LEXICOGRAPHIC PRODUCTS OF GO-SPACES

NOBUYUKI KEMOTO

ABSTRACT. It is known that lexicographic products of paracompact LOTS’s are also paracompact, see [2]. In this paper, the notion of lexicographic products of GO-spaces is defined. We characterize when a lexicographic product of GO-spaces is a LOTS. Moreover, we show that lexicographic products of paracompact GO-spaces are also paracompact. For example, we see

\[ \text{the lexicographic products } M \times \mathbb{P} \text{ and } S \times \{0,1\}_\mathbb{R} \text{ are LOTS}, \]

\[ \text{but } \mathbb{P} \times M \text{ and } S \times \{0,1\}_\mathbb{R} \text{ are not LOTS's,} \]

\[ \text{the lexicographic product } S^\gamma \text{ of the } \gamma \text{-many copies of } S \text{ is a LOTS iff } \gamma \text{ is a limit ordinal,} \]

\[ \text{the lexicographic products } M \times \mathbb{P} \text{ and } \mathbb{P} \times M \text{ are paracompact,} \]

\[ \text{the lexicographic product } S^\gamma \text{ is paracompact for every ordinal } \gamma, \]

where \( \mathbb{P}, M, S \) and \( \{0,1\}_\mathbb{R} \) denote the irrationals, the Michael line, the Sorgenfrey line and the interval \( [0,1) \) in the reals \( \mathbb{R} \), respectively.

1. Introduction

We assume all topological spaces have cardinality at least 2.

A linearly ordered set \( \langle X, <_X \rangle \) (see [1]) has a natural \( T_2 \)-topology denoted by \( \lambda_X \) or \( \lambda(<_X) \) so called the interval topology which is the topology generated by \( \{(+_X, x) : x \in X\} \cup \{(+_X, \to_X) : x \in X\} \) as a subbase, where \( (+_X, \to_X) = \{w \in X : x <_X w\}, (x, y)_X = \{w \in X : x <_X w \leq_X y\}, \ldots \) etc. Here \( w \leq_X x \) means \( w <_X x \) or \( w = x \). If the contexts are clear, we simply write \( < \) and \( (x, y) \) instead of \( <_X \) and \( (x, y)_X \) respectively. Note that this subbase induces a base by convex subsets (\( \text{e.g., the collection of all intersections of at most two members of this subbase} \)), where a subset \( B \) of \( X \) is convex if for every \( x, y \in B \) with \( x <_X y \), \( [x,y]_X \subset B \) holds. The triple \( \langle X, <_X, \lambda_X \rangle \) is called a LOTS (\( \text{= Linearly Ordered Topological Space} \)) and simply denoted by LOTS \( X \). Observe that if \( x \in U \in \lambda_X \) and \( (+_X, \neq) \neq \emptyset \), then there is

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Let \( y \in X \) such that \( y < x \) and \( (y,x) \subseteq U \). Note that for every \( x \in X \), \((\leftarrow, x]\) \( \notin \lambda_X \) iff \((x, \rightarrow) \) is non-empty and has no minimum (briefly, min), also analogously \([x, \rightarrow) \) \( \notin \lambda_X \) iff \((\leftarrow, x] \) is non-empty and has no max.

Let

\[
X_R = \{ x \in X : (\leftarrow, x] \notin \lambda_X \} \quad \text{and} \quad X_L = \{ x \in X : [x, \rightarrow) \notin \lambda_X \}.
\]

Unless otherwise stated, the real line \( \mathbb{R} \) is considered as a linearly ordered set (hence LOTS) with the usual order, similarly so are the set \( \mathbb{Q} \) of rationals, the set \( \mathbb{P} \) of irrationals and an ordinal \( \alpha \).

A \textit{generalized ordered space} \((= \text{GO-space})\) is a triple \( (X, <_X, \tau_X) \), where \( <_X \) is linear order on \( X \) and \( \tau_X \) is a \( T_2 \) topology on \( X \) which has a base consisting of convex sets, also simply denoted by GO-space \( X \). For LOTS’s and GO-spaces, see also the nice text book \([5]\). It is easy to verify that \( \tau_X \) is a GO-space.

Let \( X \) be a GO-space and \( \lambda_X \) the restricted order of \( X \). Then the GO-space \( X \) has max, in this case, \( \max X = \max X^* \).
Lexicographic Products of Go-Spaces

Note $\mathbb{S}^* = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$ with the identification $\mathbb{S} = \mathbb{R} \times \{0\}$ and $\mathbb{M}^* = \mathbb{P} \times \{-1\} \cup \mathbb{R} \times \{0\} \cup \mathbb{P} \times \{1\}$ with the identification $\mathbb{M} = \mathbb{R} \times \{0\}$.

**Definition 1.1.** Let $X_\alpha$ be a LOTS for every $\alpha < \gamma$ and $X = \prod_{\alpha < \gamma} X_\alpha$, where $\gamma$ is an ordinal. When $\gamma = 0$, we consider as $\prod_{\alpha < \gamma} X_\alpha = \{\emptyset\}$, which is a trivial LOTS, for notational conveniences. When $\gamma > 0$, every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma \rangle$. Recall that the lexicographic order $<_X$ on $X$ is defined as follows: for $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma, x \upharpoonright \alpha = x' \upharpoonright \alpha \text{ and } x(\alpha) < x'(\alpha),$$

where $x \upharpoonright \alpha = \langle x(\beta) : \beta < \alpha \rangle$. Then $X = \langle X, <_X, \lambda_X \rangle$ is a LOTS and called the lexicographic product of LOTS’s $X_\alpha$’s.

Now let $X_\alpha$ be a GO-space for every $\alpha < \gamma$ and $X = \prod_{\alpha < \gamma} X_\alpha$. Then the lexicographic product $\hat{X} = \prod_{\alpha < \gamma} X_\alpha^*$, which is a LOTS, can be defined. The lexicographic product of GO-spaces $X_\alpha$’s is the GO-space $\langle X, <_X \upharpoonright X, \lambda_X \upharpoonright X \rangle$. Obviously this definition extends the lexicographic product of LOTS’s, and is reasonable because each $X_\alpha^*$ is the smallest LOTS which contains $X_\alpha$ as a dense subspace, see [4]. When $n \in \omega$, then $\prod_{\alpha < n} X_\alpha$ is denoted by $X_0 \times \cdots \times X_{n-1}$. If all $X_\alpha$’s are $X$, then $\prod_{\alpha < \gamma} X_\alpha$ is denoted by $X^\gamma$.

Let $X$ and $Y$ be LOTS’s. A map $f : X \to Y$ is said to be $0$-order preserving if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f : X \to Y$ is said to be $1$-order preserving if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a $0$-order preserving map $f : X \to Y$ between LOTS’s $X$ and $Y$, which is onto, is a homeomorphism, i.e., both $f$ and $f^{-1}$ are continuous. But when $X = \mathbb{S}$ and $Y = \mathbb{M}$, the identity map is $0$-order preserving onto but not a homeomorphism.

So now let $X$ and $Y$ be GO-spaces. A $0$-order preserving map $f : X \to Y$ is said to be embedding if $f$ is a homeomorphism between $X$ and $f[X]$, where $f[X]$ is the subspace of the GO-space $Y$. In this case, we can identify $X$ with $f[X]$ as a GO-space. In the definition of $X^*$, the map $f : X \to X \times \{0\} \subset X^*$ defined by $f(x) = \langle x, 0 \rangle$ is a $0$-order preserving embedding, so we have identified as $X \times \{0\} = X$.

In the rest of this section, we prepare basic tools to handle the lexicographic products of GO-spaces.

**Lemma 1.2.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces and $x \in X$. The following are equivalent:

1. $x \in X^+$,
2. there is $\alpha_0 < \gamma$ such that:
(i) \( x(\alpha_0) \in X^+_{\alpha_0} \),
(ii) for every \( \alpha < \gamma \) with \( \alpha_0 < \alpha \), \( X_\alpha \) has max and \( x(\alpha) = \max X_\alpha \).

**Proof.** Let \( \hat{X} = \prod_{\alpha < \gamma} X^*_\alpha \) be the lexicographic product.

(1) \( \Rightarrow \) (2): Assume \( x \in X^+ \). Because of \( (\leftarrow, x)_X \notin \lambda_X, (x, \rightarrow)_X \) is non-empty and has no min. By \( (\leftarrow, x)_X \in \tau_X = \lambda_x \upharpoonright X \), there is \( y \in \hat{X} \) with \( x <_X y \) such that \( (\leftarrow, x)_X \supseteq [x, y)_X \cap X \), that is, \( (x, y)_X = \emptyset \). Since \( (x, \rightarrow)_X \) has no min, we have \( y \in \hat{X} \setminus X \). Let \( \alpha_0 = \min \{ \alpha < \gamma : x(\alpha) \neq y(\alpha) \} \). Then we have \( x \upharpoonright \alpha_0 = y \upharpoonright \alpha_0 \) and \( x(\alpha_0) <_{X^*_\alpha_0} y(\alpha_0) \). Since \( X_{\alpha_0} \) is dense in \( X^*_\alpha_0 \), \( (x(\alpha_0), \rightarrow)_{X_{\alpha_0}} \) is non-empty.

**Claim 1.** For every \( \alpha < \gamma \) with \( \alpha_0 < \alpha \), \( X_\alpha \) has max and \( x(\alpha) = \max X_\alpha \).

**Proof.** First assume that for some \( \alpha < \gamma \) with \( \alpha_0 < \alpha \), \( X_\alpha \) has no max. Then we can take \( v \in X_\alpha \) with \( x(\alpha) <_{X_\alpha} v \). Set \( x' = (x \upharpoonright \alpha) \wedge (v \upharpoonright (\alpha, \gamma)) \), that is,

\[
x'(\beta) = \begin{cases} 
  x(\beta) & \text{if } \beta < \alpha, \\
  v & \text{if } \beta = \alpha, \\
  x(\beta) & \text{if } \alpha < \beta < \gamma.
\end{cases}
\]

Then \( x' \in (x, y)_X \cap X \), a contradiction. Therefore for every \( \alpha < \gamma \) with \( \alpha_0 < \alpha \), \( \max X_\alpha \) exists.

Next assume that for some \( \alpha < \gamma \) with \( \alpha_0 < \alpha \), \( x(\alpha) <_{X_\alpha} \max X_\alpha \) holds. Then \( (x \upharpoonright \alpha) \wedge (\max X_\alpha) \wedge (x \upharpoonright (\alpha, \gamma)) \in (x, y)_X \cap X \), a contradiction. \( \square \)

**Claim 2.** \( (x(\alpha_0), y(\alpha_0))_{X_{\alpha_0}} = \emptyset \), therefore \( (\leftarrow, x(\alpha_0))_{X_{\alpha_0}} \in \tau_{X_{\alpha_0}} \).

**Proof.** Assume \( (x(\alpha_0), y(\alpha_0))_{X_{\alpha_0}} \neq \emptyset \). Since \( X_{\alpha_0} \) is dense in \( X^*_\alpha_0 \), take \( v \in (x(\alpha_0), y(\alpha_0))_{X_{\alpha_0}} \cap X_{\alpha_0} \). Then \( (x \upharpoonright \alpha_0) \wedge (v \wedge (x \upharpoonright (\alpha_0, \gamma))) \in (x, y)_X \cap X \), a contradiction. \( \square \)

The following claim shows \( x(\alpha_0) \in X^+_{\alpha_0} \).

**Claim 3.** \( (\leftarrow, x(\alpha_0))_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}} \).

**Proof.** Since \( x(\alpha_0) <_{X_{\alpha_0}} y(\alpha_0) \) and \( X_{\alpha_0} \) is dense in \( X^*_\alpha_0 \), we have \( (x(\alpha_0), \rightarrow)_{X_{\alpha_0}} \neq \emptyset \). Assume \( (\leftarrow, x(\alpha_0))_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}} \), then there is \( v \in X_{\alpha_0} \) such that \( x(\alpha_0) <_{X_{\alpha_0}} v \) and \( (x(\alpha_0), v)_{X_{\alpha_0}} = \emptyset \). Since \( (x(\alpha_0), v)_{X_{\alpha_0}} = \emptyset \), we have \( v = y(\alpha_0) \), thus \( y(x_0) \in X_{\alpha_0} \). Let \( \alpha_1 = \min \{ \alpha < \gamma : y(\alpha) \notin X_\alpha \} \). Because of \( y \notin Y \) and the definition of \( y \), we have \( \alpha_0 < \alpha_1 \). If
Assume $\langle x, v \rangle$ were empty, then $y(\alpha_1) = \min X_{\alpha_1}^* = \min X_{\alpha_1} \in X_{\alpha_1}$, a contradiction. Therefore we can take $v' \in (\langle - , y(\alpha_1) \rangle)_{X_{\alpha_1}^*} \cap X_{\alpha_1}$. Then $(y \uparrow \alpha_1)^\wedge (v' \wedge (x \uparrow (\alpha_1, \gamma))) \in (x, y, x, y)_{\check{X}} \cap X$, a contradiction.

$(2) \Rightarrow (1)$: Assume $(2)$. By (i), we can take $v \in X_{\alpha_0}^* \setminus X_{\alpha_0}$ such that $x(\alpha_0) \prec X_{\alpha_0}^* v$ and $(x(\alpha_0), v)'_{X_{\alpha_0}^*} = \emptyset$. Let $y = (x \uparrow \alpha_1)^\wedge (v' \wedge (x \uparrow (\alpha_1, \gamma)))$. Then we have $x < \check{X} y \in \check{X} \setminus X$ and $(x, \rightarrow)_X \neq \emptyset$. Obviously $(x, y)_{\check{X}} = \emptyset$ holds. Thus $(\langle - , x \rangle)_X = (\langle - , y \rangle)_{\check{X}} \cap X \subseteq \lambda_{\check{X}} \uparrow X = \tau_X$. The following Claim completes the proof.

**Claim 4.** $(\langle - , x \rangle)_X \notin \lambda_X$.

**Proof.** Assume $(\langle - , x \rangle)_X \in \lambda_X$. It follows from $(x, \rightarrow)_X \neq \emptyset$ that for some $x' \in X$ with $x < x', (x, x')_X = \emptyset$ holds. Let $\alpha_1 = \min \{ \alpha < \gamma : x'(\alpha) \neq x(\alpha) \}$. Then by $x(\alpha_1) \prec X_{\alpha_1}^* x'(\alpha_1)$, we have $\alpha_1 \leq \alpha_0$. Since $v \in (x(\alpha_0), \rightarrow)_{X_{\alpha_0}^*}$, we can take $u \in (x(\alpha_0), \rightarrow)_{X_{\alpha_0}^*}$. If $\alpha_1 < \alpha_0$ were true, then $(x \uparrow \alpha_0)^\wedge (u\wedge (x \uparrow (\alpha_0, \gamma))) \in (x, x')_X$, a contradiction. Thus we have $\alpha_1 = \alpha_0$.

Now by $(x(\alpha_0), v)'_{X_{\alpha_0}^*} = \emptyset$, we also have $v < X_{\alpha_0}^* x'(\alpha_0)$ moreover $(v, x'(\alpha_0))_{X_{\alpha_0}^*} \neq \emptyset$ (otherwise, $v$ is an isolated point in $X_{\alpha_0}^*$ and $v \notin X_{\alpha_0}$, a contradiction). Taking $w \in (v, x'(\alpha_0))_{X_{\alpha_0}^*} \cap X_{\alpha_0}$, we have $(x \uparrow \alpha_0)^\wedge (w\wedge (x \uparrow (\alpha_0, \gamma))) \in (x, x')_X$, a contradiction. $\square$

Similarly, we have an analogous result:

**Lemma 1.3.** Let $X = \prod_{\alpha < \gamma} X_{\alpha}$ be a lexicographic product of GO-spaces and $x \in X$. The following are equivalent:

1. $x \in X^-$,
2. there is $\alpha_0 < \gamma$ such that:
   1. $x(\alpha_0) \in X_{\alpha_0}^-$,
   2. for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $X_{\alpha}$ has min and $x(\alpha) = \min X_{\alpha}$.

From now on, we do not write down such an analogous result, we refer, for instance, Lemma 1.3 as the analogous result of Lemma 1.2.

**Corollary 1.4.** Let $X = \prod_{\alpha < \gamma} X_{\alpha}$ be a lexicographic product of GO-spaces. If $X_{\alpha}^+ = \emptyset$ for every $\alpha < \gamma$, then $X^+ = \emptyset$.

This corollary with the analogous result also shows that lexicographic products of LOTS’s are LOTS’s. However, lexicographic products of GO-spaces, some of which are not LOTS’s, can be LOTS’s. This fact will be discussed in the next section.
Now, let \( X = \prod_{\alpha < \gamma} X_{\alpha} \) be a lexicographic product of LOTS’s and \( \delta < \gamma \). For \( x \in X \), the correspondence \( x \rightarrow (x \upharpoonright \delta, x \upharpoonright [\delta, \gamma)) \) defines a 0-order preserving onto map from \( X \) to \( (\prod_{\alpha < \delta} X_{\alpha}) \times (\prod_{\delta \leq \alpha < \gamma} X_{\alpha}) \), which is a lexicographic product of two lexicographic products. So they are topologically homeomorphic, thus we can identify \( \prod_{\alpha < \gamma} X_{\alpha} \) with \( (\prod_{\alpha < \delta} X_{\alpha}) \times (\prod_{\delta \leq \alpha < \gamma} X_{\alpha}) \) as a LOTS whenever \( X_{\alpha} \)'s are LOTS’s, see [2].

Next, let \( X = \prod_{\alpha < \gamma} X_{\alpha} \) be a lexicographic product of GO-spaces and \( \delta < \gamma \). The correspondence above also defines a 0-order preserving onto map from \( X \) to \( (\prod_{\alpha < \delta} X_{\alpha}) \times (\prod_{\delta \leq \alpha < \gamma} X_{\alpha}) \). Is this map a homeomorphism between them? We show in the next lemma that the answer is positive, while the proof is not so trivial. It will be a key tool through the theory.

**Lemma 1.5.** Let \( X = \prod_{\alpha < \gamma} X_{\alpha} \) be a lexicographic product of GO-spaces and \( \delta < \gamma \). The correspondence \( x \rightarrow (x \upharpoonright \delta, x \upharpoonright [\delta, \gamma)) \) is a homeomorphism. So we can identify \( \prod_{\alpha < \gamma} X_{\alpha} \) with \( (\prod_{\alpha < \delta} X_{\alpha}) \times (\prod_{\delta \leq \alpha < \gamma} X_{\alpha}) \) as a GO-space.

**Proof.** Let \( Y_{0} = \prod_{\alpha < \delta} X_{\alpha} \) and \( Y_{1} = \prod_{\delta \leq \alpha < \gamma} X_{\alpha} \). We may identify the correspondence as \( x = (x \upharpoonright \delta, x \upharpoonright [\delta, \gamma)) \) for every \( x \in X \). By this identification, the order \( <_{X} \) coincides with the order \( <_{Y_{0} \times Y_{1}} \), where \( Y_{0} \times Y_{1} \) is the lexicographic product of the GO-spaces \( Y_{0} \) and \( Y_{1} \). It suffices to see \( \tau_{X} = \tau_{Y_{0} \times Y_{1}} \). Note that \( \tau_{X} = \lambda_{X} \upharpoonright X, \tau_{Y_{0}} = \lambda_{Y_{0}} \upharpoonright Y_{0}, \tau_{Y_{1}} = \lambda_{Y_{1}} \upharpoonright Y_{1} \) and \( \tau_{Y_{0} \times Y_{1}} = \lambda_{Y_{0} \times Y_{1}}^{*} \upharpoonright Y_{0} \times Y_{1} \) hold, where \( \hat{X} = \prod_{\alpha < \gamma} X_{\alpha}^{*}, \hat{Y}_{0} = \prod_{\alpha < \delta} X_{\alpha}^{*} \) and \( \hat{Y}_{1} = \prod_{\delta \leq \alpha < \gamma} X_{\alpha}^{*} \).

**Claim 1.** \( \tau_{X} \subseteq \tau_{Y_{0} \times Y_{1}} \).

**Proof.** It suffices to show that the subbase \( \{(\leftarrow, x)_{X} : x \in X\} \cup \{(x, \rightarrow)_{X} : x \in X\} \cup \{(\leftarrow, x)_{X} : x \in X^{+}\} \cup \{(x, \rightarrow)_{X} : x \in X^{-}\} \) is contained in \( \tau_{Y_{0} \times Y_{1}} \). Note under the identification, \( (\leftarrow, x)_{X} = (\leftarrow, x)_{Y_{0} \times Y_{1}}, (\leftarrow, x)_{X} = (\leftarrow, x)_{Y_{0} \times Y_{1}}, \ldots \), etc hold. Therefore, it only suffices to prove the following fact:

**Fact.** If \( x \in X^{+} (x \in X^{-}) \), then \( (\leftarrow, x)_{X} \in \tau_{Y_{0} \times Y_{1}} ([x, \rightarrow)_{X} \in \tau_{Y_{0} \times Y_{1}} \) respectively.

**Proof.** Let \( x \in X^{+} \). By Lemma 1.3, take \( \alpha_{0} < \gamma \) such that \( x(\alpha_{0}) \in X_{\alpha_{0}}^{+} \), and for every \( \alpha < \gamma \) with \( \alpha_{0} < \alpha \), \( x(\alpha) = \max X_{\alpha} = \max X_{\alpha}^{*} \) holds. We consider two cases.

**Case 1.** \( \alpha_{0} < \delta \).

In this case, again applying Lemma 1.2 to \( x \upharpoonright \delta \in Y_{0} \), we see \( x \upharpoonright \delta \in Y_{0}^{+} \). Therefore there is \( y_{0} \in Y_{0}^{*} \setminus Y_{0} \) such that \( x \upharpoonright \delta <_{Y_{0}^{*}} y_{0} \) and
As in Claim 1, it suffices to see that if \((x \upharpoonright \delta, y_0)_{Y_0^*} = \emptyset\), that is, \(y_0 = \langle x \upharpoonright \delta, 1 \rangle\). Let \(z = y_0 \wedge (x \upharpoonright [\delta, \gamma])\), then \(z \in Y_0^* \times Y_1 \subset Y_0^* \times Y_1^*\). Assume that there is an element \(u \in (x, z)Y_0^* \times Y_1^* \cap Y_0 \times Y_1\). Then we have \(x \upharpoonright \delta \leq Y_0 u \upharpoonright \delta\). If \(x \upharpoonright \delta = u \upharpoonright \delta\) were true, then \(x \upharpoonright [\delta, \gamma) < Y_1 u \upharpoonright [\delta, \gamma)\) has to be true. But this is a contradiction, because of \(x(\beta) = \max X_\beta\) for all \(\beta \geq \delta\). Therefore we have \(x \upharpoonright \delta < Y_0 u \upharpoonright \delta\). Since \(y_0 \notin Y_0\) and \((x \upharpoonright \delta, y_0)_{Y_0^*} = \emptyset\), we see \(z \upharpoonright \delta = y_0 < Y_1^* u \upharpoonright \delta\). Thus we have \(z < Y_0^* \times Y_1^*, z\) which contradicts \(u < Y_0^* \times Y_1^* z\), so we have seen \((x, z)_{Y_0^* \times Y_1^*} \cap (Y_0 \times Y_1) = \emptyset\). This shows \((\leftarrow, x)_{Y_0 \times Y_1} = (\leftarrow, z)_{Y_0^* \times Y_1^*} \cap Y_0 \times Y_1 \in \lambda_{Y_0^* \times Y_1^*} [\tau_{Y_0^* \times Y_1^*}] Y_0 \times Y_1 = \tau_{Y_0 \times Y_1}^\times\).

**Case 2.** \(\delta \leq \alpha_0\).

Applying Lemma 1.2 to \(Y_1\), we see \(x \upharpoonright [\delta, \gamma) \in Y_1^+\). Therefore, there is \(y_1 \in Y_1^* \setminus Y_1\) such that \(x \upharpoonright [\delta, \gamma) < Y_1^* y_1\) and \((x \upharpoonright [\delta, \gamma), y_1)_{Y_1^*} = \emptyset\). Then by \((x, x \upharpoonright \delta)^\wedge y_1)_{Y_0^* \times Y_1^*} = \emptyset\), we have \((\leftarrow, x)_{Y_0 \times Y_1} = (\leftarrow, z)_{Y_0^* \times Y_1^*} \cap Y_0 \times Y_1 \in \tau_{Y_0 \times Y_1}^\times\).

This completes the proof of Claim 1.

**Claim 2.** \(\tau_X \supset \tau_{Y_0 \times Y_1}^\times\).

**Proof.** As in Claim 1, it suffices to see that if \(x \in (Y_0 \times Y_1)^+ ( x \in (Y_0 \times Y_1)^-\), then \((\leftarrow, x)_{Y_0 \times Y_1} \in \tau_X ((x, \rightarrow)_{Y_0 \times Y_1} \in \tau_X\), respectively. Let \(x \in (Y_0 \times Y_1)^\pm\), say \(x_0 = x \upharpoonright \delta\) and \(x_1 = x \upharpoonright [\delta, \gamma)\). Apply Lemma 1.2 to \(x \in (Y_0 \times Y_1)^\pm\), we can find \(i_0 < 2\), where \(2 := \{0, 1\}\), such that \(x_{i_0} \in Y_{i_0}^+\) and for every \(i < 2\) with \(i_0 < i\), \(x_i = \max Y_i (= \max Y_i^*)\) holds.

**Case 1.** \(i_0 = 0\).

It follows from \(x_0 \in Y_0^+\) that for some \(z_0 \in Y_0^* \setminus Y_0\) with \(x_0 \leq_{Y_0^*} z_0\), \((x_0, z_0)_{Y_0^*} \) is empty. By \(x \upharpoonright [\delta, \gamma) = x_1 = \max Y_1\), we have \(x(\alpha) = \max X_\alpha\) for every \(\alpha < \gamma\) with \(\delta \leq \alpha\). It follows from \(\lambda_{Y_1^*} \upharpoonright \delta = Y_0 = \lambda_{Y_0^*} \upharpoonright \delta = x_0 \in Y_0^+\), applying Lemma 1.2, that for some \(\alpha_0 < \delta\), \(x(\alpha_0) \in X_{\alpha_0}^+\) and for every \(\alpha < \delta\) with \(\alpha_0 < \alpha\), \(x(\alpha) = \max X_\alpha\) hold. Since \(x(\alpha_0) \in X_{\alpha_0}^+\) and for every \(\alpha < \gamma\) with \(\alpha_0 < \alpha\), \(x(\alpha) = \max X_\alpha\) hold, applying Lemma 1.2 again, we have \(x \in X^+\). Thus we have \((\leftarrow, x)_{Y_0 \times Y_1} = (\leftarrow, x)_{X^+} \in \tau_X^\times\).

**Case 2.** \(i_0 = 1\).

In this case, \(x \upharpoonright [\delta, \gamma) = x_1 \in Y_1^+\). So applying Lemma 1.2, there is \(\alpha_0 < \gamma\) with \(\delta \leq \alpha_0\) such that \(x(\alpha_0) \in X_{\alpha_0}^+\) and for every \(\alpha < \gamma\) with \(\alpha_0 < \alpha\), \(x(\alpha) = \max X_\alpha\) holds. Again by Lemma 1.2, we have \((\leftarrow, x)_{Y_0 \times Y_1} = (\leftarrow, x)_{X} \in \tau_X^\times\).

The remaining case is similar. \(\square\)
This completes the proof of the lemma. □

2. When are lexicographic products of GO-spaces LOTS’s?

It is easy to verify that the lexicographic product \( S \times \mathbb{R} \) is a LOTS, while \( S \) is not a LOTS. In this section, we characterize when lexicographic products of GO-spaces are LOTS’s. Using Lemma 1.2, the following is easy to prove.

**Lemma 2.1.** Let \( X = X_0 \times X_1 \) be a lexicographic product of GO-spaces. Then the following are equivalent:

1. \( X^+ = \emptyset \) (\( X^- = \emptyset \)),
2. (i) if \( X_1 \) has max (min), then \( X_0^+ = \emptyset \) (\( X_0^- = \emptyset \)),
   (ii) \( X_1^+ = \emptyset \) (\( X_1^- = \emptyset \)).

The previous lemma shows:

**Lemma 2.2.** Let \( X = X_0 \times X_1 \) be a lexicographic product of GO-spaces. Then the following are equivalent:

1. \( X \) is a LOTS,
2. (i) if \( X_1 \) has max, then \( X_0^+ = \emptyset \),
   (ii) if \( X_1 \) has min, then \( X_0^- = \emptyset \),
   (iii) \( X_1 \) is a LOTS.

**Corollary 2.3.** Let \( X = X_0 \times X_1 \) be a lexicographic product of GO-spaces. Then:

1. if \( X_1 \) has neither min nor max, then \( X \) is a LOTS iff \( X_1 \) is a LOTS,
2. if \( X_1 \) has min (max) but has no max (min), then \( X \) is a LOTS iff \( X_0 = \emptyset \) (\( X_0^- = \emptyset \)) and \( X_1 \) is a LOTS,
3. if \( X_1 \) has both min and max, then \( X \) is a LOTS iff both \( X_0 \) and \( X_1 \) are LOTS’s.

**Example 2.4.** \( S \times \mathbb{R} \), \( S \times [0, 1] \), \( M \times P \) are LOTS’s. But \( \mathbb{R} \times S \), \( S \times (0, 1] \), \( S \times \{0, 1\} \), \( S \times [0, 1] \), \( S^2 \), \( P \times M \) are not LOTS’s.

More generally we have:

**Theorem 2.5.** Let \( X = \prod_{\alpha < \gamma} X_\alpha \) be a lexicographic product of GO-spaces. Let \( J^+ = \{ \alpha < \gamma : X_\alpha \) has no max.\} \ and \( J^- = \{ \alpha < \gamma : X_\alpha \) has no min.\}. Then the following are equivalent:

1. \( X^+ = \emptyset \) (\( X^- = \emptyset \)),
2. for every \( \alpha < \gamma \) with sup \( J^+ \leq \alpha \) (sup \( J^- \leq \alpha \), \( X_\alpha^+ = \emptyset \) (\( X_\alpha^- = \emptyset \)) holds.
Proof. Let $\alpha_0 = \sup J^+$. Note $\alpha_0 \leq \gamma$.

(1) $\Rightarrow$ (2): Let $X^+ = \emptyset$ and $\alpha_0 \leq \beta < \gamma$. Since $X = \prod_{\alpha \leq \beta} X_\alpha \times \prod_{\beta < \alpha < \gamma} X_\alpha$ and $\prod_{\beta < \alpha < \gamma} X_\alpha$ has max, by Lemma 2.1, $(\prod_{\alpha \leq \beta} X_\alpha)^+ = \emptyset$ holds. Moreover by $\prod_{\alpha \leq \beta} X_\alpha = \prod_{\alpha < \beta} X_\alpha \times X_\beta$, again by Lemma 2.1, we have $X^+_\beta = \emptyset$.

(2) $\Rightarrow$ (1): Assume that $X^+_\alpha = \emptyset$ for every $\alpha < \gamma$ with $\alpha_0 < \alpha$. If $\alpha_0 = 0$, then by Cororally 1.4, we have $X^+ = \emptyset$. So we assume $\alpha_0 > 0$.

**Case 1.** $\alpha_0 \in J^+$.

In this case, $\alpha_0 = \max J^+ < \gamma$. Since $\prod_{\alpha < \alpha_0} X_\alpha = \prod_{\alpha < \alpha_0} X_\alpha \times X_{\alpha_0}$, $X_{\alpha_0}$ has no max and $X^+_{\alpha_0} = \emptyset$, by Lemma 2.1, $(\prod_{\alpha_0 < \alpha < \gamma} X_\alpha)^+$ is empty. It follows from Corollary 1.4 that $(\prod_{\alpha_0 < \alpha < \gamma} X_\alpha)^+$ is also empty. Because of $X = \prod_{\alpha < \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha < \gamma} X_\alpha$, by the same corollary, we have $X^+ = \emptyset$.

**Case 2.** $\alpha_0 \notin J^+$.

In this case, $\alpha_0$ is a limit ordinal with $\alpha_0 \leq \gamma$.

**Claim.** $(\prod_{\alpha < \alpha_0} X_\alpha)^+ = \emptyset$.

**Proof.** If there were $x \in (\prod_{\alpha < \alpha_0} X_\alpha)^+$, then by Lemma 1.2, there is some $\alpha_1 < \alpha_0$ such that for every $\alpha < \alpha_0$ with $\alpha_1 < \alpha$, max $X_\alpha$ exists. This means sup $J^+ \leq \alpha_1 < \alpha_0$, a contradiction. $\square$

By $X = \prod_{\alpha < \alpha_0} X_\alpha \times \prod_{\alpha_0 \leq \alpha < \gamma} X_\alpha$ and the assumption $(\prod_{\alpha_0 \leq \alpha < \gamma} X_\alpha)^+ = \emptyset$, we have $X^+ = \emptyset$.

The remaining is similar. $\square$

**Corollary 2.6.** Under the same assumption of Theorem 2.5, $X$ is a LOTS if and only if the following hold:

1. for every $\alpha < \gamma$ with sup $J^+ \leq \alpha$, $X^+_\alpha = \emptyset$ holds,
2. for every $\alpha < \gamma$ with sup $J^- \leq \alpha$, $X^-_\alpha = \emptyset$ holds,

**Corollary 2.7.** Let $X = \prod_{\alpha \leq \gamma} X_\alpha$ be a lexicographic product of GO-spaces. Assume that $X_\gamma$ has neither min nor max. Then $X$ is a LOTS if and only if $X_\gamma$ is a LOTS. In particular, $\prod_{\alpha < \gamma} X_\alpha \times \mathbb{R}$ is a LOTS.

Above two corollaries show:

**Corollary 2.8.** For every non-zero ordinal $\gamma$, $S^\gamma$ is a LOTS if and only if $\gamma$ is limit.
3. WHEN IS $\prod_{\alpha<\gamma} X_{\alpha}$ DENSE IN $\prod_{\alpha<\gamma} X_{\alpha}^*$?

A GO-space $X$ is dense in the LOTS $X^*$, but generally a lexicographic product $X_0 \times X_1$ of GO-spaces need not be dense in $X_0^* \times X_1^*$. For instance, let $X_0 = [0, 1]_\mathbb{R} \cup [2, 3]_\mathbb{R}$ be the subspace of $\mathbb{R}$ and $X_1 = [0, 1]_\mathbb{R}$. Then $X_0^*$ can be considered as the subspace $[0, 1]_\mathbb{R} \cup [2, 3]_\mathbb{R}$ of $\mathbb{R}$ and obviously $X_1^* = X_1$. Now $(1, 0), (1, 1) \in X_0^* \times X_1^*$ is non-empty open in $X_0^* \times X_1^*$ but disjoint from $X_0 \times X_1$.

First we consider a special case.

**Lemma 3.1.** Let $X = X_0 \times X_1$ be a lexicographic product of GO-spaces and let $\tilde{x} = X_0^* \times X_1^*$. If $X_0$ is a LOTS, then $X$ is dense in $\tilde{x}$.

**Proof.** Let $X_0$ be a LOTS. First we prove:

**Claim 1.** If $x \in \tilde{x}$ and $(x, \to)_{\tilde{x}} \neq \emptyset$, then $(x, \to)_{\tilde{x}} \cap X \neq \emptyset$.

**Proof.** If $(x(0), \to)_{X_0^*} \neq \emptyset$, then pick $u \in (x(0), \to)_{X_0^*} \cap X_0$ and $v \in X_1$. Then $\langle u, v \rangle \in (x, \to)_{\tilde{x}} \cap X$. So let $\langle x(0), \to \rangle_{X_0^*} = \emptyset$, that is, $x(0) = \max X_0$. Take $y \in (x, \to)_{\tilde{x}}$. Then $x(0) = y(0)$ and $y(1) \in (x(1), \to)_{X_1^*}$. Since $X_1$ is dense in $X_1^*$, we can find $v \in (x(1), \to)_{X_1^*} \cap X_1$. Now we have $\langle x(0), v \rangle \in (x, \to)_{\tilde{x}} \cap X$.\hfill $\Box$

Analogously, we can prove:

**Claim 2.** If $x \in \tilde{x}$ and $(\leftarrow, x)_{\tilde{x}} \neq \emptyset$, then $(\leftarrow, x)_{\tilde{x}} \cap X \neq \emptyset$.

These two claims with the following claim complete the proof.

**Claim 3.** If $x, x' \in \tilde{x}$, $x < x'$ and $(x, x')_{\tilde{x}} \neq \emptyset$, then $(x, x')_{\tilde{x}} \cap X \neq \emptyset$.

**Proof.** Let $x, x' \in \tilde{x}$, $x < x'$ and $(x, x')_{\tilde{x}} \neq \emptyset$. Since $X_0$ is a LOTS, that is $X_0 = X_0^*$, we have $x(0), x'(0) \in X_0$.

**Case 1.** $x(0) = x'(0)$.

In this case, take $y \in (x, x')_{\tilde{x}}$. Then we have $x(0) = x'(0) = y(0)$ and $y(1) \in (x(1), x'(1))_{X_1^*}$. Since $X_1$ is dense in $X_1^*$, there is $v \in (x(1), x'(1))_{X_1^*} \cap X_1$. Now $\langle x(0), v \rangle \in (x, x')_{\tilde{x}} \cap X$.

**Case 2.** $x(0) < x'(0)$.

First assume $(x(0), x'(0))_{X_0} \neq \emptyset$. In this case, pick $u \in (x(0), x'(0))_{X_0}$ and $v \in X_1$. Then $\langle u, v \rangle \in (x, x')_{\tilde{x}} \cap X$.

Next assume $(x(0), x'(0))_{X_0} = \emptyset$. Since $(x, x')_{\tilde{x}} \neq \emptyset$, we have either $(x(1), \to)_{X_1^*} \neq \emptyset$ or $(\leftarrow, x'(1))_{X_1^*} \neq \emptyset$. In the case $(x(1), \to)_{X_1^*} \neq \emptyset$, taking $v \in (x(1), \to)_{X_1^*} \cap X_1$, we see $\langle x(0), v \rangle \in (x, x')_{\tilde{x}} \cap X$. In the case $(\leftarrow, x'(1))_{X_1^*} \neq \emptyset$, taking $v \in (\leftarrow, x'(1))_{X_1^*} \cap X_1$, we see $\langle x'(0), v \rangle \in (x, x')_{\tilde{x}} \cap X$.\hfill $\Box$
Theorem 3.2. Let \( X = \prod_{\alpha < \gamma} X_\alpha \) be a lexicographic product of GO-spaces. Then \( X \) is dense in \( \hat{X} = \prod_{\alpha < \gamma} X_\alpha^* \) if and only if for every \( \alpha < \gamma \) with \( \alpha + 1 < \gamma \), \( X_\alpha \) is a LOTS.

Proof. First assume that \( X \) is dense in \( \hat{X} \) and there is \( \alpha_0 < \gamma \) with \( \alpha_0 + 1 < \gamma \) such that \( X_{\alpha_0} \) is not a LOTS. We may assume \( X_{\alpha_0} \neq \emptyset \), so fix \( u \in X_{\alpha_0}^+ \) and take \( u' \in X_{\alpha_0}^+ \setminus X_{\alpha_0} \) such that \( u <_{X_{\alpha_0}^*} u' \) and \( (u, u')_{X_{\alpha_0}^*} = \emptyset \). Fix \( x \in X \).

Case 1. \( |\prod_{\alpha_0 < \alpha < \gamma} X_\alpha| > 2 \).

Take \( v_0, v_1, v_2 \in \prod_{\alpha_0 < \alpha < \gamma} X_\alpha \) with \( v_0 < v_1 < v_2 \). Let \( x_i = (x \upharpoonright \alpha_0)\langle u \rangle^i \upharpoonright v_i \) for \( i = 0, 1, 2 \). Then \( x_1 \in (x_0, x_2)_X \) but \( (x_0, x_2)_X \cap X = \emptyset \), a contradiction.

Case 2. \( |\prod_{\alpha_0 < \alpha < \gamma} X_\alpha| = 2 \).

In this case, note \( \gamma = \alpha_0 + 2 \) and \( \prod_{\alpha_0 < \alpha < \gamma} X_\alpha = X_{\alpha_0 + 1} \), say \( X_{\alpha_0 + 1} = \{v_0, v_1\} \) with \( v_0 < v_1 \). Let \( x_0 = (x \upharpoonright \alpha_0)\langle u \rangle^0 \upharpoonright v_1 \) and \( x_1 = (x \upharpoonright \alpha_0)\langle u \rangle^1 \upharpoonright v_1 \). Then \( (x \upharpoonright \alpha_0)\langle u \rangle^0 v_0 \in (x_0, x_1)_X \) but \( (x_0, x_1)_X \cap X = \emptyset \), a contradiction.

Next assume that for every \( \alpha < \gamma \) with \( \alpha + 1 < \gamma \), \( X_\alpha = X_\alpha^* \) holds. If \( \gamma \) is limit, then \( \prod_{\alpha < \gamma} X_\alpha = \prod_{\alpha < \gamma} X_\alpha^* \). If \( \gamma = \delta + 1 \), then \( \prod_{\alpha < \delta} X_\alpha \) is a LOTS. Therefore by the lemma above, \( X \) is dense in \( \hat{X} \).

Corollary 3.3. Let \( X = \prod_{\alpha < \gamma} X_\alpha \) be a lexicographic product of GO-spaces. Then:

1. if \( \gamma \) is limit, then \( X \) is dense in \( \hat{X} = \prod_{\alpha < \gamma} X_\alpha^* \) if and only if \( X = \hat{X} \).
2. if \( \gamma = \delta + 1 \), then \( X \) is dense in \( \hat{X} \) if and only if \( \prod_{\alpha < \delta} X_\alpha \) is a LOTS.

Note that the reverse implication of Lemma 3.1 is also true.

Example 3.4. For instance, we see:

- \( S \times X \) is not dense in \( S^* \times X \) for every GO-space \( X \).
- \( X \times S \) is dense in \( X \times S^* \) if \( X \) is a LOTS.
- \( P \times M \) is dense in \( P \times M^* \) but \( M \times P \) is not dense in \( M^* \times P \).

4. Paracompactness of lexicographic products

It is known that lexicographic products of paracompact LOTS’s are paracompact. In this section, we extend this result for paracompact GO-spaces.
Definition 4.1. Let $X$ be a GO-space. A subset $A$ of $X$ is called an initial segment or a 0-segment of $X$ if for every $x, x' \in X$ with $x \leq x'$, if $x' \in A$, then $x \in A$. Similarly a subset $A$ of $X$ is called a final segment or a 1-segment of $X$ if for every $x, x' \in X$ with $x \leq x'$, if $x \in A$, then $x' \in A$. Both $\emptyset$ and $X$ are 0-segments and 1-segments.

Let $A$ be a 0-segment of a GO-space $X$. A subset $U$ of $A$ is 0-unbounded in $A$ if for every $x \in A$, there is $x' \in U$ such that $x < x'$. Let

$$0 \text{-} \text{cf}_X A = \min \{|U| : U \text{ is 0-unbounded in } A\}.$$  

Similar notions are also defined in linearly ordered compactifications, see [3]. If the context is clear, $0 \text{-} \text{cf}_X A$ is denoted by $0 \text{-} \text{cf} A$. Obviously $A = \emptyset$ iff $0 \text{-} \text{cf} A = 0$, and $A$ has max iff $0 \text{-} \text{cf} A = 1$. Moreover we can easily check that a 0-segment $A$ has no max iff $0 \text{-} \text{cf} A \neq 0$, and in this case, $0 \text{-} \text{cf} A$ is a regular cardinal. Also remark:

- if $A$ is a 0-segment of a GO-space $X$ having no max, then $A$ is open in $X$, because of $A = \bigcup_{a \in A} (\leftarrow, a)_X$.
- if $U$ is a 0-unbounded subset of a 0-segment $A$ of a GO-space $X$, then we can define, by induction, a 0-order preserving sequence $\{x_\alpha : \alpha < \kappa\} \subset U$ (i.e., $x_\alpha < x_\alpha'$ whenever $\alpha < \alpha' < \kappa$) which is also 0-unbounded in $A$, where $\kappa = 0 \text{-} \text{cf} A$.

Analogous concepts such as 1-unbounded, 1- cf $A$, etc, are also defined.

A cut of a GO-space $X$ is a pair $\langle A_0, A_1 \rangle$ of subsets of $X$ such that $A_1 = X \setminus A_0$ and $A_0$ is a 0-segment (equivalently $A_1$ is a 1-segment). A cut $\langle A_0, A_1 \rangle$ is said to be a gap if $A_0$ has no max and $A_1$ has no min. Thus if $X$ has no max, then $\langle X, \emptyset \rangle$ is a gap. Remark that if $\langle A_0, A_1 \rangle$ is a gap, then both $A_0$ and $A_1$ are clopen in $X$. A cut $\langle A_0, A_1 \rangle$ is said to be a pseudo-gap if either “$A_0$ has max and $A_1$ has no min” or “$A_0$ has no max and $A_1$ has min”, moreover $A_0$ (equivalently $A_1$) is clopen in $X$.

The following is known:

Lemma 4.2 ([2], Theorem 2.4.6). Let $X$ be a GO-space, then the following are equivalent:

1. $X$ is paracompact,
2. for each gap and pseudo-gap $\langle A_0, A_1 \rangle$ of $X$ and for each $i \in 2$, there is a closed discrete $i$-unbounded subset of $A_i$.

Note that in the notations above:

- if $A_0 = \emptyset$, then $\emptyset$ is a closed discrete 0-unbounded subset of $A_0$,
- if $A_0$ has max, then the one element set $\{\max A_0\}$ is a closed discrete 0-unbounded subset of $A_0$.  

if $\operatorname{cf} A_0 = \omega$, then every 0-unbounded 0-order preserving sequence $\{a_n : n \in \omega\}$ in $A_0$ is closed discrete in $A_0$.

**Definition 4.3.** A GO-space $X$ is said to be 0-paracompact if for every closed 0-segment $A$ of $X$ with $0-\operatorname{cf} A \geq \omega_1$, say $\kappa = 0-\operatorname{cf} A$, there is a 0-unbounded closed discrete subset of $A$. In this case, we can take a 0-order preserving sequence $\{a_\alpha : \alpha < \kappa\}$ in $A$ which is 0-unbounded and closed discrete in $A$ (equivalently, closed discrete in $X$). 1-paracompactness is defined analogously.

Now with the consideration above, Lemma 4.2 says the following:

**Lemma 4.4.** A GO-space is paracompact if and only if it is both 0-paracompact and 1-paracompact.

Remark that Lemma 1.2 says something about pseudo-gaps in lexicographic products. On the other hand, the following says about gaps of lexicographic products.

**Lemma 4.5.** Let $X = \prod_{\alpha < \gamma} X_\alpha$ be a lexicographic product of GO-spaces. Assume that $A$ is a 0-segment with $0-\operatorname{cf} A \geq \omega$ and $1-\operatorname{cf}(X \setminus A) \geq \omega$, that is, $(A, X \setminus A)$ is a gap with $A \neq \emptyset$ and $X \setminus A \neq \emptyset$. Say $\kappa = 0-\operatorname{cf} A$, then there are $\alpha_0 < \gamma$, $y_0 \in Y_0 := \prod_{\alpha < \alpha_0} X_\alpha$ and a 0-segment $A_0$ of $X_{\alpha_0}$ such that:

1. for every $a \in A$, $a \upharpoonright \alpha_0 \leq_{Y_0} y_0$ holds,
2. for every $x \in X$,
   (i) if $x \upharpoonright \alpha_0 \leq_{Y_0} y_0$, then $x \in A$ holds,
   (ii) if $x \upharpoonright \alpha_0 >_{Y_0} y_0$, then $x \in X \setminus A$ holds,
3. for every $x \in X$ with $x \upharpoonright \alpha_0 = y_0$, $x(\alpha_0) \in A_0$ holds iff so does $x \in A$,
4. if $A_0$ is non-empty and has no max, then $\kappa = 0-\operatorname{cf}_{X_{\alpha_0}} A_0$,
5. if $A_0$ is non-empty and has max, then there is $\alpha > \alpha_0$ such that $X_\alpha$ has no max and $\kappa = 0-\operatorname{cf}_{X_{\alpha}} X_{\alpha_1}$ holds, where $\alpha_1 := \min\{\alpha < \gamma : \alpha > \alpha_0 \text{ and } X_\alpha \text{ has no max}\}$,
6. if $A_0$ is empty, then:
   (i) for every $a \in A$, $a \upharpoonright \alpha_0 <_{Y_0} y_0$ holds,
   (ii) $\alpha_0$ is limit,
   (iii) there is $\alpha \geq \alpha_0$ such that $X_\alpha$ has no min.
   (iv) $A = (\leftarrow, y_0)_{Y_0} \times Y_1$, where $Y_1 := \prod_{\alpha \leq \alpha < \gamma} X_\alpha$.
   (v) $\leftarrow, y_0)_{Y_0}$ has no max,
   (vi) $\kappa = 0-\operatorname{cf}_{Y_0}(\leftarrow, y_0)_{Y_0} = \operatorname{cf} \alpha_0$.
   (vii) for every $\beta < \alpha_0$, there is $a \in A$ satisfying $\beta < \min\{\alpha < \alpha_0 : a(\alpha) \neq y_0(\alpha)\}$. 

Proof. Set \( B = X \setminus A \). For each \( a \in A \) and \( b \in B \), let \( \alpha(a, b) = \min \{ \alpha < \gamma : a(\alpha) \neq b(\alpha) \} \) and \( \alpha_0 = \sup \{ \alpha(a, b) : a \in A, b \in B \} \). Note \( \alpha_0 \leq \gamma \).

**Claim 1.** Let \( a_0, a_1 \in A \) and \( b_0, b_1 \in B \). If \( \alpha(a_0, b_0) \leq \alpha(a_1, b_1) \), then \( a_0 \upharpoonright \alpha(a_0, b_0) = a_1 \upharpoonright \alpha(a_0, b_0) \).

**Proof.** Assume that there is \( \beta < \alpha(a_0, b_0) \) such that \( a_0(\beta) \neq a_1(\beta) \). Let \( \beta_0 = \min \{ \beta < \alpha(a_0, b_0) : a_0(\beta) \neq a_1(\beta) \} \). Then \( b_0 \upharpoonright \beta_0 = a_0 \upharpoonright \beta_0 = a_1 \upharpoonright \beta_0 = b_1 \upharpoonright \beta_0 \) and \( b_0(\beta_0) = a_0(\beta_0) \neq a_1(\beta_0) = b_1(\beta_0) \). If \( a_0(\beta_0) < a_1(\beta_0) \), then we have \( b_0 < b_1 \), \( b_0, b_1 \in B \) and \( a_0, a_1 \in A \), a contradiction. If \( a_0(\beta_0) > a_1(\beta_0) \), then we have \( a_0 > b_1 \), \( b_1 \in B \) and \( a_0 \in A \), a contradiction. \( \square \)

This claim ensures that the function \( y_0 := \bigcup \{ a \upharpoonright \alpha(a, b) : a \in A, b \in B \} \) is well-defined and \( y_0 \in \prod_{\alpha < \alpha_0} X_\alpha \).

**Claim 2.** \( \alpha_0 < \gamma \).

**Proof.** Assume \( \alpha_0 = \gamma \). Then \( y_0 \in X = A \cup B \). If \( y_0 \in A \), then there is \( \alpha_0 \in A \) with \( y_0 <_X \alpha_0 \). Letting \( \beta_0 = \min \{ \beta < \gamma : y_0(\beta) \neq \alpha_0(\beta) \} \), take \( a \in A \) and \( b \in B \) with \( \beta_0 < \alpha(a, b) \). Then we have \( b <_X \alpha_0 \), a contradiction. When \( y_0 \in B \), similarly we can get a contradiction. \( \square \)

By a similar argument of the proof above, we can check the clauses (1) and (2). Now let \( A_0 = \{ a(\alpha_0) : a \in A, a \upharpoonright \alpha_0 = y_0 \} \) and \( B_0 = X_{\alpha_0} \setminus A_0 \). Obviously \( A_0 \) is a 0-segment of \( X_{\alpha_0} \) and \( B_0 \) is a 1-segment of \( X_{\alpha_0} \).

**Claim 3.** \( B_0 = \{ a(\alpha_0) : a \in B, a \upharpoonright \alpha_0 = y_0 \} \) holds.

**Proof.** The inclusion “\( \subseteq \)” is obvious.

To see the other inclusion, let \( b \in B \) with \( b \upharpoonright \alpha_0 = y_0 \). If \( b(\alpha_0) \in A_0 \) were true, then there is \( a \in A \) with \( a \upharpoonright \alpha_0 = y_0 \) and \( b(\alpha_0) = a(\alpha_0) \). This means \( a \upharpoonright (\alpha_0 + 1) = b \upharpoonright (\alpha_0 + 1) \), thus \( \alpha_0 < \alpha(a, b) \), a contradiction. We have \( b(\alpha_0) \in B_0 \). \( \square \)

This claim shows the clause (3).

**Claim 4.** The clause (4) holds.

**Proof.** Assume that \( A_0 \neq \emptyset \) and \( A_0 \) has no max. To see \( \kappa \geq 0\)-cf \( A_0 \), let \( U \) be a 0-unbounded subset of \( A \). Fix \( u_0 \in A_0 \) and \( a_0 \in A \) with \( a_0 \upharpoonright \alpha_0 = y_0 \) and \( a_0(\alpha_0) = u_0 \). Then it is easy to check that \( V := \{ a(\alpha_0) : a_0 <_X a \in U \} \) is 0-unbounded in \( A_0 \).

To see \( \kappa \leq 0\)-cf \( A_0 \), let \( V \) be a 0-unbounded in \( A_0 \). For every \( u \in V \), we can fix \( a_u \in A \) with \( a_u \upharpoonright \alpha_0 = y_0 \) and \( a_u(\alpha_0) = u \). Then \( U := \{ a_u : u \in V \} \) is 0-unbounded in \( A \). \( \square \)
Claim 5. The clause (5) holds.

Proof. Assume that $A_0 \neq \emptyset$ and $A_0$ has max $u_0$. If for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $X_{\alpha}$ has max, then $y_0^{\wedge}(u_0)^{\wedge}(\max X_{\alpha} : \alpha_0 < \alpha < \gamma) = \max A$, a contradiction. Therefore there is $\alpha < \gamma$ with $\alpha_0 < \alpha$ such that $X_{\alpha}$ has no max. Let $\alpha_1$ be such a smallest one. By a similar argument in Claim 4, we see $\kappa = 0$-cf$_{X_{\alpha_1}} X_{\alpha_1}$.

Claim 6. The clause (6) holds.

Proof. Let $A_0 = \emptyset$. If there is $a \in A$ with $a \cap \alpha_0 = y_0$, then $a(\alpha_0) \in A_0$, a contradiction. This shows (i).

If $\alpha_0 = \beta + 1$ for some ordinal $\beta$, then we can find $a \in A$ and $b \in B$ with $\beta < \alpha(a, b) \leq \alpha_0$, so $a(\alpha, b) = \alpha_0$. Now we have $y_0 = a \cap \alpha_0$, this contradicts (i). This shows (ii).

If $Y_1 = \prod_{\alpha_0 < \alpha < \gamma} X_{\alpha}$ has min, then we have $b_0 := y_0^{\wedge}(\min X_{\alpha} : \alpha_0 \leq \alpha < \gamma) \in B$ by $A_0 = \emptyset$. If $a \in X$ and $a < b_0$, then $a \uparrow \alpha_0 < b \uparrow \alpha_0 = y_0$, thus $a \in A$ by (i). This shows $b_0 = \min B$, a contradiction. We see (iii). (2-i) and (i) show (iv).

To see (v), assume that $y_1 := \max(\leftarrow, y_0)_{\gamma_0}$ exists. Let $\alpha_1 = \min\{\alpha < \alpha_0 : y_1(\alpha) \neq y_0(\alpha)\}$, moreover take $a \in A$ and $b \in B$ with $\alpha_1 < \alpha(a, b)$. By (i), we have $a \uparrow \alpha_0 < y_0$, therefore $a \uparrow \alpha_0 \leq y_1$. By $y_1 \uparrow \alpha_1 = y_0 \uparrow \alpha_1 = a \uparrow \alpha_1$ and $y_1(\alpha_1) < y_0(\alpha_1) = a(\alpha_1)$, we have $y_1 < a \uparrow \alpha_0$, a contradiction.

(vi) can be similarly proved as in Claim 4. (vii) follows from the definition of $\alpha_0$.

Theorem 4.6. If $X_{\alpha}$ is a 0-paracompact GO-space for every $\alpha < \gamma$, then the lexicographic product $X = \prod_{\alpha < \gamma} X_{\alpha}$ is also 0-paracompact.

Proof. Let $A$ be a closed 0-segment of $X$ with $0$-cf $A \geq \omega_1$, set $\kappa = 0$-cf $A$. We will find a 0-unbounded 0-order preserving sequence $\{a_\delta : \delta < \kappa\} \subset A$ which is closed discrete in $A$. We have to consider several cases. Let $B = X \setminus A$.

Case 1. $B$ has min $b_0$.

In this case, since $A$ is closed and has no max, $b_0$ belongs to $X^-$. From Lemma 1.3, we can find $\alpha_0 < \gamma$ such that $b_0(\alpha_0) \in X^-_{\alpha_0}$ and for every $\alpha < \gamma$ with $\alpha_0 < \alpha$, $b_0(\alpha) = \min X_{\alpha}$ holds. Let $A_0 = (\leftarrow, b_0(\alpha_0))_{X^-_{\alpha_0}}$. Then $A_0$ is a closed 0-segment of $X_{\alpha_0}$. By a similar argument of Claim 4 in the previous lemma, we see $\kappa = 0$-cf$_{X^-_{\alpha_0}} A_0$. Since $X_{\alpha_0}$ is 0-paracompact, we can take a 0-unbounded 0-order preserving sequence
\{u_\delta : \delta < \kappa\} in \(A_0\) which is closed discrete in \(A_0\) and \((\leftarrow, u_0)X_{\alpha_0} \neq \emptyset\). For each \(\delta < \kappa\), let \(a_\delta = (b_0 \upharpoonright \alpha_0)^\langle u_\delta \rangle \langle b_0 \upharpoonright (\alpha_0, \gamma)\rangle\).

**Claim 1.** The sequence \(F = \{a_\delta : \delta < \kappa\}\) is 0-unbounded, 0-order preserving and closed discrete in \(A\).

**Proof.** Obviously \(F\) is 0-order preserving. Let \(a \in A\). Then we have \(a \upharpoonright \alpha_0 \leq b_0 \upharpoonright \alpha_0\). If \(a \upharpoonright \alpha_0 < b_0 \upharpoonright \alpha_0\), then \(a < a_0\). If \(a \upharpoonright \alpha_0 = b_0 \upharpoonright \alpha_0\), then we can take \(\delta < \kappa\) with \(a(\alpha_0) < u_\delta\) (otherwise, \(a \geq b_0\), a contradiction). Then we have \(a < a_\delta\). Thus \(F\) is 0-unbounded in \(A\). To see the closed discreteness of \(F\), take the smallest \(\delta_0 < \kappa\) with \(a < a_{\delta_0}\). If \(\delta_0 = 0\), then \((\leftarrow, a_0)X\) is a neighborhood of \(a\) disjoint from \(F\). If \(\delta_0 > 0\), then we have \(a \upharpoonright \alpha_0 = b_0 \upharpoonright \alpha_0\) and \(a(\alpha_0) \in A_0\). Note \(u_0 \leq a(\alpha_0)\) because of \(a_0 \leq a\). Since \(\{u_\delta : \delta < \kappa\}\) is closed discrete in \(X_{\alpha_0}\), we can find \(u^* \in X_{\alpha_0}^*\) with \(u^* < X_{\alpha_0}^* a(\alpha_0)\) such that \((u^*, a(\alpha_0)]_{X_{\alpha_0}^*} \cap X_{\alpha_0}\) contains at most one \(u_\delta\). Let \(a^* = (b_0 \upharpoonright \alpha_0)^\langle u^* \rangle \langle b_0 \upharpoonright (\alpha_0, \gamma)\rangle\). Then \((a^*, a_{\delta_0}]_{X} \cap X\) is a neighborhood of \(a\) witnessing the closed discreteness of \(F\) at \(a\). \(\square\)

**Case 2.** \(B \neq \emptyset\) and has no min.

This case is a modification of Theorem 4.2.2 in [2]. In this case, take \(\alpha_0 < \gamma\), \(y_0 \in \prod_{\alpha < \alpha_0} X_{\alpha}\) and the 0-segment \(A_0\) of \(X_{\alpha_0}\) in Lemma 4.5. Further we divide Case 2 into several subcases.

**Case 2-1.** \(A_0 = \emptyset\).

In this case, we use (6) of Lemma 4.5. By induction using (i) and (vi) in (6), define \(\{a_\delta : \delta < \kappa\} \subset A\) such that \(\{\min\{\alpha < \alpha_0 : a_\delta(\alpha) \neq y_0(\alpha)\} : \delta < \kappa\}\) is 0-unbounded and 0-order preserving in \(\alpha_0\).

**Claim 2.** The sequence \(F = \{a_\delta : \delta < \kappa\}\) is 0-unbounded, 0-order preserving and closed discrete in \(A\).

**Proof.** The proof that \(F\) is 0-unbounded and 0-order preserving is easy. Let \(a \in A\) and \(\delta_0 < \kappa\) be the smallest \(\delta < \kappa\) with \(a < a_\delta\). By (6-iiii) in Lemma 4.5, \(Y_1 := \prod_{\alpha_0 \leq \alpha < \gamma} X_{\alpha}\) has no min, so take \(y_1 \in Y_1\) with \(y_1 < Y_1 a \upharpoonright (\alpha_0, \gamma)\). Then \(((a \upharpoonright \alpha_0)^\langle y_1 \rangle, a_{\delta_0}]_{X}\) is a neighborhood of \(a\) witnessing the closed discreteness of \(F\) at \(a\). \(\square\)

**Case 2-2.** \(A_0 \neq \emptyset\).

We further divide this case into several cases.

**Case 2-2-1.** \(A_0\) has no max and \(B_0 := X_{\alpha_0} \setminus A_0\) has min.

Note that in this case, \(A_0\) need not be closed in \(X_{\alpha_0}\). We can find \(\alpha > \alpha_0\) such that \(X_{\alpha}\) has no min (otherwise, \(B\) has min). Let \(\alpha_1\) be
such a smallest one. By (4) in Lemma 4.5, we can find a 0-unbounded 0-order preserving sequence \( \{u_\delta : \delta < \kappa\} \) in \( A_0 \). But remark that in general, \( \{u_\delta : \delta < \kappa\} \) cannot be closed discrete in \( A_0 \). For each \( \delta < \kappa \), take \( a_\delta \in X \) with \( a_\delta \restriction (\alpha_0 + 1) = y_0 \wedge \langle u_\delta \rangle \), then \( a_\delta \in A \).

**Claim 3.** The sequence \( F = \{a_\delta : \delta < \kappa\} \) is 0-unbounded, 0-order preserving and closed discrete in \( A \).

**Proof.** Obviously \( F \) is 0-unbounded and 0-order preserving in \( A \). Let \( a \in A \) and \( \delta_0 < \kappa \) be the smallest \( \delta < \kappa \) with \( a < a_\delta \). If \( \delta_0 = 0 \), then \( (\leftarrow, a_0)_X \) is a neighborhood of \( a \) disjoint from \( F \).

Let \( \delta_0 > 0 \), then we have \( a \restriction \alpha_0 = y_0 \). Since \( Y_1 := \prod_{\alpha_0 < \alpha < \gamma} X_\alpha \) has no min, take \( y_1 \in Y_1 \) with \( y_1 < a \restriction (\alpha_0, \gamma) \). Then \((a \restriction (\alpha_0 + 1))^\wedge y_1, a_{\delta_0}) \) is a neighborhood of \( a \) witnessing the closed discreteness of \( F \) at \( a \).

**Case 2-2-2.** \( A_0 \) has no max and \( B_0 := X_{\alpha_0} \setminus A_0 \) has no min.

In this case \( A_0 \) is a closed 0-segment in the 0-paracompact GO-space \( X_{\alpha_0} \). Using (4) in Lemma 4.5, take a 0-unbounded 0-order preserving sequence \( \{u_\delta : \delta < \kappa\} \) which is closed discrete in \( A_0 \) and \((\leftarrow, u_0)_{X_{\alpha_0}} \neq \emptyset \). For each \( \delta < \kappa \), take \( a_\delta \in X \) with \( a_\delta \restriction (\alpha_0 + 1) = y_0 \wedge \langle u_\delta \rangle \), then \( a_\delta \in A \). \( \square \)

**Claim 4.** The sequence \( F = \{a_\delta : \delta < \kappa\} \) is 0-unbounded, 0-order preserving and closed discrete in \( A \).

**Proof.** Obviously \( F \) is 0-unbounded and 0-order preserving in \( A \). Let \( a \in A \) and \( \delta_0 < \kappa \) be the smallest \( \delta < \kappa \) with \( a < a_\delta \). As in the proof of the claim above, when \( \delta_0 = 0 \), then \((\leftarrow, a_0)_X \) witnesses the closed discreteness of \( F \) at \( a \). When \( \delta_0 > 0 \), we have \( a \restriction \alpha_0 = y_0 \) and \( a(\alpha_0) \in A_0 \). Since \( \{u_\delta : \delta < \kappa\} \) is closed discrete in \( X_{\alpha_0} \), we can take \( u^* \in X_{\alpha_0}^* \) with \( u^* < a(\alpha_0) \), \((u^*, a(\alpha_0)]_{X_{\alpha_0}^*} \cap X_{\alpha_0} \) contains at most one \( u_\delta \). Take \( a^* \in X \) with \( a^* \restriction (\alpha_0 + 1) = (a \restriction \alpha_0)^\wedge \langle u^* \rangle \). Then \((a^*, a_{\delta_0})_X \cap X \) is a neighborhood of \( a \) witnessing the closed discreteness of \( F \) at \( a \). \( \square \)

**Case 2-2-3.** \( A_0 \) has max.

In this case, by (5) of Lemma 4.5, there is \( \alpha > \alpha_0 \) such that \( X_\alpha \) has no max. Let \( \alpha_1 \) be such a smallest one. Since \( \kappa = 0 \cdot \text{cf}_{X_{\alpha_1}} X_{\alpha_1} \) and \( X_{\alpha_1} \) is 0-paracompact, the 0-segment \( X_{\alpha_1} \) has a 0-unbounded 0-order preserving sequence \( \{u_\delta : \delta < \kappa\} \subset X_{\alpha_1} \) which is closed discrete in \( X_{\alpha_1} \) and \((\leftarrow, u_0)_{X_{\alpha_1}} \neq \emptyset \). For each \( \delta < \kappa \), take \( a_\delta \in X \) with \( a_\delta \restriction (\alpha_1 + 1) = y_0 \wedge \langle \max A_0 \rangle \wedge \langle \max X_\alpha : \alpha_0 < \alpha < \alpha_1 \rangle \wedge \langle u_\delta \rangle \). Note \( a_\delta \in A \). As in Claim 4, we see:
Claim 5. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in $A$.

Case 3. $B = \emptyset$, i.e., $A = X$.

Since $X$ has no max, let $\alpha_0 = \min\{\alpha < \gamma : X_\alpha \text{ has no max}\}$. Then as in Claim 4 in Lemma 4.5, we see $\kappa = 0$-cf$X_{\alpha_0}$ $X_{\alpha_0}$. Since $X_{\alpha_0}$ is 0-paracompact, we can find a 0-unbounded 0-order preserving sequence $\{u_\delta : \delta < \kappa\} \subset X_{\alpha_0}$ which is closed discrete in $X_{\alpha_0}$ and $(\leftarrow, u_0)X_{\alpha_0} \neq \emptyset$. For every $\delta < \kappa$, take $a_\delta \in X$ with $a_\delta \upharpoonright (\alpha_0 + 1) = (\max X_\alpha : \alpha < \alpha_0)^\wedge (u_\delta)$. Note $a_\delta \in A$. Similarly we can see:

Claim 6. The sequence $F = \{a_\delta : \delta < \kappa\}$ is 0-unbounded, 0-order preserving and closed discrete in $A$. □

With the analogy of the theorem above, we extends the result Theorem 4.2.2 in [2] as follows:

Corollary 4.7. Lexicographic products of paracompact GO-spaces are paracompact.

Example 4.8. For example we see:

- the lexicographic products $S^\gamma$ and $M^\gamma$ are paracompact for every ordinal $\gamma$.
- the lexicographic products $M \times P$ and $P \times M$ are paracompact.
- lexicographic products of metrizable GO-spaces are paracompact. For instance, the lexicographic product $([0, 1]_R \cup [2, 3]_R)^\omega_1$ is paracompact.

However, there is a paracompact lexicographic product of non-paracompact LOTS’s, see Example in page 73 in [2]. We end this paper with the following question.


References
