LEXICOGRAPHIC PRODUCTS OF GO-SPACES

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Abstract. It is known that lexicographic products of paracompact LOTS's are also paracompact, see [2]. In this paper, the notion of lexicographic products of GO-spaces is defined. We characterize when a lexicographic product of GO-spaces is a LOTS. Moreover, we show that lexicographic products of paracompact GO-spaces are also paracompact. For example, we see

- the lexicographic products $\mathbb{M}\times\mathbb{P}$ and $\mathbb{S}\times[0,1)_{\mathbb{R}}$ are LOTS's, but $\mathbb{P}\times\mathbb{M}$ and $\mathbb{S}\times(0,1]_{\mathbb{R}}$ are not LOTS's,
- the lexicographic product $\mathbb{S}^\gamma$ of the γ-many copies of $\mathbb{S}$ is a LOTS iff γ is a limit ordinal,
- the lexicographic products $\mathbb{M}\times\mathbb{P}$ and $\mathbb{P}\times\mathbb{M}$ are paracompact,
- the lexicographic product $\mathbb{S}^\gamma$ is paracompact for every ordinal γ,

where $\mathbb{P}$, $\mathbb{M}$, $\mathbb{S}$ and $[0,1)_{\mathbb{R}}$ denote the irrationals, the Michael line, the Sorgenfrey line and the interval $[0,1)$ in the reals $\mathbb{R}$, respectively.

1. Introduction

We assume all topological spaces have cardinality at least 2.

A linearly ordered set $\langle X,<_X\rangle$ (see [1]) has a natural T_2-topology denoted by λ_X or $\lambda(<_X)$ so called the *interval topology* which is the topology generated by $\{(\leftarrow,x)_X : x\in X\}\cup\{(x,\rightarrow)_X : x\in X\}$ as a subbase, where $(x,\rightarrow)_X=\{w\in X : x<_X w\}$, $(x,y]_X=\{w\in X : x<_X w\leq_X y\}$, ..., etc. Here $w\leq_X x$ means $w<_X x$ or $w=x$. If the contexts are clear, we simply write $<$ and $(x,y]$ instead of $<_X$ and $(x,y]_X$ respectively. Note that this subbase induces a base by convex subsets (e.g., the collection of all intersections of at most two members of this subbase), where a subset B of X is *convex* if for every $x,y\in B$ with $x<_X y$, $[x,y]_X\subset B$ holds. The triple $\langle X,<_X,\lambda_X\rangle$ is called a *LOTS* (= Linearly Ordered Topological Space) and simply denoted by LOTS X. Observe that if $x\in U\in\lambda_X$ and $(\leftarrow,x)\neq\emptyset$, then there is

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$y \in X$ such that $y < x$ and $(y, x] \subset U$. Note that for every $x \in X$, $(\leftarrow, x] \notin \lambda_X$ iff $(x, \rightarrow)$ is non-empty and has no minimum (briefly, min), also analogously $[x, \rightarrow) \notin \lambda_X$ iff $(\leftarrow, x)$ is non-empty and has no max. Let

$$X_R = \{x \in X : (\leftarrow, x] \notin \lambda_X\} \text{ and } X_L = \{x \in X : [x, \rightarrow) \notin \lambda_X\}.$$

Unless otherwise stated, the real line $\mathbb{R}$ is considered as a linearly ordered set (hence LOTS) with the usual order, similarly so are the set $\mathbb{Q}$ of rationals, the set $\mathbb{P}$ of irrationals and an ordinal α.

A *generalized ordered space* (= GO-space) is a triple $\langle X, <_X, \tau_X \rangle$, where $<_X$ is linear order on X and τ_X is a T_2 topology on X which has a base consisting of convex sets, also simply denoted by GO-space X. For LOTS's and GO-spaces, see also the nice text book [5]. It is easy to verify that τ_X is stronger than λ_X. Also let

$$X^+_{\tau_X} = \{x \in X : (\leftarrow, x]_X \in \tau_X \setminus \lambda_X\},$$

$$X^-_{\tau_X} = \{x \in X : [x, \rightarrow)_X \in \tau_X \setminus \lambda_X\}.$$

Obviously $X^+_{\tau_X} \subset X_R$ and $X^-_{\tau_X} \subset X_L$. When contexts are clear, we usually simply write X^+ and X^- instead of $X^+_{\tau_X}$ and $X^-_{\tau_X}$, respectively. Note that X is a LOTS iff $X^+ \cup X^- = \emptyset$. For $A \subset X_R$ and $B \subset X_L$, let $\tau(A, B)$ be the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in A\} \cup \{[x, \rightarrow)_X : x \in B\}$ as a subbase. Obviously $\tau_X = \tau(X^+, X^-)$ whenever X is a GO-space, and also $\tau(A, B)$ defines a GO-space topology on X whenever X is a LOTS with $A \subset X_R$ and $B \subset X_L$. The Sorgenfrey line $\mathbb{S}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\mathbb{R}, \emptyset) \rangle$ and the Michael line $\mathbb{M}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\mathbb{P}, \mathbb{P}) \rangle$. These spaces are GO-spaces but not LOTS's.

Let X be a GO-space and $Y \subset X$, then "the subspace Y of a GO-space X" means the GO-space $\langle Y, <_X \restriction Y, \lambda_X \restriction Y \rangle$, where $<_X \restriction Y$ is the restricted order of $<_X$ on Y and $\lambda_X \restriction Y := \{U \cap Y : U \in \lambda_X\}$, that is, $\lambda_X \restriction Y$ is the subspace topology of λ_X.

Now for a given GO-space X, let

$$X^* = \big(X^- \times \{-1\}\big) \cup \big(X \times \{0\}\big) \cup \big(X^+ \times \{1\}\big)$$

and consider the lexicographic order $<_{X*}$ on X^* induced by the lexicographic order on $X \times \{-1, 0, 1\}$, here of course $-1 < 0 < 1$. We usually identify X as $X = X \times \{0\}$ in the obvious way (i.e., $x = \langle x, 0 \rangle$), thus we may consider $X^* = \big(X^- \times \{-1\}\big) \cup X \cup \big(X^+ \times \{1\}\big)$. Note $(\leftarrow, x]_X = (\leftarrow, \langle x, 1 \rangle)_{X^*} \cap X \in \lambda(<_{X^*}) \restriction X$ whenever $x \in X^+$, and also its analogy. Then the GO-space X is a dense subspace of the LOTS X^*, and X has max iff X^* has max, in this case, $\max X = \max X^*$

(and similarly for min). Note $\mathbb{S}^* = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$ with the identification $\mathbb{S} = \mathbb{R} \times \{0\}$ and $\mathbb{M}^* = \mathbb{P} \times \{-1\} \cup \mathbb{R} \times \{0\} \cup \mathbb{P} \times \{1\}$ with the identification $\mathbb{M} = \mathbb{R} \times \{0\}$.

Definition 1.1. Let X_α be a LOTS for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$, where γ is an ordinal. When $\gamma = 0$, we consider as $\prod_{\alpha<\gamma} X_\alpha = \{\emptyset\}$, which is a trivial LOTS, for notational conveniences. When $\gamma > 0$, every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma \rangle$. Recall that the lexicographic order $<_X$ on X is defied as follows: for $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) < x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha \rangle$. Then $X = \langle X, <_X, \lambda_X \rangle$ is a LOTS and called the lexicographic product of LOTS's X_α's.

Now let X_α be a GO-space for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$. Then the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, which is a LOTS, can be defined. The *lexicographic product of GO-spaces* X_α's is the GO-space $\langle X, <_{\hat{X}} \restriction X, \lambda_{\hat{X}} \restriction X \rangle$. Obviously this definition extends the lexicographic product of LOTS's, and is reasonable because each X^*_α is the smallest LOTS which contains X_α as a dense subspace, see [4]. When $n \in \omega$, then $\prod_{i<n} X_i$ is denoted by $X_0 \times \cdots \times X_{n-1}$. If all X_α's are X, then $\prod_{\alpha<\gamma} X_\alpha$ is denoted by X^γ.

Let X and Y be LOTS's. A map $f : X \to Y$ is said to be 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map $f : X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. But when $X = \mathbb{S}$ and $Y = \mathbb{M}$, the identity map is 0-order preserving onto but not a homeomorphism.

So now let X and Y be GO-spaces. A 0-order preserving map $f : X \to Y$ is said to be *embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we can identify X with $f[X]$ as a GO-space. In the definition of X^*, the map $f : X \to X \times \{0\} \subset X^*$ defined by $f(x) = \langle x, 0 \rangle$ is a 0-order preserving embedding, so we have identified as $X \times \{0\} = X$.

In the rest of this section, we prepare basic tools to handle the lexicographic products of GO-spaces.

Lemma 1.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and $x \in X$. The following are equivalent:*

(1) $x \in X^+$,
(2) *there is $\alpha_0 < \gamma$ such that:*

(i) $x(\alpha_0) \in X^+_{\alpha_0}$,
(ii) *for every* $\alpha < \gamma$ *with* $\alpha_0 < \alpha$, X_α *has max and* $x(\alpha) = \max X_\alpha$.

Proof. Let $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ be the lexicographic product.

(1) $\Rightarrow$ (2): Assume $x \in X^+$. Because of $(\leftarrow, x]_X \notin \lambda_X$, $(x, \rightarrow)_X$ is non-empty and has no min. By $(\leftarrow, x]_X \in \tau_X = \lambda_{\hat{X}} \restriction X$, there is $y \in \hat{X}$ with $x <_{\hat{X}} y$ such that $(\leftarrow, x]_X \supset [x, y)_{\hat{X}} \cap X$, that is, $(x, y)_{\hat{X}} = \emptyset$. Since $(x, \rightarrow)_X$ has no min, we have $y \in \hat{X} \setminus X$. Let $\alpha_0 = \min\{\alpha < \gamma : x(\alpha) \neq y(\alpha)\}$. Then we have $x \restriction \alpha_0 = y \restriction \alpha_0$ and $x(\alpha_0) <_{X^*_{\alpha_0}} y(\alpha_0)$. Since X_{α_0} is dense in $X^*_{\alpha_0}$, $(x(\alpha_0), \rightarrow)_{X_{\alpha_0}}$ is non-empty.

Claim 1. For every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has max and $x(\alpha) = \max X_\alpha$

Proof. First assume that for some $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has no max. Then we can take $v \in X_\alpha$ with $x(\alpha) <_{X_\alpha} v$. Set $x' = (x \restriction \alpha)^\wedge\langle v\rangle^\wedge(x \restriction (\alpha, \gamma))$, that is,

$$x'(\beta) = \begin{cases} x(\beta) & \text{if } \beta < \alpha, \\ v & \text{if } \beta = \alpha, \\ x(\beta) & \text{if } \alpha < \beta < \gamma. \end{cases}$$

Then $x' \in (x, y)_{\hat{X}} \cap X$, a contradiction. Therefore for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $\max X_\alpha$ exists.

Next assume that for some $\alpha < \gamma$ with $\alpha_0 < \alpha$, $x(\alpha) <_{X_\alpha} \max X_\alpha$ holds. Then $(x \restriction \alpha)^\wedge\langle \max X_\alpha\rangle^\wedge(x \restriction (\alpha, \gamma)) \in (x, y)_{\hat{X}} \cap X$, a contradiction. □

Claim 2. $(x(\alpha_0), y(\alpha_0))_{X^*_{\alpha_0}} = \emptyset$, therefore $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \in \tau_{X_{\alpha_0}}$.

Proof. Assume $(x(\alpha_0), y(\alpha_0))_{X^*_{\alpha_0}} \neq \emptyset$. Since X_{α_0} is dense in $X^*_{\alpha_0}$, take $v \in (x(\alpha_0), y(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0}$. Then $(x \restriction \alpha_0)^\wedge\langle v\rangle^\wedge(x \restriction (\alpha_0, \gamma)) \in (x, y)_{\hat{X}} \cap X$, a contradiction. □

The following claim shows $x(\alpha_0) \in X^+_{\alpha_0}$.

Claim 3. $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$.

Proof. Since $x(\alpha_0) <_{X^*_{\alpha_0}} y(\alpha_0)$ and X_{α_0} is dense in $X^*_{\alpha_0}$, we have $(x(\alpha_0), \rightarrow)_{X_{\alpha_0}} \neq \emptyset$. Assume $(\leftarrow, x(\alpha_0)]_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, then there is $v \in X_{\alpha_0}$ such that $x(\alpha_0) <_{X_{\alpha_0}} v$ and $(x(\alpha_0), v)_{X_{\alpha_0}} = \emptyset$. Since $(x(\alpha_0), v)_{X^*_{\alpha_0}} = \emptyset$, we have $v = y(\alpha_0)$, thus $y(\alpha_0) \in X_{\alpha_0}$. Let $\alpha_1 = \min\{\alpha < \gamma : y(\alpha) \notin X_\alpha\}$. Because of $y \notin Y$ and the definition of y, we have $\alpha_0 < \alpha_1$. If

$(\leftarrow, y(\alpha_1))_{X^*_{\alpha_1}}$ were empty, then $y(\alpha_1) = \min X^*_{\alpha_1} = \min X_{\alpha_1} \in X_{\alpha_1}$, a contradiction. Therefore we can take $v' \in (\leftarrow, y(\alpha_1))_{X^*_{\alpha_1}} \cap X_{\alpha_1}$. Then $(y \restriction \alpha_1)^\wedge\langle v'\rangle^\wedge(x \restriction (\alpha_1, \gamma)) \in (x, y)_{\hat{X}} \cap X$, a contradiction. □

(2) ⇒ (1): Assume (2). By (i), we can take $v \in X^*_{\alpha_0} \setminus X_{\alpha_0}$ such that $x(\alpha_0) <_{X^*_{\alpha_0}} v$ and $(x(\alpha_0), v)_{X^*_{\alpha_0}} = \emptyset$. Let $y = (x \restriction \alpha_0)^\wedge\langle v\rangle^\wedge(x \restriction (\alpha_0, \gamma))$. Then we have $x <_{\hat{X}} y \in \hat{X} \setminus X$ and $(x, \rightarrow)_X \neq \emptyset$. Obviously $(x, y)_{\hat{X}} = \emptyset$ holds. Thus $(\leftarrow, x]_X = (\leftarrow, y)_{\hat{X}} \cap X \in \lambda_{\hat{X}} \restriction X = \tau_X$. The following Claim completes the proof.

Claim 4. $(\leftarrow, x]_X \notin \lambda_X$.

Proof. Assume $(\leftarrow, x]_X \in \lambda_X$. It follows from $(x, \rightarrow)_X \neq \emptyset$ that for some $x' \in X$ with $x <_X x'$, $(x, x')_X = \emptyset$ holds. Let $\alpha_1 = \min\{\alpha < \gamma : x'(\alpha) \neq x(\alpha)\}$. Then by $x(\alpha_1) <_{X_{\alpha_1}} x'(\alpha_1)$, we have $\alpha_1 \leq \alpha_0$. Since $v \in (x(\alpha_0), \rightarrow)_{X^*_{\alpha_0}}$, we can take $u \in (x(\alpha_0), \rightarrow)_{X_{\alpha_0}}$. If $\alpha_1 < \alpha_0$ were true, then $(x \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(x \restriction (\alpha_0, \gamma)) \in (x, x')_X$, a contradiction. Thus we have $\alpha_1 = \alpha_0$.

Now by $(x(\alpha_0), v)_{X^*_{\alpha_0}} = \emptyset$, we also have $v <_{X^*_{\alpha_0}} x'(\alpha_0)$ moreover $(v, x'(\alpha_0))_{X^*_{\alpha_0}} \neq \emptyset$ (otherwise, v is an isolated point in $X^*_{\alpha_0}$ and $v \notin X_{\alpha_0}$, a contradiction). Taking $w \in (v, x'(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0}$, we have $(x \restriction \alpha_0)^\wedge\langle w\rangle^\wedge(x \restriction (\alpha_0, \gamma)) \in (x, x')_X$, a contradiction. □

□

Similarly, we have an analogous result:

Lemma 1.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and $x \in X$. The following are equivalent:*

1. *$x \in X^-$,*
2. *there is $\alpha_0 < \gamma$ such that:*
   - (i) *$x(\alpha_0) \in X^-_{\alpha_0}$,*
   - (ii) *for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has min and $x(\alpha) = \min X_\alpha$.*

From now on, we do not write down such an analogous result, we refer, for instance, Lemma 1.3 as the analogous result of Lemma 1.2.

Corollary 1.4. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $X^+_\alpha = \emptyset$ for every $\alpha < \gamma$, then $X^+ = \emptyset$.*

This corollary with the analogous result also shows that lexicographic products of LOTS's are LOTS's. However, lexicographic products of GO-spaces, some of which are not LOTS's, can be LOTS's. This fact will be discussed in the next section.

Now, let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of LOTS's and $\delta < \gamma$. For $x \in X$, the correspondence $x \to \langle x \restriction \delta, x \restriction [\delta, \gamma)\rangle$ defines a 0-order preserving onto map from X to $(\prod_{\alpha<\delta} X_\alpha) \times (\prod_{\delta\leq\alpha<\gamma} X_\alpha)$, which is a lexicographic product of two lexicographic products. So they are topologically homeomorphic, thus we can identify $\prod_{\alpha<\gamma} X_\alpha$ with $(\prod_{\alpha<\delta} X_\alpha) \times (\prod_{\delta\leq\alpha<\gamma} X_\alpha)$ as a LOTS whenever X_α's are LOTS's, see [2].

Next, let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and $\delta < \gamma$. The correspondence above also defines a 0-order preserving onto map from X to $(\prod_{\alpha<\delta} X_\alpha) \times (\prod_{\delta\leq\alpha<\gamma} X_\alpha)$. Is this map a homeomorphism between them? We show in the next lemma that the answer is positive, while the proof is not so trivial. It will be a key tool through the theory.

Lemma 1.5. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces and $\delta < \gamma$. The correspondence $x \to \langle x \restriction \delta, x \restriction [\delta, \gamma)\rangle$ is a homeomorphism. So we can identify $\prod_{\alpha<\gamma} X_\alpha$ with $(\prod_{\alpha<\delta} X_\alpha) \times (\prod_{\delta\leq\alpha<\gamma} X_\alpha)$ as a GO-space.*

Proof. Let $Y_0 = \prod_{\alpha<\delta} X_\alpha$ and $Y_1 = \prod_{\delta\leq\alpha<\gamma} X_\alpha$. We may identify the correspondence as $x = \langle x \restriction \delta, x \restriction [\delta, \gamma)\rangle$ for every $x \in X$. By this identification, the order $<_X$ coincides with the order $<_{Y_0\times Y_1}$, where $Y_0 \times Y_1$ is the lexicographic product of the GO-spaces Y_0 and Y_1 . It suffices to see $\tau_X = \tau_{Y_0\times Y_1}$. Note that $\tau_X = \lambda_{\hat{X}} \restriction X$, $\tau_{Y_0} = \lambda_{\hat{Y}_0} \restriction Y_0$, $\tau_{Y_1} = \lambda_{\hat{Y}_1} \restriction Y_1$ and $\tau_{Y_0\times Y_1} = \lambda_{Y_0^*\times Y_1^*} \restriction Y_0 \times Y_1$ hold, where $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, $\hat{Y}_0 = \prod_{\alpha<\delta} X^*_\alpha$ and $\hat{Y}_1 = \prod_{\delta\leq\alpha<\gamma} X^*_\alpha$.

Claim 1. $\tau_X \subset \tau_{Y_0\times Y_1}$.

Proof. It suffices to show that the subbase $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \to)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in X^+\} \cup \{[x, \to)_X : x \in X^-\}$ is contained in $\tau_{Y_0\times Y_1}$. Note under the identification, $(\leftarrow, x)_X = (\leftarrow, x)_{Y_0\times Y_1}$, $(\leftarrow, x]_X = (\leftarrow, x]_{Y_0\times Y_1} \cdots$, etc hold. Therefore, it only suffices to prove the following fact:.

Fact. If $x \in X^+$ ($x \in X^-$), then $(\leftarrow, x]_X \in \tau_{Y_0\times Y_1}$ ($[x, \to)_X \in \tau_{Y_0\times Y_1}$, respectively).

Proof. Let $x \in X^+$. By Lemma 1.3, take $\alpha_0 < \gamma$ such that $x(\alpha_0) \in X^+_{\alpha_0}$, and for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $x(\alpha) = \max X_\alpha = \max X^*_\alpha$ holds. We consider two cases.

Case 1. $\alpha_0 < \delta$.

In this case, again applying Lemma 1.2 to $x \restriction \delta \in Y_0$, we see $x \restriction \delta \in Y_0^+$. Therefore there is $y_0 \in Y_0^* \setminus Y_0$ such that $x \restriction \delta <_{Y_0^*} y_0$ and

$(x \restriction \delta, y_0)_{Y_0^*} = \emptyset$, that is, $y_0 = \langle x \restriction \delta, 1\rangle$. Let $z = y_0{}^\wedge(x \restriction [\delta,\gamma))$, then $z \in Y_0^* \times Y_1 \subset Y_0^* \times Y_1^*$. Assume that there is an element $u \in (x,z)_{Y_0^*\times Y_1^*} \cap Y_0 \times Y_1$. Then we have $x \restriction \delta \leq_{Y_0} u \restriction \delta$. If $x \restriction \delta = u \restriction \delta$ were true, then $x \restriction [\delta,\gamma) <_{Y_1} u \restriction [\delta,\gamma)$ has to be true. But this is a contradiction, because of $x(\beta) = \max X_\beta$ for all $\beta \geq \delta$. Therefore we have $x \restriction \delta <_{Y_0} u \restriction \delta$. Since $y_0 \notin Y_0$ and $(x \restriction \delta, y_0)_{Y_0^*} = \emptyset$, we see $z \restriction \delta = y_0 <_{Y_0^*} u \restriction \delta$. Thus we have $z <_{Y_0^*\times Y_1^*} u$ which contradicts $u <_{Y_0^*\times Y_1^*} z$, so we have seen $(x,z)_{Y_0^*\times Y_1^*} \cap (Y_0 \times Y_1) = \emptyset$. This shows $(\leftarrow, x]_{Y_0\times Y_1} = (\leftarrow, z)_{Y_0^*\times Y_1^*} \cap Y_0 \times Y_1 \in \lambda_{Y_0^*\times Y_1^*} \restriction Y_0 \times Y_1 = \tau_{Y_0\times Y_1}$.

Case 2. $\delta \leq \alpha_0$.

Applying Lemma 1.2 to Y_1, we see $x \restriction [\delta,\gamma) \in Y_1^+$. Therefore, there is $y_1 \in Y_1^* \setminus Y_1$ such that $x \restriction [\delta,\gamma) <_{Y_1^*} y_1$ and $(x \restriction [\delta,\gamma), y_1)_{Y_1^*} = \emptyset$. Then by $(x, (x \restriction \delta)^\wedge y_1)_{Y_0^*\times Y_1^*} = \emptyset$, we have $(\leftarrow, x]_{Y_0\times Y_1} = (\leftarrow, (x \restriction \delta)^\wedge y_1)_{Y_0^*\times Y_1^*} \cap Y_0 \times Y_1 \in \tau_{Y_0\times Y_1}$. □

This completes the proof of Claim 1. □

Claim 2. $\tau_X \supset \tau_{Y_0\times Y_1}$.

Proof. As in Claim 1, it suffices to see that if $x \in (Y_0 \times Y_1)^+$ ($x \in (Y_0 \times Y_1)^-$), then $(\leftarrow x]_{Y_0\times Y_1} \in \tau_X$ ($[x, \rightarrow)_{Y_0\times Y_1} \in \tau_X$, respectively). Let $x \in (Y_0 \times Y_1)^+$, say $x_0 = x \restriction \delta$ and $x_1 = x \restriction [\delta,\gamma)$. Apply Lemma 1.2 to $x \in (Y_0 \times Y_1)^+$, we can find $i_0 < 2$, where $2 := \{0,1\}$, such that $x_{i_0} \in Y_{i_0}^+$ and for every $i < 2$ with $i_0 < i$, $x_i = \max Y_i$ $(= \max Y_i^*)$ holds.

Case 1. $i_0 = 0$.

It follows from $x_0 \in Y_0^+$ that for some $z_0 \in Y_0^* \setminus Y_0$ with $x_0 <_{Y_0^*} z_0$, $(x_0, z_0)_{Y_0^*}$ is empty. By $x \restriction [\delta,\gamma) = x_1 = \max Y_1$, we have $x(\alpha) = \max X_\alpha$ for every $\alpha < \gamma$ with $\delta \leq \alpha$. It follows from $\lambda_{Y_0^*} \restriction Y_0 = \tau_{Y_0} = \lambda_{\hat{Y}_0} \restriction Y_0$ and $x_0 \in Y_0^+$, applying Lemma 1.2, that for some $\alpha_0 < \delta$, $x(\alpha_0) \in X_{\alpha_0}^+$ and for every $\alpha < \delta$ with $\alpha_0 < \alpha$, $x(\alpha) = \max X_\alpha$ hold. Since $x(\alpha_0) \in X_{\alpha_0}^+$ and for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $x(\alpha) = \max X_\alpha$ hold, applying Lemma 1.2 again, we have $x \in X^+$. Thus we have $(\leftarrow, x]_{Y_0\times Y_1} = (\leftarrow, x]_X \in \tau_X$.

Case 2. $i_0 = 1$.

In this case, $x \restriction [\delta,\gamma) = x_1 \in Y_1^+$. So applying Lemma 1.2, there is $\alpha_0 < \gamma$ with $\delta \leq \alpha_0$ such that $x(\alpha_0) \in X_{\alpha_0}^+$ and for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $x(\alpha) = \max X_\alpha$ holds. Again by Lemma 1.2, we have $(\leftarrow, x]_{Y_0\times Y_1} = (\leftarrow, x]_X \in \tau_X$.

The remaining case is similar. □

This completes the proof of the lemma. □

2. When are lexicographic products of GO-spaces LOTS's?

It is easy to verify that the lexicographic product $\mathbb{S} \times \mathbb{R}$ is a LOTS, while $\mathbb{S}$ is not a LOTS. In this section, we characterize when lexicographic products of GO-spaces are LOTS's. Using Lemma 1.2, the following is easy to prove.

Lemma 2.1. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) $X^+ = \emptyset$ *(*$X^- = \emptyset$*),*
(2) (i) *if* X_1 *has max (min), then* $X_0^+ = \emptyset$ *(*$X_0^- = \emptyset$ *),*
 (ii) $X_1^+ = \emptyset$ *(*$X_1^- = \emptyset$ *).*

The previous lemma shows:

Lemma 2.2. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) X *is a LOTS,*
(2) (i) *if* X_1 *has max, then* $X_0^+ = \emptyset$*,*
 (ii) *if* X_1 *has min, then* $X_0^- = \emptyset$*,*
 (iii) X_1 *is a LOTS.*

Corollary 2.3. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces. Then:*

(1) *if* X_1 *has neither min nor max, then* X *is a LOTS iff* X_1 *is a LOTS,*
(2) *if* X_1 *has min (max) but has no max (min), then* X *is a LOTS iff* $X_0^- = \emptyset$ *(*$X_0^+ = \emptyset$*) and* X_1 *is a LOTS,*
(3) *if* X_1 *has both min and max, then* X *is a LOTS iff both* X_0 *and* X_1 *are LOTS's.*

Example 2.4. $\mathbb{S} \times \mathbb{R}$, $\mathbb{S} \times [0,1)_{\mathbb{R}}$, $\mathbb{M} \times \mathbb{P}$ are LOTS's. But $\mathbb{R} \times \mathbb{S}$, $\mathbb{S} \times (0,1]_{\mathbb{R}}$, $\mathbb{S} \times \{0,1\}$, $\mathbb{S} \times [0,1]_{\mathbb{R}}$, $\mathbb{S}^2$, $\mathbb{P} \times \mathbb{M}$ are not LOTS's.

More generally we have:

Theorem 2.5. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Let $J^+ = \{\alpha < \gamma : X_\alpha \text{ has no max.}\}$ and $J^- = \{\alpha < \gamma : X_\alpha \text{ has no min.}\}$. Then the following are equivalent:*

(1) $X^+ = \emptyset$ *(*$X^- = \emptyset$*),*
(2) *for every* $\alpha < \gamma$ *with* $\sup J^+ \leq \alpha$ *(* $\sup J^- \leq \alpha$*),* $X_\alpha^+ = \emptyset$ *(*$X_\alpha^- = \emptyset$*) holds.*

Proof. Let $\alpha_0 = \sup J^+$. Note $\alpha_0 \leq \gamma$.

(1) $\Rightarrow$ (2): Let $X^+ = \emptyset$ and $\alpha_0 \leq \beta < \gamma$. Since $X = \prod_{\alpha\leq\beta} X_\alpha \times \prod_{\beta<\alpha<\gamma} X_\alpha$ and $\prod_{\beta<\alpha<\gamma} X_\alpha$ has max, by Lemma 2.1, $(\prod_{\alpha\leq\beta} X_\alpha)^+ = \emptyset$ holds. Moreover by $\prod_{\alpha\leq\beta} X_\alpha = \prod_{\alpha<\beta} X_\alpha \times X_\beta$, again by Lemma 2.1, we have $X_\beta^+ = \emptyset$.

(2) $\Rightarrow$ (1): Assume that $X_\alpha^+ = \emptyset$ for every $\alpha < \gamma$ with $\alpha_0 \leq \alpha$. If $\alpha_0 = 0$, then by Cororally 1.4, we have $X^+ = \emptyset$. So we assume $\alpha_0 > 0$.

Case 1. $\alpha_0 \in J^+$.

In this case, $\alpha_0 = \max J^+ < \gamma$. Since $\prod_{\alpha\leq\alpha_0} X_\alpha = \prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$, X_{α_0} has no max and $X_{\alpha_0}^+ = \emptyset$, by Lemma 2.1, $(\prod_{\alpha\leq\alpha_0} X_\alpha)^+$ is empty. It follows from Corollary 1.4 that $(\prod_{\alpha_0<\alpha<\gamma} X_\alpha)^+$ is also empty. Because of $X = \prod_{\alpha\leq\alpha_0} X_\alpha \times \prod_{\alpha_0<\alpha<\gamma} X_\alpha$, by the same corollary, we have $X^+ = \emptyset$.

Case 2. $\alpha_0 \notin J^+$.

In this case, α_0 is a limit ordinal with $\alpha_0 \leq \gamma$.

Claim. $(\prod_{\alpha<\alpha_0} X_\alpha)^+ = \emptyset$.

Proof. If there were $x \in (\prod_{\alpha<\alpha_0} X_\alpha)^+$, then by Lemma 1.2, there is some $\alpha_1 < \alpha_0$ such that fore every $\alpha < \alpha_0$ with $\alpha_1 < \alpha$, $\max X_\alpha$ exists. This means $\sup J^+ \leq \alpha_1 < \alpha_0$, a contradiction. □

By $X = \prod_{\alpha<\alpha_0} X_\alpha \times \prod_{\alpha_0\leq\alpha<\gamma} X_\alpha$ and the assumption $(\prod_{\alpha_0\leq\alpha<\gamma} X_\alpha)^+ = \emptyset$, we have $X^+ = \emptyset$.

The remaining is similar. □

Corollary 2.6. *Under the same assumption of Theorem 2.5, X is a LOTS if and only if the following hold:*

1. *for every $\alpha < \gamma$ with $\sup J^+ \leq \alpha$, $X_\alpha^+ = \emptyset$ holds,*
2. *for every $\alpha < \gamma$ with $\sup J^- \leq \alpha$, $X_\alpha^- = \emptyset$ holds,*

Corollary 2.7. *Let $X = \prod_{\alpha\leq\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Assume that X_γ has neither min nor max. Then X is a LOTS if and only if X_γ is a LOTS. In particular, $\prod_{\alpha<\gamma} X_\alpha \times \mathbb{R}$ is a LOTS.*

Above two corollaries show:

Corollary 2.8. *For every non-zero ordinal γ, $\mathbb{S}^\gamma$ is a LOTS if and only if γ is limit.*

3. When is $\prod_{\alpha<\gamma} X_\alpha$ dense in $\prod_{\alpha<\gamma} X^*_\alpha$?

A GO-space X is dense in the LOTS X^*, but generally a lexicographic product $X_0 \times X_1$ of GO-spaces need not be dense in $X_0^* \times X_1^*$. For instance, let $X_0 = [0,1)_{\mathbb{R}} \cup [2,3]_{\mathbb{R}}$ be the subspace of $\mathbb{R}$ and $X_1 = [0,1]_{\mathbb{R}}$. Then X_0^* can be considered as the subspace $[0,1]_{\mathbb{R}} \cup [2,3]_{\mathbb{R}}$ of $\mathbb{R}$ and obviously $X_1^* = X_1$. Now $(\langle 1,0\rangle, \langle 1,1\rangle)_{X_0^*\times X_1^*}$ is non-empty open in $X_0^* \times X_1^*$ but disjoint from $X_0 \times X_1$.

First we consider a special case.

Lemma 3.1. *Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and let $\hat{X} = X_0^* \times X_1^*$. If X_0 is a LOTS, then X is dense in $\hat{X}$.*

Proof. Let X_0 be a LOTS. First we prove:

Claim 1. If $x \in \hat{X}$ and $(x, \to)_{\hat{X}} \neq \emptyset$, then $(x, \to)_{\hat{X}} \cap X \neq \emptyset$.

Proof. If $(x(0), \to)_{X_0^*} \neq \emptyset$, then pick $u \in (x(0), \to)_{X_0^*} \cap X_0$ and $v \in X_1$. Then $\langle u, v\rangle \in (x, \to)_{\hat{X}} \cap X$. So let $(x(0), \to)_{X_0^*} = \emptyset$, that is, $x(0) = \max X_0$. Take $y \in (x, \to)_{\hat{X}}$. Then $x(0) = y(0)$ and $y(1) \in (x(1), \to)_{X_1^*}$. Since X_1 is dense in X_1^*, we can find $v \in (x(1), \to)_{X_1^*} \cap X_1$. Now we have $\langle x(0), v\rangle \in (x, \to)_{\hat{X}} \cap X$. □

Analogously, we can prove:

Claim 2. If $x \in \hat{X}$ and $(\leftarrow, x)_{\hat{X}} \neq \emptyset$, then $(\leftarrow, x)_{\hat{X}} \cap X \neq \emptyset$.

These two claims with the following claim complete the proof.

Claim 3. If $x, x' \in \hat{X}, x <_{\hat{X}} x'$ and $(x, x')_{\hat{X}} \neq \emptyset$, then $(x, x')_{\hat{X}} \cap X \neq \emptyset$.

Proof. Let $x, x' \in \hat{X}, x <_{\hat{X}} x'$ and $(x, x')_{\hat{X}} \neq \emptyset$. Since X_0 is a LOTS, that is $X_0 = X_0^*$, we have $x(0), x'(0) \in X_0$.

Case 1. $x(0) = x'(0)$.

In this case, take $y \in (x, x')_{\hat{X}}$. Then we have $x(0) = x'(0) = y(0)$ and $y(1) \in (x(1), x'(1))_{X_1^*}$. Since X_1 is dense in X_1^*, there is $v \in (x(1), x'(1))_{X_1^*} \cap X_1$. Now $\langle x(0), v\rangle \in (x, x')_{\hat{X}} \cap X$.

Case 2. $x(0) < x'(0)$.

First assume $(x(0), x'(0))_{X_0} \neq \emptyset$. In this case, pick $u \in (x(0), x'(0))_{X_0}$ and $v \in X_1$. Then $\langle u, v\rangle \in (x, x')_{\hat{X}} \cap X$.

Next assume $(x(0), x'(0))_{X_0} = \emptyset$. Since $(x, x')_{\hat{X}} \neq \emptyset$, we have either $(x(1), \to)_{X_1^*} \neq \emptyset$ or $(\leftarrow, x'(1))_{X_1^*} \neq \emptyset$. In the case $(x(1), \to)_{X_1^*} \neq \emptyset$, taking $v \in (x(1), \to)_{X_1^*} \cap X_1$, we see $\langle x(0), v\rangle \in (x, x')_{\hat{X}} \cap X$. In the case $(\leftarrow, x'(1))_{X_1^*} \neq \emptyset$, taking $v \in (\leftarrow, x'(1))_{X_1^*} \cap X_1$, we see $\langle x'(0), v\rangle \in (x, x')_{\hat{X}} \cap X$. □

□

Theorem 3.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then X is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ if and only if for every $\alpha < \gamma$ with $\alpha + 1 < \gamma$, X_α is a LOTS.*

Proof. First assume that X is dense in $\hat{X}$ and there is $\alpha_0 < \gamma$ with $\alpha_0 + 1 < \gamma$ such that X_{α_0} is not a LOTS. We may assume $X^+_{\alpha_0} \neq \emptyset$, so fix $u \in X^+_{\alpha_0}$ and take $u' \in X^*_{\alpha_0} \setminus X_{\alpha_0}$ such that $u <_{X^*_{\alpha_0}} u'$ and $(u, u')_{X^*_{\alpha_0}} = \emptyset$. Fix $x \in X$.

Case 1. $|\prod_{\alpha_0<\alpha<\gamma} X_\alpha| > 2$.

Take $v_0, v_1, v_2 \in \prod_{\alpha_0<\alpha<\gamma} X_\alpha$ with $v_0 < v_1 < v_2$. Let $x_i = (x \restriction \alpha_0)^\wedge\langle u'\rangle^\wedge v_i$ for $i = 0, 1, 2$. Then $x_1 \in (x_0, x_2)_{\hat{X}}$ but $(x_0, x_2)_{\hat{X}} \cap X = \emptyset$, a contradiction.

Case 2. $|\prod_{\alpha_0<\alpha<\gamma} X_\alpha| = 2$.

In this case, note $\gamma = \alpha_0 + 2$ and $\prod_{\alpha_0<\alpha<\gamma} X_\alpha = X_{\alpha_0+1}$, say $X_{\alpha_0+1} = \{v_0, v_1\}$ with $v_0 < v_1$. Let $x_0 = (x \restriction \alpha_0)^\wedge\langle u\rangle^\wedge v_1$ and $x_1 = (x \restriction \alpha_0)^\wedge\langle u'\rangle^\wedge v_1$. Then $(x \restriction \alpha_0)^\wedge\langle u'\rangle^\wedge v_0 \in (x_0, x_1)_{\hat{X}}$ but $(x_0, x_1)_{\hat{X}} \cap X = \emptyset$, a contradiction.

Next assume that for every $\alpha < \gamma$ with $\alpha + 1 < \gamma$, $X_\alpha = X^*_\alpha$ holds. If γ is limit, then $\prod_{\alpha<\gamma} X_\alpha = \prod_{\alpha<\gamma} X^*_\alpha$. If $\gamma = \delta + 1$, then $\prod_{\alpha<\delta} X_\alpha$ is a LOTS. Therefore by the lemma above, X is dense in $\hat{X}$. □

Corollary 3.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then:*

(1) *if γ is limit, then X is dense in $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$ if and only if $X = \hat{X}$,*
(2) *if $\gamma = \delta + 1$, then X is dense in $\hat{X}$ if and only if $\prod_{\alpha<\delta} X_\alpha$ is a LOTS.*

Note that the reverse implication of Lemma 3.1 is also true.

Example 3.4. For instance, we see:

- $\mathbb{S} \times X$ is not dense in $\mathbb{S}^* \times X$ for every GO-space X.
- $X \times \mathbb{S}$ is dense in $X \times \mathbb{S}^*$ if X is a LOTS.
- $\mathbb{P} \times \mathbb{M}$ is dense in $\mathbb{P} \times \mathbb{M}^*$ but $\mathbb{M} \times \mathbb{P}$ is not dense in $\mathbb{M}^* \times \mathbb{P}$.

4. Paracompactness of lexicographic products

It is known that lexicographic products of paracompact LOTS's are paracompact. In this section, we extend this result for paracompact GO-spaces.

Definition 4.1. Let X be a GO-space. A subset A of X is called an *initial segment* or a 0-*segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. Similarly a subset A of X is called a *final segment* or a 1-*segment* of X if for every $x, x' \in X$ with $x \leq x'$, if $x \in A$, then $x' \in A$. Both $\emptyset$ and X are 0-segments and 1-segments.

Let A be a 0-segment of a GO-space X. A subset U of A is 0-*unbounded in* A if for every $x \in A$, there is $x' \in U$ such that $x \leq x'$. Let

$$0\text{-}\operatorname{cf}_X A = \min\{|U| : U \text{ is 0-unbounded in } A.\}.$$

Similar notions are also defined in linearly ordered compactifications, see [3]. If the context is clear, $0\text{-}\operatorname{cf}_X A$ is denoted by $0\text{-}\operatorname{cf} A$. Obviously $A = \emptyset$ iff $0\text{-}\operatorname{cf} A = 0$, and A has max iff $0\text{-}\operatorname{cf} A = 1$. Moreover we can easily check that a 0-segment A has no max iff $0\text{-}\operatorname{cf} A \geq \omega$, and in this case, $0\text{-}\operatorname{cf} A$ is a regular cardinal. Also remark:

- if A is a 0-segment of a GO-space X having no max, then A is open in X, because of $A = \bigcup_{a \in A} (\leftarrow, a)_X$,
- if U is a 0-unbounded subset of a 0-segment A of a GO-space X, then we can define, by induction, a 0-order preserving sequence $\{x_\alpha : \alpha < \kappa\} \subset U$ (i.e., $x_\alpha <_X x_{\alpha'}$ whenever $\alpha < \alpha' < \kappa$) which is also 0-unbounded in A, where $\kappa = 0\text{-}\operatorname{cf} A$.

Analogous concepts such as 1-unbounded, $1\text{-}\operatorname{cf} A, \ldots$ etc, are also defined.

A *cut* of a GO-space X is a pair $\langle A_0, A_1 \rangle$ of subsets of X such that $A_1 = X \setminus A_0$ and A_0 is a 0-segment (equivalently A_1 is a 1-segment). A cut $\langle A_0, A_1 \rangle$ is said to be a *gap* if A_0 has no max and A_1 has no min. Thus if X has no max, then $\langle X, \emptyset \rangle$ is a gap. Remark that if $\langle A_0, A_1 \rangle$ is a gap, then both A_0 and A_1 are clopen in X. A cut $\langle A_0, A_1 \rangle$ is said to be a *pseudo-gap* if either "A_0 has max and A_1 has no min" or "A_0 has no max and A_1 has min", moreover A_0 (equivalently A_1) is clopen in X.

The following is known:

Lemma 4.2 ([2], Theorem 2.4.6)**.** *Let X be a GO-space, then the following are equivalent:*

1. *X is paracompact,*
2. *for each gap and pseudo-gap $\langle A_0, A_1 \rangle$ of X and for each $i \in 2$, there is a closed discrete i-unbounded subset of A_i.*

Note that in the notations above:

- if $A_0 = \emptyset$, then $\emptyset$ is a closed discrete 0-unbounded subset of A_0,
- if A_0 has max, then the one element set $\{\max A_0\}$ is a closed discrete 0-unbounded subset of A_0,

- if $0\text{-}\operatorname{cf} A_0 = \omega$, then every 0-unbounded 0-order preserving sequence $\{a_n : n \in \omega\}$ in A_0 is closed discrete in A_0.

Definition 4.3. A GO-space X is said to be 0-*paracompact* if for every closed 0-segment A of X with $0\text{-}\operatorname{cf} A \geq \omega_1$, say $\kappa = 0\text{-}\operatorname{cf} A$, there is a 0-unbounded closed discrete subset of A. In this case, we can take a 0-order preserving sequence $\{a_\alpha : \alpha < \kappa\}$ in A which is 0-unbounded and closed discrete in A (equivalently, closed discrete in X). 1-*paracompactness* is defined analogously.

Now with the consideration above, Lemma 4.2 says the following:

Lemma 4.4. *A GO-space is paracompact if and only if it is both* 0-*paracompact and* 1-*paracompact.*

Remark that Lemma 1.2 says something about pseudo-gaps in lexicographic products. On the other hand, the following says about gaps of lexicographic products.

Lemma 4.5. *Let* $X = \prod_{\alpha<\gamma} X_\alpha$ *be a lexicographic product of GO-spaces. Assume that* A *is a 0-segment with* $0\text{-}\operatorname{cf} A \geq \omega$ *and* $1\text{-}\operatorname{cf}(X \setminus A) \geq \omega$, *that is,* $\langle A, X \setminus A\rangle$ *is a gap with* $A \neq \emptyset$ *and* $X \setminus A \neq \emptyset$. *Say* $\kappa = 0\text{-}\operatorname{cf} A$, *then there are* $\alpha_0 < \gamma$, $y_0 \in Y_0 := \prod_{\alpha<\alpha_0} X_\alpha$ *and a* 0-*segment* A_0 *of* X_{α_0} *such that:*

(1) *for every* $a \in A$, $a \restriction \alpha_0 \leq_{Y_0} y_0$ *holds,*
(2) *for every* $x \in X$,
  (i) *if* $x \restriction \alpha_0 <_{Y_0} y_0$, *then* $x \in A$ *holds,*
  (ii) *if* $x \restriction \alpha_0 >_{Y_0} y_0$, *then* $x \in X \setminus A$ *holds,*
(3) *for every* $x \in X$ *with* $x \restriction \alpha_0 = y_0$, $x(\alpha_0) \in A_0$ *holds iff so does* $x \in A$,
(4) *if* A_0 *is non-empty and has no max, then* $\kappa = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0$,
(5) *if* A_0 *is non-empty and has max, then there is* $\alpha > \alpha_0$ *such that* X_α *has no max and* $\kappa = 0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1}$ *holds, where* $\alpha_1 := \min\{\alpha < \gamma : \alpha > \alpha_0$ *and* X_α *ha no max.*$\}$,
(6) *if* A_0 *is empty, then:*
  (i) *for every* $a \in A$, $a \restriction \alpha_0 <_{Y_0} y_0$ *holds,*
  (ii) α_0 *is limit,*
  (iii) *there is* $\alpha \geq \alpha_0$ *such that* X_α *has no min.*
  (iv) $A = (\leftarrow, y_0)_{Y_0} \times Y_1$, *where* $Y_1 := \prod_{\alpha_0 \leq \alpha < \gamma} X_\alpha$.
  (v) $(\leftarrow, y_0)_{Y_0}$ *has no max,*
  (vi) $\kappa = 0\text{-}\operatorname{cf}_{Y_0}(\leftarrow, y_0)_{Y_0} = \operatorname{cf} \alpha_0$,
  (vii) *for every* $\beta < \alpha_0$, *there is* $a \in A$ *satisfying* $\beta < \min\{\alpha < \alpha_0 : a(\alpha) \neq y_0(\alpha)\}$.

Proof. Set $B = X \setminus A$. For each $a \in A$ and $b \in B$, let $\alpha(a,b) = \min\{\alpha < \gamma : a(\alpha) \neq b(\alpha)\}$ and $\alpha_0 = \sup\{\alpha(a,b) : a \in A, b \in B\}$. Note $\alpha_0 \leq \gamma$.

Claim 1. Let $a_0, a_1 \in A$ and $b_0, b_1 \in B$. If $\alpha(a_0,b_0) \leq \alpha(a_1,b_1)$, then $a_0 \upharpoonright \alpha(a_0,b_0) = a_1 \upharpoonright \alpha(a_0,b_0)$.

Proof. Assume that there is $\beta < \alpha(a_0,b_0)$ such that $a_0(\beta) \neq a_1(\beta)$. Let $\beta_0 = \min\{\beta < \alpha(a_0,b_0) : a_0(\beta) \neq a_1(\beta)\}$. Then $b_0 \upharpoonright \beta_0 = a_0 \upharpoonright \beta_0 = a_1 \upharpoonright \beta_0 = b_1 \upharpoonright \beta_0$ and $b_0(\beta_0) = a_0(\beta_0) \neq a_1(\beta_0) = b_1(\beta_0)$. If $a_0(\beta_0) < a_1(\beta_0)$, then we have $b_0 < a_1$, $b_0 \in B$ and $a_1 \in A$, a contradiction. If $a_0(\beta_0) > a_1(\beta_0)$, then we have $a_0 > b_1$, $b_1 \in B$ and $a_0 \in A$, a contradiction. □

This claim ensures that the function $y_0 := \bigcup\{a \upharpoonright \alpha(a,b) : a \in A, b \in B\}$ is well-defined and $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$.

Claim 2. $\alpha_0 < \gamma$.

Proof. Assume $\alpha_0 = \gamma$. Then $y_0 \in X = A \cup B$. If $y_0 \in A$, then there is $a_0 \in A$ with $y_0 <_X a_0$. Letting $\beta_0 = \min\{\beta < \gamma : y_0(\beta) \neq a_0(\beta)\}$, take $a \in A$ and $b \in B$ with $\beta_0 < \alpha(a,b)$. Then we have $b <_X a_0$, a contradiction. When $y_0 \in B$, similarly we can get a contradiction. □

By a similar argument of the proof above, we can check the clauses (1) and (2). Now let $A_0 = \{a(\alpha_0) : a \in A, a \upharpoonright \alpha_0 = y_0\}$ amd $B_0 = X_{\alpha_0} \setminus A_0$. Obviously A_0 is a 0-segment of X_{α_0} and B_0 is a 1-segment of X_{α_0}.

Claim 3. $B_0 = \{a(\alpha_0) : a \in B, a \upharpoonright \alpha_0 = y_0\}$ holds.

Proof. The inclusion "$\subset$" is obvious.

To see the other inclusion, let $b \in B$ with $b \upharpoonright \alpha_0 = y_0$. If $b(\alpha_0) \in A_0$ were true, then there is $a \in A$ with $a \upharpoonright \alpha_0 = y_0$ and $b(\alpha_0) = a(\alpha_0)$. This means $a \upharpoonright (\alpha_0 + 1) = b \upharpoonright (\alpha_0 + 1)$, thus $\alpha_0 < \alpha(a,b)$, a contradiction. We have $b(\alpha_0) \in B_0$. □

This claim shows the clause (3).

Claim 4. The clause (4) holds.

Proof. Assume that $A_0 \neq \emptyset$ and A_0 has no max. To see $\kappa \geq 0\text{-}\operatorname{cf} A_0$, let U be a 0-unbounded subset of A. Fix $u_0 \in A_0$ and $a_0 \in A$ with $a_0 \upharpoonright \alpha_0 = y_0$ and $a_0(\alpha_0) = u_0$. Then it is easy to check that $V := \{a(\alpha_0) : a_0 <_X a \in U\}$ is 0-unbounded in A_0.

To see $\kappa \leq 0\text{-}\operatorname{cf} A_0$, let V be a 0-unbounded in A_0. For every $u \in V$, we can fix $a_u \in A$ with $a_u \upharpoonright \alpha_0 = y_0$ and $a_u(\alpha_0) = u$. Then $U := \{a_u : u \in V\}$ is 0-unbounded in A. □

Claim 5. The clause (5) holds.

Proof. Assume that $A_0 \neq \emptyset$ and A_0 has max u_0. If for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, X_α has max, then $y_0{}^\wedge\langle u_0\rangle{}^\wedge\langle \max X_\alpha : \alpha_0 < \alpha < \gamma\rangle = \max A$, a contradiction. Therefore there is $\alpha < \gamma$ with $\alpha_0 < \alpha$ such that X_α has no max. Let α_1 be such a smallest one. By a similar argument in Claim 4, we see $\kappa = 0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1}$ □

Claim 6. The clause (6) holds.

Proof. Let $A_0 = \emptyset$. If there is $a \in A$ with $a \restriction \alpha_0 = y_0$, then $a(\alpha_0) \in A_0$, a contradiction. Tis shows (i).

If $\alpha_0 = \beta + 1$ for some ordinal β, then we can find $a \in A$ and $b \in B$ with $\beta < \alpha(a, b) \leq \alpha_0$, so $\alpha(a, b) = \alpha_0$. Now we have $y_0 = a \restriction \alpha_0$, this contradicts (i). This shows (ii).

If $Y_1 = \prod_{\alpha_0 \leq \alpha < \gamma} X_\alpha$ has min, then we have $b_0 := y_0{}^\wedge\langle \min X_\alpha : \alpha_0 \leq \alpha < \gamma\rangle \in B$ by $A_0 = \emptyset$. If $a \in X$ and $a < b_0$, then $a \restriction \alpha_0 < b \restriction \alpha_0 = y_0$, thus $a \in A$ by (i). This shows $b_0 = \min B$, a contradiction. We see (iii). (2-i) and (i) show (iv).

To see (v), assume that $y_1 := \max(\leftarrow, y_0)_{Y_0}$ exists. Let $\alpha_1 = \min\{\alpha < \alpha_0 : y_1(\alpha) \neq y_0(\alpha)\}$, moreover take $a \in A$ and $b \in B$ with $\alpha_1 < \alpha(a, b)$. By (i), we have $a \restriction \alpha_0 < y_0$, therefore $a \restriction \alpha_0 \leq y_1$. By $y_1 \restriction \alpha_1 = y_0 \restriction \alpha_1 = a \restriction \alpha_1$ and $y_1(\alpha_1) < y_0(\alpha_1) = a(\alpha_1)$, we have $y_1 < a \restriction \alpha_0$, a contradiction.

(vi) can be similarly proved as in Claim 4. (vii) follows from the definition of α_0 □

□

Theorem 4.6. *If X_α is a 0-paracompact GO-space for every $\alpha < \gamma$, then the lexicographic product $X = \prod_{\alpha<\gamma} X_\alpha$ is also 0-paracompact.*

Proof. Let A be a closed 0-segment of X with $0\text{-}\operatorname{cf} A \geq \omega_1$, set $\kappa = 0\text{-}\operatorname{cf} A$. We will find a 0-unbounded 0-order preserving sequence $\{a_\delta : \delta < \kappa\} \subset A$ which is closed discrete in A. We have to consider several cases. Let $B = X \setminus A$.

Case 1. B has min b_0.

In this case, since A is closed and has no max, b_0 belongs to X^-. From Lemma 1.3, we can find $\alpha_0 < \gamma$ such that $b_0(\alpha_0) \in X^-_{\alpha_0}$ and for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $b_0(\alpha) = \min X_\alpha$ holds. Let $A_0 = (\leftarrow, b_0(\alpha_0))_{X_{\alpha_0}}$. Then A_0 is a closed 0-segment of X_{α_0}. By a similar argument of Claim 4 in the previous lemma, we see $\kappa = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0$. Since X_{α_0} is 0-paracompact, we can take a 0-unbounded 0-order preserving sequence

$\{u_\delta : \delta < \kappa\}$ in A_0 which is closed discrete in A_0 and $(\leftarrow, u_0)_{X_{\alpha_0}} \neq \emptyset$. For each $\delta < \kappa$, let $a_\delta = (b_0 \restriction \alpha_0)^\wedge\langle u_\delta\rangle^\wedge(b_0 \restriction (\alpha_0, \gamma))$.

Claim 1. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in A.

Proof. Obviously F is 0-order preserving. Let $a \in A$. Then we have $a \restriction \alpha_0 \leq b_0 \restriction \alpha_0$. If $a \restriction \alpha_0 < b_0 \restriction \alpha_0$, then $a < a_0$. If $a \restriction \alpha_0 = b_0 \restriction \alpha_0$, then we can take $\delta < \kappa$ with $a(\alpha_0) < u_\delta$ (otherwise, $a \geq b_0$, a contradiction). Then we have $a < a_\delta$. Thus F is 0-unbounded in A. To see the closed discreteness of F, take the smallest $\delta_0 < \kappa$ with $a < a_{\delta_0}$. If $\delta_0 = 0$, then $(\leftarrow, a_0)_X$ is a neighborhood of a disjoint from F. If $\delta_0 > 0$, then we have $a \restriction \alpha_0 = b_0 \restriction \alpha_0$ and $a(\alpha_0) \in A_0$. Note $u_0 \leq a(\alpha_0)$ because of $a_0 \leq a$. Since $\{u_\delta : \delta < \kappa\}$ is closed discrete in X_{α_0}, we can find $u^* \in X^*_{\alpha_0}$ with $u^* <_{X^*_{\alpha_0}} a(\alpha_0)$ such that $(u^*, a(\alpha_0)]_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ contains at most one u_δ. Let $a^* = (b_0 \restriction \alpha_0)^\wedge\langle u^*\rangle^\wedge(b_0 \restriction (\alpha_0, \gamma))$. Then $a^* \in \hat{X}$ and $(a^*, a_{\delta_0})_{\hat{X}} \cap X$ is a neighborhood of a witnessing the closed discreteness of F at a. □

Case 2. $B \neq \emptyset$ and has no min.

This case is a modification of Theorem 4.2.2 in [2]. In this case, take $\alpha_0 < \gamma$, $y_0 \in \prod_{\alpha<\alpha_0} X_\alpha$ and the 0-segment A_0 of X_{α_0} in Lemma 4.5. Further we divide Case 2 into several subcases.

Case 2-1. $A_0 = \emptyset$.

In this case, we use (6) of Lemma 4.5. By induction using (i) and (vi) in (6), define $\{a_\delta : \delta < \kappa\} \subset A$ such that $\{\min\{\alpha < \alpha_0 : a_\delta(\alpha) \neq y_0(\alpha)\} : \delta < \kappa\}$ is 0-unbounded and 0-order preserving in α_0.

Claim 2. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in A.

Proof. The proof that F is 0-unbounded and 0-order preserving is easy. Let $a \in A$ and $\delta_0 < \kappa$ be the smallest $\delta < \kappa$ with $a < a_\delta$. By (6-iii) in Lemma 4.5, $Y_1 := \prod_{\alpha_0 \leq \alpha < \gamma} X_\alpha$ has no min, so take $y_1 \in Y_1$ with $y_1 <_{Y_1} a \restriction [\alpha_0, \gamma)$. Then $((a \restriction \alpha_0)^\wedge y_1, a_{\delta_0})_X$ is a neighborhood of a witnessing the closed discreteness of F at a. □

Case 2-2. $A_0 \neq \emptyset$.

We further divide this case into several cases.

Case 2-2-1. A_0 has no max and $B_0 := X_{\alpha_0} \setminus A_0$ has min.

Note that in this case, A_0 need not be closed in X_{α_0}. We can find $\alpha > \alpha_0$ such that X_α has no min (otherwise, B has min). Let α_1 be such a smallest one. By (4) in Lemma 4.5, we can find a 0-unbounded 0-order preserving sequence $\{u_\delta : \delta < \kappa\}$ in A_0. But remark that in general, $\{u_\delta : \delta < \kappa\}$ cannot be closed discrete in A_0. For each $\delta < \kappa$, take $a_\delta \in X$ with $a_\delta \restriction (\alpha_0 + 1) = y_0{}^\wedge\langle u_\delta\rangle$, then $a_\delta \in A$.

Claim 3. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in A.

Proof. Obviously F is 0-unbounded and 0-order preserving in A. Let $a \in A$ and $\delta_0 < \kappa$ be the smallest $\delta < \kappa$ with $a < a_\delta$. If $\delta_0 = 0$, then $(\leftarrow, a_0)_X$ is a neighborhood of a disjoint from F.

Let $\delta_0 > 0$, then we have $a \restriction \alpha_0 = y_0$. Since $Y_1 := \prod_{\alpha_0<\alpha<\gamma} X_\alpha$ has no min, take $y_1 \in Y_1$ with $y_1 < a \restriction (\alpha_0, \gamma)$. Then $((a \restriction (\alpha_0+1))^\wedge y_1, a_{\delta_0})$ is a neighborhood of a witnessing the closed discreteness of F at a.

Case 2-2-2. A_0 has no max and $B_0 := X_{\alpha_0} \setminus A_0$ has no min.

In this case A_0 is a closed 0-segment in the 0-paracompact GO-space X_{α_0}. Using (4) in Lemma 4.5, take a 0-unbounded 0-oder preserving sequence $\{u_\delta : \delta < \kappa\}$ which is closed discrete in A_0 and $(\leftarrow, u_0)_{X_{\alpha_0}} \neq \emptyset$. For each $\delta < \kappa$, take $a_\delta \in X$ with $a_\delta \restriction (\alpha_0 + 1) = y_0{}^\wedge\langle u_\delta\rangle$, then $a_\delta \in A$. □

Claim 4. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in A.

Proof. Obviously F is 0-unbounded and 0-order preserving in A. Let $a \in A$ and $\delta_0 < \kappa$ be the smallest $\delta < \kappa$ with $a < a_\delta$. As in the proof of the claim above, when $\delta_0 = 0$, then $(\leftarrow, a_0)_X$ witnesses the closed discreteness of F at a. When $\delta_0 > 0$, we have $a \restriction \alpha_0 = y_0$ and $a(\alpha_0) \in A_0$. Since $\{u_\delta : \delta < \kappa\}$ is closed discrete in X_{α_0}, we can take $u^* \in X^*_{\alpha_0}$ with $u^* < a(\alpha_0)$, $(u^*, a(\alpha_0)]_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ contains at most one u_δ. Take $a^* \in \hat{X}$ with $a^* \restriction (\alpha_0+1) = (a \restriction \alpha_0)^\wedge\langle u^*\rangle$. Then $(a^*, a_{\delta_0})_{\hat{X}} \cap X$ is a neighborhood of a witnessing the closed discreteness of F at a. □

Case 2-2-3. A_0 has max.

In this case, by (5) of Lemma 4.5, there is $\alpha > \alpha_0$ such that X_α has no max. Let α_1 be such a smallest one. Since $\kappa = 0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1}$ and X_{α_1} is 0-paracompact, the 0-segment X_{α_1} has a 0-unbounded 0-order preserving sequence $\{u_\delta : \delta < \kappa\} \subset X_{\alpha_1}$ which is closed discrete in X_{α_1} and $(\leftarrow, u_0)_{X_{\alpha_1}} \neq \emptyset$. For each $\delta < \kappa$, take $a_\delta \in X$ with $a_\delta \restriction (\alpha_1 + 1) = y_0{}^\wedge\langle \max A_0\rangle^\wedge\langle \max X_\alpha : \alpha_0 < \alpha < \alpha_1\rangle^\wedge\langle u_\delta\rangle$. Note $a_\delta \in A$. As in Claim 4, we see:

Claim 5. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in A.

Case 3. $B = \emptyset$, i.e., $A = X$.

Since X has no max, let $\alpha_0 = \min\{\alpha < \gamma : X_\alpha$ has no max.$\}$. Then as in Claim 4 in Lemma 4.5, we see $\kappa = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0}$. Since X_{α_0} is 0-paracompact, we can find a 0-unbounded 0-order preserving sequence $\{u_\delta : \delta < \kappa\} \subset X_{\alpha_0}$ which is closed discrete in X_{α_0} and $(\leftarrow, u_0)_{X_{\alpha_0}} \neq \emptyset$. For every $\delta < \kappa$, take $a_\delta \in X$ with $a_\delta \restriction (\alpha_0 + 1) = \langle \max X_\alpha : \alpha < \alpha_0 \rangle^\wedge \langle u_\delta \rangle$. Note $a_\delta \in A$. Similarly we can see:

Claim 6. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in A. □

With the analogy of the theorem above, we extends the result Theorem 4.2.2 in [2] as follows:

Corollary 4.7. *Lexicographic products of paracompact GO-spaces are paracompact.*

Example 4.8. For example we see:

- the lexicographic products $\mathbb{S}^\gamma$ and $\mathbb{M}^\gamma$ are paracompact for every ordinal γ.
- the lexicographic products $\mathbb{M} \times \mathbb{P}$ and $\mathbb{P} \times \mathbb{M}$ are paracompact.
- lexicographic products of metrizable GO-spaces are paracompact. For instance, the lexicographic product $([0, 1)_\mathbb{R} \cup [2, 3]_\mathbb{R})^{\omega_1}$ is paracompact.

However, there is a paracompact lexicographic product of non-paracompact LOTS's, see Example in page 73 in [2]. We end this paper with the following question.

Question 4.9. Characterize paracompactness of lexicographic products of GO-spaces.

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