b(\alpha_0) : b \in B, b \upharpoonright \alpha_0 = y_0\}$.

Claim 7. The following hold:

(1) for every $a \in A$, $a \upharpoonright \alpha_0 \leq y_0$ holds,

(2) for every $x \in X$, if $x \upharpoonright \alpha_0 < y_0$, then $x \in A$.

Proof. (1) Assume $a \upharpoonright \alpha_0 > y_0$ for some $a \in A$. Letting $\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}$, we see $B \ni b_{\beta_0} < a \in A$, a contradiction.

(2) Assume $x \upharpoonright \alpha_0 < y_0$. Letting $\beta_0 = \min\{\beta < \alpha_0 : x(\beta) \neq y_0(\beta)\}$, we see $x < a_{\beta_0} \in A$. Since $A$ is a 0-segment, we have $x \in A$. □

Similarly we have:

Claim 8. The following hold:

(1) for every $b \in B$, $b \upharpoonright \alpha_0 \geq y_0$ holds,

(2) for every $x \in X$, if $x \upharpoonright \alpha_0 > y_0$, then $x \in B$.

Claim 9. $A_0$ is a 0-segment of $X_{\alpha_0}$ and $B_0 = X_{\alpha_0} \setminus A_0$.

Proof. To see that $A_0$ is a 0-segment, let $u' < u \in A_0$. Pick $a \in A$ with $a \upharpoonright \alpha_0 = y_0$ and $u = a(\alpha_0)$. Let $a' = (a \upharpoonright \alpha_0)^\upharpoonright \langle u' \rangle^\upharpoonright (a \upharpoonright (\alpha_0, \gamma))$. Since $A$ is a 0-segment and $a' < a \in A$, we have $a' \in A$. Now we see $u' = a'(\alpha_0) \in A_0$ because of $a' \upharpoonright \alpha_0 = y_0$.

To see $B_0 = X_{\alpha_0} \setminus A_0$, first let $u \in B_0$. Take $b \in B$ with $b \upharpoonright \alpha_0 = y_0$ and $b(\alpha_0) = u$. If $u \in A_0$ were true, then by taking $a \in A$ with
Assume one inclusion follows from Claim 7 (2). To see the other inclusion, let \(u \in X_{\alpha_0} \setminus A_0\). Take \(x \in X\) with \(x \uparrow (\alpha_0 + 1) = y_0 \wedge (u)\). If \(x \in A\) were true, then by \(x \uparrow \alpha_0 = y_0\), we have \(u = x(\alpha_0) \in A_0\), a contradiction. So we have \(x \in B\), therefore \(u \in B_0\).

\[\text{Claim 10. } A_0 \neq \emptyset.\]

Proof. Assume \(A_0 = \emptyset\). We prove the following facts.

**Fact 1.** \((\leftarrow, y_0) Y_0 \times Y_1 = A\).

Proof. One inclusion follows from Claim 7 (2). To see the other inclusion, let \(a \in A\). Claim 7 (1) shows \(a \uparrow \alpha_0 \leq y_0\). If \(a \uparrow \alpha_0 = y_0\) were true, then we have \(a(\alpha_0) \in A_0\), a contradiction. So we have \(a \uparrow \alpha_0 < y_0\) therefore \(a \in (\leftarrow, y_0) Y_0 \times Y_1\).

**Fact 2.** \(\alpha_0 > 0\) and \(\alpha_0\) is limit.

Proof. If \(\alpha_0 = 0\) were true, then by taking \(a \in A\), we have \(a(\alpha_0) \in A_0\), a contradiction. Therefore we have \(\alpha_0 > 0\). Next if \(\alpha_0 = \beta_0 + 1\) were true for some ordinal \(\beta_0\), then by \(\beta_0 \in \alpha_0\) and Claim 5, we have \(y_0 \uparrow \alpha_0 = y_0 \uparrow (\beta_0 + 1) = a_{\beta_0} \uparrow (\beta_0 + 1) = a_{\beta_0} \uparrow \alpha_0\), thus we have \(a_{\beta_0}(\alpha_0) \in A_0\), a contradiction. Thus \(\alpha_0\) is limit.

Now Claim 6 and Fact 2 show \(\omega \leq \alpha_0 < \gamma\), so Lemma 3.2 shows \(\omega \in L^+\). Moreover the condition (2b) shows \(J^- \subset \min L^+ \leq \omega \leq \alpha_0\), in particular, \(X_\alpha\) has a minimal element for every \(\alpha \geq \alpha_0\). This means \(Y_1 = \prod_{\alpha_0 \leq \alpha} X_\alpha\) has a minimal element. Now by Fact 1, we see \(y_0 \wedge \min Y_1 = \min(X \setminus A)\), which contradicts our case.

Next let \(Z_0 = \prod_{\alpha_0 \leq \alpha} X_\alpha\), \(Z_1 = \prod_{\alpha_0 < \alpha} X_\alpha\) and

\[A^* = \{z \in Z_0 : z \uparrow \alpha_0 < y_0\ \text{or}\ (z \uparrow \alpha_0 = y_0, z(\alpha_0) \in A_0)\}.\]

Note \(A^* = ((\leftarrow, y_0) X_{\alpha_0}) \cup \{y_0\} \times A_0\).

**Claim 11.** \(A^*\) is a 0-segment of \(Z_0\) and \(A = A^* \times Z_1\).

Proof. Since \(A_0\) is a 0-segment of \(X_{\alpha_0}\), \(A^*\) is obviously a 0-segment of \(Z_0\). To see \(A \subset A^* \times Z_1\), let \(a \in A\). Claim 7 (1) shows \(a \uparrow \alpha_0 \leq y_0\). When \(a \uparrow \alpha_0 < y_0\), obviously we have \(a \uparrow (\alpha_0 + 1) \in A^*\). When \(a \uparrow \alpha_0 = y_0\), \(a \in A\) shows \(a(\alpha_0) \in A_0\) thus \(a \uparrow (\alpha_0 + 1) \in A^*\). To see \(A \supset A^* \times Z_1\), let \(a \in A^* \times Z_1\). Then note \(a \uparrow (\alpha_0 + 1) \in A^*\). When \(a \uparrow \alpha_0 < y_0\), let \(\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}\), we see \(a < a_{\beta_0} \in A\) thus \(a \in A\). When \(a \uparrow \alpha_0 = y_0\) and \(a(\alpha_0) \in A_0\), Claim 9 shows \(a \in A\).
Since \( \{y_0\} \times A_0 \) is unbounded in the 0-segment \( A^* \), we see \( 1 \leq 0-\text{cf}_{Z_0} A^* = 0-\text{cf}_{X_{\alpha_0}} A_0 \). We divide Case 3 into two subcases.

**Case 3-1.** \( 0-\text{cf}_{Z_0} A^* \geq \omega \).

In this case, Claim 11 and Lemma 2.1 show \( \omega = 0-\text{cf}_X A = 0-\text{cf}_{Z_0} A^* = 0-\text{cf}_{X_{\alpha_0}} A_0 \).

**Claim 12.** \( A_0 \neq X_{\alpha_0} \).

**Proof.** Assume \( A_0 = X_{\alpha_0} \). \( 0-\text{cf}_{X_{\alpha_0}} X_{\alpha_0} = 0-\text{cf}_{X_{\alpha_0}} A_0 = \omega \) shows \( \alpha_0 \in J^+ \). Assume \( \alpha_0 = \beta_0 + 1 \) for some ordinal \( \beta_0 \). Then \( \beta_0 < \alpha_0 = I \) shows \( b_{\beta_0} \in B \). Now from \( b_{\beta_0} \downarrow \alpha_0 = b_{\beta_0} \downarrow (\beta_0 + 1) = y_0 \downarrow (\beta_0 + 1) = y_0 \downarrow \alpha_0 \), we have \( b_{\beta_0}(\alpha_0) \in B_0 = X_{\alpha_0} \setminus A_0 \), a contradiction. Thus we see that \( \alpha_0 = 0 \) or \( \alpha_0 \) is limit, that is, \( [I(\alpha_0), \alpha_0) = \emptyset \). Now the condition (2ci) shows \( 0-\text{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega \), a contradiction.

**Claim 13.** \( A_0 \) is closed in \( X_{\alpha_0} \).

**Proof.** When \( B_0 \) has no minimal element, obviously \( A_0 \) is closed. So assume that \( B_0 \) has a minimal element, say \( u = \min B_0 \). It suffices to find a neighborhood of \( u \) disjoint from \( A_0 \). \( A^* = (\leftarrow, y_0 \uparrow \langle u \rangle)_{Z_0} \) and \( 0-\text{cf}_{Z_0} A^* = \omega \) show \( \alpha_0 + 1 \in L^+ \), therefore \( \min L^+ \leq \alpha_0 + 1 \). The condition (2b) ensures \( J^+ \subseteq \min L^+ \leq \alpha_0 + 1 \), so \( J^+ \subseteq [0, \alpha_0] \). Therefore \( X_{\alpha} \) has a minimal element for every \( \alpha > \alpha_0 \). Let \( b = y_0 \uparrow \langle u \rangle \uparrow (\min X_{\alpha} : \alpha < \alpha) \). Since \( b \in B (= X \setminus A) \) and \( A \) is closed in \( X \), there is \( b^* \in X \) such that \( b^* < b \) and \( (b^*, b)_X \cap A = \emptyset \). Set \( \beta_0 = \min\{\beta < \gamma : b^*(\beta) \neq b(\beta)\} \), then obviously \( \beta_0 \leq \alpha_0 \). If \( \beta_0 < \alpha_0 \) were true, we have \( a_{\beta_0} \in (b^*, b)_X \cap A \), a contradiction. Thus we have \( \beta_0 = \alpha_0 \), so \( b^* \uparrow \alpha_0 = y_0 \) and \( b^*(\alpha_0) < u \). If there were \( v \in (b^*(\alpha_0), \rightarrow)_{X_{\alpha_0}} \cap A_0 \), then \( v < u \) shows \( y_0 \uparrow \langle v \rangle \uparrow (\min X_{\alpha} : \alpha < \alpha) \in (b^*, b) \cap A \), a contradiction. Therefore \( (b^*(\alpha_0), \rightarrow)_{X_{\alpha_0}} \cap X_{\alpha_0} \) is a neighborhood of \( u \) disjoint from \( A_0 \).

These claims above show that \( A_0 \) is a bounded closed 0-segment of \( X_{\alpha_0} \). Now the condition (2a) shows \( 0-\text{cf}_{X_{\alpha_0}} A_0 \neq \omega \), a contradiction.

**Case 3-2.** \( 0-\text{cf}_{Z_0} A^* = 1 \).

Since \( A = A^* \times Z_1 \), \( A^* \) has a maximal element but \( A \) has no maximal element, we see that \( Z_1 \) has no maximal element. Therefore \( X_{\alpha} \) has no maximal element for some \( \alpha > \alpha_0 \), in particular \( (\alpha_0, \gamma) \neq \emptyset \). Let \( \alpha_1 = \min\{\alpha > \alpha_0 : X_{\alpha} \) has no maximal element. \}. Then we have \( \alpha_0 < \alpha_1 \in J^+ \) and \( (\alpha_0, \alpha_1) \cap J^+ = \emptyset \). Since \( A = A^* \times Z_1 = A^* \times (\prod_{\alpha_0<\alpha_\leq \alpha_1} X_{\alpha} \times \prod_{\alpha_1<\alpha} X_{\alpha}) = (A^* \times \prod_{\alpha_0<\alpha_\leq \alpha_1} X_{\alpha}) \times \prod_{\alpha_1<\alpha} X_{\alpha} \) and \( A^* \times \prod_{\alpha_0<\alpha_\leq \alpha_1} X_{\alpha} \) is a 0-segment in \( \prod_{\alpha \leq \alpha_1} X_{\alpha} \) with no maximal
element, Lemma 2.1 shows $\omega = 0\text{-}cf X A = 0\text{-}cf (A^* \times \prod_{a_0 < a < a_1} X_a) = 0\text{-}cf_{X_{a_1}} X_{a_1}$ (that $\{y_0 \wedge (\max A_0) \wedge (\max X_{a}: a_0 < a < a_1)\} \times X_{a_1}$ is unbounded in the 0-segment $A^* \times \prod_{a_0 < a < a_1} X_a$ witnesses the last equality).

**Claim 14.** $l(\alpha_1) \leq \alpha_0$ and $J^+ \cap [l(\alpha_1), \alpha_0] \neq \emptyset$ hold, in particular $J^+ \cap [l(\alpha_1), \alpha_1) \neq \emptyset$.

**Proof.** First assume $\alpha_0 < l(\alpha_1)$. Then $J^+ \cap [l(\alpha_1), \alpha_1) \subset J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ and the condition (2ci) show $0\text{-}cf_{X_{a_1}} X_{a_1} \neq \omega$, a contradiction. Thus we have $l(\alpha_1) \leq \alpha_0$.

Next assume $J^+ \cap [l(\alpha_1), \alpha_0] = \emptyset$, then we have $J^+ \cap [l(\alpha_1), \alpha_1) = \emptyset$ because of $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$. Therefore the condition (2ci) shows $0\text{-}cf_{X_{a_1}} X_{a_1} \neq \omega$, a contradiction. Thus $J^+ \cap [l(\alpha_1), \alpha_0] \neq \emptyset$. \hfill \Box

Using the above claim, set $\alpha_2 = \max(J^+ \cap [l(\alpha_1), \alpha_1))$. Note $0 \leq l(\alpha_1) \leq \alpha_2 \leq \alpha_0 < \alpha_1$ and $J^+ \cap (\alpha_2, \alpha_1) = \emptyset$.

**Claim 15.** $B_0$ has a minimal element.

**Proof.** First we check $B_0 \neq \emptyset$, so assume $B_0 = \emptyset$, i.e., $A_0 = X_{a_0}$. $1 = 0\text{-}cf_{Z_0} A^* = 0\text{-}cf_{X_{a_0}} A_0 = 0\text{-}cf_{X_{a_0}} X_{a_0}$ shows $\alpha_0 \notin J^+$. Also $\alpha_2 \leq \alpha_0$ and $\alpha_2 \in J^+$ show $0 \leq \alpha_2 < \alpha_0$. Assume that $\alpha_0 = \beta_0 + 1$ for some ordinal $\beta_0$, then by $\beta_0 < \alpha_0 = I$, we have $b_{\beta_0} \in B$ and $b_{\beta_0} \upharpoonright \alpha_0 = b_{\beta_0} \upharpoonright (\beta_0 + 1) = y_0 \upharpoonright (\beta_0 + 1) = y_0 \upharpoonright \alpha_0$. Therefore we have $b_{\beta_0}(\alpha_0) \in B_0$, a contradiction. So we have $0 < \alpha_0$ and $\alpha_0$ is limit, therefore $\alpha_0 \leq l(\alpha_1) \leq \alpha_2$, which contradicts $\alpha_2 < \alpha_0$. We have seen $B_0 \neq \emptyset$.

Next we check that $B_0$ has a minimal element. Assume that $B_0$ has no minimal element, then max $A_0$ witnesses $\alpha_0 \in [\alpha_2, \alpha_1) \cap K^+$. The definition of $\alpha_2$ and the condition (2ciii) show $0\text{-}cf_{X_{a_1}} X_{a_1} \neq \omega$, a contradiction. \hfill \Box

Now since $B$ has no minimal element, by the claim above, there is $\alpha > \alpha_0$ such that $X_\alpha$ has no minimal element. So let $\alpha_3 = \min\{\alpha > \alpha_0 : X_\alpha \text{ has no minimal element.}\}$. Then we have $\alpha_0 < \alpha_3 \in J^-$. When $\omega \leq \gamma$, Lemma 3.2 and the condition (2b) show $J^- \subset \min L^+ \leq \omega$. When $\gamma < \omega$, obviously $J^- \subset \omega$. So in any case we have $J^- \subset \omega$. Therefore $l(\alpha_1) \leq \alpha_0 < \alpha_3 \in \omega$ so we have $\alpha_1 \in \omega$.

**Claim 16.** $\alpha_3 \leq \alpha_1$.

**Proof.** Assume $\alpha_1 < \alpha_3$, then $X_\alpha$ has a minimal element for every $\alpha \in (\alpha_0, \alpha_1]$. So let $y = y_0 \wedge (\min B_0) \wedge (\min X_{\alpha} : \alpha_0 < \alpha < \alpha_1 \in \alpha_1)$. Note $y \in \prod_{\alpha \leq \alpha_1} X_\alpha$ and consider the interval $(\leftarrow, y)$ in $\prod_{\alpha \leq \alpha_1} X_\alpha$. The definition of $\alpha_2$ and $\alpha_2 \leq \alpha_0$ show that $X_\alpha$ has a maximal element for
every \( \alpha \in (\alpha_0, \alpha_1) \). Since \( \{y_0 \wedge \langle \max A_0 \rangle \wedge (\max X_\alpha : \alpha_0 < \alpha < \alpha_1) \} \times X_{\alpha_1} \) is unbounded in \((\langle \cdot \rangle, y)\), we have \( 0-\text{cf}(\langle \cdot \rangle, y) = 0-\text{cf}_{X_{\alpha_1}} X_{\alpha_1} = \omega \). Thus \( y \) witnesses \( \alpha_1 + 1 \in L^+ \). The condition (2b) ensures \( J^- \subset \min L^+ \leq \alpha_1 + 1 \), thus \( \alpha_3 \in J^- \subset [0, \alpha_1] \), a contradiction. Now we have \( \alpha_3 \leq \alpha_1 \).

Now \( \alpha_3 \in (\alpha_0, \alpha_1] \cap J^- \subset (\alpha_2, \alpha_1] \cap J^- \), \( \alpha_2 = \max(J^+ \cap [l(\alpha_1), \alpha_1]) \) and the condition (2cii) show \( 0-\text{cf}_{X_{\alpha_1}} X_{\alpha_1} \neq \omega \), a contradiction. This completes the proof of the theorem.

Analogously we can see:

**Theorem 3.4.** Let \( X = \prod_{\alpha<\gamma} X_\alpha \) be a lexicographic product of GO-spaces. Then the following are equivalent:

1. \( X \) is countably \( 1 \)-compact,
2. the following clauses hold:
   a. \( X_\alpha \) is boundedly countably \( 1 \)-compact for every \( \alpha < \gamma \),
   b. if \( L^- \neq \emptyset \), then \( J^+ \subset \min L^- \),
   c. for every \( \alpha < \gamma \), if any one of the following cases holds,
      then \( 1-\text{cf}_{X_\alpha} X_\alpha \neq \omega \) holds,
      i. \( J^- \cap [l(\alpha), \alpha] = \emptyset \),
      ii. \( J^- \cap [l(\alpha), \alpha] \neq \emptyset \) and \( (\alpha_0, \alpha] \cap J^+ \neq \emptyset \), where \( \alpha_0 = \max(J^+ \cap [l(\alpha), \alpha]) \),
      iii. \( J^- \cap [l(\alpha), \alpha] \neq \emptyset \) and \( [\alpha_0, \alpha) \cap K^- \neq \emptyset \), where \( \alpha_0 = \max(J^- \cap [l(\alpha), \alpha]) \).

4. Applications

In this section, we apply the theorems in the previous section

**Corollary 4.1.** Let \( X = \prod_{\alpha<\gamma} X_\alpha \) be a lexicographic product of GO-spaces. Then the following hold:

1. if \( X \) is countably \( 0 \)-compact, then \( J^- \subset \omega \),
2. if \( X \) is countably \( 1 \)-compact, then \( J^+ \subset \omega \),
3. if \( X \) is countably \( 0 \)-compact, then for every \( \delta < \gamma \), the lexicographic product \( \prod_{\alpha<\delta} X_\alpha \) is countably \( 0 \)-compact, in particular \( X_0 \) is countably \( 0 \)-compact,
4. if \( X \) is countably \( 1 \)-compact, then for every \( \delta < \gamma \), the lexicographic product \( \prod_{\alpha<\delta} X_\alpha \) is countably \( 1 \)-compact, in particular \( X_0 \) is countably \( 1 \)-compact,

**Proof.** Lemma 3.2 and the condition (2b) in Theorem 3.3 show (1). (3) obviously follows from Theorem 3.3 or Lemma 2.2 directly. The remaining is similar.
Corollary 4.2. Let $X$ be a GO-space. Then the lexicographic product $X^\omega_1$ is countably compact if and only if $X$ is countably compact and has both a minimal and a maximal element.

Proof. That $X^\omega_1$ is countably compact implies that $X$ is countably compact and has both a minimal and a maximal element follows from the corollary above. The other implication follows from the theorems in the previous section because of $J^+ = J^- = \emptyset$.

Corollary 4.3. Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of countably compact GO-spaces. Then the following are equivalent:

1. $X$ is countably compact,
2. the following clauses hold:
   a. if $L^+ \neq \emptyset$, then $J^- \subset \min L^+$,
   b. if $L^- \neq \emptyset$, then $J^+ \subset \min L^-$.

Proof. Since all $X_\alpha$’s are countably compact, (2a)+(2c) in Theorems 3.3 and 3.4 of the previous section are true.

Example 4.4. Let $[0,1)_R$ denote the unit half open interval in the real line $\mathbb{R}$ with the usual order. Let $X$ be the lexicographic product $[0,1)_R \times \omega_1$. Since $[0,1)_R$ is not countably 0-compact, Corollary 4.1 shows that $X$ is not countably 0-compact. Both $[0,1)_R$ and $\omega_1$ are countably 1-compact. Considering $X_0 = [0,1)_R$ and $X_1 = \omega_1$, we see $1 \in L^-$ (0 in $[0,1)_R$ witnesses this) therefore $1 = \min L^-$. Moreover by $1 \in J^+$, (2b) in Theorem 3.4 does not hold. Therefore $X$ is neither countably 0-compact nor countably 1-compact. Note that $X$ is not paracompact, see [7, Example 4.6].

Example 4.5. Let $X$ be the lexicographic product $\omega_1 \times [0,1)_R$. Checking all clauses in the theorems in the previous section, we see that $X$ is countably compact. Since it is not compact, it is not paracompact. The lexicographic product $\omega_1 \times [0,1)_R$ is called the long line of length $\omega_1$ and denoted by $L(\omega_1)$.

Example 4.6. Let $S$ be the Sorgenfrey line, where half open intervals $[a,b)_R$’s are declared to be open. Then it is known that $\omega_1 \times S$ is paracompact but $S \times \omega_1$ is not paracompact, see [7]. On the other hand, both lexicographic products $\omega_1 \times S$ and $S \times \omega_1$ are not countably compact, because $S$ is not boundedly 0-compact.

Example 4.7. Let $X$ be the lexicographic product $\omega_1 \times [0,1)_R \times \omega_1$, and consider as $X_0 = \omega_1$, $X_1 = [0,1)_R$ and $X_2 = \omega_1$. Then $\text{1-cf}_{\omega_1 \times [0,1)_R}((0,0), \to) = \omega$ shows $2 \in L^-$. Since $0,1 \notin L^-$, we have $\min L^- = 2$. Now $2 \in J^+$ implies $J^+ \notin \min L^-$. Thus Theorem 3.4
shows that $X$ is not countably (1-) compact. On the other hand, we will later see that the lexicographic product $\omega_1 \times \omega \times \omega_1$ is countably compact.

**Corollary 4.8.** There is a countably compact LOTS $X$ whose lexicographic square $X^2$ is not countably compact.

**Proof.** $X = L(\omega_1)$ is such an example, because $L(\omega_1)^2 = (\omega_1 \times [0, 1]_{\mathbb{R}} \times \omega_1) \times [0, 1]_{\mathbb{R}}$ (use Example 4.7). We will later see that the lexicographic product $X = \omega_1^\omega$ is also such an example. 

In the rest of the paper, we consider countable compactness of lexicographic products whose all factors have minimal elements. In the following, apply theorems with $J^- = \emptyset$.

**Corollary 4.9.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of G sober spaces. If all $X_\alpha$’s have minimal elements, then the following are equivalent:

1. $X$ is countably 0-compact,
2. the following clauses hold:
   - (a) $X_\alpha$ is (boundedly) countably 0-compact for every $\alpha < \gamma$,
   - (b) for every $\alpha < \gamma$, if either one of the following cases holds, then $0-\text{cf}_{X_\alpha} X_\alpha \neq \omega$ holds,
     - (i) $J^+ \cap [l(\alpha), \alpha) = \emptyset$,
     - (ii) $J^+ \cap [l(\alpha), \alpha) \neq 0$ and $[\alpha_0, \alpha) \cap K^+ \neq \emptyset$, where $\alpha_0 = \max(J^+ \cap [l(\alpha), \alpha))$.

**Corollary 4.10.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of G sober spaces. If all $X_\alpha$’s have minimal elements, then the following are equivalent:

1. $X$ is countably 1-compact,
2. the following clauses hold:
   - (a) $X_\alpha$ is (boundedly) countably 1-compact for every $\alpha < \gamma$,
   - (b) if $L^- \neq \emptyset$, then $J^+ \subset \min L^-$.

Now we consider the case that all factors are subspaces of ordinals. First let $X$ be a subspace of an ordinal. Since $X$ is well-ordered, the following hold:

- $X$ is countably 1-compact,
- $X$ has a minimal element,
- for every $u \in X$ with $(u, \rightarrow) \neq \emptyset$, $(u, \rightarrow)$ has a minimal element,
- there is $u \in X$ such that $(\leftarrow, u)$ is non-empty and has no maximal element if and only if the order type of $X$ is greater than $\omega$. 

Note that a subspace $X$ of $\omega_1$ is countably compact if and only if it is closed in $\omega_1$, and also note that the subspace $X = \{ \alpha < \omega_2 : \text{cf} \alpha \leq \omega \}$ is countably compact but not closed in $\omega_2$.

Next let $X_\alpha$ be a subspace of an ordinal for every $\alpha < \gamma$ and $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product. Then using the notation in section 3, we see:

- $J^- = \emptyset$,
- $K^+ = \emptyset$,
- $\alpha \in K^-$ iff the order type of $X_\alpha$ is greater than $\omega$.

Remarking these facts with Corollaries above, we see:

**Corollary 4.11.** Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product. If all $X_\alpha$’s are subspaces of ordinals, then the following are equivalent:

1. $X$ is countably $0$-compact,
2. the following clauses hold:
   a. $X_\alpha$ is boundedly countably $0$-compact for every $\alpha < \gamma$,
   b. for every $\alpha < \gamma$ with $J^+ \cap (l(\alpha), \alpha) = \emptyset$, $\text{0-cf}_{X_\alpha} X_\alpha \neq \omega$ holds,

**Corollary 4.12.** Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product. If all $X_\alpha$’s are subspaces of ordinals, then the following are equivalent:

1. $X$ is countably $1$-compact,
2. $J^+ \subseteq \omega$.

**Proof.** (1) $\Rightarrow$ (2) Assume that $X$ is countably $1$-compact. By Corollary 4.10, if $L^- \neq \emptyset$, then $J^+ \subseteq \text{min} L^-$. When $\gamma \geq \omega$, because of $\omega \in L^-$, we see $J^+ \subseteq \text{min} (\omega, L^-) \leq \omega$. When $\gamma < \omega$, obviously we see $J^+ \subseteq \gamma < \omega$.

(2) $\Rightarrow$ (1) Assume $J^+ \subseteq \omega$. It suffices to check (2a) and (2b) in Corollary 4.10. (2a) is obvious. To see (2b), let $L^- \neq \emptyset$. Now assume $\omega \cap L^- \neq \emptyset$, and take $n \in \omega \cap L^-$. Then we can take $u \in \prod_{m<n} X_m$ with $1-\text{cf} (u, \rightarrow) = \omega$. But this is a contradiction, because a lexicographic product of finite length of subspaces of ordinals are also a subspace of ordinal, see [7, Lemma 4.3]. Therefore we have $\omega \cap L^- = \emptyset$. $L^- \neq \emptyset$ and Lemma 3.2 show $J^+ \subseteq \omega = \text{min} L^-$. \qed

If $X$ is an ordinal, then it is boundedly countably $0$-compact and $0-\text{cf}_X X = \text{cf} X$. Therefore we have:

**Corollary 4.13.** Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of ordinals. Then the following are equivalent:

1. $X$ is countably compact,
2. the following clauses hold:
   a. if $J^+ \neq \emptyset$, then $\text{cf} X_{\min J^+} \geq \omega_1$, 

Corollary 4.14. [4] The following clauses hold:

1. the lexicographic product \( \omega_1^\gamma \) is countably \( 0 \)-compact for every ordinal \( \gamma \),

2. the lexicographic product \( \omega_1^\gamma \) is countably \( 1 \)-compact iff \( \gamma \leq \omega \).

Example 4.15. Using Corollary 4.13, we see:

1. lexicographic products \( \omega_2, \omega_1 \times \omega, (\omega + 1) \times (\omega_1 + 1) \times \omega_1 \times \omega, \omega_1 \times \omega \times \omega_1, \omega_1 \times \omega \times \omega_1 \times \omega \times \cdots, \omega_1 \times \omega^\omega, \omega_1 \times \omega^\omega \times (\omega + 1), \omega_1^\omega, \omega_1^\omega \times (\omega_1 + 1) \) and \( \prod_{n \in \omega} \omega_{n+1} \) are countably compact,

2. lexicographic products \( \omega \times \omega_1, (\omega + 1) \times (\omega_1 + 1) \times \omega \times \omega_1, \omega \times \omega_1 \times \omega \times \omega_1 \times \cdots, \omega \times \omega^\omega, \omega_1 \times \omega^\omega \times \omega_1, \omega_1^\omega \times \omega, \omega_1^\omega \times \omega, \prod_{n \in \omega} \omega_n \) and \( \prod_{n \leq \omega} \omega_{n+1} \) are not countably compact,

3. let \( X = \omega_1^\omega \), then the lexicographic product \( X^2 \) is not countably compact because of \( X^2 = \omega_1^\omega \times \omega_1^\omega = \omega_1^{\omega+\omega} \), so this shows also Corollary 4.8.

For a GO-space \( X = \langle X, <_X, \tau_X \rangle \), \( -X \) denotes the reverse of \( X \), that is, the GO-space \( \langle X, >_X, \tau_X \rangle \), see [7]. Note that \( X \) and \( -X \) are topologically homeomorphic.

Example 4.16. As above, the lexicographic product \( \omega_1^2 \) was countably compact. But the lexicographic product \( \omega_1 \times (\omega_1) \) is not countably compact. Indeed, let \( X = \omega_1 \times (\omega_1), X_0 = \omega_1 \) and \( X_1 = -\omega_1 \), \( \omega \in X_0 \) with \( 0\text{-}cf_{X_0}(\omega, -\omega) = cf \omega = \omega \) witnesses \( 1 \in L^+, \) therefore \( \min L^+ = 1 \).

On the other hand \( -\omega_1 \) has no minimal element, so we have \( 1 \in J^- \).

Therefore (2b) of Theorem 3.3 does not hold, thus \( X \) is not countably \( (0-)\)-compact.

Also note that \( (\omega_1) \times (\omega_1) \) is countably compact but \( (\omega_1) \times \omega_1 \) is not countably compact, because \( (\omega_1) \times (\omega_1) \) and \( (\omega_1) \times \omega_1 \) are topologically homeomorphic to \( \omega_1^2 \) and \( \omega_1 \times (\omega_1) \) respectively, see [7].

Moreover \( \omega_1 \times (\omega) \) is directly shown not to be countably \( (1-)\)-compact, because the \( 1 \)-order preserving sequence \( \{ \langle 0, n \rangle : n \in \omega \} \) has no cluster point in \( \omega_1 \times (\omega) \).

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