PARACOMPACTNESS OF LEXICOGRAPHIC PRODUCTS OF GO-SPACES

NOBUYUKI KEMOTO

Abstract. It is known that lexicographic products of paracompact GO-spaces are also paracompact, see [4, 5]. On the other hand, a paracompact lexicographic product of non-paracompact LOTS's is known, see [4]. In [5], it is asked when lexicographic products of GO-spaces are paracompact.

In this paper, paracompactness of lexicographic products of two GO-spaces is characterized. This characterization correct a misstated result in [11]. Using this characterization, for instance we see about lexicographic products:

- $\omega_1 \times \mathbb{S}$ and $(-\omega_1) \times \mathbb{S}$ are paracompact, but $\mathbb{S} \times \omega_1$ is not paracompact,
- $(-\omega_1) \times [0,1)_{\mathbb{R}}$ and $\omega_1 \times (0,1]_{\mathbb{R}}$ are paracompact but $\omega_1 \times [0,1)_{\mathbb{R}}$ is not paracompact,
- for a GO-space X, X is paracompact iff so is X^n for every (some) $n \in \omega$ with $1 \le n$ iff so is $(-X) \times X$,
- for ordinals α and β, $\alpha \times \beta$ is paracompact iff so is $(-\beta) \times \alpha$,
- for subspaces X_0 and X_1 of ω_1, $X_0 \times X_1$ is paracompact iff so is $(-X_1) \times X_0$,

where $\mathbb{S}$ and $[0,1)_{\mathbb{R}}$ denote the Sorgenfrey line and the interval $[0,1)$ in the real line $\mathbb{R}$, respectively, moreover $-X$ denotes the reverse GO-space $\langle X, >_X, \tau_X \rangle$ of X when $X = \langle X, <_X, \tau_X \rangle$ is a GO-space.

Also we characterize paracompactness of lexicographic products of any length of ordinal subspaces, as corollaries, we see about lexicographic products of ordinal subspaces:

- $\omega^2 \times \omega_1^\omega$, $\omega^2 \times \omega_1^\omega \times (\omega_1 + 1) \times \omega \times \omega_1$, $\omega^{\omega_1+1} \times \omega_1^\omega$, $\omega \times \omega_1 \times \omega \times \omega_1 \times \cdots$ and $\prod_{\alpha<\omega_1} \omega_\alpha$ are paracompact,
- $\omega_1^\omega \times \omega^2$, $\omega^2 \times \omega_1^\omega \times (\omega_1+1) \times \omega_1$, $\omega_1 \times \omega \times \omega_1 \times \omega \times \cdots$, $\omega^\omega \times \omega_1^\omega$, $\omega_1^\omega \times \omega^\omega$ and $\prod_{\alpha \le \omega_1} \omega_\alpha$ are not paracompact.
- whenever each X_α is an uncountable subspace of ω_1, the lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ is paracompact iff X_α is not stationary for every α with $\alpha = 0$ or limit α.

Date: February 23, 2018.

2010 *Mathematics Subject Classification.* Primary 54F05, 54B10, 54B05 . Secondary 54C05.

Key words and phrases. lexicographic product, GO-space, LOTS, paracompact, ordinal subspace.

1. Introduction

All spaces are assumed to be regular T_1 and have cardinality at least 2. In particular, about a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α are assumed to have cardinality at least 2. $\mathrm{Cl}_X B$ denotes the closure of a subset B in a topological space X. ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. For a subset S of a regular uncountable cardinal κ, $\mathrm{Lim}(S)$ denotes the set $\{\alpha < \kappa : \sup(S \cap \alpha) = \alpha\}$, that is, the set of all cluster points of S in κ, where $\sup \emptyset$ is defined to be -1, which is considered to be the immediate predecessor of the ordinal 0. Note that $\mathrm{Lim}(\kappa)$ is the set of all limit ordinals less than κ. $\mathrm{Succ}(\kappa)$ denotes the set of all non-limit ordinals in κ, that is, $\mathrm{Succ}(\kappa) = \kappa \setminus \mathrm{Lim}(\kappa)$.

It is known that lexicographic products of paracompact LOTS's are also paracompact, see [4]. In [5], the notion of lexicographic products of GO-spaces is defined and the result above is extended for lexicographic products of paracompact GO-spaces. Obviously, if the usual Tychonoff product $\prod_{\alpha<\gamma} X_\alpha$ of topological spaces is paracompact, then each factor X_α is paracompact. However this is not true for the lexicographic products, see Example in page 73 in [4]. In [5], it is asked when lexicographic products of GO-spaces are paracompact. In this paper, paracompactness of lexicographic products of two GO-spaces is characterized, also paracompactness of lexicographic products of any length of ordinal subspaces is characterized.

In the remaining of this section, we prepare various notions which will be used. A linearly ordered set $\langle X, <_X \rangle$ (see [2]) has a natural T_2-topology, so called the *interval topology*, denoted by λ_X or $\lambda(<_X)$ which is the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\}$ as a subbase, where $(x, \rightarrow)_X = \{w \in X : x <_X w\}$, $(x, y]_X = \{w \in X : x <_X w \leq_X y\}$, ..., etc and $w \leq_X x$ means $w <_X x$ or $w = x$. We usually write $<$ and $(x, y]$ instead of $<_X$ and $(x, y]_X$ respectively. The triple $\langle X, <_X, \lambda_X \rangle$ is called a *LOTS* (= Linearly Ordered Topological Space) and simply denoted by LOTS X.

Unless otherwise stated, the real line $\mathbb{R}$ is considered as a linearly ordered set (hence LOTS) with the usual order, similarly so are the set $\mathbb{Q}$ of rationals, the set $\mathbb{P}$ of irrationals and an ordinal α.

A *generalized ordered space* (= GO-space) is a triple $\langle X, <_X, \tau_X \rangle$, where $<_X$ is linear order on X and τ_X is a T_2 topology on X which has a base consisting of convex sets, also simply denoted by GO-space X, where a subset B of X is *convex* if for every $x, y \in B$ with $x <_X y$, $[x, y]_X \subset B$ holds. For LOTS's and GO-spaces, see also [10]. It is easy to verify that the topology τ_X as described above is stronger than the interval topology λ_X. Let

$$X_R = \{x \in X : (\leftarrow, x] \notin \lambda_X\}, X_L = \{x \in X : [x, \rightarrow) \notin \lambda_X\}.$$

Also let

$$X^+_{\tau_X} = \{x \in X : (\leftarrow, x]_X \in \tau_X \setminus \lambda_X\},$$

$$X^-_{\tau_X} = \{x \in X : [x, \rightarrow)_X \in \tau_X \setminus \lambda_X\}.$$

Obviously $X^+_{\tau_X} \subset X_R$ and $X^-_{\tau_X} \subset X_L$. When contexts are clear, we usually simply write X^+ and X^- instead of $X^+_{\tau_X}$ and $X^-_{\tau_X}$. Note that X is a LOTS iff $X^+ \cup X^- = \emptyset$. For $A \subset X_R$ and $B \subset X_L$, let $\tau(A, B)$ be the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in A\} \cup \{[x, \rightarrow)_X : x \in B\}$ as a subbase. Obviously $\tau_X = \tau(X^+, X^-)$ whenever X is a GO-space, and also $\tau(A, B)$ defines a GO-space topology on X whenever X is a LOTS with $A \subset X_R$ and $B \subset X_L$. The Sorgenfrey line $\mathbb{S}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\emptyset, \mathbb{R})\rangle$ (i.e., the half open intervals of type $[a, b)_{\mathbb{R}}$ are declared to be open) and the Michael line $\mathbb{M}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\mathbb{P}, \mathbb{P})\rangle$. These spaces are GO-spaces but not LOTS's.

Let X be a GO-space $\langle X, <_X, \tau_X\rangle$ and $Y \subset X$, then "the *subspace* Y of a GO-space X" means the GO-space $\langle Y, <_X\restriction Y, \tau_X \restriction Y\rangle$, where $<_X\restriction Y$ is the restricted order of $<_X$ on Y and $\tau_X \restriction Y = \{U \cap Y : U \in \tau_X\}$.

Now for a given GO-space X, let

$$X^* = \big(X^- \times \{-1\}\big) \cup \big(X \times \{0\}\big) \cup \big(X^+ \times \{1\}\big)$$

and consider the lexicographic order $<_{X^*}$ on X^* induced by the lexicographic order on $X \times \{-1, 0, 1\}$, here of course $-1 < 0 < 1$. We usually identify X as $X = X \times \{0\}$ in the obvious way (i.e., $x = \langle x, 0\rangle$), thus we may consider $X^* = \big(X^- \times \{-1\}\big) \cup X \cup \big(X^+ \times \{1\}\big)$. Note $(\leftarrow, x]_X = (\leftarrow, \langle x, 1\rangle)_{X^*} \cap X \in \lambda(<_{X^*}) \restriction X$ whenever $x \in X^+$, and also its analogy. Then the GO-space X is a dense subspace of the LOTS X^*, and X has a maximal element (for short, we say "X has max") iff X^* has max, in this case, $\max X = \max X^*$ (and similarly for min). It is known that X^* is the smallest LOTS which contains the GO-space X as a dense subspace, see [9, 5, 6]. Note $\mathbb{S}^* = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{-1\}$ with the identification $\mathbb{S} = \mathbb{R} \times \{0\}$ and $\mathbb{M}^* = \mathbb{P} \times \{-1\} \cup \mathbb{R} \times \{0\} \cup \mathbb{P} \times \{1\}$ with the identification $\mathbb{M} = \mathbb{R} \times \{0\}$.

Definition 1.1. [5] Let X_α be a LOTS for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$, where γ is an ordinal. When $\gamma = 0$, we consider as $\prod_{\alpha<\gamma} X_\alpha = \{\emptyset\}$, which is a trivial LOTS, for notational conveniences. When $\gamma > 0$, every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma\rangle$.

Recall that the lexicographic order $<_X$ on X is defied as follows: for $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) < x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha \rangle$. Then $X = \langle X, <_X, \lambda_X \rangle$ is a LOTS and called the lexicographic product of LOTS's X_α's.

Now let X_α be a GO-space for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$. Then the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X_\alpha^*$, which is a LOTS, can be defined. The *lexicographic product of GO-spaces* X_α's is the GO-space $\langle X, <_{\hat{X}} \restriction X, \lambda_{\hat{X}} \restriction X \rangle$. The lexicographic product of two GO-spaces is defined in [11] in a defferent manner, but it is not difficult to see that these notions are equivalent when $\gamma = 2$. When $n \in \omega$, then $\prod_{i<n} X_i$ is denoted by $X_0 \times X_1 \times \cdots \times X_{n-1}$. $\prod_{i\in\omega} X_i$ is also denoted by $X_0 \times X_1 \times \cdots$. If all X_α's are X, then $\prod_{\alpha<\gamma} X_\alpha$ is denoted by X^γ.

Let X and Y be LOTS's. A map $f : X \to Y$ is said to be *order preserving* or 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be *order reversing* or 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map $f : X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. But when $X = \mathbb{S}$ and $Y = \mathbb{M}$, the identity map is 0-order preserving onto but not a homeomorphism.

So now let X and Y be GO-spaces. A 0-order preserving map $f : X \to Y$ is said to be 0-*order preserving embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we can identify X with $f[X]$ as a GO-space and write $X = f[X]$.

2. Paracompactness of GO-spaces

Remark the following result:

Lemma 2.1. *Let X be a GO-space, then the following are equivalent:*

(1) *X is paracompact,*
(2) *for every gap and pseudo-gap $\langle A_0, A_1 \rangle$, both A_0 and A_1 have an unbounded (see the definition below) closed discrete subset, see* [4]*,*
(3) *there is no closed subspace X which is homeomorphic to a stationary set of a regular uncountable cardinal, see* [3].

In this section, for later use, we investigate the relationship between (2) and (3) in the lemma above. A subset of a regular uncountable cardinal κ is called *stationary* if it intersects with all closed unbounded (=club) sets in κ.

Definition 2.2. Let X be a GO-space. A subset A of X is called an *initial segment* or a 0*-segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. Similarly a subset A of X is called a *final segment* or a 1*-segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x \in A$, then $x' \in A$. Both $\emptyset$ and X are 0-segments and 1-segments. A 0*-segment* A is said to be *bounded* if $X \setminus A$ is non-empty. Note that if A is a 0-segment, then $X \setminus A$ is a 1-segment.

Let A be a 0-segment of a GO-space X. A subset U of A is *unbounded in* A if for every $x \in A$, there is $x' \in U$ such that $x \leq x'$. Let

$$0\text{-}\operatorname{cf}_X A = \min\{|U| : \ U \text{ is unbounded in } A.\}.$$

$0\text{-}\operatorname{cf}_X A$ can be $0, 1$ or regular infinite cardinals. Obviously if U is an unbounded subset of a 0-segment A with $\kappa = 0\text{-}\operatorname{cf}_X A$, then by induction, we can construct a 0-order preserving (i.e., $\alpha < \beta \to a_\alpha < a_\beta$) unbounded sequence $\{a_\alpha : \alpha < \kappa\} \subset U$ such that for each $\alpha < \kappa$, $\sup_X\{a_\beta : \beta < \alpha\} < a_\alpha$ if $\sup_X\{a_\beta : \beta < \alpha\}$ exists, where $\sup_X B$ denotes the least upper bound of a subset B of X in X. $1\text{-}\operatorname{cf}_X A$ can be similarly defined whenever A is 1-segment. If contexts are clear, $0\text{-}\operatorname{cf}_X A$ is denoted by $0\text{-}\operatorname{cf} A$. For more details, see [5].

Definition 2.3. A 0-segment A of a GO-space X is said to be *stationary* if $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$ and there are a stationary set S of κ and a continuous map $\pi : S \to A$ such that $\pi[S]$ is unbounded in A (we say such a π "an unbounded continuous map"). Stationarity of 1-segment is similarly defined.

Lemma 2.4. *Let A be a* 0*-segment of a GO-space X with $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$. If there are a stationary set S of κ and an unbounded continuous map $\pi : S \to A$, then:*

(1) *for every $x \in A$, $S_x := \{\alpha \in S : \pi(\alpha) \leq x\}$ is non-stationary,*
(2) *for every club set C in κ, $\pi[S \cap C]$ is unbounded in A,*
(3) *there is a club set C in κ such that $\pi \restriction (S \cap C) : S \cap C \to A$ is a* 0*-order preserving embedding such that $\pi[S \cap C] \subset \{x \in A : x \in \operatorname{Cl}_X(\leftarrow, x)\}$.*

Proof. Note that $x \notin \operatorname{Cl}_X(\leftarrow, x)$ iff $[x, \to)$ is open.

(1) Let $x \in A$ and fix $\alpha_0 \in S$ with $x < \pi(\alpha_0)$. Then obviously the size of $S' := \{\alpha \in S : \pi(\alpha_0) < \pi(\alpha)\}$ is κ. Assume that S_x is stationary in κ, then $S_x \cap \operatorname{Lim}(S')$ is stationary. Take $\alpha \in S_x \cap \operatorname{Lim}(S')$ with $\alpha_0 < \alpha$. Since $\pi(\alpha) \leq x < \pi(\alpha_0)$ and π is continuous, there is $\beta^* < \alpha$ with

$\alpha_0 \leq \beta^*$ with $\pi[S \cap (\beta^*, \alpha]] \subset (\leftarrow, \pi(\alpha_0))$. Take $\beta \in S' \cap (\beta^*, \alpha)$, then $\pi(\beta) < \pi(\alpha_0) < \pi(\beta)$, a contradiction.

(2): Assume that there is a club set C in κ such that $\pi[S \cap C]$ is not unbounded in A. Take $\alpha \in S$ with $\pi[S \cap C] \subset (\leftarrow, \pi(\alpha))_X$. Then we have $S \cap C \subset S_{\pi(\alpha)}$, this contradicts (1).

(3): From (1), take a club set C_α disjoint from $S_{\pi(\alpha)}$, for every $\alpha \in S$. The following claim is easy to prove.

Claim 1. $\pi \restriction (S \cap \triangle_{\alpha \in S} C_\alpha)$ is 0-order preserving, where $\triangle_{\alpha \in S} C_\alpha$ denotes the club set $\{\beta < \kappa : \beta \in \bigcap_{\alpha \in S \cap \beta} C_\alpha\}$.

Let $C^* = (\triangle_{\alpha \in S} C_\alpha) \cap \mathrm{Lim}(S \cap \triangle_{\alpha \in S} C_\alpha)$. Then C^* is a club set.

Claim 2. $\pi' := \pi \restriction (S \cap C^*) : S \cap C^* \to A$ is an embedding.

Proof. It suffices to see that π'^{-1} is continuous. Let $\alpha \in S \cap C^*$ and $\alpha^* < \alpha$. We will see $\pi[S \cap C^* \cap (\alpha^*, \alpha]]$ is a neighborhood of $\pi(\alpha)$ in $\pi[S \cap C^*]$. It follows from $\alpha^* < \alpha \in C^* \subset \mathrm{Lim}(S \cap \triangle_{\alpha \in S} C_\alpha)$ that there is $\alpha_0 \in S \cap \triangle_{\alpha \in S} C_\alpha \cap (\alpha^*, \alpha)$. By $\alpha \in S \cap C^* \subset S \cap \triangle_{\alpha \in S} C_\alpha$ and Claim 1, we have $\pi(\alpha_0) < \pi(\alpha)$. Let $\alpha_1 = \min\{\beta \in S \cap C^* : \alpha < \beta\}$, then by Claim 1 $\pi(\alpha) < \pi(\alpha_1)$ holds. Then it is straightforward to see $(\pi(\alpha_0), \pi(\alpha_1)) \cap \pi[S \cap C^*] \subset \pi[S \cap C^* \cap (\alpha^*, \alpha]]$. □

Now let $B = \{\alpha \in S \cap C^* : \pi(\alpha) \notin \mathrm{Cl}_X(\leftarrow, \pi(\alpha))\}$. For every $\alpha \in B$, since $\pi(\alpha)$ is an isolated point in $\pi[S \cap C^*]$ and $\pi \restriction (S \cap C^*)$ is embedding, α is also an isolated point of $S \cap C^*$. Therefore there is a club set D disjoint from B, then $C := C^* \cap D$ satisfies (3). □

Lemma 2.5. *Let A be a* 0*-segment of a GO-space X with $\kappa :=$ 0- $\mathrm{cf}\, A \geq \omega_1$. If there are a stationary set S of κ and a* 0*-order preserving and unbounded embedding $\pi : S \to A$, then there is a subset T of κ with $S \subset T$ and a* 0*-order preserving embedding $\sigma : T \to A$ extending π such that $\sigma[T]$ is closed in A*

Proof. Let $T_0 = \{\alpha \in \mathrm{Lim}(S) \setminus S : \sup_X \pi[S \cap \alpha]$ exists and $\sup_X \pi[S \cap \alpha] \in \mathrm{Cl}_X \pi[S \cap \alpha]\}$. Moreover let $T = S \cup T_0$ and for each $\alpha \in T$, let

$$\sigma(\alpha) = \begin{cases} \pi(\alpha) & \text{if } \alpha \in S, \\ \sup_X \pi[S \cap \alpha] & \text{if } \alpha \in T_0. \end{cases}$$

Note $\mathrm{Lim}(T) = \mathrm{Lim}(S)$. We will check that T and σ are as desired.

Claim 1. $\sigma : T \to A$ is 0-order preserving

Proof. First let $\alpha \in T_0$. Pick $\alpha_0 \in S$ with $\alpha < \alpha_0$. Then $\pi(\alpha_0)$ is an upper bound of $\pi[S \cap \alpha]$, therefore we have $\sigma(\alpha) \leq \pi(\alpha_0) \in A$. Since A is 0-segment, $\sigma(\alpha) \in A$ holds. This shows that σ maps into A.

Next, to see that σ is 0-order preserving, let $\alpha_0, \alpha_1 \in T$ with $\alpha_0 < \alpha_1$.

Case 1. For some $\beta \in S$, $\alpha_0 \leq \beta < \alpha_1$.

In this case obviously, we have $\sigma(\alpha_0) \leq \pi(\beta) < \sigma(\alpha_1)$.

Case 2. Otherwise.

In this case, obviously we have $\alpha_0 \in T_0$. If $\alpha_1 \in T_0$ were true, then for some $\beta \in S \cap \alpha_1$, $\alpha_0 < \beta < \alpha_1$, a contradiction. Thus we have $\alpha_1 \in S$. Since $\pi(\alpha_1)$ is an upper bound of $\pi[S \cap \alpha_0]$, we have $\sigma(\alpha_0) \leq \pi(\alpha_1) = \sigma(\alpha_1)$. If $\sigma(\alpha_0) = \sigma(\alpha_1)$ were true, then $\pi(\alpha_1) = \sigma(\alpha_0) \in \mathrm{Cl}_X \pi[S \cap \alpha_0]$ holds, thus $\pi(\alpha_1) \in \mathrm{Cl}_{\pi[S]} \pi[S \cap \alpha_0]$. Since π is an embedding, we have $\alpha_1 \in \mathrm{Cl}_S(S \cap \alpha_0)$, a contradiction. Thus we have $\sigma(\alpha_0) < \sigma(\alpha_1)$. $\square$

Claim 2. For every $\alpha \in T$, $\alpha \in S \setminus \mathrm{Lim}(S)$ if and only if $\sigma(\alpha)$ is isolated in $\sigma[T]$.

Proof. Note $T \setminus \mathrm{Lim}(T) = S \setminus \mathrm{Lim}(S)$.

The "if" part: Assume that $\alpha \in T$ and $\sigma(\alpha)$ is isolated in $\sigma[T]$, then by the construction, we have $\alpha \notin T_0 \cup (S \cap \mathrm{Lim}(S))$ thus $\alpha \in S \setminus \mathrm{Lim}(S)$.

The "only if" part: Let $\alpha \in S \setminus \mathrm{Lim}(S)$ and set $\alpha_0 = \sup(S \cap \alpha)$. Then $\alpha_0 < \alpha$ and $T \cap (\alpha_0, \alpha) = \emptyset$. Let $\alpha_1 = \min(T \cap (\alpha, \kappa))$, then by Claim 1, we have $\sigma(\alpha) < \sigma(\alpha_1)$.

Case 1. $\alpha_0 \in S$.

In this case $\{\sigma(\alpha)\} = (\sigma(\alpha_0), \sigma(\alpha_1)) \cap \sigma[T]$.

Case 2. $\alpha_0 \notin S$.

In this case, $S \cap \alpha = S \cap \alpha_0$ and $\alpha_0 \in \mathrm{Lim}(S)$ hold. There are two subcases.

Case 2-1. $\alpha_0 \in T_0$.

In this case, by Claim 1, $\sigma(\alpha) > \sigma(\alpha_0) = \sup_X \pi[S \cap \alpha_0] = \sup_X \pi[S \cap \alpha]$. Therefore we have $\{\sigma(\alpha)\} = (\sigma(\alpha_0), \sigma(\alpha_1)) \cap \sigma[T]$.

Case 2-2. $\alpha_0 \notin T_0$.

Moreover we consider two subcases.

Case 2-2-1. $\sup_X \pi[S \cap \alpha_0]$ does not exist.

Since $\pi(\alpha)$ is an upper bound of $\pi[S \cap \alpha_0]$, there is $x < \pi(\alpha)$ such that x is upper bound of $\pi[S \cap \alpha_0]$. Then we have $\{\sigma(\alpha)\} = (x, \sigma(\alpha_1)) \cap \sigma[T]$.

Case 2-2-2. $\sup_X \pi[S \cap \alpha_0]$ exists.

In this case, it follows from $\alpha_0 \notin T_0$ that $\sup_X \pi[S \cap \alpha_0] \notin \mathrm{Cl}_X \pi[S \cap \alpha_0]$. Let $x = \sup_X \pi[S \cap \alpha_0]$, then $x \leq \pi(\alpha)$. When $x < \pi(\alpha)$, we have $\{\sigma(\alpha)\} = (x, \sigma(\alpha_1)) \cap \sigma[T]$. When $x = \pi(\alpha)$, by $x \notin \mathrm{Cl}_X \pi[S \cap \alpha]$, $[x, \rightarrow)$ is open in X. Therefore we have $\{\sigma(\alpha)\} = [x, \sigma(\alpha_1)) \cap \sigma[T]$. $\square$

Claim 3. $\sigma[T]$ is closed in A.

Proof. Let $x \in A \setminus \sigma[T]$ and $\alpha_1 = \min\{\alpha \in T : x \leq \sigma(\alpha)\}$. Note $x < \sigma(\alpha_1)$. When $T \cap \alpha_1 = \emptyset$, $(\leftarrow, \sigma(\alpha_1))$ is a neighborhood of x disjoint from $\sigma[T]$. So we may assume $T \cap \alpha_1 \neq \emptyset$, then note $S \cap \alpha_1 \neq \emptyset$. Since x is an upper bound of $\sigma[T \cap \alpha_1]$, x is also an upper bound of $\pi[S \cap \alpha_1]$. Since $\sigma(\alpha_1)$ is isolated in $\sigma[T]$ because of $\{\sigma(\alpha_1)\} = (x, \min(T \cap (\alpha_1, \kappa))) \cap \sigma[T]$, by Claim 2 we have $\alpha_1 \in S \setminus \mathrm{Lim}(S)$. Let $\alpha = \sup(S \cap \alpha_1)$, then note $\alpha < \alpha_1$ and $(\alpha, \alpha_1) \cap T = \emptyset$.

Case 1. $\alpha \in T$.

Since $\sigma(\alpha) < x$, $(\sigma(\alpha), \sigma(\alpha_1))_X$ is a neighborhood of x disjoint from $\sigma[T]$.

Case 2. $\alpha \notin T$.

By $\alpha \notin T \supset S$, we have $S \cap \alpha = S \cap \alpha_1$ and $\alpha = \sup(S \cap \alpha_1) = \sup(S \cap \alpha)$, therefore we have $\alpha \in \mathrm{Lim}(S) \setminus S$ and $\alpha \notin T_0$.

Case 2-1. $\sup_X \pi[S \cap \alpha]$ does not exist.

In this case, we can take $y < x$ such that y is an upper bound of $\pi[S \cap \alpha]$. Then $(y, \sigma(\alpha_1))$ is a neighborhood of x disjoint from $\sigma[T]$.

Case 2-2. $y := \sup_X \pi[S \cap \alpha]$ exists.

If $y < x$, then $(y, \sigma(\alpha_1))$ is a neighborhood of x disjoint from $\sigma[T]$. If $y = x$, then from $\alpha \notin T_0$, $[x, \rightarrow)$ is open in X and $[x, \sigma(\alpha_1))$ is a neighborhood of x disjoint from $\sigma[T]$. □

The following claim completes the proof.

Claim 4. σ is embedding.

Proof. To see that σ is continuous, let $\alpha \in T$ and U be a convex open neighborhood of $\sigma(\alpha)$ in X. We may assume $\alpha \in \mathrm{Lim}(T)$ $(= \mathrm{Lim}(S))$. By Claim 2, $\sigma(\alpha)$ is not isolated in $\sigma[T]$. Then there is $\alpha_0 \in T \cap \alpha$ such that $\sigma(\alpha_0) \in U$. Since U is convex, we have $\sigma[T \cap (\alpha_0, \alpha]] \subset U$.

To see that σ^{-1} is continuous, let $\alpha \in T$ and V be a neighborhood of α in T. If $\alpha \in T \setminus \mathrm{Lim}(T)$, then by Claim 2, $\sigma(\alpha)$ is isolated in $\sigma[T]$. So we assume $\alpha \in \mathrm{Lim}(T)$. Take $\alpha_0 \in T \cap \alpha$ with $T \cap [\alpha_0, \alpha] \subset V$ and set $\alpha_1 = \min(T \cap (\alpha, \kappa))$. Then letting $U = (\sigma(\alpha_0), \sigma(\alpha_1)) \cap \sigma[T]$, which is a neighborhood of $\sigma(\alpha)$ in $\sigma[T]$, since σ is 0-order preserving, we have $\sigma^{-1}[U] \subset V$. □

□

A subset H of a GO-space X is said to be 0-*closed* in X if for every $x \in X \setminus H$, there is an open neighborhood U of x such that $H \cap (U \cap (\leftarrow, x]) = \emptyset$. 1-closedness is similarly defined. Obviously H is closed if and only if it is both 0-closed and 1-closed.

Lemma 2.6. *Let A be a 0-segment of a GO-space X with $\kappa := 0\text{-}\mathrm{cf}\, A \geq \omega_1$ and H a 0-closed and unbounded (we call it "0-club") subset in A. If there are a stationary set S of κ and an unbounded continuous map $\pi : S \to A$, then there is a club set C in κ such that $\pi[S \cap C] \subset H$*

Proof. By Lemma 2.4 (3), there is a club set $C_0 \subset \kappa$ such that $\pi \restriction (S \cap C_0)$ is a 0-order preserving embedding. Let $S_0 = S \cap C_0$. Let M be an elementary submodel of $H(\theta)$, where θ is large enough, with $X, A, \kappa, H, S_0, \pi, \dots \in M$ such that $|M| < \kappa$, see [1, 8] for elementary submodels. Since $|M| < \kappa$ and S_0 is stationary in κ, by using the usual closure argument, we may assume that $\kappa \cap M$ is an ordinal, say $\alpha_0 = \kappa \cap M$, and $\alpha_0 \in S_0$.

Claim 1. $\alpha_0 \in \mathrm{Lim}(S_0)$.

Proof. Since $\forall \alpha \in \kappa \exists \beta \in S_0(\alpha < \beta)$ holds, by elementarity, we see $M \models \forall \alpha \in \kappa \exists \beta \in S_0(\alpha < \beta)$. Therefore we have $\forall \alpha \in \kappa \cap M \exists \beta \in S_0 \cap M(\alpha < \beta)$, that is, $\forall \alpha < \alpha_0 \exists \beta \in S_0 \cap \alpha_0(\alpha < \beta)$. This means $\alpha_0 \in \mathrm{Lim}(S_0)$. □

Claim 2. $\pi(\alpha_0) \in H$.

Proof. Assume $\pi(\alpha_0) \notin H$, then there is an open neighborhood U of $\pi(\alpha_0)$ such that $H \cap (U \cap (\leftarrow, \pi(\alpha_0)]) = \emptyset$. We may assume that U is a convex. Since π is continuous, there is $\beta^* < \alpha_0$ with $\pi[S_0 \cap (\beta^*, \alpha_0]] \subset U$. By Claim 1, we can take $\alpha \in (\beta^*, \alpha_0) \cap S_0$. Then $\alpha \in M$ and $\pi(\alpha) \in U$ hold. By the unboundedness of H, there are $x \in H$ and $\beta \in S_0$ with $\pi(\alpha) \leq x \leq \pi(\beta)$. By $\alpha, \pi, H, S_0 \in M$ and the elementarity, we see $x, \beta \in M$ thus $\beta < \alpha_0$. Convexity of U and $\pi(\alpha), \pi(\alpha_0) \in U$ ensure $x \in [\pi(\alpha), \pi(\alpha_0)] \cap H \subset H \cap (U \cap (\leftarrow, \pi(\alpha_0)])$, a contradiction. □

Now since $\pi[S_0]$ is unbounded in A, we have $\forall x \in A \exists \alpha \in S_0(x \leq \pi(\alpha))$. By elementarity, $\forall x \in A \cap M \exists \alpha \in S_0 \cap M(x \leq \pi(\alpha))$. By letting $y = \pi(\alpha_0)$, since $\pi(\alpha) < \pi(\alpha_0)$ whenever $\alpha \in S_0 \cap M$, we have $\forall x \in A \cap M \exists y \in H \cap \pi[S_0](x < y)$. Thus $M \models \forall x \in A \exists y \in H \cap \pi[S_0](x < y)$ and therefore $\forall x \in A \exists y \in H \cap \pi[S_0](x < y)$. Since H is 0-closed in A and every subspace of $\pi[S_0]$ is 1-closed in $\pi[S_0]$, $H \cap \pi[S_0]$ is closed in $\pi[S_0]$. Therefore there is a club set C_1 in κ with $H \cap \pi[S_0] = \pi[S_0 \cap C_1]$. By letting $C = C_0 \cap C_1$, ,we have $H \supset \pi[S \cap C]$. □

Compare the following lemma with Lemma 2.1.

Lemma 2.7. *Let A be a 0-segment of a GO-space X with $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$. The following are equivalent:*

(1) *there are a stationary set S of κ and an unbounded continuous map $\pi : S \to A$,*
(2) *there are a stationary set S of κ and a 0-order preserving unbounded embedding $\pi : S \to A$ such that $\pi[S] \subset \{x \in A : x \in \mathrm{Cl}_X(\leftarrow, x)\}$,*
(3) *there are a stationary set S of κ and a 0-order preserving unbounded embedding $\pi : S \to A$ such that $\pi[S]$ is closed in A,*
(4) *every (0-)closed unbounded subset in A is not discrete, where a subspace is said to be discrete if every element in it is an isolated point in it.*

Proof. By the lemmas above, (1) , (2) and (3) are equivalent.

(3) $\Rightarrow$ (4): Assume (3) and take S and π in (3). If there were a 0-closed discrete unbounded set H in A, then by Lemma 2.6, there is a club set $C \subset \kappa$ with $\pi[S \cap C] \subset H \cap \pi[S]$. Note $H \cap \pi[S]$ is 1-closed in $\pi[S]$, it is closed in $\pi[S]$ (thus in A). Since π is embedding and $\pi[S \cap C]$ is closed discrete in in $\pi[S]$, the stationary set $S \cap C$ is also closed discrete in S, a contradiction.

(4) $\Rightarrow$ (1): Let $\{a_\alpha : \alpha < \kappa\}$ be a 0-order preserving unbounded sequence in A such that for each $\alpha < \kappa$,

$$(*)\ \sup\{a_\beta : \beta < \alpha\} < a_\alpha \text{ if } \sup\{a_\beta : \beta < \alpha\} \text{ exists.}$$

Let

$$S = \{\alpha < \kappa : \ \sup\{a_\beta : \beta < \alpha\} \text{ exists and}$$
$$\sup\{a_\beta : \beta < \alpha\} \in \mathrm{Cl}_X\{a_\beta : \beta < \alpha\}.\ \}.$$

Moreover Let $\pi : S \to A$ be the map defined by for every α in S, $\pi(\alpha) = \sup_X\{a_\beta : \beta < \alpha\}$. Obviously if α is a non-limit ordinal in κ, say $\alpha = \gamma + 1$, then $\sup_X\{a_\beta : \beta < \alpha\} = a_\gamma$ and $a_\gamma \in \mathrm{Cl}_X\{a_\beta : \beta < \alpha\}$, thus $\mathrm{Succ}(\kappa) \subset S$. Obviously π is a 0-order preserving (use (*)) unbounded map. It suffices to see that π is continuous and S is stationary.

Claim 1. π is continuous.

Proof. Let $\alpha \in S$ and U be a convex open neighborhood of $\pi(\alpha)$. We will find a neighborhood V of α with $\pi[V] \subset U$. We may assume $\alpha \in \mathrm{Lim}(S)$, otherwise obvious. Since $\pi(\alpha) \in \mathrm{Cl}_X\{a_\beta : \beta < \alpha\}$, Find $\beta < \alpha$ with $a_\beta \in U$. Then because $\pi(\beta + 1) = a_\beta, \pi(\alpha) \in U$ and U is convex, we have $[\pi(\beta+1), \pi(\alpha)] \subset U$. Since π is 0-order preserving, by letting $V = (\beta, \alpha] \cap S$, we have $\pi[V] \subset U$. ☐

Claim 2. S is stationary.

Proof. Assuming that S is non-stationary, take a club set C disjoint from S. Set $J = \{\alpha + 1 : \alpha \in C\}$. Obviously $J \subset \mathrm{Succ}(\kappa) \subset S$ and $\pi[J]$ is unbounded in A. It suffices to see the following fact.

Fact. $\pi[J]$ is closed discrete in A.

Proof. Let $x \in A$ and set $\alpha_1 = \min\{\alpha \in C : x \leq a_\alpha\}$, $\alpha_2 = \min(C \cap (\alpha_1, \kappa)$. If $C \cap \alpha_1 = \emptyset$, then $(\leftarrow, a_{\alpha_2})$ is a neighborhood of x meeting with $\pi[J]$ at most one point, that is, $\pi(\alpha_1 + 1)$ $(= a_{\alpha_1})$. So we may assume $C \cap \alpha_1 \neq \emptyset$ and set $\alpha_0 = \sup(C \cap \alpha_1)$. Then $\alpha_0 \in C$ therefore $\alpha_0 \notin S$, and $\alpha_0 \leq \alpha_1$.

Case 1. $\alpha_0 < \alpha_1$.

In this case, it follows from $\pi(\alpha_0+1) = a_{\alpha_0} < x \leq a_{\alpha_1} < a_{\alpha_2} = \pi(\alpha_2+1)$ that $(a_{\alpha_0}, a_{\alpha_2})$ is a neighborhood of x meeting with $\pi[J]$ at most one point.

Case 2. $\alpha_0 = \alpha_1$.

In this case, x is an upper bound of $\{a_\beta : \beta < \alpha_1\}$. We consider further two subcases.

Case 2-1. $\sup\{a_\beta : \beta < \alpha_1\}$ does not exist.

In this case, since x is an upper bound of $\{a_\beta : \beta < \alpha_1\}$, there is $y < x$ such that y is an upper bound of $\{a_\beta : \beta < \alpha_1\}$. Then (y, a_{α_2}) is a neighborhood of x meeting with $\pi[J]$ at most one point.

Case 2-2. $z := \sup\{a_\beta : \beta < \alpha_1\}$ exists.

In this case, when $z < x$, (z, a_{α_2}) is a neighborhood of x meeting with $\pi[J]$ at most one point. So assume $z = x$. It follows from $\alpha_0 = \alpha_1 \notin S$ that $x = z \notin \mathrm{Cl}_X\{a_\beta : \beta < \alpha_1\}$. Therefore $[x, \rightarrow)$ is open in X. Then $[x, a_{\alpha_2})$ is a neighborhood of x meeting with $\pi[J]$ at most one point. □

□

□

Recall that a GO-space X is said to be 0-paracompact if for every closed 0-segment A with $\kappa := 0\text{-}\operatorname{cf} A \geq \omega_1$, A does not satisfy the clause (4) in Lemma 2.7, see [5]. Also 1-paracomapctness is defined similarly. Note that the equivalence (1) and (2) in Lemma 2.1 essentially proves the following, see also [5].

Proposition 2.8. *A GO-space X is paracompact iff it is 0-paracomapct and 1-paracompact.*

By the lemma above, we can also define the 0-paracompactness as follows:

Definition 2.9. A GO-space X is said to be (*boundedly*) 0-*paracompact* if every (bounded) closed 0-segment is not stationary. (Bounded) 1-paracomapctness is defined similarly.

Obviously a GO-space X is 0-paracompact iff it is boundedly 0-paracompact and the 0-segment X is not stationary. Also obviously, whenever a GO-space X has max, X is 0-paracompact iff it is boundedly 0-paracompact. Note that all subspaces of ordinals are 1-paracompact, because they are well-ordered. Also note that every ordinal α is boundedly 0-paracompact, because for every $\beta < \alpha$, $[0, \beta]$ is compact. Further note that an ordinal α is paracompact iff $\mathrm{cf}\alpha \leq \omega$. If X is a subspace of ω_1, then X is boundedly 0-paracompact, because for every $\beta < \omega_1$, $X \cap \beta$ is countable. Moreover such a X is paracompact iff it is non-stationary in ω_1.

3. Paracompactness of lexicographic products of two GO-spaces

Remember the following result.

- If $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic product of LOTS's, then X is compact iff all X_α's are compact, see [4, Theorem 4.2.1].

First we remark that this result can be extended for lexicographic products of GO-spaces.

Proposition 3.1. *If $X = \prod_{\alpha<\gamma} X_\alpha$ is a lexicographic product of GO-spaces, then X is compact iff all X_α's are compact*

Proof. Since compact GO-spaces are LOTS's, one direction is obvious. To see the other direction, assume that X is compact. By the result above, it suffices to see that all X_α's are LOTS's. So assume that some X_α is not a LOTS. We may assume $X_\alpha^- \neq \emptyset$, so take $u \in X_\alpha^-$. Then $(\leftarrow, u)_{X_\alpha}$ is closed in X_α and has no max. Since X is compact, $\min X$ and $\max X$ exist. Therefore for each $\beta < \gamma$, $\min X_\beta$ and $\max X_\beta$ exist. Define $x \in X$ by

$$x(\beta) = \begin{cases} \min X_\beta & \text{if } \beta \neq \alpha, \\ u & \text{otherwise.} \end{cases}$$

Then $(\leftarrow, x)_X$ is closed in X with no max, thus X is not compact, a contradiction. □

In this section, we characterize paracompactness of lexicographic products of two GO-spaces, then we will see that even if $\gamma = 2$, the proposition above cannot be extended for paracompactness.

Lemma 3.2. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Let $\pi : S \to X_0$ be a 0-order preserving map for some subset S of an ordinal λ, here π need not be continuous. Then the following hold:*

1. *Assume that X_1 has min and let $h : X_0 \to X_0 \times \{\min X_1\}$ be the map defined by $h(u) = \langle u, \min X_1 \rangle$. Then:*
   (a) *h is 0-order preserving onto and h^{-1} is continuous.*
   (b) *$h \restriction \pi[S] : \pi[S] \to X_0 \times \{\min X_1\}$ is a 0-order preserving embedding.*
2. *Assume that X_1 has max and let $h : X_0 \to X_0 \times \{\max X_1\}$ be the map defined by $h(u) = \langle u, \max X_1 \rangle$. Then $h[\pi[S]]$ is closed discrete in $X_0 \times \{\max X_1\}$.*

Proof. (1): First to see (a), let $u \in X_0$ and V be an open neighborhood u. We may assume that there are $u_0^*, u_1^* \in X_0^*$ such that $u_0^* < u < u_1^*$ and $(u_0^*, u_1^*)_{X_0^*} \cap X_0 \subset V$ (other cases are similar). Fix $v \in X_1$ with $\min X_1 < v$. Let $U = (\langle u_0^*, \min X_1 \rangle, \langle u, v \rangle)_{X_0^* \times X_1^*} \cap X_0 \times \{\min X_1\}$. Then U is a neighborhood of $h(u)$ in $X_0 \times \{\min X_1\}$ with $U \subset h[V]$. That h is 0-order preserving onto is obvious.

Next we check (b). Obviously $h \restriction \pi[S]$ is 0-order preserving. It suffices to see the continuity of $h \restriction \pi[S]$. Let $\alpha \in S$ and U be an open neighborhood of $h(\pi(\alpha))$ in $X_0 \times \{\min X_1\}$. We may assume that there are $x_0^*, x_1^* \in X_0^* \times X_1^*$ such that $x_0^* < h(\pi(\alpha)) < x_1^*$ and $(x_0^*, x_1^*)_{X_0^* \times X_1^*} \cap X_0 \times \{\min X_1\} \subset U$. Let

$$u_1 = \begin{cases} \pi(\min(S \cap (\alpha, \lambda)) & \text{if } S \cap (\alpha, \lambda) \neq \emptyset, \\ \to & \text{otherwise.} \end{cases}$$

By $x_0^* < h(\pi(\alpha)) = \langle \pi(\alpha), \min X_1 \rangle$, we have $x_0^*(0) < \pi(\alpha)$. Then $V := (x_0^*(0), u_1)_{X_0^*} \cap X_0$ is a neighborhood of $\pi(\alpha)$ with $h[V \cap \pi[S]] \subset U$.

(2): Let $x \in X_0 \times \{\max X_1\}$, say $x = \langle u, \max X_1 \rangle$. Fix $v \in X_1$ with $v < \max X_1$. Let $y =$

$$\begin{cases} \langle \pi(\min\{\alpha \in S : u < \pi(\alpha)\}), \max X_1 \rangle & \text{if } \{\alpha \in S : u < \pi(\alpha)\} \neq \emptyset, \\ \to & \text{otherwise.} \end{cases}$$

Then $(\langle u, v \rangle, y)_{X_0 \times X_1}$ is a neighborhood of x meeting with $h[\pi[S]]$ at most one member. $\square$

Example 3.3. Let $X_0 = \mathbb{R}$ and $X_1 = 2$, where $2 = \{0, 1\}$. Define $x_n \in \mathbb{R}$ by:

$$x_n = \begin{cases} -\frac{1}{n} & \text{if } n \in \omega, \\ 0 & \text{if } n = \omega, \end{cases}$$

Then $\{x_n : n \leq \omega\}$ is a 0-order preserving sequence in $\mathbb{R}$ which is a convergent sequence in $\mathbb{R}$. It follows from (1) in the lemma above that it is homeomorphic to $\{x_n : n \leq \omega\} \times \{0\}$ in $\mathbb{R} \times 2$. However from (2) of the lemma above, $\{x_n : n \leq \omega\} \times \{1\}$ is closed discrete in $\mathbb{R} \times \{1\}$. Note $\mathrm{Cl}_{\mathbb{R}\times 2}\{x_n : n \leq \omega\} \times \{1\} = \{x_n : n \leq \omega\} \times \{1\} \cup \{\langle 0, 0\rangle\}$. Also note that $\mathbb{R}$ itself is not homeomorphic to $\mathbb{R} \times \{0\}$, because the latter subspace is topologically homeomorphic to $\mathbb{S}$.

It is easy to check the following lemma.

Lemma 3.4. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and $u \in X_0$. Then the map $k_u : X_1 \to \{u\} \times X_1$ by $k_u(v) = \langle u, v\rangle$ is a 0-order preserving homeomorphism.*

Remark 3.5. In general, $\{u\} \times X_1$ is not closed in $X_0 \times X_1$ in the situation above. For example, $\{0\} \times [0, 1)_{\mathbb{R}}$ is not closed in the lexicographic product $2 \times [0, 1)_{\mathbb{R}}$. Also note that if X_0 has max, then the 0-segment X is stationary iff the 0-segment X_1 is stationary.

The following two lemmas as well as Lemma 2.6 will be main tools when we discuss paracompactness of lexicographic products.

Lemma 3.6. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and A_0 a 0-segment of X_0. Put $A = A_0 \times X_1$. Then the following hold:*

1. *A is a 0-segment of X,*
2. *if $0\text{-}\mathrm{cf}_{X_0} A_0 = 1$, then*
   - (a) $0\text{-}\mathrm{cf}_X A = 0\text{-}\mathrm{cf}_{X_1} X_1$,
   - (b) *the 0-segment A is stationary if and only if the 0-segment X_1 is stationary,*
   - (c) *A is closed in X if and only if either X_1 has max, $X_0 \setminus A_0$ has no min or X_1 has no min,*
3. *if $0\text{-}\mathrm{cf}_{X_0} A_0 \geq \omega$, then*
   - (a) $0\text{-}\mathrm{cf}_X A = 0\text{-}\mathrm{cf}_{X_0} A_0$,
   - (b) *the 0-segment A is stationary if and only if X_1 has min and the 0-segment A_0 is stationary,*
   - (c) *A is closed in X if and only if either X_1 has no min or A_0 is closed in X_0.*

Proof. (1) is obvious.

(2): Let $0\text{-}\mathrm{cf}_{X_0} A_0 = 1$ and $u_0 := \max A_0$.

(a) and (b) follow from Lemma 3.4.

(c): To see the "only if" part, assume that A is closed in X, X_1 has no max, $X_0 \setminus A_0$ has min u_1 and X_1 has min. Then we have $\langle u_1, \min X_1 \rangle = \min(X \setminus A) \notin A$. Since A is closed, we can find $x^* \in X_0^* \times X_1^*$ such that $x^* < \langle u_1, \min X_1 \rangle$ and $U := (x^*, \rightarrow)_{X_0^* \times X_1^*} \cap X$ is a neighborhood of $\langle u_1, \min X_1 \rangle$ disjoint from A. Note $x^*(0) < u_1$. Since $(u_0, u_1)_{X_0} = \emptyset$ and X_0 is dense in X_0^*, we have $(u_0, u_1)_{X_0^*} = \emptyset$, therefore $x^*(0) \leq u_0$. Since X_1 has no max, take $v \in X_1$ with $x^*(1) < v$. Then $\langle u_0, v \rangle \in U \cap A$ holds, a contradiction.

To see the "if" part, let $x \in X \setminus A$. Then $x(0) > u_0$ and $x(0) \in X_0 \setminus A_0$. If X_1 has max, then $(\langle u_0, \max X_1 \rangle, \rightarrow)_X$ is a neighborhood of x disjoint from A. If $X_0 \setminus A_0$ has no min, then by taking $u \in X_0 \setminus A_0$ with $u < x(0)$ and any $v \in X_1$, we see that $(\langle u, v \rangle, \rightarrow)_X$ is a neighborhood of x disjoint from A. If X_1 has no min, then by taking $v \in X_1$ with $v < x(1)$, we see that $(\langle x(0), v \rangle, \rightarrow)_X$ is a neighborhood of x disjoint from A.

(3): Let $\kappa = 0\text{-}\mathrm{cf}_{X_0} A_0 \geq \omega$.

(a): If U is unbounded in the 0-segment A, then $\{x(0) : x \in U\}$ is also unbounded in A_0, so we have $0\text{-}\mathrm{cf}_{X_0} A_0 \leq 0\text{-}\mathrm{cf}_X A$. Conversely, if V is unbounded in A_0, then by fixing $v \in X_1$, $\{\langle u, v \rangle : u \in V\}$ is unbounded in A, thus $0\text{-}\mathrm{cf}_{X_0} A_0 \geq 0\text{-}\mathrm{cf}_X A$.

(b): Assume that A is stationary. Take a stationary set S in κ and an unbounded continuous map $\pi : S \to A$. From Lemma 2.7, we may assume that π is 0-order preserving.

Claim 1. X_1 has min.

Proof. Assume that X_1 has no min. For each $\alpha \in S \cap \mathrm{Lim}(S)$, let $f(\alpha) = \min\{\beta \in S : \pi(\beta)(0) = \pi(\alpha)(0)\}$. By the continuity of π, we have $f(\alpha) < \alpha$ for each $\alpha \in S \cap \mathrm{Lim}(S)$. Let M be an elementary submodel of $H(\theta)$, where θ is large enough, with $X, A, \kappa, f, S, \pi, \cdots \in M$ such that $|M| < \kappa$ and $\alpha_0 := \kappa \cap M \in S$. Since $\alpha_0 \in S \cap \lim(S)$ (see Lemma 2.6 Claim 1), $f(\alpha_0) < \alpha_0$, therefore $\beta_0 := f(\alpha_0) \in M$. Then for every $\alpha \in S \cap M$ with $\beta_0 < \alpha$, we have $f(\alpha) = \beta_0$, therefore $M \models$ "for every $\alpha \in S$ with $\beta_0 < \alpha$, $f(\alpha) = \beta_0$ holds." . By $\beta_0, S, f \in M$, elementarity ensures that in the real world, for every $\alpha \in S$ with $\beta_0 < \alpha$, $f(\alpha) = \beta_0$ holds. This means $\pi[S \cap (\beta_0, \kappa)] \subset \{\pi(\beta_0)(0)\} \times X_1$, this contradicts the unboundedness of π. □

Claim 2. $H := A_0 \times \{\min X_1\}$ is 0-closed in A.

Proof. Let $x \in A \setminus H$, then $x(1) > \min X_1$. Now $U := (\langle x(0), \min X_1 \rangle, \rightarrow)_X$ is a neighborhood of x with $H \cap (U \cap (\leftarrow, x]_X) = \emptyset$. □

It follows from Lemma 2.6 that for some club set C of κ, $\pi[S\cap C] \subset H$. Now Lemma 3.2 shows that the composition $h^{-1}\circ(\pi \restriction (S\cap C))$ is an unbounded continuous map on $S\cap C$ into A_0, thus A_0 is stationary.

To see the other direction, assume that X_1 has min and A_0 is stationary. Then there are a stationary set $S \subset \kappa$ and a 0-order preserving unbounded continuous map $\pi : S \to A_0$. Then the composition $h\circ\pi : S \to A_0 \times \{\min X_1\} \subset A$ witnesses the stationarity of A by Lemma 3.2.

(c): First, assume that A is closed in X and X_1 has min. We will prove that A_0 is closed in X_0. To see this, let $u \in X_0 \setminus A_0$. When $u' < u$ for some $u' \in X_0 \setminus A_0$, $(u', \to)$ is a neighborhood of u disjoint from A_0. When $u \leq u'$ for every $u' \in X_0 \setminus A_0$, that is, $u = \min(X_0 \setminus A_0)$. Then $\min(X \setminus A) = \langle u, \min X_1\rangle$. Since A is closed and $\langle u, \min X_1\rangle \notin A$, there is $x^* \in X_0^* \times X_1^*$ with $x^* < \langle u, \min X_1\rangle$ such that $U := (x^*, \to)_{X_0^*\times X_1^*} \cap X$ is disjoint from A. Then $V := (x^*(0), \to)_{X_0^*} \cap X_0$ is a neighborhood of u disjoint from A_0.

Next, to see the other direction, let $x \in X \setminus A$. When X_1 has no min, by taking $v \in X_1$ with $v < x(1)$, $(\langle x(0), v\rangle, \to)_{X_0\times X_1}$ is a neighborhood of x disjoint from A. Now assume that A_0 is closed in X_0. It follows from $x(0) \in X_0 \setminus A_0$ that for some $u^* \in X_0^*$ with $u^* < x(0)$, $V := (u^*, \to)_{X_0^*} \cap X_0$ is a neighborhood of $x(0)$ disjoint from A_0. Fix $v \in X_1$, then $\langle u^*, v\rangle < x$ and $U := (\langle u^*, v\rangle, \to)_{X_0^*\times X_1^*} \cap X_0 \times X_1$ is a neighborhood of x which is disjoint from A. □

Lemma 3.7. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces, A_1 be a 0-segment of X_1 with $0\text{-}\operatorname{cf}_{X_1} A_1 \geq 1$ and $u \in X_0$. Let $A = \{x \in X :$ for some $v \in A_1$, $x \leq \langle u, v\rangle$ holds.$\}$. Then the following hold:*

1. *A is a 0-segment of X,*
2. *$0\text{-}\operatorname{cf}_{X_1} A_1 = 0\text{-}\operatorname{cf}_X A$,*
3. *A is stationary if and only if A_1 is stationary,*
4. *if A_1 is a bounded closed 0-segment in X_1, then A is closed in X.*

Proof. (1) is obvious. For (2) and (3), use Lemma 3.4.

(4): Let A_1 be a bounded closed 0-segment in X_1. Take $v_0 \in X_1 \setminus A_1$. To see that A is closed, let $x \in X \setminus A$. When $\langle u, v_0\rangle < x$, $(\langle u, v_0\rangle, \to)$ is a neighborhood of x disjoint from A. When $x \leq \langle u, v_0\rangle$, we have $x(0) = u$, $x(1) \in X_1 \setminus A_1$ and $x(1) \leq v_0$. Since A_1 is closed, take $v^* \in X_1^*$ with $v^* < x(1)$ such that $(v^*, \to)_{X_1^*} \cap X_1$ is a neighborhood of $x(1)$ which is disjoint from A_1. Now $(\langle u, v^*\rangle, \to)_{X_0^*\times X_1^*} \cap X_0 \times X_1$ is a neighborhood of x which is disjoint from A. □

Note that in (4) in the lemma above, the "bounded" in the assumption is essential, see Remark 3.5. Now we have prepared to see the following theorem.

Theorem 3.8. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. The following are equivalent:*

(1) *X is 0-paracompact,*
(2) (a) *X_1 is boundedly 0-paracompact,*
  (b) *if either $(u, \rightarrow)_{X_0}$ has no min for some $u \in X_0$ or X_1 has no min, then the 0-segment X_1 is not stationary,*
  (c) *if X_1 has min, then X_0 is 0-paracompact.*

Proof. Note that if X_0 has max, then $(\max X_0, \rightarrow)_{X_0}$ has no min because it is empty.

(1) $\Rightarrow$ (2): Let X be 0-paracompact. (a) follows from Lemma 3.7.

To see (b), first assume that $(u, \rightarrow)_{X_0}$ has no min for some $u \in X_0$. Let $A_0 = (\leftarrow, u]_{X_0}$ and $A = A_0 \times X_1$, then $0\text{-}\operatorname{cf}_{X_0} A_0 = 1$ and $X_0 \setminus A_0$ has no min. By Lemma 3.6 (2-c), A is a closed 0-segment of the 0-paracompact GO-space X, therefore A is not stationary. Now by Lemma 3.6 (2-b), the 0-segment X_1 is not stationary. Next assume that X_1 has no min. Fix $u \in X_0$ and let $A_0 = (\leftarrow, u]_{X_0}$ and $A = A_0 \times X_1$. Then similarly using Lemma 3.6, we see that the 0-segment X_1 is not stationary.

To see (c), assume that X_1 has min. If X_0 were not 0-paracompact, then there is a closed 0-segment A_0 of X_0 which is stationary. Then by Lemma 3.6 (3), $A := A_0 \times X_1$ is a closed 0-segment of X that is stationary, this contradicts the 0-paracompactness of X.

(2) $\Rightarrow$ (1): Assuming that X is not 0-paracompact, let A be a stationary closed 0-segment of X. Moreover letting $A_0 = \{u \in X_0 : \text{ for some } v \in X_1, \langle u, v\rangle \in A \text{ holds. }\}$, we obviously see that A_0 is a non-empty 0-segment of X_0 and $A \subset A_0 \times X_1$. We consider two cases.

Case 1. $A \subsetneq A_0 \times X_1$.

Fix $\langle u, v_1\rangle \in A_0 \times X_1 \setminus A$. By $u \in A_0$, we can find $v_0 \in X_1$ with $\langle u, v_0\rangle \in A$. Let $A_1 = \{v \in X_1 : \langle u, v\rangle \in A\}$. Then A_1 is a bounded non-empty 0-segment of X_1 and $A = \{x \in X : \text{ for some } v \in A_1, x \leq \langle u, v\rangle \text{ holds.}\}$.

Claim 1. A_1 is closed in X_1.

Proof. Let $v \in X_1 \setminus A_1$. When $v_1 < v$, $(v_1, \rightarrow)_{X_1}$ is a neighborhood of v disjoint from A_1. When $v \leq v_1$, by the closedness of A and $\langle u, v\rangle \notin A$, we can find $x^* \in X_0^* \times X_1^*$ with $x^* < \langle u, v\rangle$ such that $(x^*, \rightarrow)_{X_0^* \times X_1^*} \cap X_0 \times X_1$ is disjoint form A. Then $x^*(0) = u$ and $(x^*(1), \rightarrow)_{X_1^*} \cap X_1$ is a neighborhood of v disjoint from A_1. ☐

Thus by Lemma 3.7, we see that A_1 is a bounded stationary closed 0-segment of X_1 which contradicts the clause (a).

Case 2. $A = A_0 \times X_1$.

Further, we consider the following two cases.

Case 2-1. $0\text{-}\operatorname{cf} A_0 = 1$.

Let $u = \max A_0$. It follows from Lemma 3.6 (2-a) and (2-b) that $0\text{-}\operatorname{cf}_{X_1} X_1 = 0\text{-}\operatorname{cf}_X A$ and the 0-segment X_1 is stationary, in particular X_1 has no max. Moreover, since A is closed, it follows from Lemma 3.6 (2-c) that $X_0 \setminus A_0$ $(= (u, \rightarrow))$ has no min or X_1 has no min. Then by the condition (2-b), we see the 0-segment X_1 is not stationary, a contradiction.

Case 2-2. $0\text{-}\operatorname{cf} A_0 \geq \omega$.

Using Lemma 3.6 (3), similarly as above, we have a contradiction. □

For later use, we also write down the analogous result:

Theorem 3.9. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. The following are equivalent:*

(1) *X is 1-paracompact,*
(2) (a) *X_1 is boundedly 1-paracompact,*
  (b) *if either $(\leftarrow, u)_{X_0}$ has no max for some $u \in X_0$ or X_1 has no max, then the 1-segment X_1 is not stationary,*
  (c) *if X_1 has max, then X_0 is 1-paracompact.*

The GO-space (in fact, LOTS) $\{\dots, -3, -2, -1, 0\}$, which will be denoted by $-\omega$, of all non-positive integers with the usual order is topologically homeomorphic to the space ω. But as GO-spaces, they are different, for instance, ω has min but $-\omega$ has no min. First, we formulate $-X$ of a GO-space X.

Definition 3.10. Let $X = \langle X, <_X, \tau_X \rangle$ be a GO-space. $-X$ denotes the GO-space $\langle X, >_X, \tau_X \rangle$, that is, the order $<_{-X}$ is the reverse order of $<_X$, but the underlying set X and the topology τ_X are unchanged. Thus a GO-space X has a topological property P iff so does $-X$. $-X$ is said to be the *reverse* of X.

The following are easy to see for GO-spaces, where $X = Y$ means the existence of the identification between GO-spaces X and Y:

- $-(-X) = X$,
- $(-X)^* = -X^*$,
- $-((-X_0) \times (-X_1)) = X_0 \times X_1$,
- X is 0-paracompact, then $-X$ is 1-paracompact.

Thus $X_0 \times X_1$ and $(-X_0) \times X_1$ are homeomorphic to $(-X_0) \times (-X_1)$ and $X_0 \times (-X_1)$, respectively.

The two theorems above yield the following corollaries, where remark that the equivalence (a) and (c) in (1) ((2)) in the corollary below is Theorem 3.2 (Theorem 3.3, respectively) in [11].

Corollary 3.11. [11] *Let X_0 and X_1 be GO-spaces.*

(1) *if X_1 has neither min nor max, then the following are equivalent:*
   (a) *the lexicographic product $X_0 \times X_1$ is paracompact,*
   (b) *the lexicographic product $(-X_0) \times X_1$ is paracompact,*
   (c) *X_1 is paracompact,*

(2) *if X_1 has min and max, then the following are equivalent:*
   (a) *the lexicographic product $X_0 \times X_1$ is paracompact,*
   (b) *the lexicographic product $(-X_0) \times X_1$ is paracompact,*
   (c) *both X_0 and X_1 are paracompact.*

Remark 3.12. Here we point out that Theorem 3.4 in [11] is misstated. Theorem 3.4 in [11] says that whenever X_0 and X_1 are GO-spaces such that X_0 has no neighbor points (i.e., there are no pair u, v with $u < v$ and $(u, v)_{X_0} = \emptyset$), the lexicographic product $X_0 \times X_1$ is paracompact iff both X_0 and X_1 are paracompact. But let X_0 be the long line $\mathbb{L}(\omega_1)$ of length ω_1 and X_1 the usual real line $\mathbb{R}$, where the long line $\mathbb{L}(\omega_1)$ means the LOTS such that the unit open interval $(0, 1)_{\mathbb{R}}$ in $\mathbb{R}$ is inserted between α and $\alpha+1$ for every $\alpha < \omega_1$. Since $\mathbb{R}$ is paracompact having neither min nor max, by (1) of the corollary above, $\mathbb{L}(\omega_1) \times \mathbb{R}$ is paracompact. But $\mathbb{L}(\omega_1)$ is obviously not paracompat (because, it has ω_1 as a closed subspace), moreover it has no neighbor points.

The case "X_1 has min but has no max" is somewhat complicated but is the most interesting case.

Corollary 3.13. *Let X_0 and X_1 be GO-spaces and X_1 has min but has no max.*

(1) *the lexicographic product $X_0 \times X_1$ is paracompact iff*
   (a) *X_0 is 0-paracompact,*
   (b) *X_1 is boundedly 0-paracompact,*
   (c) *if $(u, \rightarrow)_{X_0}$ has no min for some $u \in X_0$, then the 0-segment X_1 is not stationary,*
   (d) *X_1 is 1-paracompact,*

(2) *the lexicographic product $(-X_0) \times X_1$ is paracompact iff*
   (a) *X_0 is 1-paracompact,*
   (b) *X_1 is boundedly 0-paracompact,*
   (c) *if $(\leftarrow, u)_{X_0}$ has no max for some $u \in X_0$, then the 0-segment X_1 is not stationary,*

(d) X_1 *is* 1*-paracompact,*

In (1) of the corollary above, (a)+(b)+(c) is equivalent to 0-paracompactness of $X_0 \times X_1$, (d) is equivalent to 1-paracompactness of $X_0 \times X_1$. (2) is an easy consequence of (1).

The case "X_1 has no min but has max" is analogous as follows:

Corollary 3.14. *Let X_0 and X_1 be GO-spaces and X_1 has no min but has max.*

(1) *the lexicographic product $X_0 \times X_1$ is paracompact iff*
  (a) *X_0 is 1-paracompact,*
  (b) *X_1 is boundedly 1-paracompact,*
  (c) *if $(\leftarrow, u)_{X_0}$ has no max for some $u \in X_0$, then the 1-segment X_1 is not stationary,*
  (d) *X_1 is 0-paracompact,*
(2) *the lexicographic product $(-X_0) \times X_1$ is paracompact iff*
  (a) *X_0 is 0-paracompact,*
  (b) *X_1 is boundedly 1-paracompact,*
  (c) *if $(u, \rightarrow)_{X_0}$ has no min for some $u \in X_0$, then the 1-segment X_1 is not stationary,*
  (d) *X_1 is 0-paracompact,*

Using the corollaries above, we see:

Corollary 3.15. *Let X be a GO-space and $n \in \omega$ with $1 \leq n$. Then the following are equivalent:*

(1) *the lexicographic product X^n is paracompact,*
(2) *X is paracompact.*

Proof. The implication (2) $\Rightarrow$ (1) follows from the result of [5].

(1) $\Rightarrow$ (2): Assume that X^n is paracompact and $n \geq 2$. When X has neither min nor max, because of $X^n = (X^{n-1}) \times X$ (see, [5]), by Corollary 3.11 (1), X is paracompact. When X has both min and max, by Corollary 3.11 (2), X is paracompact. Let X have min but have no max. By $X^n = (X^{n-1}) \times X$ and Corollary 3.13 (1-d), X is 1-paracompact. Similarly by $X^n = X \times (X^{n-1})$ and Corollary 3.13 (1-a), X is 0-paracompact. Thus X is paracompact. The remaining case is similar. □

Corollary 3.16. *Let X be a GO-space. Then the following are equivalent:*

(1) *the lexicographic product $(-X) \times X$ is paracompact,*
(2) *X is paracompact.*

Example 3.17. Using the corollaries above, for lexicographic products, we see:

(1) $\omega_1 \times \mathbb{S}$ and $\omega_1 \times \mathbb{M}$ are paracompact,
(2) $\mathbb{S} \times \omega_1$ and $\mathbb{M} \times \omega_1$ are not paracompact,
(3) $(-\omega_1) \times \mathbb{S}$ and $(-\omega_1) \times \mathbb{M}$ are paracompact,
(4) $(-\mathbb{S}) \times \omega_1$ and $(-\mathbb{M}) \times \omega_1$ are not paracompact,
(5) $\omega_1 \times [0,1)_{\mathbb{R}}$ is not paracompact,
(6) $(-\omega_1) \times [0,1)_{\mathbb{R}}$ is paracompact,
(7) $[0,1)_{\mathbb{R}} \times \omega_1$ is not paracompact,
(8) $(-[0,1)_{\mathbb{R}}) \times \omega_1$ is not paracompact,
(9) $\omega_1 \times (0,1]_{\mathbb{R}}$ is paracompact,
(10) $(-\omega_1) \times (0,1]_{\mathbb{R}}$ is not paracompact,
(11) $(0,1]_{\mathbb{R}} \times \omega_1$ is not paracompact,
(12) $(-(0,1]_{\mathbb{R}}) \times \omega_1$ is not paracompact,

It is known that for subspaces X_0 and X_1 of ordinals, the usual Tychonoff product $X_0 \times X_1$ is paracompact if and only if both X_0 and X_1 are paracompact, see [7]. Now we consider paracompactness of lexicographic products of two subspaces of ordinals. Paracompactness of lexicographic products of infinite length of subspaces of ordinals will be discussed in the next section. Let X be a subspace of an ordinal (of course, $|X| \geq 2$). Since ordinals are well-order, note again:

- $(u, \rightarrow)_X$ has no min for some $u \in X_0$ if and only if X has max.
- X has min, thus " $(\leftarrow, u)_X$ has no max for some $u \in X$" is true.
- X is 1-paracompact,

Using these facts and the theorems above, we see:

Corollary 3.18. *Let X_0 and X_1 be subspaces of ordinals.*
(1) *the lexicographic product $X_0 \times X_1$ is paracompact iff*
  (a) *X_0 is (0-)paracompact,*
  (b) *X_1 is boundedly 0-paracompact,*
  (c) *if X_0 has max, then the 0-segment X_1 is not stationary,*
(2) *the lexicographic product $(-X_0) \times X_1$ is paracompact iff*
  (a) *X_1 is (0-)paracompact,*
  (b) *if X_1 has max, then X_0 is (0-)paracompact,*

In (2) of the corollary above, (a) is equivalent to 0-paracompactness of $(-X_0) \times X_1$, and (b) is equivalent to 1-paracompactness of $(-X_0) \times X_1$. Using this corollary, for lexicographic products of ordinals, we have:

Corollary 3.19. *Let α and β be ordinals.*
(1) *the lexicographic product $\alpha \times \beta$ is paracompact iff*
  (a) $\mathrm{cf}\alpha \leq \omega$,
  (b) *if* $\mathrm{cf}\alpha = 1$, *then* $\mathrm{cf}\beta \leq \omega$,

(2) *the lexicographic product* $(-\alpha) \times \beta$ *is paracompact iff*
 (a) $\mathrm{cf}\beta \leq \omega$,
 (b) *if* $\mathrm{cf}\beta = 1$, *then* $\mathrm{cf}\alpha \leq \omega$,

therefore:

(3) $\alpha \times \beta$ *is paracompact iff so is* $(-\beta) \times \alpha$.

For lexicographic products of subspaces of ω_1, we have:

Corollary 3.20. *Let* X_0 *and* X_1 *be subspaces of* ω_1.

(1) *the lexicographic product* $X_0 \times X_1$ *is paracompact iff*
 (a) X_0 *is not stationary in* ω_1,
 (b) *if* X_0 *has max, then* X_1 *is not stationary in* ω_1,

(2) *the lexicographic product* $(-X_0) \times X_1$ *is paracompact iff*
 (a) X_1 *is not stationary in* ω_1,
 (b) *if* X_1 *has max, then* X_0 *is not stationary in* ω_1,

therefore:

(3) $X_0 \times X_1$ *is paracompact iff so is* $(-X_1) \times X_0$.

Example 3.21. Using the corollaries above, we see:

(1) $\omega \times \omega_1$ and $\omega \times \omega_2$ are paracompact,
(2) $\omega_1 \times \omega$ and $\omega_2 \times \omega$ are not paracompact,
(3) $(-\omega) \times \omega_1$ is not paracompact,
(4) $(-\omega_1) \times \omega$ is paracompact,
(5) $(\omega + 1) \times \omega_1$ is not paracompact,
(6) $\omega_1 \times (\omega + 1)$ is not paracompact,
(7) $(-(\omega + 1)) \times \omega_1$ is not paracompact,
(8) $(-\omega_1) \times (\omega + 1)$ is not paracompact,
(9) $\mathrm{Succ}(\omega_1) \times \omega_1$ is paracompact,
(10) $(-\mathrm{Succ}(\omega_1)) \times \omega_1$ is not paracompact,
(11) $\omega_1 \times \mathrm{Succ}(\omega_1)$ is not paracompact,
(12) $(-\omega_1) \times \mathrm{Succ}(\omega_1)$ is paracompact.

4. Paracompactness of lexicographic products of ordinal subspaces

Remember that a linearly order $<_X$ on a set X is said to be a *well-order* if every non-empty subset A of X has min. Obviously if $<_{X_i}$ is a well-order on X_i for every $i \in n$, where $n \in \omega$, then the lexicographic order $<_X$ on the product $X = \prod_{i \in n} X_i$ is also a well-order. Moreover if $<_X$ is a well-order on X and $Z \subset X$, then the restriction $<_X \restriction Z$ is also a well-order on Z.

Now let $X = \langle X, <_X, \tau_X \rangle$ is a GO-space such that the order $<_X$ is a well-order (equivalently, X is a subspace of an ordinal), then by

$X^* \subset X \times \{-1, 0, 1\}$, we see that the order $<_{X^*}$ on X^* is also a well-order. Thus whenever X is a subspace of an ordinal, we may identify the LOTS X^* with an ordinal.

Now we have:

Lemma 4.1. *Let X_i be a subspace of an ordinal for every $i \in n$, where $n \in \omega$. Then the lexicographic product $X = \prod_{i \in n} X_i$ is also a subspace of an ordinal.*

Proof. Since the order on $\prod_{i \in n} X_i^*$ is a well-order, we may consider $\prod_{i \in n} X_i^*$ as an ordinal. Thus X is a subspace of an ordinal. $\square$

Remark 4.2. In the lexicographic product $X = 2^\omega$, where 2 is the ordinal $\{0, 1\}$, let for each $n \in \omega$,

$$x_n(i) = \begin{cases} 0 & \text{if } i < n, \\ 1 & \text{if } i \geq n. \end{cases}$$

Then $\{x_n : n \in \omega\}$ is 1-order preserving (= strictly decreasing) sequence in X. Therefore the lemma above cannot be relaxed for infinite length of ordinal subspaces.

Let α be an ordinal and let

$$l(\alpha) = \begin{cases} 0 & \text{if } \alpha < \omega, \\ \sup\{\beta \leq \alpha : \beta \text{ is limit.}\} & \text{if } \alpha \geq \omega. \end{cases}$$

Note that $l(\alpha)$ is the largest limit ordinal less than or equal to α whenever $\alpha \geq \omega$. So the interval $[l(\alpha), \alpha)$ of ordinals is finite, thus every ordinal α can be uniquely represented as $l(\alpha) + n(\alpha)$ for some $n(\alpha) \in \omega$.

Definition 4.3. Let γ be an ordinal and $A \subset \gamma$. Put

$$\tilde{A} = \{\alpha \in A : [l(\alpha), \alpha) \cap A = \emptyset\}.$$

Note:

- if $A \neq \emptyset$, then $\min A \in \tilde{A}$,
- if $\omega \leq \alpha \in A$ and α is limit, then $\alpha \in \tilde{A}$.

Lemma 4.4. *Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of subspaces of ordinals and*

$$J^+ = \{\alpha < \gamma : \ X_\alpha \textit{ has no max.}\}.$$

Then for every $\alpha \in J^+$ with $\alpha > 0$, the following are equivalent, where $Y = \prod_{\beta < \alpha} X_\beta$,

(1) *there is $y \in Y$ such that $(y, \rightarrow)_Y$ has no min,*
(2) *$\alpha \in \tilde{J^+}$.*

Proof. Let $0 < \alpha \in J^+$ and set $Y_0 = \prod_{\beta < l(\alpha)} X_\beta$, $Y_1 = \prod_{l(\alpha) \le \beta < \alpha} X_\beta$. Then Y can be identified with $Y_0 \times Y_1$, see [5]. Note $Y = Y_1$ whenever $l(\alpha) = 0$.

(1) $\to$ (2): Assume that there is $y \in Y$ such that $(y, \to)_Y$ has no min.

Case 1. $(y, \to)_Y = \emptyset$.

Note in this case, $y = \max Y$. Therefore $\max X_\beta$ exists for every $\beta < \alpha$ and $y = \langle \max X_\beta : \beta < \alpha \rangle$. Now we have $[l(\alpha), \alpha) \cap J^+ \subset [0, \alpha) \cap J^+ = \emptyset$, so $\alpha \in \tilde{J}^+$.

Case 2. $(y, \to)_Y \neq \emptyset$.

Assuming the negation of (2), take $\beta \in [l(\alpha), \alpha) \cap J^+$. Then $Y_1 \neq \emptyset$ and Y_1 has no max. Since $[l(\alpha), \alpha)$ is finite, by Lemma 4.1, Y_1 is a subspace of an ordinal. One can pick $y_1 \in Y_1$ with $y \restriction [l(\alpha), \alpha) <_{Y_1} y_1$ and $(y \restriction [l(\alpha), \alpha), y_1)_{Y_1} = \emptyset$, that is, y_1 is the immediate successor of $y \restriction [l(\alpha), \alpha)$ in Y_1. Then $(y \restriction l(\alpha))^\wedge y_1$ is the minimal element of $(y, \to)_Y$, this contradicts (1), where $(y \restriction l(\alpha))^\wedge y_1$ denotes the element z in Y defined by

$$z(\beta) = \begin{cases} y(\beta) & \text{if } \beta < l(\alpha), \\ y_1(\beta) & \text{if } l(\alpha) \le \beta < \alpha. \end{cases}$$

(2) $\to$ (1): Let $\alpha \in \tilde{J}^+$. Then for every $\beta \in [l(\alpha), \alpha)$, X_β has max. When $\alpha < \omega$, because of $Y = Y_1$, Y has max therefore $y = \max Y$ satisfies (1). Assume $\alpha \ge \omega$, then $l(\alpha)$ is limit. Note $\min Y_0 = \langle \min X_\beta : \beta < l(\alpha) \rangle$.

Claim. $(\min Y_0, \to)_{Y_0}$ has no min.

Proof. To see this, let $z \in Y_0$ with $\min Y_0 < z$ and $\beta_0 = \min\{\beta < l(\alpha) : \min X_\beta \neq z(\beta)\}$. Fix $u \in X_{\beta_0+1}$ with $\min X_{\beta_0+1} < u$. Let $z' = \langle \min X_\beta : \beta \le \beta_0 \rangle^\wedge \langle u \rangle^\wedge \langle \min X_\beta : \beta_0 + 1 < \beta < l(\alpha) \rangle$, then $\min Y_0 < z' < z$. □

When $l(\alpha) = \alpha$, it follows from $Y_0 = Y$ that $y = \min Y_0$ satisfies (1). When $l(\alpha) < \alpha$, Y_1 is non-empty and has max. Therefore $y = \min Y_0 {}^\wedge \max Y_1$ satisfies (1). □

In [5], it is proved that if for every $\alpha < \gamma$, X_α is a 0-paracompact GO-space, then the lexicographic product $X = \prod_{\alpha < \gamma} X_\alpha$ is also 0-paracompact. But there is a 0-paracompact lexicographic product $X = \prod_{\alpha < \gamma} X_\alpha$ such that some X_α is not 0-paracompact, for instance $X = \omega \times \omega_1$. About bounded 0-paracompactness, we see the following:

Lemma 4.5. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces such that all X_α's have min. If X is boundedly 0-paracompact, then all X_α's are also boundedly 0-paracompact.*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$. Assuming that some X_{α_0} is not boundedly 0-paracompact, take a bounded closed stationary 0-segment A_0 of X_{α_0} with $\kappa = 0\text{-}\mathrm{cf}_{X_{\alpha_0}} A_0 \geq \omega_1$. Then there are a stationary set S of κ and an unbounded continuous map $\pi : S \to A_0$. By Lemma 2.7, we may assume that π is 0-order preserving. Set $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha$ and fix $y_0 \in Y_0$.

Claim 1. $A = \{a \in X : a \restriction \alpha_0 <_{Y_0} y_0$ or $(a \restriction \alpha_0 = y_0$ and $a(\alpha_0) \in A_0)\}$ is a bounded closed 0-segment of X.

Proof. By the boundedness of A_0, take $u \in X_{\alpha_0} \setminus A_0$. Then $y_0 {}^\frown\langle u\rangle^\frown \langle \min X_\alpha : \alpha_0 < \alpha\rangle$ witnesses the boundedness of A. That A is a 0-segment is easy to see. To prove that A is closed in X, let $x \in X \setminus A$. Then note $x \restriction \alpha_0 \geq_{Y_0} y_0$. When $x \restriction \alpha_0 >_{Y_0} y_0$, $(y_0 {}^\frown\langle u\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle, \to)_X$ is a neighborhood of x disjoint from A. Now let us consider the case $x \restriction \alpha_0 = y_0$. By $x \notin A$, we have $x(\alpha_0) \notin A_0$. Since A_0 is closed and non-empty in X_{α_0}, there is $u^* \in X^*_{\alpha_0}$ such that $u^* <_{X^*_{\alpha_0}} x(\alpha_0)$ and $((u^*, \to)_{X^*_{\alpha_0}} \cap X_{\alpha_0}) \cap A_0 = \emptyset$. Then $(y_0 {}^\frown\langle u^*\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle, \to)_{\hat{X}} \cap X$ is a neighborhood of x disjoint from A. Thus A is closed in X. $\square$

For every $\beta \in S$, let $\sigma(\beta) = y_0 {}^\frown\langle \pi(\beta)\rangle^\frown\langle \min X_\alpha : \alpha_0 < \alpha\rangle$. Then obviously $\sigma : S \to A$ is 0-order preserving and unbounded. The following claim completes the proof of the lemma.

Claim 2. σ is continuous.

Proof. Let $\beta \in S$ and U be an open neighborhood of $\sigma(\beta)$ in X. We may assume $\beta \in \mathrm{Lim}(S)$. Then by $\min X_{\alpha_0} < \pi(\beta)$, we have $y_0 {}^\frown\langle \min X_\alpha : \alpha_0 \leq \alpha\rangle <_X \sigma(\beta)$, thus $(\leftarrow, \sigma(\beta))_X \neq \emptyset$. Since U is an open neighborhood of $\sigma(\beta)$, there is $x^* \in \hat{X}$ such that $x^* <_{\hat{X}} \sigma(\beta)$ and $(x^*, \sigma(\beta)]_{\hat{X}} \cap X \subset U$. When $x^* \restriction \alpha_0 <_{\hat{Y}_0} y_0$, obviously $\sigma[S \cap [0, \beta]] \subset U$ holds (use the fact that σ is 0-order preserving), where $\hat{Y}_0 = \prod_{\alpha<\alpha_0} X^*_\alpha$. So let assume $x^* \restriction \alpha_0 = y_0$. In this case, since $\sigma(\beta)(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$, we have $x^*(\alpha_0) < \sigma(\beta)(\alpha_0) = \pi(\beta)$. Using the continuity of π at β, find $\beta_0 < \beta$ such that $\pi[S \cap (\beta_0, \beta]] \subset (x^*(\alpha_0), \pi(\beta)]_{X^*_{\alpha_0}} \cap X_{\alpha_0}$. By $\beta \in \mathrm{Lim}(S)$, we may assume $\beta_0 \in S$. It suffices to see $\sigma[S \cap (\beta_0, \beta]] \subset U$. To see this, let $\beta' \in S \cap (\beta_0, \beta]$. From $\beta_0, \beta', \beta \in S$ and $\beta_0 < \beta' \leq \beta$, we have $\sigma(\beta_0) < \sigma(\beta') \leq \sigma(\beta)$. Now by $x^* \restriction \alpha_0 = y_0 = \sigma(\beta') \restriction \alpha_0$ and

$x^*(\alpha_0) < \pi(\beta') = \sigma(\beta')(\alpha_0)$, we have $x^* <_X \sigma(\beta') \leq_X \sigma(\beta)$, therefore $\sigma[S \cap (\beta_0, \beta]] \subset (x^*, \sigma(\beta)]_{\hat{X}} \cap X \subset U$. □

□

Example 4.6. The reverse implication of Lemma 4.5 does not hold. Because, consider the lexicographic product $X = [0,1)_{\mathbb{R}} \times \omega_1$ and the 0-segment $A = \{0\} \times \omega_1$. Both $[0,1)_{\mathbb{R}}$ and ω_1 are boundedly 0-paracompact. But A is a bounded closed stationary 0-segment of X, thus $[0,1)_{\mathbb{R}} \times \omega_1$ is not boundedly 0-paracompact.

Example 4.7. In Lemma 4.5, the assumption "all X_α's have min" cannot be removed. To see this, let $Y = \{\alpha < \omega_2 : \mathrm{cf}\alpha = \omega\}$ be the subspace of ω_2. Obviously, Y is not boundedly 0-paracompact (the closed 0-segment $A = Y \cap \omega_1$ witnesses this). But the lexicographic product $Y \times \mathbb{R}$ is paracompact.

Theorem 4.8. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of subspaces of ordinals with $\gamma \geq 2$. Then the following are equivalent,*

(1) *X is paracompact,*
(2) (a) *for each $\alpha < \gamma$, X_α is boundedly 0-paracompact,*
 (b) *for each $\alpha \in \tilde{J}^+$, the 0-segment X_α is not stationary.*

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$.

(1) $\to$ (2): Assume that X is (0-)paracompact. (a) follows from the lemma above. To see (b), let $\alpha \in \tilde{J}^+$. Assuming on the contrary that the 0-segment X_α is stationary, take a stationary set S in κ and a 0-order preserving unbounded continuous map $\pi : S \to X_\alpha$, where $\kappa := 0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha \geq \omega_1$.

Case 1. $\alpha = 0$.

By letting $\sigma(\beta) = \langle \pi(\beta) \rangle {}^\wedge \langle \min X_\delta : 0 < \delta \rangle$ for every $\beta \in S$, as in the previous lemma, it is easy to check that $\sigma : S \to X$ is a 0-order preserving unbounded continuous map to the 0-segment X, this contradicts (1).

Case 2. $\alpha > 0$.

It follows from Lemma 4.4 that for some $y \in Y$, where $Y = \prod_{\beta<\alpha} X_\beta$, $(y, \to)_Y$ has no min. Let $A = \{x \in X : x \restriction \alpha \leq_Y y\}$. Obviously A is a 0-segment of X and it is closed, because $(y, \to)_Y$ has no min. Then by letting $\sigma(\beta) = y {}^\wedge \langle \pi(\beta) \rangle {}^\wedge \langle \min X_\delta : \alpha < \delta \rangle$ for every $\beta \in S$, $\sigma : S \to A$ is a 0-order preserving unbounded continuous map, this contradicts (1):

(2) $\to$ (1): Assume (2). Since all X_α's are 1-paracompact, $X = \prod_{\alpha<\gamma} X_\alpha$ is also 1-paracompact, see [5]. Thus it suffices to see that X is 0-paracompact. Assuming on the contrary that X is not 0-paracompact, take a stationary closed 0-segment A with $\kappa := 0\text{-}\operatorname{cf}_X A \geq \omega_1$. Then there are a stationary set S in κ and a 0-order preserving unbounded continuous map $\pi : S \to A$.

Case 1. $A = X$.

Since $A = X$ has no max, let $\alpha_1 = \min\{\alpha < \gamma : X_\alpha \text{ has no max.}\}$. Then by $\alpha_1 \in \tilde{J}^+$ and (b), X_{α_1} is not stationary.

Claim 1. $H = \{x \in X : x \restriction \alpha_1 = \langle \max X_\alpha : \alpha < \alpha_1 \rangle, x \restriction (\alpha_1, \gamma) = \langle \min X_\alpha : \alpha_1 < \alpha < \gamma \rangle\}$ is 0-club in X (=A).

Proof. The unboundedness of H in X is easy. To see that H is 0-closed in X, let $x \in X \setminus H$.

Case 1. $x \restriction \alpha_1 \neq \langle \max X_\alpha : \alpha < \alpha_1 \rangle$.

In this case, let $\alpha_0 = \min\{\alpha < \alpha_1 : x(\alpha) \neq \max X_\alpha\}$ and $y = \langle \max X_\alpha : \alpha \leq \alpha_0 \rangle^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$. Then $U = (\leftarrow, y)_X$ is a neighborhood of x disjoint from H.

Case 2. $x \restriction \alpha_1 = \langle \max X_\alpha : \alpha < \alpha_1 \rangle$.

It follows from $x \notin H$ that for some $\alpha > \alpha_1$, $x(\alpha) \neq \min X_\alpha$. Let $\alpha_0 = \{\alpha > \alpha_1 : x(\alpha) \neq \min X_\alpha\}$ and $y = (x \restriction \alpha_0)^\wedge \langle \min X_\alpha : \alpha_0 \leq \alpha \rangle$. Then $U = (y, \to)_X$ is a neighborhood of x with $(U \cap (\leftarrow, x]_X) \cap H = \emptyset$. $\square$

Now by Lemma 2.6, we can find a club set C in κ with $\pi[S \cap C] \subset H$. Define $\sigma : S \cap C \to X_{\alpha_1}$ by $\sigma(\beta) = \pi(\beta)(\alpha_1)$ for every $\beta \in S \cap C$. Since π is 0-order preserving and $\pi[S \cap C] \subset H$, σ is also 0-order preserving unbounded in the 0-segment X_{α_1}. Since the 0-segment X_{α_1} is not stationary, the following claim completes Case1.

Claim 2. σ is continuous.

Proof. Let $\beta \in S \cap C$ and U be an open neighborhood of $\sigma(\beta)$ in X_{α_1}. We may assume $\beta \in \operatorname{Lim}(S \cap C)$, then $(\leftarrow, \sigma(\beta))_{X_{\alpha_1}} \neq \emptyset$. Then we can take $u^* \in X^*_{\alpha_1}$ such that $u^* < \sigma(\beta)$ and $(u^*, \sigma(\beta)]_{X^*_{\alpha_1}} \cap X_{\alpha_1} \subset U$. Define $x^* \in \hat{X}$ by $x^* = \langle \max X_\alpha : \alpha < \alpha_1 \rangle^\wedge \langle u^* \rangle^\wedge \langle \min X_\alpha : \alpha_1 < \alpha \rangle$. Then $x^* < \pi(\beta)$, so $(x^*, \to)_{\hat{X}} \cap X$ is an open neighborhood of $\pi(\beta)$ in X. It follows from the continuity of π that for some $\beta_1 < \beta$, $\pi[S \cap (\beta_1, \beta]] \subset (x^*, \to)_{\hat{X}} \cap X$ holds. By $\beta \in \operatorname{Lim}(S \cap C)$, we may assume $\beta_1 \in S \cap C$. To see $\sigma[S \cap C \cap (\beta_1, \beta]] \subset U$, let $\beta' \in S \cap C \cap (\beta_1, \beta]$. Then by $\beta_1 < \beta' \leq \beta$ and $\beta_1, \beta', \beta \in S \cap C$, we have $\sigma(\beta_1) < \sigma(\beta') \leq \sigma(\beta)$.

It follows from $x^* < \pi(\beta') \leq \pi(\beta)$ that $u^* < \sigma(\beta') \leq \sigma(\beta)$, thus $\sigma(\beta') \in (u^*, \sigma(\beta)]_{X^*_{\alpha_1}} \cap X_{\alpha_1} \subset U$. □

Case 2. $A \neq X$ and $X \setminus A$ has no min.

Set $B = X \setminus A$ and $K = \{\alpha < \gamma : \exists a \in A \exists b \in B(a \restriction (\alpha+1) = b \restriction (\alpha+1)\}$. Note that for every pair $a \in A$ and $b \in B$, $a <_X b$ holds and K is an initial segment (i.e., 0-segment) in γ, therefore $K = \alpha_0$ for some $\alpha_0 \leq \gamma$. For every $\alpha < \alpha_0$, fix a pair $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$.

Claim 3. If $\alpha' \leq \alpha < \alpha_0$, then $a_{\alpha'} \restriction (\alpha'+1) = a_\alpha \restriction (\alpha'+1)$.

Proof. Assuming that for some $\beta \leq \alpha'$, $a_{\alpha'}(\beta) \neq a_\alpha(\beta)$, let $\beta_0 = \min\{\beta \leq \alpha' : a_{\alpha'}(\beta) \neq a_\alpha(\beta)\}$. Note $a_{\alpha'} \restriction \beta_0 = a_\alpha \restriction \beta_0$. When $a_{\alpha'}(\beta_0) <_{X_{\beta_0}} a_\alpha(\beta_0)$, we have $b_{\alpha'} <_X a_\alpha$, a contradiction. When $a_{\alpha'}(\beta_0) >_{X_{\beta_0}} a_\alpha(\beta_0)$, we have $a_{\alpha'} >_X b_\alpha$, a contradiction. □

Define $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$. By the claim above, we have $y_0 \restriction (\alpha+1) = a_\alpha \restriction (\alpha+1) = b_\alpha \restriction (\alpha+1)$ for every $\alpha < \alpha_0$. Let $Y_0 = \prod_{\alpha<\alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0 \leq \alpha} X_\alpha$.

Claim 4. $0 < \alpha_0 < \gamma$.

Proof. If $0 \notin K$ were true, then for every pair $a \in A$ and $b \in B$, $a(0) \neq b(0)$ holds. Let $u_0 = \min\{b(0) : b \in B\}$ and $b_0 = \langle u_0 \rangle ^\wedge \langle \min X_\alpha : 0 < \alpha \rangle$. If $b_0 \in A$ were true, then by taking $b \in B$ with $b(0) = u_0$, $b_0 \in A$ and $b \in B$ witness $0 \in K$, a contradiction. Therefore $b_0 \in B$ and obviously $b_0 = \min B$, which contradicts this case. Therefore we see $0 < \alpha_0$.

Now assume $\alpha_0 = \gamma$, then $y_0 \in X = A \cup B$. First assume $y_0 \in A$. Since A has no max, take $a \in A$ with $y_0 <_X a$ and set $\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq a(\beta)\}$. By $\beta_0 < \gamma = \alpha_0$, we have $y_0 \restriction (\beta_0+1) = a_{\beta_0} \restriction (\beta_0+1) = b_{\beta_0} \restriction (\beta_0+1)$. Now by $y_0 \restriction \beta_0 = a \restriction \beta_0$ and $y_0(\beta_0) < a(\beta_0)$, we have $b_{\beta_0} <_X a$, a contradiction. Next assume $y_0 \in B$. Since B ha no min, take $b \in B$ with $b <_X y_0$ and set $\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq b(\beta)\}$. By a similar argument as above, we have $b <_X a_{\beta_0}$, a contradiction. □

Let $A_0 = \{a(\alpha_0) : a \in A, a \restriction \alpha_0 = y_0\}$. Since A is a 0-segment of X, it is easy to verify that A_0 is also a 0-segment of X_{α_0}.

Claim 5. $A_0 = X_{\alpha_0}$ and $\{b \in B : b \restriction \alpha_0 = y_0\} = \emptyset$.

Proof. $A_0 \subset X_{\alpha_0}$ is obvious. To see $A_0 \supset X_{\alpha_0}$, assume $X_{\alpha_0} \setminus A_0 \neq \emptyset$. Let $u = \min(X_{\alpha_0} \setminus A_0)$ and $b = y_0 {}^\wedge \langle u \rangle ^\wedge \langle \min X_\alpha : \alpha_0 < \alpha \rangle$. Then by $u \notin A_0$, $b \in B$ holds. Since B has no min, take $b' \in B$ with $b' < b$. Let $\beta_0 = \min\{\beta < \gamma : b'(\beta) \neq b(\beta)\}$. Since $b(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$, we have $\beta_0 \leq \alpha_0$, $b' \upharpoonright \beta_0 = b \upharpoonright \beta_0$ and $b'(\beta_0) < b(\beta_0)$. When $\beta_0 < \alpha_0$, we have $b' < a_{\beta_0}$, a contradiction. When $\beta_0 = \alpha_0$, by $b'(\alpha_0) < u$, there is $a \in A$ with $a \upharpoonright \alpha_0 = y_0$ and $b'(\alpha_0) = a(\alpha_0)$. Since $b' \upharpoonright (\alpha_0+1) = a \upharpoonright (\alpha_0+1)$ holds, we have $\alpha_0 \in K = \alpha_0$, a contradiction.

If there is $b \in B$ with $b \upharpoonright \alpha_0 = y_0$, then by $b(\alpha_0) \in X_{\alpha_0} = A_0$, for some $a \in A$ with $a \upharpoonright \alpha_0 = y_0$, $a(\alpha_0) = b(\alpha_0)$ holds. This means $\alpha_0 \in K = \alpha_0$, a contradiction. Thus $\{b \in B : b \upharpoonright \alpha_0 = y_0\} = \emptyset$. □

Claim 6. α_0 is limit.

Proof. Assuming that α_0 is not limit, let $\alpha_0 = \beta + 1$ for some β. Then by $\beta \in \alpha_0 = K$, we have $a_\beta \upharpoonright (\beta + 1) = b_\beta \upharpoonright (\beta + 1)$. It follows from $b_\beta(\alpha_0) \in A_0$ that for some $a \in A$ with $a \upharpoonright \alpha_0 = y_0$, $a(\alpha_0) = b_\beta(\alpha_0)$ holds. Then we have $a \upharpoonright (\alpha_0 + 1) = b_\beta \upharpoonright (\alpha_0 + 1)$, this means $\alpha_0 \in K = \alpha_0$, a contradiction. □

Claim 7. $A = \{a \in X : a \upharpoonright \alpha_0 \leq_{Y_0} y_0\}$.

Proof. "$\subset$": Let $a \in A$. If $a \upharpoonright \alpha_0 > y_0$ were true, then by letting $\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}$, we see $a > b_{\beta_0}$, a contradiction.

"$\supset$": Let $a \in X$ with $a \upharpoonright \alpha_0 \leq y_0$. When $a \upharpoonright \alpha_0 < y_0$, by letting $\beta_0 = \min\{\beta < \alpha_0 : a(\beta) \neq y_0(\beta)\}$, we see $a < a_{\beta_0}$. Since A is a 0-segment and $a_{\beta_0} \in A$, we have $a \in A$. When $a \upharpoonright \alpha_0 = y_0$, by Claim 5, we see $a \in A$. □

Claim 8. Y_1 has no max.

Proof. If Y_1 has max y_1, then by the claim above, $y_0 {}^\wedge y_1$ is the maximal element of A, a contradiction. □

Using Claim 8, let $\alpha_1 = \min\{\alpha \geq \alpha_0 :\ X_\alpha$ has no max.$\}$. It follows from $\alpha_0 \leq \alpha_1$ and Claim 6, we see $\alpha_1 \in \tilde{J}^+$. By the condition (b) in (2), the 0-segment X_{α_1} is not stationary.

Claim 9. $H = \{x \in X : x \upharpoonright \alpha_1 = y_0 {}^\wedge\langle \max X_\alpha : \alpha_0 \leq \alpha < \alpha_1\rangle, x \upharpoonright (\alpha_1, \gamma) = \langle \min X_\alpha : \alpha_1 < \alpha < \gamma\rangle\}$ is 0-club in A.

Proof. This proof is similar to the proof of Claim 1 but somewhat complicated, so we give its abstract proof. $H \subset A$ is obvious.

First to see the unboundedness of H in A, let $a \in A$. Then $a \upharpoonright \alpha_1 \leq y_0 {}^\wedge\langle \max X_\alpha : \alpha_0 \leq \alpha < \alpha_1\rangle$. When $a \upharpoonright \alpha_1 < y_0 {}^\wedge\langle \max X_\alpha : \alpha_0 \leq \alpha < \alpha_1\rangle$, $a < x$ holds for every $x \in H$. When $a \upharpoonright \alpha_1 = y_0 {}^\wedge\langle \max X_\alpha : \alpha_0 \leq \alpha < \alpha_1\rangle$, pick $u \in X_{\alpha_1}$ with $a(\alpha_1) < u$. Then we have $a < y_0 {}^\wedge\langle \max X_\alpha : \alpha_0 \leq \alpha < \alpha_1\rangle {}^\wedge\langle u\rangle {}^\wedge\langle \min X_\alpha : \alpha_1 < \alpha\rangle \in H$.

Next to see that H is 0-closed in A, let $a \in A \setminus H$, then $a \upharpoonright \alpha_1 \leq y_0 {}^\wedge\langle \max X_\alpha : \alpha_0 \leq \alpha < \alpha_1\rangle$ as above. When $a \upharpoonright \alpha_1 < y_0 {}^\wedge\langle \max X_\alpha :$

$\alpha_0 \le \alpha < \alpha_1\rangle$, let $\beta_0 = \min\{\alpha < \alpha_1 : a(\alpha) \ne (y_0 {}^\frown\langle \max X_\alpha : \alpha_0 \le \alpha < \alpha_1\rangle)(\alpha)\}$ and define $y \in X$ by

$$y(\alpha) = \begin{cases} (y_0 {}^\frown\langle \max X_\alpha : \alpha_0 \le \alpha < \alpha_1\rangle)(\alpha) & \text{if } \alpha \le \beta_0, \\ \min X_\alpha & \text{if } \alpha > \beta_0, \end{cases}$$

for every $\alpha < \gamma$. Then $U = (\leftarrow, y)_X$ is a neighborhood of a disjoint from H. When $a \restriction \alpha_1 = y_0 {}^\frown\langle \max X_\alpha : \alpha_0 \le \alpha < \alpha_1\rangle$, by $a \notin H$, there is $\alpha < \gamma$ with $\alpha_1 < \alpha$ and $\min X_\alpha \ne a(\alpha)$. So let $\alpha_2 = \min\{\alpha > \alpha_1 : \min X_\alpha \ne a(\alpha)\}$. Note $\alpha_1 < \alpha_2$. Let for each $\alpha < \gamma$,

$$y(\alpha) = \begin{cases} a(\alpha) & \text{if } \alpha < \alpha_2, \\ \min X_\alpha & \text{if } \alpha \ge \alpha_2, \end{cases}$$

Then $U = (y, \rightarrow)_X$ is a neighborhood of a satisfying $(U \cap (\leftarrow, a]_X) \cap H = \emptyset$. $\square$

Now take a club set C in κ with $\pi[S \cap C] \subset H$ and define $\sigma : S \cap C \to X_{\alpha_1}$ by $\sigma(\beta) = \pi(\beta)(\alpha_1)$ for every $\beta \in S \cap C$. Also using a similar argument with Claim 2, we see that σ is 0-order preserving unbounded in X_{α_1} and continuous. This contradicts to the non-stationarity of the 0-segment of X_{α_1}. This completes Case 2.

Case 3. $A \ne X$ and $X \setminus A$ has min.

Let $B = X \setminus A$ and $b = \min B$. Since A is non-empty closed and $B = [b, \rightarrow)_X$, there is $b^* \in \hat{X}$ such that $b^* <_{\hat{X}} b$ and $(b^*, b)_{\hat{X}} \cap X = \emptyset$. Because A has no max, we see $b^* \notin X$. Let $\alpha_0 = \min\{\alpha < \gamma : b^*(\alpha) \ne b(\alpha)\}$.

Claim 10. $b(\alpha) = \min X_\alpha$ for every $\alpha > \alpha_0$.

Proof. Otherwise, let $\alpha_1 = \min\{\alpha > \alpha_0 : b(\alpha) > \min X_\alpha\}$. Let $b' = (b \restriction \alpha_1)^\frown\langle \min X_{\alpha_1}\rangle^\frown(b \restriction (\alpha_1, \gamma))$. Then we have $b' \in (b^*, b)_{\hat{X}} \cap X$, a contradiction. $\square$

Claim 11. $(b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0} = \emptyset$.

Proof. Otherwise, take $u \in (b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ and let $b' = (b \restriction \alpha_0)^\frown\langle u\rangle^\frown(b \restriction (\alpha_0, \gamma))$. Then we have $b' \in (b^*, b)_{\hat{X}} \cap X$, a contradiction. $\square$

Claim 11 says that $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}}$ is open in X_{α_0}, that is, $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \in \tau_{X_{\alpha_0}}$.

Claim 12. $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$.

Proof. Since X_{α_0} is dense in $X^*_{\alpha_0}$ and $b^*(\alpha_0) \in (\leftarrow, b(\alpha_0))_{X^*_{\alpha_0}}$, we have $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}} \neq \emptyset$. Assuming $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, take $u \in X_{\alpha_0}$ such that $u < b(\alpha_0)$ and $(u, b(\alpha_0))_{X_{\alpha_0}} = \emptyset$. Then u has to be $b^*(\alpha_0)$, therefore $b^*(\alpha_0) = u \in X_{\alpha_0}$. Let $\alpha_1 = \min\{\alpha < \gamma : b^*(\alpha) \notin X_\alpha\}$. Since $b^* \notin X$, α_1 is well-defined and $\alpha_0 < \alpha_1$. If $(b^*(\alpha_1), \rightarrow)_{X^*_{\alpha_1}} = \emptyset$ were true, then we have $b^*(\alpha_1) = \max X^*_{\alpha_1} = \max X_{\alpha_1} \in X_{\alpha_1}$, a contradiction. So taking $v \in (b^*(\alpha_1), \rightarrow)_{X^*_{\alpha_1}} \cap X_{\alpha_1}$, let $b' = (b^* \restriction \alpha_1)^\wedge\langle v\rangle^\wedge\langle \min X_\alpha : \alpha_1 < \alpha\rangle$. Then we have $b' \in (b^*, b)_{\hat{X}} \cap X$, a contradiction. $\square$

It follows from Claims 11 and 12 that $A_0 = (\leftarrow, b(\alpha_0))_{X_{\alpha_0}}$ is a bounded closed 0-segment in X_{α_0} with no max. By the assumption (a) in (2), A_0 is not stationary. As in Claim 1 or 9, we see:

Claim 13. $H = \{x \in X : x \restriction \alpha_0 = b \restriction \alpha_0, x(\alpha_0) \in A_0$ and $x \restriction (\alpha_0, \gamma) = \langle \min X_\alpha : \alpha_0 < \alpha < \gamma\rangle\}$ is 0-club in A.

Now taking a club set C in κ with $\pi[S \cap C] \subset H$, let $\sigma : S \cap C \to A_0$ by $\sigma(\beta) = \pi(\beta)(\alpha_0)$ for every $\beta \in S \cap C$. By a similar argument with Claim 2, we see that σ is 0-order preserving unbounded in A_0 and continuous. This contradicts to the non-stationarity of A_0. This completes Case 3. $\square$

Example 4.9. Using the theorem, about the following lexicographic products, we see:

(1) $\omega^2 \times \omega_1^\omega$ is paracompact, where $\omega^2 \times \omega_1^\omega$ can be considered as the lexicographic product $\prod_{n\in\omega} X_n$ with $X_0 = X_1 = \omega$, $X_2 = X_3 = \cdots = \omega_1$, see [5],
(2) $\omega_1^\omega \times \omega^2$ not paracompact,
(3) $\omega \times \omega_1^\omega \times \omega^2$ is paracompact,
(4) $\omega \times \omega_1^{\omega+2}$ is not paracompact,
(5) $(\omega_1 + 1)^2 \times \omega \times \omega_1^\omega$ is paracompact,
(6) $(\omega_1 + 1)^2 \times \omega_1^\omega$ is not paracompact,
(7) $\omega^2 \times \omega_1^\omega \times (\omega_1 + 1) \times \omega_1$ is not paracompact,
(8) $\omega^2 \times \omega_1^\omega \times (\omega_1 + 1) \times \omega \times \omega_1$ is paracompact,
(9) $\mathrm{Succ}(\omega_1)^2 \times \omega_1^\omega \times \mathrm{Succ}(\omega_1) \times \omega_1$ is paracompact,
(10) when all X_α's are uncountable subspaces of ω_1, $X = \prod_{\alpha<\gamma} X_\alpha$ is paracompact iff X_α is non-stationary for every $\alpha < \gamma$ with $\alpha = 0$ or limit α,
(11) $\omega \times \omega_1 \times \omega \times \omega_1 \times \cdots$ is paracompact,
(12) $\omega_1 \times \omega \times \omega_1 \times \omega \times \cdots$ is not paracompact,
(13) $(\omega + 1) \times \omega_1 \times (\omega + 1) \times \omega_1 \times \cdots$ is not paracompact,
(14) $\omega^{\omega_1+1} \times \omega_1^\omega$ is paracompact,
(15) $\omega^{\omega_1} \times \omega_1^\omega$ is not paracompact,

(16) $\prod_{\alpha<\omega}\omega_\alpha$, $\prod_{\alpha\leq\omega}\omega_\alpha$ and $\prod_{\alpha<\omega_1}\omega_\alpha$ are paracompact,
(17) $\prod_{\alpha<\omega}\omega_{\alpha+1}$, $\prod_{\alpha<\omega_1}\omega_{\alpha+1}$ and $\prod_{\alpha\leq\omega_1}\omega_\alpha$ are not paracompact.

Acknowledgement. The author would like to thank the reviewer of this paper for his valuable comments and suggestions.

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Department of Mathematics, Oita University, Oita, 870-1192 Japan
E-mail address: `nkemoto@cc.oita-u.ac.jp`