A CHARACTERIZATION OF PARACOMPACTNESS OF LEXICOGRAPHIC PRODUCTS

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Abstract. It was known that lexicographic products of paracompact LOTS's are also paracompact, see [2]. After then in [3], the notion of the lexicographic products of GO-spaces is defined and the result above is extended for lexicographic products of GO-spaces and it is asked when lexicographic products of GO-spaces are paracompact. For this question, paracompactness of lexicographic products of some special cases below are characterized in [4]:

- lexicographic products of two GO-spaces,
- lexicographic products of any length of ordinal subspaces.

In this paper, we will give a complete answer of the question above in [3]. As corollaries, we will see :

- if γ is a limit ordinal and X_α has neither minimal nor maximal elements for every $\alpha < \gamma$, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact,
- the lexicographic product $\mathbb{S} \times \omega_1$ is not paracompact but the lexicographic products $\omega_1 \times \mathbb{S}$ and $\mathbb{S} \times \omega_1 \times \mathbb{S} \times \omega_1 \times \cdots$ are paracompact, where $\mathbb{S}$ denotes the Sorgenfrey line,
- the lexicographic products $\omega \times \omega_1 \times \mathbb{I}$ and $\mathbb{I} \times \omega \times \omega_1$ are paracompact but the lexicographic product $\omega \times \mathbb{I} \times \omega_1$ is not paracompact, where $\mathbb{I}$ denotes the unit interval $[0, 1]$ in the real line $\mathbb{R}$.
- the lexicographic product $\omega_1 \times (-\omega_1) \times \omega_1 \times (-\omega_1) \times \cdots$ is paracompact, where for a GO-space $X = \langle X, <_X, \tau_X\rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X\rangle$.

1. Introduction

All spaces are assumed to be regular T_1 and when we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α are assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminology follow [5] and [1].

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A linearly ordered set $\langle L, <_L \rangle$ has a natural topology λ_L, which is called an interval topology, generated by $\{(\leftarrow, x)_L : x \in L\} \cup \{(x, \rightarrow)_L : x \in L\}$ as a subbase, where $(x, \rightarrow)_L = \{z \in L : x <_L z\}$, $(x, y)_L = \{z \in L : x <_L z <_L y\}$, $(x, y]_L = \{z \in L : x <_L z \leq_L y\}$ and so on. The triple $\langle L, <_L, \lambda_L \rangle$, which is simply denoted by L, is called a *LOTS*.

A triple $\langle X, <_X, \tau_X \rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X \rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subset C$ holds. For more information on LOTS's or GO-spaces, see [6]. Usually $<_L$, $(x, y)_L$, λ_L or τ_X are written simply $<$, (x, y), λ or τ if contexts are clear.

ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \cdots$, are considered to be LOTS's with the usual intereval topology. For a subset A of an ordinal α, $\mathrm{Lim}(A)$ denotes the set $\{\beta < \alpha : \beta = \sup(A \cap \beta)\}$, that is, the set of all cluster points of A in the topological space α.

For GO-spaces $X = \langle X, <_X, \tau_X \rangle$ and $Y = \langle Y, <_Y, \tau_Y \rangle$, X is said to be a *subspace* of Y if $X \subset Y$, the linear order $<_X$ is the restriction $<_Y \restriction X$ of the order $<_Y$ and the topology τ_X is the subspace topology $\tau_Y \restriction X$ $(= \{U \cap X : U \in \tau_Y\})$ on X of the topology τ_Y. So a subset of a GO-space is naturally considered as a GO-space. For every GO-space X, there is a LOTS X^* such that X is a dense subspace of X^* and X^* has the property that if L is a LOTS containing X as a dense subspace, then L also contains the LOTS X^* as a subspace, see [7]. Such a X^* is called the *minimal d-extension of a GO-space* X. The construction of X^* is also shown in [3]. Obviously, we can see:

- if X is a LOTS, then $X^* = X$,
- X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

For every $\alpha < \gamma$, let X_α be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma \rangle$. The lexicographic order $<_X$ on X is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha \rangle$ and $<_{X_\alpha}$ is the order on X_α. Now for every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace X of the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ is said to be the *lexicographic product of GO-spaces* X_α's, for more details see [3].

$\prod_{i\in\omega} X_i$ ($\prod_{i\leq n} X_i$ where $n \in \omega$) is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let X and Y be LOTS's. A map $f : X \to Y$ is said to be *order preserving* or 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be *order reversing* or 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map (also 1-order preserving map) $f : X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. Now let X and Y be GO-spaces. A 0-order preserving map $f : X \to Y$ is said to be 0-*order preserving embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we identify X with $f[X]$ as a GO-space and write $X = f[X]$.

Recall that a subset of a regular uncountable cardinal κ is called *stationary* if it intersects with all closed unbounded (= club) sets in κ.

Let X be a GO-space. A subset A of X is called a 0-*segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. A 0-*segment* A is said to be *bounded* if $X \setminus A$ is non-empty. Similarly the notion of (bounded) 1-segment can be defined. Both $\emptyset$ and X are 0-segments and 1-segments.

Let A be a 0-segment of a GO-space X. A subset U of A is *unbounded in* A if for every $x \in A$, there is $x' \in U$ such that $x \leq x'$. Let

$$0\text{-}\operatorname{cf}_X A = \min\{|U| : U \text{ is unbounded in } A.\}.$$

$0\text{-}\operatorname{cf}_X A$ can be $0, 1$ or regular infinite cardinals. If contexts are clear, $0\text{-}\operatorname{cf}_X A$ is denoted by $0\text{-}\operatorname{cf} A$. A 0-segment A of a GO-space X is said to be *stationary* if $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$ and there are a stationary set S of κ and a continuous map $\pi : S \to A$ such that $\pi[S]$ is unbounded in A (we say such a π "an unbounded continuous map").

A GO-space X is said to be (*boundedly*) 0-*paracompact* if every (bounded, respectively) closed 0-segment is not stationary. Similarly the notions of $1\text{-}\operatorname{cf} A$, stationarity of a 1-segment and (bounded) 1-paracompactness are defined. Note that a GO-space X is 0-paracompact if and only if it is boundedly 0-paracompact and the 0-segment X is not stationary. Remember that a GO-space is paracompact if and only if it is both 0-paracompact and 1-paracompact, see [3]. We frequently use the following basic 4 lemmas from [4], where a subset H of a GO-space X is 0-*closed* if for every $x \in X \setminus H$, there is an open neighborhood U of x such that $(U \cap (\leftarrow, x]) \cap H = \emptyset$.

Lemma 1.1. [4, Lemma 2.7] *Let A be a 0-segment of a GO-space X with $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$. If there are a stationary set S of κ and an unbounded continuous map $\pi : S \to A$, then:*

(1) *there is a club set C in κ such that $\pi \restriction (S \cap C) : S \cap C \to A$ is 0-order preserving embedding,*
(2) *if H is 0-club (= 0-closed and unbounded) in A, then there is a club set C in κ such that $\pi[S \cap C] \subset H$.*

Lemma 1.2. [4, Lemma 3.4] *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and $u \in X_0$. Then the map $k_u : X_1 \to \{u\} \times X_1$ by $k_u(v) = \langle u, v\rangle$ is a 0-order preserving homeomorphism.*

This lemma shows that the 0-segment X is stationary iff the 0-segment X_1 is stationary, whenever X_0 has a maximal element.

Lemma 1.3. [4, Lemma 3.6] *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and A_0 a 0-segment of X_0. Put $A = A_0 \times X_1$. Then the following hold:*

(1) *A is a 0-segment of X,*
(2) *if $0\text{-}\operatorname{cf}_{X_0} A_0 = 1$, then*
    (a) $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{X_1} X_1$,
    (b) *A is stationary if and only if the 0-segment X_1 is stationary,*
    (c) *A is closed in X if and only if either X_1 has a maximal element, $X_0 \setminus A_0$ has no minimal element or X_1 has no minimal element,*
(3) *if $0\text{-}\operatorname{cf}_{X_0} A_0 \geq \omega$, then*
    (a) $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{X_0} A_0$,
    (b) *A is stationary if and only if X_1 has a minimal element and A_0 is stationary,*
    (c) *A is closed in X if and only if either X_1 has no minimal element or A_0 is closed in X_0.*

Lemma 1.4. [4, Theorem 3.8] *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. The following are equivalent:*

(1) *X is 0-paracompact,*
(2) (a) *X_1 is boundedly 0-paracompact,*
    (b) *if either $(u, \to)_{X_0}$ has no minimal element for some $u \in X_0$ or X_1 has no minimal element, then the 0-segment X_1 is not stationary,*
    (c) *if X_1 has a minimal element, then X_0 is 0-paracompact.*

In this lemma, if X_0 has a maximal element, then $(\max X_0, \to)_{X_0}$ is considered to have no minimal element because of $(\max X_0, \to)_{X_0} = \emptyset$.

Using the lemmas above, we will give a complete characterization of paracompactness of lexicographic products.

2. A characterization

Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. We use the following special notations.

$$J^+ = \{\alpha < \gamma : \ X_\alpha \text{ has no maximal element.}\},$$
$$J^- = \{\alpha < \gamma : \ X_\alpha \text{ has no minimal element.}\},$$
$$K^+ = \{\alpha < \gamma : \text{ there is } x \in X_\alpha \text{ such that } (x, \rightarrow)_{X_\alpha} \text{ is non-empty and has no minimal element.}\},$$
$$K^- = \{\alpha < \gamma : \text{ there is } x \in X_\alpha \text{ such that } (\leftarrow, x)_{X_\alpha} \text{ is non-empty and has no maximal element.}\}$$

Let α be an ordinal and let

$$l(\alpha) = \begin{cases} 0 & \text{if } \alpha < \omega, \\ \sup\{\beta \leq \alpha : \beta \text{ is limit.}\} & \text{if } \alpha \geq \omega. \end{cases}$$

Note that $l(\alpha)$ is the largest limit ordinal less than or equal to α whenever $\alpha \geq \omega$. So the interval $[l(\alpha), \alpha)$ of ordinals is finite, thus every ordinal α can be uniquely represented as $l(\alpha)+n(\alpha)$ for some $n(\alpha) \in \omega$.

Theorem 2.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

1. *X is 0-paracompact,*
2. *for every ordinal $\alpha < \gamma$ with $\sup J^- \leq \alpha$, the following hold:*
   - (a) *X_α is boundedly 0-paracompact,*
   - (b) *in each of the following cases, the 0-segment X_α is not stationary,*
     - (i) $J^+ \cap [l(\alpha), \alpha) = \emptyset$,
     - (ii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $J^- \cap (\alpha', \alpha] \neq \emptyset$,
     - (iii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^+ \cap [\alpha', \alpha) \neq \emptyset$,

     where $\alpha' = \max(J^+ \cap [l(\alpha), \alpha))$ *in case* $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$.

Notice that in (ii) or (iii) above, α' is well-defined since $[l(\alpha), \alpha)$ is a finite set.

Proof. (1) $\Rightarrow$ (2): Let X be 0-paracompact and α_0 an ordinal with $\sup J^- \leq \alpha_0 < \gamma$ and $Y = \prod_{\alpha \leq \alpha_0} X_\alpha$. Note that if $(\alpha_0, \gamma) \neq \emptyset$, then $\prod_{\alpha_0<\alpha} X_\alpha$ has a minimal element because of $\sup J^- \leq \alpha_0$. Remark that when $(\alpha_0, \gamma) = \emptyset$, $\prod_{\alpha_0<\alpha} X_\alpha$ is considered as the trivial one point LOTS $\{\emptyset\}$. Also for a GO-space Z, $Z \times \{\emptyset\}$ and $\{\emptyset\} \times Z$ are identified with

Z. When $(\alpha_0, \gamma) = \emptyset$, then $\prod_{\alpha \leq \alpha_0} X_\alpha$ ($= X$) is itself 0-paracompact. When $(\alpha_0, \gamma) \neq \emptyset$, then by $X = (\prod_{\alpha \leq \alpha_0} X_\alpha) \times (\prod_{\alpha_0 < \alpha} X_\alpha)$ (see [3, Lemma 1.5]) and Lemma 1.4 (2c), $\prod_{\alpha \leq \alpha_0} X_\alpha$ is 0-paracompact. Thus in both cases Y is 0-paracompact.

(a): By $Y = (\prod_{\alpha < \alpha_0} X_\alpha) \times X_{\alpha_0}$, it follows from Lemma 1.4 (2a) that X_{α_0} is boundedly 0-paracompact, where $\prod_{\alpha < \alpha_0} X_\alpha = \{\emptyset\}$ whenever $\alpha_0 = 0$.

(b): In each cases of (i), (ii) and (iii) above, we will see that the 0-segment X_{α_0} is non-stationary. In case $\alpha_0 \notin J^+$, it is trivial that X_{α_0} is non-stationary since $0\text{-}\operatorname{cf} X_{\alpha_0} = 1$. So we consider the case that $\alpha_0 \in J^+$. Let $Y_0 = \prod_{\alpha < \alpha_0} X_\alpha$.

(i): $J^+ \cap [l(\alpha_0), \alpha_0) = \emptyset$.

When $\alpha_0 = 0$, X_{α_0} ($= Y$) is 0-paracompact thus the 0-segment X_{α_0} is non-stationary. So let $\alpha_0 > 0$. Then $|Y_0| \geq 2$ and $Y = Y_0 \times X_{\alpha_0}$.

Case 1. $l(\alpha_0) = 0$, i.e., $0 < \alpha_0 < \omega$.

In our case (i), X_α has a maximal element for every $\alpha < \alpha_0$. So let $y_0 = \langle \max X_\alpha : \alpha < \alpha_0 \rangle$, that is, y_0 is the element of Y_0 with $y_0(\alpha) = \max X_\alpha$ for every $\alpha < \alpha_0$. Then $y_0 = \max Y_0$ thus $(y_0, \rightarrow)_{Y_0}$ is empty and has no minimal element. Therefore by Lemma 1.4 (2b), the 0-segment X_{α_0} is non-stationary.

Case 2. $l(\alpha_0) \geq \omega$.

In this case, X_α has a maximal element for every $\alpha \in [l(\alpha_0), \alpha_0)$. For every $\alpha < l(\alpha_0)$, fix $x_0(\alpha), x_1(\alpha) \in X_\alpha$ with $x_0(\alpha) <_{X_\alpha} x_1(\alpha)$. Let $y_0 = \langle x_0(\alpha) : \alpha < l(\alpha_0) \rangle ^\wedge \langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0 \rangle$, that is, y_0 is the element of Y_0 such that $y_0(\alpha) = x_0(\alpha)$ for every $\alpha < l(\alpha_0)$, and $y_0(\alpha) = \max X_\alpha$ for every $\alpha < \alpha_0$ with $l(\alpha_0) \leq \alpha$. Here, when $l(\alpha_0) = \alpha_0$, i.e., α_0 is limit, y_0 is considered as $\langle x_0(\alpha) : \alpha < l(\alpha_0) \rangle$. More generally, $z^\wedge \emptyset$ and $\emptyset^\wedge z$ will be identified with z whenever z is a sequence. Moreover for every $\beta < l(\alpha_0)$, let $z_\beta = \langle x_0(\alpha) : \alpha < \beta \rangle ^\wedge \langle x_1(\alpha) : \beta \leq \alpha < l(\alpha_0) \rangle ^\wedge \langle \max X_\alpha : l(\alpha_0) \leq \alpha < \alpha_0 \rangle$. Then the sequence $\{z_\beta : \beta < l(\alpha_0)\}$ is 1-order preserving (i.e., strictly decreasing) and unbounded in the 1-segment $(y_0, \rightarrow)_{Y_0}$. Therefore $(y_0, \rightarrow)_{Y_0}$ has no minimal element, so again by Lemma 1.4 (2b), the 0-segment X_{α_0} is non-stationary.

(ii): $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $J^- \cap (\alpha_1, \alpha_0] \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Note that $J^+ \cap [l(\alpha_0), \alpha_0)$ is non-empty and finite, therefore α_1 is well-defined. Let $\alpha_2 = \max((\alpha_1, \alpha_0] \cap J^-)$. Then $0 \leq l(\alpha_0) \leq \alpha_1 < \alpha_2 \leq \alpha_0$ holds, in particular we have $[0, \alpha_2) \neq \emptyset$. We consider two cases.

Case 1. $\alpha_2 = \alpha_0$.

In this case, since $Y = (\prod_{\alpha<\alpha_0} X_\alpha) \times X_{\alpha_0}$ $(= Y_0 \times X_{\alpha_0})$, Y is 0-paracompact and X_{α_0} has no minimal element, by Lemma 1.4 (2b), the 0-segment X_{α_0} is non-stationary.

Case 2. $\alpha_2 < \alpha_0$.

In this case, note that X_α has a minimal element for every $\alpha \in (\alpha_2, \alpha_0]$. Now let $Z_0 = \prod_{\alpha<\alpha_2} X_\alpha$ and fix an element $z_0 \in Z_0$. Noting that X_α has a maximal element for every $\alpha \in [\alpha_2, \alpha_0)$, let $y_0 = z_0 {}^\wedge\langle \max X_\alpha : \alpha_2 \leq \alpha < \alpha_0\rangle$, which is an element of Y_0. We prove:

Claim 1. $(y_0, \rightarrow)_{Y_0}$ is non-empty and has no minimal element.

Proof. From $\alpha_1 \in J^+$, fixing $u \in X_{\alpha_1}$ with $y_0(\alpha_1) < u$, let $y_1 = (y_0 \restriction \alpha_1){}^\wedge\langle u\rangle{}^\wedge(y_0 \restriction (\alpha_1, \alpha_0))$. Then $y_0 <_{Y_0} y_1$ holds, so $(y_0, \rightarrow)_{Y_0} \neq \emptyset$. To see that $(y_0, \rightarrow)_{Y_0}$ has no minimal element, let $y \in (y_0, \rightarrow)_{Y_0}$. Then we have $y_0 \restriction \alpha_2 <_{Z_0} y \restriction \alpha_2$. Since X_{α_2} has no minimal element, fixing $v \in X_{\alpha_2}$ with $v < y(\alpha_2)$, we see $y_0 <_{Y_0} (y \restriction \alpha_2){}^\wedge\langle v\rangle{}^\wedge(y \restriction (\alpha_2, \alpha_0)) <_{Y_0} y$. □

Since $Y_0 \times X_{\alpha_0}$ $(= Y)$ is 0-paracompact, it follows from the claim above and Lemma 1.4 (2b) that X_{α_0} is non-stationary.

(iii): $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $K^+ \cap [\alpha_1, \alpha_0) \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Let $\alpha_2 = \max([\alpha_1, \alpha_0) \cap K^+)$ and $Z_0 = \prod_{\alpha<\alpha_2} X_\alpha$. Then $l(\alpha_0) \leq \alpha_1 \leq \alpha_2 < \alpha_0$ holds. By $\alpha_2 \in K^+$, one can fix $u \in X_{\alpha_2}$ such that $(u, \rightarrow)_{X_{\alpha_2}}$ is non-empty and has no minimal element. Fixing $z_0 \in Z_0$, let $y_0 = z_0 {}^\wedge\langle u\rangle{}^\wedge\langle \max X_\alpha : \alpha_2 < \alpha < \alpha_0\rangle$. Then $y_0 \in Y_0$, $(y_0, \rightarrow)_{Y_0}$ is non-empty and has no minimal element. Since $(\prod_{\alpha<\alpha_0} X_\alpha) \times X_{\alpha_0}$ $(= Y)$ is 0-paracompact, from Lemma 1.4 (2b), the 0-segment X_{α_0} is non-stationary.

(2) ⇒ (1): Assume that (2) holds but X is not 0-paracompact. Then there is a closed satationary 0-segment A of X. Letting $\kappa = 0\text{-}\mathrm{cf}_X A$ $(\geq \omega_1)$, fix a stationary set $S \subset \kappa$ and an unbounded continuous map $\pi : S \to A$. By Lemma 1.1 (1), we may assume that π is 0-order preserving. We consider several cases, and in all cases we will derive a contradiction.

Case 1. $A = X$.

Since A has no maximal element and $A = X$, J^+ is non-empty, so let $\alpha_0 = \min J^+$. It follows from $\alpha_0 \in J^+$ and $J^+ \cap [l(\alpha_0), \alpha_0) \subset J^+ \cap [0, \alpha_0) = \emptyset$ that (bi) in this theorem is satisfied, therefore $\sup J^- \leq \alpha_0$ implies that the 0-segment X_{α_0} is non-stationary. When $(\alpha_0, \gamma) = \emptyset$ (i.e., $\gamma = \alpha_0 + 1$), the 0-segment X $(= \prod_{\alpha \leq \alpha_0} X_\alpha)$ in X is stationary and $\sup J^- \leq \alpha_0$. When $(\alpha_0, \gamma) \neq \emptyset$, noting that $\prod_{\alpha \leq \alpha_0} X_\alpha$ has no maximal element, by Lemma 1.3 (3b), we see that $\prod_{\alpha_0 < \alpha} X_\alpha$ has a minimal element (therefore $\sup J^- \leq \alpha_0$) and the 0-segment $\prod_{\alpha \leq \alpha_0} X_\alpha$ in $\prod_{\alpha \leq \alpha_0} X_\alpha$ is stationary. In either cases, $\sup J^- \leq \alpha_0$ and the 0-segment $\prod_{\alpha \leq \alpha_0} X_\alpha$ in $\prod_{\alpha \leq \alpha_0} X_\alpha$ is stationary. Since X_{α_0} is non-stationary, we have $\alpha_0 > 0$. Now it follows from $\prod_{\alpha \leq \alpha_0} X_\alpha = (\prod_{\alpha < \alpha_0} X_\alpha) \times X_{\alpha_0}$ and the minimality of α_0 that $\prod_{\alpha < \alpha_0} X_\alpha$ has a maximal element. Therefore from Lemma 1.2, the 0-segment X_{α_0} in X_{α_0} is stationary, a contradiction.

Case 2. $A \neq X$ and $X \setminus A$ has a minimal element.

Let $B = X \setminus A$ and $b = \min B$. Since A is non-empty closed and $B = [b, \rightarrow)_X$, there is $b^* \in \hat{X}$ such that $b^* <_{\hat{X}} b$ and $(b^*, b)_{\hat{X}} \cap X = \emptyset$, where $\hat{X} = \prod_{\alpha < \gamma} X^*_\alpha$. Since A has no maximal element, we have $b^* \notin X$. Let $\alpha_0 = \min\{\alpha < \gamma : b^*(\alpha) \neq b(\alpha)\}$, then $b^* \upharpoonright \alpha_0 = b \upharpoonright \alpha_0$ and $b^*(\alpha_0) < b(\alpha_0)$ hold.

Claim 2. For every $\alpha > \alpha_0$, X_α has a minimal element and $b(\alpha) = \min X_\alpha$ holds, in particular, $\sup J^- \leq \alpha_0$.

Proof. Assume that there are $\alpha > \alpha_0$ and $u \in X_\alpha$ with $b(\alpha) > u$. Let $\alpha_1 = \min\{\alpha > \alpha_0 : b(\alpha) > u \text{ for some } u \in X_\alpha.\}$ and take $u \in X_{\alpha_1}$ with $b(\alpha) > u$. Then we have $b^* <_{\hat{X}} (b \upharpoonright \alpha_1)^\wedge\langle u\rangle^\wedge(b \upharpoonright (\alpha_1, \gamma)) <_X b$, which means $(b^*, b)_{\hat{X}} \cap X \neq \emptyset$, a contradiction. □

Claim 3. $(b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0} = \emptyset$.

Proof. If there were $u \in (b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0}$, then we have $(b \upharpoonright \alpha_0)^\wedge\langle u\rangle^\wedge(b \upharpoonright (\alpha_0, \gamma)) \in (b^*, b)_{\hat{X}} \cap X$, a contradiction. □

Claim 4. $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$.

Proof. Since $b^*(\alpha_0) <_{X^*_{\alpha_0}} b(\alpha_0)$ and X_{α_0} is dense in $X^*_{\alpha_0}$, we have $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}} \neq \emptyset$. Assuming $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, find $u \in X_{\alpha_0}$ with $u < b(\alpha_0)$ and $(u, b(\alpha_0))_{X_{\alpha_0}} = \emptyset$. It follows from Claim 3 that $b^*(\alpha_0) = u \in X_{\alpha_0}$. Now a similar argument of the proof of Claim 2 shows that for every $\alpha > \alpha_0$, $b^*(\alpha) = \max X_\alpha$ which implies $b^* \in X$, a contradiction. □

Let $A_0 = (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$. Then Claim 4 with $A_0 = (\leftarrow, b^*(\alpha_0)]_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ shows that A_0 is a bounded closed 0-segment of X_{α_0} with no maximal element. By our assumption (2a) and $\sup J^- \leq \alpha_0$, the 0-segment A_0 is non-stationary.

Claim 5. $H = \{x \in X : x \upharpoonright \alpha_0 = b \upharpoonright \alpha_0, x(\alpha_0) \in A_0, x \upharpoonright (\alpha_0, \gamma) = b \upharpoonright (\alpha_0, \gamma)\}$ is 0-club in A.

Proof. $H \subset A$ is obvious.

To see that H is unbounded in the 0-segment A in X, let $a \in A$ $(= (\leftarrow, b)_X)$. Put $\beta_0 = \min\{\alpha < \gamma : a(\alpha) \neq b(\alpha)\}$. Since $b(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$, we have $\beta_0 \leq \alpha_0$. When $\beta_0 < \alpha_0$, fix $u \in A_0$. When $\beta_0 = \alpha_0$, fix $u \in A_0$ with $a(\alpha_0) < u$. In both cases, we have $a < (b \upharpoonright \alpha_0)^\wedge\langle u\rangle^\wedge(b \upharpoonright (\alpha_0, \gamma)) \in H$, thus H is unbounded in the 0-segment A.

To see that H is 0-closed, let $a \in X \setminus H$. Since A is closed in X and $H \subset A$, we may assume $a \in A \setminus H$. Put $\beta_0 = \min\{\alpha < \gamma : a(\alpha) \neq b(\alpha)\}$, then we have $\beta_0 \leq \alpha_0$ as above. When $\beta_0 < \alpha_0$, letting $U = X$, we see $H \cap (U \cap (\leftarrow, a]) = \emptyset$. So let $\beta_0 = \alpha_0$. By $a \upharpoonright \alpha_0 = b \upharpoonright \alpha_0, a(\alpha_0) \in A_0$ and $a \notin H$, we can find $\alpha > \alpha_0$ such that $a(\alpha) \neq b(\alpha)$ $(= \min X_\alpha)$. Let $\alpha_1 = \min\{\alpha > \alpha_0 : a(\alpha) \neq b(\alpha)\}$, $a' = (a \upharpoonright \alpha_1)^\wedge\langle \min X_\alpha : \alpha_1 \leq \alpha\rangle$ and $U = (a', \rightarrow)_X$. It follows from $a' < a$ that U is a neighborhood of a. Obviously $H \cap (U \cap (\leftarrow, a]) = \emptyset$, thus we see that H is 0-club. □

By Claim 5 and Lemma 1.1 (2), we can find a club set C in κ with $\pi[S \cap C] \subset H$. Define a map $\sigma : S \cap C \to A_0$ by $\sigma(\beta) = \pi(\beta)(\alpha_0)$ for every $\beta \in S \cap C$.

Claim 6. $\sigma[S \cap C]$ is unbounded in the 0-segment A_0.

Proof. Let $u \in A_0$ and $a = (b \upharpoonright \alpha_0)^\wedge\langle u\rangle^\wedge(b \upharpoonright (\alpha_0, \gamma))$, then $a \in A$. Since π is 0-order preserving and $\pi[S]$ is unbounded in the 0-segment A, we can find $\beta \in S \cap C$ with $a \leq \pi(\beta)$. Noting $\pi(\beta) \in H$, we see $u \leq \sigma(\beta)$. □

Claim 7. σ is continuous.

Proof. Let $\beta \in S \cap C$ and U be a neighborhood of $\sigma(\beta)$ in X_{α_0}. We may assume $\beta \in \mathrm{Lim}(S \cap C)$, then note $(\leftarrow, \sigma(\beta))_{X_{\alpha_0}} \neq \emptyset$. We can find $u^* \in X^*_{\alpha_0}$ such that $u^* < \sigma(\beta)$ and $(u^*, \sigma(\beta)]_{X^*_{\alpha_0}} \cap X_{\alpha_0} \subset U$. Let $x^* = (b \upharpoonright \alpha_0)^\wedge\langle u^*\rangle^\wedge(b \upharpoonright (\alpha_0, \gamma)$, then $x^* \in \hat{X}$ and $x^* < \pi(\beta)$. Since

$(x^*, \to)_{\hat{X}} \cap X$ is a neighborhood of $\pi(\beta)$, by continuity at $\pi(\beta)$, we can find $\beta_1 < \beta$ such that $\pi[S \cap (\beta_1, \beta]] \subset (x^*, \to)_{\hat{X}} \cap X$. We may assume $\beta_1 \in S \cap C$ because of $\beta \in \mathrm{Lim}(S \cap C)$. Then we can easily verify $\sigma[S \cap C \cap (\beta_1, \beta]] \subset U$, which shows that σ is continuous. □

Now Claims 6 and 7 contradict that the 0-segment A_0 is not stationary.

Case 3. $A \neq X$ and $X \setminus A$ has no minimal element.

This case is the most complicated case. Let $B = X \setminus A$ and

$$I = \{\alpha < \gamma : \exists a \in A \exists b \in B (a \restriction (\alpha + 1) = b \restriction (\alpha + 1))\}.$$

Obviously I is an initial segment (i.e., 0-segment) in γ. Therefore for some $\alpha_0 \leq \gamma$, $I = \alpha_0$ holds. For every $\alpha < \alpha_0$, fix $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$ and consider the lexicographic products $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0 \leq \alpha} X_\alpha$. Note that $X = Y_0 \times Y_1$ (see [3, Lemma 1.5]), in particular $Y_0 = X$ ($Y_1 = X$) whenever $\alpha_0 = \gamma$ ($\alpha_0 = 0$, respectively). Define $y_0 \in Y_0$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$.

Claim 8. For every $\alpha < \alpha_0$, $y_0 \restriction (\alpha + 1) = a_\alpha \restriction (\alpha + 1) = b_\alpha \restriction (\alpha + 1)$ holds.

Proof. It suffices to see the first equality. Assuming $y_0 \restriction (\alpha + 1) \neq a_\alpha \restriction (\alpha + 1)$ for some $\alpha < \alpha_0$, let $\alpha_1 = \min\{\alpha < \alpha_0 : y_0 \restriction (\alpha + 1) \neq a_\alpha \restriction (\alpha + 1)\}$. Moreover let $\alpha_2 = \min\{\alpha \leq \alpha_1 : y_0(\alpha) \neq a_{\alpha_1}(\alpha)\}$, then note $\alpha_2 < \alpha_1$ (because of $y_0(\alpha_1) = a_{\alpha_1}(\alpha_1)$), $y_0 \restriction \alpha_2 = a_{\alpha_1} \restriction \alpha_2$ and $y_0(\alpha_2) \neq a_{\alpha_1}(\alpha_2)$. By the minimality of α_1, also note $y_0 \restriction (\alpha_2 + 1) = a_{\alpha_2} \restriction (\alpha_2 + 1)$ $(= b_{\alpha_2} \restriction (\alpha_2 + 1))$. When $y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_2} < a_{\alpha_1} \in A$, a contradiction. When $y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)$, we also have $B \ni b_{\alpha_1} < a_{\alpha_2} \in A$, a contradiction. □

Claim 9. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$, then $y_0 \in Y_0 = X = A \cup B$. First assume $y_0 \in A$. Since A has no maximal element, we can take $a \in A$ with $y_0 < a$ and set $\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq a(\beta)\}$. Applying Claim 8 with $\alpha = \beta_0$, by $y_0 \restriction \beta_0 = a \restriction \beta_0$ and $y_0(\beta_0) < a(\beta_0)$, we see $B \ni b_{\beta_0} < a \in A$, a contradiction. Next assume $y_0 \in B$. Since B has no minimal element, take $b \in B$ with $b < y_0$. By a similar argument, we also get a contradiction. □

Let $A_0 = \{a(\alpha_0) : a \in A, a \restriction \alpha_0 = y_0\}$ and $B_0 = \{b(\alpha_0) : b \in B, b \restriction \alpha_0 = y_0\}$. Of course, whenever $\alpha_0 = 0$, y_0 is considered as $\emptyset$, $A_0 = \{a(\alpha_0) : a \in A\}$ and $B_0 = \{b(\alpha_0) : b \in B\}$.

Claim 10. The following properties hold:

(1) for every $a \in A$, $a \restriction \alpha_0 \leq_{Y_0} y_0$ holds,
(2) for every $x \in X$, if $x \restriction \alpha_0 <_{Y_0} y_0$, then $x \in A$.

Proof. (1): Assuming $a \restriction \alpha_0 >_{Y_0} y_0$ for some $a \in A$, let $\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}$. Then we have $B \ni b_{\beta_0} < a \in A$, a contradiction.

(2): Assuming $x \restriction \alpha_0 <_{Y_0} y_0$, let $\beta_0 = \min\{\beta < \alpha_0 : x(\beta) \neq y_0(\beta)\}$. Then we have $x < a_{\beta_0} \in A$. Now since A is a 0-segment, we see $x \in A$. □

Similarly we see:

Claim 11. The following properties hold:

(1) for every $b \in B$, $b \restriction \alpha_0 \geq_{Y_0} y_0$ holds,
(2) for every $x \in X$, if $x \restriction \alpha_0 >_{Y_0} y_0$, then $x \in B$.

Claim 12. A_0 is a 0-segment of X_{α_0} and $B_0 = X_{\alpha_0} \setminus A_0$.

Proof. To see that A_0 is a 0-segment, let $u' < u \in A_0$. Taking $a \in A$ with $a \restriction \alpha_0 = y_0$ and $u = a(\alpha_0)$, let $a' = (a \restriction \alpha_0)^\wedge\langle u' \rangle^\wedge(a \restriction (\alpha_0, \gamma))$. Since A is a 0-segment and $a' < a$, we have $a' \in A$, thus $u' = a'(\alpha_0) \in A_0$.

Now we prove $B_0 = X_{\alpha_0} \setminus A_0$. First let $u \in B_0$. Take $b \in B$ with $b \restriction \alpha_0 = y_0$ and $u = b(\alpha_0)$. If $u \in A_0$ were true, then by taking $a \in A$ with $a \restriction \alpha_0$ and $a(\alpha_0) = u$, we have $a \restriction (\alpha_0 + 1) = b \restriction (\alpha_0 + 1)$ thus $\alpha_0 \in I = \alpha_0$, a contradiction. So we have $u \in X_{\alpha_0} \setminus A_0$. Conversely let $u \in X_{\alpha_0} \setminus A_0$. Take $x \in X$ with $x \restriction (\alpha_0 + 1) = y_0 {}^\wedge\langle u \rangle$. If $x \in A$ were true, then by $x \restriction \alpha_0 = y_0$ and $x(\alpha_0) = u$, we see $u \in A_0$, a contradiction. Thus we have $x \in B$. Now since $x \restriction \alpha_0 = y_0$, we see $u = x(\alpha_0) \in B_0$. □

Claim 13. $A_0 \neq \emptyset$.

Proof. Assume $A_0 = \emptyset$. We prove the following three facts.

Fact 1. $(\leftarrow, y_0)_{Y_0} \times Y_1 = A$.

Proof. The inclusion "$\subset$" follows from Claim 10 (2). To see the other inclusion, let $a \in A$, then by Claim 10(1), we have $a \restriction \alpha_0 \leq y_0$. If $a \restriction \alpha_0 = y_0$ were true, then $a(\alpha_0) \in A_0$ holds, which contradicts $A_0 = \emptyset$. □

Fact 2. $\alpha_0 > 0$ and α_0 is limit.

Proof. If $\alpha_0 = 0$ were true, then taking $a \in A$, we see $a(\alpha_0) \in A_0$, a contradiction. If for some ordinal β_0, $\alpha_0 = \beta_0 + 1$ were true, then by $a_{\beta_0} \restriction \alpha_0 = a_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$, we see $a_{\beta_0}(\alpha_0) \in A_0$, a contradiction. □

Fact 3. $0\text{-}\mathrm{cf}_{Y_0}(\leftarrow, y_0)_{Y_0} \geq \omega$.

Proof. By $A \neq \emptyset$ and Fact 1, we see $(\leftarrow, y_0)_{Y_0} \neq \emptyset$ thus $0\text{-}\mathrm{cf}_{Y_0}(\leftarrow, y_0)_{Y_0} \geq 1$. If $0\text{-}\mathrm{cf}_{Y_0}(\leftarrow, y_0)_{Y_0} = 1$ were true, then letting $y_1 = \max(\leftarrow, y_0)_{Y_0}$ and $\beta_0 = \min\{\beta < \alpha_0 : y_1(\beta) \neq y_0(\beta)\}$, we see $y_1 <_{Y_0} a_{\beta_0} \restriction \alpha_0 <_{Y_0} y_0$, a contradiction. □

Now Fact 1, 3 and Lemma 1.3 (3) show that Y_1 $(= \prod_{\alpha_0 \leq \alpha} X_\alpha)$ has a minimal element and the 0-segment $(\leftarrow, y_0)$ in Y_0 is stationary. Then by Claim 11 (1), $y_0 {}^\wedge \langle \min X_\alpha : \alpha_0 \leq \alpha \rangle$ is the minimal element of B in X, which contradicts our case "Case 3". □

Let $Z_0 = \prod_{\alpha \leq \alpha_0} X_\alpha$, $Z_1 = \prod_{\alpha_0 < \alpha} X_\alpha$ and

$$A^* = \{z \in Z_0 : z \restriction \alpha_0 <_{Y_0} y_0 \text{ or } (z \restriction \alpha_0 = y_0 \text{ and } z(\alpha_0) \in A_0)\},$$

that is, $A^* = (\leftarrow, y_0)_{Y_0} \times X_{\alpha_0} \cup \{y_0\} \times A_0$. Note that when $(\alpha_0, \gamma) = \emptyset$, Z_0 is identified with X and also A^* is identified with A.

Claim 14. A^* is a 0-segment of Z_0 and $A = A^* \times Z_1$.

Proof. It is straightforward to see that A^* is a 0-segment of Z_0. We prove $A = A^* \times Z_1$. First let $a \in A$, then by Claim 10(1), we have $a \restriction \alpha_0 \leq y_0$. When $a \restriction \alpha_0 < y_0$, obviously we have $a \restriction (\alpha_0 + 1) \in A^*$. When $a \restriction \alpha_0 = y_0$, by $a \in A$, we have $a(\alpha_0) \in A_0$ therefore $a \restriction (\alpha_0 + 1) \in A^*$. In either cases, we see $a \in A^* \times Z_1$. Next let $a \in A^* \times Z_1$. When $a \restriction \alpha_0 < y_0$, letting $\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}$, we see $a < a_{\beta_0} \in A$ thus $a \in A$. When $a \restriction \alpha_0 = y_0$, by $a \restriction (\alpha_0 + 1) \in A^*$, we have $a(\alpha_0) \in A_0$. Now if $a \in B$ were true, then by $a \restriction \alpha_0 = y_0$, we have $a(\alpha_0) \in B_0$, which contradicts Claim 12. Thus we have $a \in A$. □

Claim 15. The following properties hold:

1. $0\text{-}\mathrm{cf}_{Z_0} A^* = 0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0 \geq 1$,
2. the 0-segment A^* in Z_0 is stationary iff the 0-segment A_0 in X_{α_0} is stationary.

Proof. Claim 13 shows $0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0 \geq 1$. It follows from $A^* = (\leftarrow, y_0)_{Y_0} \times X_{\alpha_0} \cup \{y_0\} \times A_0$ that $\{y_0\} \times A_0$ is a 1-segment (final segment) of A^*. So other properties are almost obvious. □

Note that $0\text{-}\mathrm{cf}_{Z_0} A^* = 0\text{-}\mathrm{cf}_X A$ is generally not true. The case "$0\text{-}\mathrm{cf}_{Z_0} A^* = 1$ and $0\text{-}\mathrm{cf}_X A \geq \omega$" can happen. But if $0\text{-}\mathrm{cf}_{Z_0} A^* \geq \omega$, then $0\text{-}\mathrm{cf}_{Z_0} A^* = 0\text{-}\mathrm{cf}_X A$, see Lemma 1.3 (3a). So we divide Case 3 into two cases.

Case 3-1. $0\text{-}\mathrm{cf}_{Z_0} A^* \geq \omega$.

When $(\alpha_0, \gamma) = \emptyset$, the 0-segment A $(= A^*)$ is closed stationary. When $(\alpha_0, \gamma) \neq \emptyset$, by Lemma 1.3 (3b), the 0-segment A^* in Z_0 is closed stationary and Z_1 has a minimal element. Therefore we have:

Claim 16. The following properties hold:

(1) the 0-segment A^* in Z_0 is closed stationary,
(2) if $(\alpha_0, \gamma) \neq \emptyset$, then Z_1 has a minimal element.

Thus we see $\sup J^- \leq \alpha_0$.

Claim 17. $A_0 \neq X_{\alpha_0}$.

Proof. Assume $A_0 = X_{\alpha_0}$. Note $\alpha_0 \in J^+$ from $0\text{-}\mathrm{cf}_{X_{\alpha_0}} X_{\alpha_0} = 0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0$ $= 0\text{-}\mathrm{cf}_{Z_0} A^* \geq \omega$. Now it follows from Claims 15 and 16 that the 0-segment X_{α_0} is stationary. If $\alpha_0 = \beta_0 + 1$ were true for some ordinal β_0, then it follows from $b_{\beta_0} \in B$ and $b_{\beta_0} \restriction \alpha_0 = b_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$ that $b_{\beta_0}(\alpha_0) \in B_0 = X_{\alpha_0} \setminus A_0 = \emptyset$, a contradiction. Therefore we see that α_0 is 0 or a limit ordinal. This shows $l(\alpha_0) = \alpha_0 \in J^+$ and so $[l(\alpha_0), \alpha_0) \cap J^+ = \emptyset$. Now by our assumption (2bi), the 0-segment X_{α_0} is non-stationary, a contradiction. □

Claim 18. A_0 is closed in X_{α_0}.

Proof. Let $u \in X_{\alpha_0} \setminus A_0$ $(= B_0)$ and set $b = y_0{}^\wedge\langle u\rangle^\wedge\langle \min X_\alpha : \alpha_0 < \alpha\rangle$. Since $b \in B$ and B is open in X, we can find $b^* \in \hat{X}$ such that $b^* <_{\hat{X}} b$ and $(b^*, b)_{\hat{X}} \cap A = \emptyset$. Let $\beta_0 = \min\{\beta < \gamma : b^*(\beta) \neq b(\beta)\}$. Then we have $\beta_0 \leq \alpha_0$ because of $b \restriction (\alpha_0, \gamma) = \langle \min X_\alpha : \alpha_0 < \alpha\rangle$. If $\beta_0 < \alpha_0$ were true, then $a_{\beta_0} \in (b^*, b)_{\hat{X}} \cap A$, a contradiction. So we have $\beta_0 = \alpha_0$, that is $b^* \restriction \alpha_0 = y_0$ and $b^*(\alpha_0) < u$. If there were $v \in (b^*(\alpha_0), \rightarrow)_{X^*_{\alpha_0}} \cap A_0$, then $y_0{}^\wedge\langle v\rangle^\wedge\langle \min X_\alpha : \alpha_0 < \alpha\rangle \in (b^*, b)_{\hat{X}} \cap A$, a contradiction. Therefore $(b^*(\alpha_0), \rightarrow)_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ is a neighborhood of u disjoint from A_0. □

It follows from Claims 15, 16, 17 and 18 that A_0 is a bounded closed stationary 0-segment of X_{α_0}, which contradicts our assumption (2a) because of $\sup J^- \leq \alpha_0$.

Case 3-2. $0\text{-}\mathrm{cf}_{Z_0} A^* = 1$, that is, $\max A^*$ exists.

In this case, note $(\alpha_0, \gamma) \neq \emptyset$, otherwise $A = A^*$ and A has no maximal element, a contradiction. Also note $\max A^* = y_0 {}^\frown\langle \max A_0\rangle$ because of $A^* = (\leftarrow, y_0)_{Y_0} \times X_{\alpha_0} \cup \{y_0\} \times A_0$. Since $A = A^* \times Z_1$, A has no maximal element but A^* has a maximal element, we see Z_1 has no maximal element. So let $\alpha_1 = \min\{\alpha_0 < \alpha : X_\alpha \text{ has no maximal element.}\}$, then note $\alpha_1 \in J^+$ and $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$. Also note that $A = (A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha) \times \prod_{\alpha_1 < \alpha} X_\alpha$ and $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$ is a 0-segment of $\prod_{\alpha \leq \alpha_1} X_\alpha$ having no maximal element. Since A is closed stationary in X, it follows from Lemma 1.3 (3) that $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$ is stationary, moreover $\prod_{\alpha_1 < \alpha} X_\alpha$ has a minimal element whenever $(\alpha_1, \gamma) \neq \emptyset$. So we have $\sup J^- \leq \alpha_1$. Since $\{y_0 {}^\frown\langle \max A_0\rangle^\frown\langle \max X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle\} \times X_{\alpha_1}$ is a 1-segment (i.e., final segment) in $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$, the 0-segment X_{α_1} in X_{α_1} is also stationary.

Claim 19. $l(\alpha_1) \leq \alpha_0$ and $J^+ \cap [l(\alpha_1), \alpha_0] \neq \emptyset$, therefore $J^+ \cap [l(\alpha_1), \alpha_1) \neq \emptyset$.

Proof. If $\alpha_0 < l(\alpha_1)$ were true, then it follows from $J^+ \cap [l(\alpha_1), \alpha_1) \subset J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ and our assumption (2bi) that X_{α_1} is not stationary, a contradiction. So we see $l(\alpha_1) \leq \alpha_0$.

Next assume $J^+ \cap [l(\alpha_1), \alpha_0] = \emptyset$. It follows from $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ that $J^+ \cap [l(\alpha_1), \alpha_1) = \emptyset$. Now by our assumption (2bi), X_{α_1} has to be non-stationary, a contradiction. □

Noting that $[l(\alpha_1), \alpha_1)$ is finite, let $\alpha_2 = \max(J^+ \cap [l(\alpha_1), \alpha_1))$. It follows from Claim 19 and $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ that $\alpha_2 \leq \alpha_0$.

Claim 20. B_0 has a minimal element.

Proof. In case $\alpha_0 = \alpha_2$, we see that $A_0 \neq X_{\alpha_0}$ since $\alpha_0 = \alpha_2 \in J^+$ and A_{α_0} has a maximal element. In case $\alpha_0 \neq \alpha_2$, it follows that $\alpha_0 = \beta_0 + 1$ for some ordinal β_0 since $l(\alpha_1) \leq \alpha_2 < \alpha_0 < \alpha_1$, and we see that $A_0 \neq X_{\alpha_0}$ in a similar way of Claim 17. In either case, we have $A_0 \neq X_{\alpha_0}$, so B_0 is non-empty. Assume that B_0 $(= (\max A_0, \rightarrow)_{X_{\alpha_0}})$ has no minimal element. Then we have $\alpha_0 \in K^+$, therefore $\alpha_0 \in [\alpha_2, \alpha_1) \cap K^+$. So by our assumption (2biii) and $\sup J^- \leq \alpha_1$, X_{α_1} has to be non-stationary, a contradiction. □

Now since B has no minimal element but B_0 has a minimal element, there is $\alpha < \gamma$ with $\alpha_0 < \alpha$ such that X_α has no minimal element (otherwise, $\min B = y_0 {}^\frown\langle \min B_0\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$). So let $\alpha_3 = \min\{\alpha > \alpha_0 : X_\alpha \text{ has no minimal element.}\}$. Then $\alpha_3 \in J^-$, so $\alpha_2 \leq \alpha_0 < \alpha_3 \leq \sup J^- \leq \alpha_1$ and $\alpha_3 \in J^-$, i.e., $\alpha_3 \in J^- \cap (\alpha_2, \alpha_1]$. It follows from $\sup J^- \leq \alpha_1$ and the assumption (2bii) that X_{α_1} is non-stationary, a contradiction. This completes the proof of the Theorem. □

The theorem above with its analogy below gives a characterization of paracompactness of lexicographic products.

Theorem 2.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) *X is 1-paracompact,*

(2) *for every ordinal $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, the following hold:*

  (a) *X_α is boundedly 1-paracompact,*

  (b) *in each of the following cases, the 1-segment X_α is not stationary,*

    (i) *$J^- \cap [l(\alpha), \alpha) = \emptyset$,*

    (ii) *$J^- \cap [l(\alpha), \alpha) \neq \emptyset$ and $J^+ \cap (\alpha', \alpha] \neq \emptyset$,*

    (iii) *$J^- \cap [l(\alpha), \alpha) \neq \emptyset$ and $K^- \cap [\alpha', \alpha) \neq \emptyset$,*

    where $\alpha' = \max(J^- \cap [l(\alpha), \alpha))$ in case $J^- \cap [l(\alpha), \alpha) \neq \emptyset$.

3. Applications

In this section, we apply the theorems in the previous section. We first show the case that all GO-spaces X_α's have both a minimal and a maximal elements.

Corollary 3.1. *Let X_α be a GO-space having both a minimal and a maximal elements for every $\alpha < \gamma$. Then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is 0-paracompact if and only if for every $\alpha < \gamma$, X_α is 0-paracomapct.*

Proof. Note that if all GO-spaces X_α's have both a minimal and a maximal elements, then $J^- = \emptyset$ and "X_α is boundedly 0-paracompact iff it is 0-paracompact". Then the proof is almost obvious. □

This corollary with its analogous result shows:

Corollary 3.2. *Let X_α be a GO-space having both a minimal and a maximal elements for every $\alpha < \gamma$. Then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact if and only if for every $\alpha < \gamma$, X_α is paracomapct.*

Corollary 3.3. *Let X_α be a GO-space for every $\alpha < \gamma$. If γ is limit and $\sup J^- = \sup J^+ = \gamma$, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact.*

Proof. Let γ be limit. An ordinal $\alpha < \gamma$ with $\sup J^- \leq \alpha$ and $\sup J^+ \leq \alpha$ cannot be exist when $\sup J^- = \sup J^+ = \gamma$. Then apply the theorems in the previous section. □

This Corollary yields the following strange result, see also the example described page 73 in [2].

Corollary 3.4. *Let X_α be a GO-space having neither a minimal nor a maximal elements for every $\alpha < \gamma$. If γ is limit, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact.*

Corollary 3.5. *Let X_α be a GO-space for every $\alpha < \gamma$. If $\gamma = \beta+1$ for some ordinal β and X_β has neither a minimal nor a maximal elements, then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ ($= \prod_{\alpha\leq\beta} X_\alpha$) is paracompact iff X_β is paracompact.*

Proof. Let $\gamma = \beta + 1$ and X_β have neither a minimal nor a maximal elements. Note $\sup J^- = \sup J^+ = \beta$. Apply Theorems 2.1 and 2.2, noting (2a), (2bi) and (2bii) of them. □

Corollary 3.6. *Let X_α be a GO-space having a minimal element for every $\alpha < \gamma$. Then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact if and only if the following clauses hold:*

(1) *for every $\alpha < \gamma$, X_α is boundedly 0-paracompact,*
(2) *for every $\alpha < \gamma$ in each of the following cases, the 0-segment X_α is not stationary,*
  - $J^+ \cap [l(\alpha), \alpha) = \emptyset$,
  - $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $K^+ \cap [\alpha', \alpha) \neq \emptyset$, *where* $\alpha' = \max(J^+ \cap [l(\alpha), \alpha))$,

(3) *for every $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, X_α is 1-paracompact.*

Proof. Apply Theorems 2.1 and 2.2, noting that $J^- = \emptyset$ and therefore the 1-segment X_α is non-stationary for every $\alpha < \gamma$. Also remark that (1)+(2) is equivalent to 0-paracompactness of $\prod_{\alpha<\gamma} X_\alpha$ and that (3) is equivalent to 1-paracompactness of $\prod_{\alpha<\gamma} X_\alpha$. □

If all X_α's are subspaces of ordinals, then note $J^- = \emptyset$, $K^+ = \emptyset$ and X_α's are (boundedly) 1-paracompact (because X_α's are well-order). So we have the result in [4].

Corollary 3.7. [4, Theorem 4.8] *Let X_α be a subspace of an ordinal for every $\alpha < \gamma$. Then the lexicographic product $\prod_{\alpha<\gamma} X_\alpha$ is paracompact if and only if the following clauses hold:*

(1) *for every $\alpha < \gamma$, X_α is boundedly 0-paracompact,*
(2) *for every $\alpha < \gamma$ with $J^+ \cap [l(\alpha), \alpha) = \emptyset$, the 0-segment X_α is not stationary.*

In [4], it is shown that a GO-space X is paracompact iff the lexicographic product X^n is paracompact for every (some) $1 \leq n \in \omega$.

Also this result can be shown from the corollaries above. The situation of the lexicographic product X^ω is somewhat different from the finite case. Applying the theorems in the previous section, we easily see the following corollaries:

Corollary 3.8. *Let X be a GO-space and γ a limit ordinal. Then the following hold:*

(1) *if X has both a minimal and a maximal elements, then the lexicographic product X^γ is paracompact iff X is paracompact,*
(2) *if X has neither a minimal nor a maximal elements, then the lexicographic product X^γ is paracompact,*
(3) *if X has a minimal element but has no maximal element, then the lexicographic product X^γ is paracompact iff X is 0-paracompact.*

Corollary 3.9. *Let X be a GO-space and γ a successor ordinal. Then the lexicographic product X^γ is paracompact iff X is paracompact,*

For two LOTS's X_0 and X_1, $X_0 + X_1$ denotes the LOTS $\langle X_0 \cup X_1, <_{X_0+X_1}, \lambda_{X_0+X_1}\rangle$, where the linear order $<_{X_0+X_1}$ extends both $<_{X_0}$ and $<_{X_1}$, moreover satisfies $x <_{X_0+X_1} x'$ for every $x \in X_0$ and $x' \in X_1$. That is, $X_0 + X_1$ is the resulting LOTS such that X_1 is added after X_0. Also for a GO-space $X = \langle X, <_X, \tau_X\rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X\rangle$ which is called the reverse of X, see [4]. $-X$ is topologically homeomorphic to X, because the identity map on X to $-X$ $(= X)$ is 1-order preserving and homeomorphism.

Example 3.10. Note that the lexicographic product $2\times\omega_1$ is identified with $\omega_1 + \omega_1$, on the other hand the lexicographic product $\omega_1 \times 2$ is identified with ω_1. Note that $\omega_1+\omega_1$ is not topologically homeomorphic to ω_1, because ω_1 is first countable but $\omega_1+\omega_1$ is not so. Also note that $(-\omega_1)+\omega_1$ is not topologically homeomorphic to ω_1, because $(-\omega_1)+\omega_1$ has two disjoint uncountable closed subsets $-\omega_1$ and ω_1 but there are no two disjoint club sets in ω_1. Obviously $(-\omega_1) + \omega_1$ is topologically homeomorphic but not order-isomorphic to $\omega_1 + (-\omega_1)$.

Since ω_1, $(-\omega_1) + \omega_1$ and $\omega_1 + (-\omega_1)$ are not paracompact, for every $n \in \omega$ with $1 \leq n$, the lexicographic products ω_1^n, $((-\omega_1) + \omega_1)^n$ and $(\omega_1 + (-\omega_1))^n$ are not paracompact. Also note that the lexicographic products $\omega_1 \times \mathbb{S}$ is paracompact but $\mathbb{S} \times \omega_1$ is not paracompact. Now from the corollaries and theorems above, about lexicographic products we see (for products of ordinals, see [4] or Corollary 3.7) :

- ω_1^ω is not paracompact but both $\omega_1 \times (-\omega_1) \times \omega_1 \times (-\omega_1) \times \cdots$ and $(-\omega_1) \times \omega_1 \times (-\omega_1) \times \omega_1 \times \cdots$ are paracompact,
- $(\omega_1 + \omega)^\omega$ is paracompact but $(\omega_1 + \omega_1)^\omega$ is not paracompact

- $((-\omega_1)+\omega_1)^\omega$ is paracompact but $(\omega_1+(-\omega_1))^\omega$ is not paracompact,
- both $\omega_1\times\mathbb{S}\times\omega_1\times\mathbb{S}\times\cdots$ and $\mathbb{S}\times\omega_1\times\mathbb{S}\times\omega_1\times\cdots$ are paracompact,
- $\omega\times\omega_1\times\omega_1$ is paracompact but both $\omega\times(-\omega_1)\times\omega_1$ and $\omega\times\omega_1\times(-\omega_1)$ are not paracompact,
- both $\omega\times\omega_1\times\mathbb{I}$ and $\mathbb{I}\times\omega\times\omega_1$ are paracompact but both $\omega\times\mathbb{I}\times\omega_1$ and $(-\omega)\times\mathbb{I}\times\omega_1$ are not paracompact.

Question 3.11. We consider the following property $(^*)_P$, where P is a closed hereditary property, that is, a topological property so that if a topological space X has the property P, then all closed subspaces of X have also the property P.

$(^*)_P$: For every ordinal γ, if X_α is a GO-space having the property P for every $\alpha<\gamma$, then the lexicographic product $\prod_{\alpha<\gamma}X_\alpha$ also has the property P.

Note that if P is "compact" or "paracompact", then P is closed hereditarily and $(^*)_P$ is true. We ask:

Find other closed hereditary properties P's which make $(^*)_P$ true.

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