NORMALITY, ORTHOCOMPACTNESS AND COUNTABLE PARACOMPACTNESS OF PRODUCTS OF GO-SPACES

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Abstract. In [4], it is asked whether orthocompact products of two GO-spaces are normal or not. In this paper we discuss when products of GO-spaces are normal iff they are orthocompact. As a corollary, we see that products of two countably compact GO-spaces are normal iff they are orthocompact. Also we discuss when normal products of two GO-spaces are countably paracompact.

1. Introduction

It is known:

- If X and Y are subspaces of ordinals, then normality and orthocompactness of $X \times Y$ are equivalent, see [10].
- If X and Y are subspaces of ordinals and $X \times Y$ is normal, then $X \times Y$ is countably paracompact, see [8].

Note that subspaces of ordinals with the usual order are GO-spaces. Recently the first result above is extended as:

- If X is a subspace of an ordinal and Y is a GO-space, then normality and orthocompactness of $X \times Y$ are equivalent, see [4].

In the same paper, the following are asked:

(1) If X and Y are GO-spaces, then is $X \times Y$ normal whenever it is orthocompact?
(2) If X and Y are GO-spaces, then is $X \times Y$ rectangular whenever it is codecop product?
(3) If X and Y are GO-spaces, then is $X \times Y$ countably paracompact whenever it is codecop product?

We will remark that the Sorgenfrey square $\mathbb{S}^2$ answers these problems negatively, moreover we will discuss when products of two GO-spaces

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are normal iff they are orthocompact, and when normal products of two GO-spaces are countably paracompact.

Recall that a triple $\langle X, <, \tau \rangle$ is called a *GO-space* (= *Generalized Ordered space*) if $<$ is a linear order on X and τ is a T_2-topology on X which has a base by convex subsets by $<$, where a subset C of X is convex if b belongs to C whenever $a < b < c$, $b \in X$ and $a, c \in C$. A linearly ordered set $\langle X, < \rangle$ has a natural T_2-topology $\lambda(<)$ so called the *interval topology* which is the topology generated by $\{(\leftarrow, a) : a \in X\} \cup \{(a, \rightarrow) : a \in X\}$ as a subbase, where $(a, \rightarrow) = \{x \in X : a < x\}$, $(a, b) = \{x \in X : a < x < b\}$, ..., etc. If necessary, we write as $(a, b)_X$ instead of (a, b). Note that this subbase induce a base by convex subsets. The triple $\langle X, <, \lambda(<) \rangle$ is called a LOTS (= *Linearly Ordered Topological Space*). Obviously if $\langle X, <, \tau \rangle$ is a GO-space, then the topology τ is stronger than $\lambda(<)$, that is, $\lambda(<) \subset \tau$. If there are no confusion, then a GO-space X means the triple $\langle X, <, \tau \rangle$ and a LOTS X means the triple $\langle X, <, \lambda(<) \rangle$. Obviously LOTS' are GO-spaces but not vice versa, for example, the Sorgenfrey line $\mathbb{S}$ is such an example.

We need some tools handling GO-spaces which are appeared in [6]. For reader's convenience, we give their abstracts here. At first, recall a well-known lemma below.

Lemma 1.1. [2, Problem 3.12.3(a)] *Let $\langle L, <, \lambda(<) \rangle$ be a LOTS. Then the following are equivalent:*

1. *The space $\langle L, \lambda(<) \rangle$ is compact.*
2. *For every subset A of L, A has the least upper bound $\sup_L A$ in $\langle L, < \rangle$.*
3. *For every subset A of L, A has the greatest lower bound $\inf_L A$ in $\langle L, < \rangle$.*

Note that $\sup_L \emptyset = \min L$ (=the smallest element of L) and $\inf_L \emptyset = \max L$ (=the largest element of L) whenever L is a compact LOTS.

Definition 1.2. Let L be a compact LOTS and $x \in L$. A subset $A \subset (\leftarrow, x)$ is said to be 0-*unbounded* for x in L if for every $y < x$, there is $a \in A$ with $y \leq a$. Similarly for a subset $A \subset (x, \rightarrow)$, "1-unbounded for x" is defined. Now 0-cofinality $0\text{-}\operatorname{cf}_L x$ of x in L is defined by:

$$0\text{-}\operatorname{cf}_L x = \min\{|A| : A \text{ is 0-unbounded for } x \text{ in } L.\}.$$

Also $1\text{-}\operatorname{cf}_L x$ is defined. If there are no confusion, we write simply $0\text{-}\operatorname{cf} x$ and $1\text{-}\operatorname{cf} x$. Observe that

- if x is the smallest element of L, then $0\text{-}\operatorname{cf} x = 0$,
- if x has the immediate predecessor in L, then $0\text{-}\operatorname{cf} x = 1$,

- otherwise, then $0\text{-}\operatorname{cf} x$ is a regular infinite cardinal.

Moreover, remark:

- Whenever $\min L < x$, $\omega \leq 0\text{-}\operatorname{cf} x$ iff $\sup_L(\leftarrow, x) = x$ iff $x \in \operatorname{Cl}_L(\leftarrow, x)$.

Let $x \in L$ and $\kappa = 0\text{-}\operatorname{cf} x$. We can take a sequence $c : \kappa \to L$ which is strictly increasing and continuous as a function, and the range $\{c(\alpha) : \alpha \in \kappa\}$ is a subset of $(\leftarrow, x)$ which is 0-unbounded for x in L. We call such c a 0-*normal sequence* for x in L. Similarly, a 1-*normal sequence* for x in L is defined.

Lemma 1.3. [6, Lemma 3.3] *Let x be a point in a compact LOTS L with $\kappa = 0\text{-}\operatorname{cf} x \geq \omega_1$. Let $\{c(\alpha) : \alpha \in \kappa\}$ and $\{c'(\alpha) : \alpha < \kappa\}$ be two 0-normal sequences for x. Then $\{\alpha \in \kappa : c(\alpha) = c'(\alpha)\}$ is club(= closed and unbounded) in κ.*

In our discussion, whenever X is a GO-space, we apply these methods for $L = lX$ below, and consider $0\text{-}\operatorname{cf}_{lX} a$ or $1\text{-}\operatorname{cf}_{lX} a$ for $a \in lX$.

Lemma 1.4. [6, Lemma 2.1] *Let X be a GO-space. Then there is a unique (up to order isomorphisms) compact LOTS lX such that $X \subset lX$, the order on lX extends the order on X, the original GO-topology on X coincides with the subspace topology on X with respect to the interval topology on lX, X is dense in lX, and for every $a, b \in lX \setminus X$ with $a < b$, $(a, b) \neq \emptyset$ holds.*

In particular, if X is a subspace of an ordinal, say $X \subset [0, \gamma]$, with the usual order, then using Lemma 1.4, we can easily check $lX = \operatorname{Cl}_{[0,\gamma]} X$. Moreover in this case, for every $a \in lX$, obviously $1\text{-}\operatorname{cf} a$ is 0 or 1, furthermore we can easily check that $0\text{-}\operatorname{cf} a$ is equal to $\operatorname{cf} a$ in the usual sense whenever a is a cluster point of X.

Let C be a subset of a regular uncountable cardinal κ. Define $p_C(\alpha) = \sup(C \cap \alpha)$ for $\alpha < \kappa$, $\operatorname{Lim}(C) = \{\alpha \in \kappa : \alpha = p_C(\alpha)\}$ and $\operatorname{Succ}(C) = C \setminus \operatorname{Lim}(C)$, where for notational convenience we consider that -1 is the immediate predecessor of the ordinal 0 and $\sup \emptyset = -1$. Note that $\operatorname{Lim}(C)$ is the set of all cluster points of C in κ therefore it is club in κ whenever C is unbounded in κ, also note that $\operatorname{Succ}(C)$ is the set of isolated points in the subspace C. In particular, $\operatorname{Succ}(\kappa)$ is the set of all successor ordinals of κ. A subset of κ is *stationary* if it intersects all club sets of κ.

Let X be a GO-space, $a \in lX$ and $\kappa = 0\text{-}\operatorname{cf} a$. We can always fix a 0-normal sequence $\{a_0(\alpha) : \alpha \in \kappa\}$ for a(also we can fix a 1-normal sequence $\{a_1(\alpha) : \alpha \in \kappa\}$ for a where $\kappa = 1\text{-}\operatorname{cf}_{lX} a$). Observe that by Lemma 1.3, the stationarity of $\{\alpha \in \kappa : a_0(\alpha) \in X\}$ does not depend on the choices of 0-normal sequences whenever $\kappa \geq \omega_1$. It follows from Lemmas 1.3 and 1.4 that we may assume that $\mathrm{Succ}(\kappa) \subset \{\alpha \in \kappa : a_0(\alpha) \in X\}$ whenever $\kappa \geq \omega$ (by induction, redefine another 0-normal sequence for a). We say that X is 0-*stationary at* a if $\kappa = 0\text{-}\operatorname{cf} a$ is uncountable and $\{\alpha \in \kappa : a_0(\alpha) \in X\}$ is stationary in κ. By the argument above, 0-stationarity at a does not depend on choices of 0-normal sequences for a. Similarly 1-*stationarity at* a is (well-)defined. If $\{\alpha \in \kappa : a_0(\alpha) \in X\}$ is non-stationary in κ, then we can take a club set C in κ such that $\{a_0(\alpha) : \alpha \in C\} \subset lX \setminus X$. Then remark that $(\leftarrow, a) \cap X$ can be represented as the disjoint sum $\bigcup_{\alpha \in \mathrm{Succ}(C)}((a_0(p_C(\alpha)), a_0(\alpha)) \cap X)$ of open subspaces, where $a_0(-1) = \leftarrow$.

Definition 1.5. A GO-space X is said to be *countably* 0-*compact* (*countably* 1-*compact*) if each strictly increasing (decreasing) sequence $\{x_n : n \in \omega\} \subset X$ of length ω has a cluster point in X, equivalently for every $a \in lX \setminus X$, $0\text{-}\operatorname{cf} a \neq \omega$ ($1\text{-}\operatorname{cf} a \neq \omega$) holds.

Remark that a GO-space is countably compact iff it is both countably 0-compact and countably 1-compact, moreover that subspaces of ordinals are countably 1-compact but in general not countably 0-compact.

Lemma 1.6. *Let X be a countably* 0-*compact GO-space with $a \in lX \setminus X$. Then X is* 0-*stationary at a.*

Proof. Let $\lambda_0 = 0\text{-}\operatorname{cf} a$ and $S_0 = \{\alpha \in \lambda_0 : a_0(\alpha) \in X\}$, where $\{a_0(\alpha) : \alpha \in \lambda_0\}$ is the fixed 0-normal sequence for a. By the assumption, obviously $\lambda_0 > \omega$. If S_0 were non-stationary in λ_0, then there is a club set C in λ_0 disjoint from S_0. Take $\alpha \in \mathrm{Lim}(C)$ with $\operatorname{cf} \alpha = \omega$. Then $0\text{-}\operatorname{cf} a_0(\alpha) = \omega$. Since X is countably 0-compact, we have $a_0(\alpha) \in X$, a contradiction because of $\alpha \in \mathrm{Lim}(C) \subset C$. $\square$

Also there is an analogous result:

- Let X be a countably 1-compact GO-space with $a \in lX \setminus X$. Then X is 1-stationary at a.

But here after, we do not write down such analogous results.

Let $\mathcal{U}$ be an open cover of a topological space X. A collection $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ of closed sets in X indexed by $\mathcal{U}$ is said to be a *partial closed shrinking* of $\mathcal{U}$ in X if $F(U) \subset U$ holds for every $U \in \mathcal{U}$. When $Z \subset X$, $\mathcal{F} \restriction Z$ denotes $\{F(U) \cap Z : U \in \mathcal{U}\}$. A partial closed shrinking $\mathcal{F}$ of $\mathcal{U}$ is said to be a *closed shrinking* if it covers X. Recall that a space X is normal iff every binary open cover has a closed shrinking in X, where a binary open cover means an open cover of size at most 2. Remark that if for every $\lambda \in \Lambda$, $\mathcal{F}_\lambda = \{F_\lambda(U) : U \in \mathcal{U}\}$ is a partial closed shrinking of $\mathcal{U}$ in X, moreover $\{\bigcup \mathcal{F}_\lambda : \lambda \in \Lambda\}$ is locally finite in X, then

$$\bigvee_{\lambda\in\Lambda} \mathcal{F}_\lambda = \{\bigcup_{\lambda\in\Lambda} F_\lambda(U) : U \in \mathcal{U}\}$$

is also a partial closed shrinking of $\mathcal{U}$ in X. $\bigvee_{i\in\{0,1\}} \mathcal{F}_i$ is denoted by $\mathcal{F}_0 \vee \mathcal{F}_1$. A collection $\mathcal{V}$ of open sets in a topological space X is said to be *interior preserving* if $\bigcap \mathcal{V}'$ is open for every $\mathcal{V}' \subset \mathcal{V}$. Observe that a collection $\mathcal{V}$ of open sets is interior preserving iff $\bigcap(\mathcal{V})_x$ is a neighborhood of x in X for every $x \in X$, where $(\mathcal{V})_x = \{V \in \mathcal{V} : x \in V\}$. Here note that $\bigcap \emptyset = X$ and $\bigcup \emptyset = \emptyset$ by the usual sense of $\bigcap$ and $\bigcup$. A space X is said to be *orthocompact* if every open cover $\mathcal{U}$ has an interior preserving open refinement $\mathcal{V}$, that is, $\bigcup \mathcal{V} = X$, $\mathcal{V}$ is an interior preserving collection of open sets, moreover every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. When we do not require "$\bigcup \mathcal{V} = X$", then we say such a $\mathcal{V}$ as a *partial* interior preserving open refinement of $\mathcal{U}$. Remark that if for every $\lambda \in \Lambda$, $\mathcal{V}_\lambda$ is an interior preserving (a point finite) collection of open sets, moreover $\{\bigcup \mathcal{V}_\lambda : \lambda \in \Lambda\}$ is point finite, then $\bigcup_{\lambda\in\Lambda} \mathcal{V}_\lambda$ is also interior preserving (point finite). When $Z \subset X$, $\mathcal{V} \restriction Z$ denotes $\{V \cap Z : V \in \mathcal{V}\}$.

In the product theory, the normality has been compared with the orthocompactness, for example:

- If a space X is normal, then $X \times \mathbb{I}$ is normal iff X is countably paracompact [1], where $\mathbb{I}$ denotes the unit interval.
- If a space X is orthocompact, then $X \times \mathbb{I}$ is orthocompact iff X is countably metacompact [12].

Here recall that a space is *countably paracompact* (*countably metacompact*) if every countable open cover has a locally finite (a point finite) open refinement. Note that normal countably metacompact spaces are countably paracompact. It is well-known that the Sorgenfrey square $\mathbb{S}^2$ is neither normal nor countably paracompact but it is countably metacompact. Recently in [3], it is known that $\mathbb{S}^2$ is orthocompact, thus the question (1) above is negative. Also it was pointed out in [14] that the product of the Michael line and the irrationals is orthocompact but not normal. Also it is known that $\mathbb{S}^2$ is not rectangular, see Remark after Theorem 7 in [5]. Since $\mathbb{S}$ is hereditarily separable and first countable, it is easy to see that $\mathbb{S}^2$ is codecop product (for this definition see [4]). Therefore the questions (2) and (3) above are also negative. In this paper, we will prove:

Theorem 1.7. *Let X and Y be GO-spaces satisfying:*

(1) *X is countably 1-compact,*

(2) *if there is* $c \in lX$ *with* $1\text{-}\operatorname{cf} c \geq \omega$ *(equivalently,* $\langle X < \rangle$ *is not well-ordered), then* Y *is countably compact.*

Then $X \times Y$ *is normal if and only if it is orthocompact.*

If X is a subspace of an ordinal, then it is countably 1-compact and the assumption in the assumption (2) above is not true (therefore (2) is true). Therefore we have:

Corollary 1.8. [4] *Let* X *be a subspace of an ordinal and* Y *a GO-space. Then* $X \times Y$ *is normal if and only if it is orthocompact.*

Moreover, we have:

Corollary 1.9. *Let* X *be a countably* 1*-compact GO-space and* Y *a countably compact GO-space. Then* $X \times Y$ *is normal if and only if it is orthocompact.*

In particular:

Corollary 1.10. *Let* X *and* Y *be countably compact GO-spaces. Then* $X \times Y$ *is normal if and only if it is orthocompact.*

Here note that $\omega_1 + 1$ is compact and ω_1 is countably compact but not compact. Also note that ω_1^2 is normal but $\omega_1 \times (\omega_1 + 1)$ is not normal. In a different line, it is known that $X \times Y$ is normal if and only if it is orthocompact, whenever X and Y are locally compact GO-spaces, see [13, 14].

Moreover we show:

Theorem 1.11. *Let* X *and* Y *be GO-spaces satisfying:*

(1) X *is countably* 1*-compact,*

(2) *if there is* $c \in lX$ *with* $1\text{-}\operatorname{cf} c \geq \omega$*, then* Y *is countably compact.*

If $X \times Y$ *is normal, then it is countably paracompact.*

Therefore we have:

Corollary 1.12. *Let* X *be a subspace of an ordinal and* Y *a GO-space. If* $X \times Y$ *is normal, then it is countably paracompact.*

Corollary 1.13. *Let* X *be a countably* 1*-compact GO-space and* Y *a countably compact GO-space. If* $X \times Y$ *is normal, then it is countably paracompact.*

2. Basic lemmas

In this section, we prepare basic lemmas for proving the theorem. We frequently use the following lemma.

Lemma 2.1. [7, 10] *Let λ be a regular uncountable cardinal, S and T subsets of λ. Then:*

1. *If $S \times T$ is normal (orthocompact), then S is non-stationary in λ, T is non-stationary in λ or $S \cap T$ is stationary.*
2. *If $S \times (T \cup \{\lambda\})$ is normal (orthocompact), then S is non-stationary in λ or T is not unbounded in λ.*

In the rest of this section, let X and Y be GO-spaces with $a \in lX$ and $b \in lY$. Moreover, for $i \in 2 = \{0, 1\}$, let $\lambda_i = i\text{-}\operatorname{cf} a$ and $\{a_i(\alpha) : \alpha \in \lambda_i\}$ be the fixed i-normal sequence for a such that $\operatorname{Succ}(\lambda_i) \subset \{\alpha \in \lambda_i : a_i(\alpha) \in X\}$ whenever $\lambda_i \geq \omega$. Similarly, for $i \in 2$, let $\mu_i = i\text{-}\operatorname{cf} b$ and $\{b_i(\beta) : \beta \in \mu_i\}$ be the fixed i-normal sequence for b such that $\operatorname{Succ}(\mu_i) \subset \{\beta \in \mu_i : b_i(\beta) \in Y\}$ whenever $\mu_i \geq \omega$.

Lemma 2.2. *If $a \in X$, $b \in Y$ and $\mathcal{U}$ is an open cover of $X \times Y$, then there are $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\alpha_1 \in \lambda_1 \cup \{-1\}$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and $U_0 \in \mathcal{U}$ such that*

$$((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U_0.$$

Therefore there is a closed shrinking $\mathcal{F}$ of $\mathcal{U}$ in $((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)$, moreover there is an interior preserving (a point finite) partial open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup \mathcal{V} = ((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)$.

The proof is almost obvious (note, if $\lambda_0 = 0$ then let $\alpha_0 = -1$), moreover $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is defined by

$$F(U) = \begin{cases} ((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) & \text{if } U = U_0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Also define $\mathcal{V} = \{((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)\}$.

Lemma 2.3. *If $a \in lX \setminus X$, $b \in Y$, X is 0-stationary at a, $\mathcal{U}$ is an open cover of $X \times Y$ and $X \times Y$ is normal, then there are $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ such that*

$$\bigcup \mathcal{V} = ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y).$$

Proof. Let $S_0 = \{\alpha \in \lambda_0 : a_0(\alpha) \in X\}$. By the assumption, S_0 is stationary in λ_0. For every $\alpha \in S_0 \cap \operatorname{Lim}(S_0)$, it follows from $\langle a_0(\alpha), b\rangle \in X \times Y$ that for some $U(\alpha) \in \mathcal{U}$, $f(\alpha) < \alpha$ and $g_i(\alpha) \in \mu_i \cup \{-1\}$ $(i \in 2)$,

$$((a_0(f(\alpha)), a_0(\alpha)] \cap X) \times ((b_0(g_0(\alpha)), b_1(g_1(\alpha))) \cap Y) \subset U(\alpha)$$

holds.

Claim. $\lambda_0 \neq \mu_0$ and $\lambda_0 \neq \mu_1$ hold.

Proof. Assume that $\lambda_0 = \mu_0$ is true. Note that $S_0 \times (T_0 \cup \{\mu_0\})$ is homeomorphic to a closed subspace of the normal space $X \times Y$, where $T_0 = \{\beta \in \mu_0 : b_0(\beta) \in Y\}$. Since $\mathrm{Succ}(\mu_0) \subset T_0$, T_0 is unbounded in $\mu_0 = \lambda_0$. This contradicts Lemma 2.1 (2). The remaining is similar. □

Applying the Pressing Down Lemma (PDL), we find a stationary set $S_0' \subset S_0 \cap \mathrm{Lim}(S_0)$ and $\alpha_0 \in \lambda_0$ such that $f(\alpha) \leq \alpha_0$ for every $\alpha \in S_0'$. Whenever $\lambda_0 < \mu_0$, take $\beta_0 \in \mu_0$ with $\sup\{g_0(\alpha) : \alpha \in S_0'\} \leq \beta_0$ and set $S_0'' = S_0'$. Whenever $\lambda_0 > \mu_0$, also applying PDL, it follows from $|\{g_0(\alpha) : \alpha \in S_0'\}| < \lambda_0$ that for some stationary set $S_0'' \subset S_0'$ and $\beta_0 \in \mu_0 \cup \{-1\}$, $g_0(\alpha) \leq \beta_0$ holds for every $\alpha \in S_0''$. Note that in either cases $f(\alpha) \leq \alpha_0$ and $g_0(\alpha) \leq \beta_0$ hold for every $\alpha \in S_0''$. By $\lambda_0 \neq \mu_1$, similarly we can find a stationary set $S_0''' \subset S_0''$ and $\beta_1 \in \mu_1 \cup \{-1\}$ such that $g_1(\alpha) \leq \beta_1$ holds for every $\alpha \in S_0'''$. Then for every $\alpha \in S_0'''$, we have

$$((a_0(\alpha_0), a_0(\alpha)] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U(\alpha).$$

Set $V(\alpha) = ((a_0(\alpha_0), a_0(\alpha)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)$ for every $\alpha \in S_0'''$ and let $\mathcal{V} = \{V(\alpha) : \alpha \in S_0'''\}$. Obviously $\mathcal{V}$ is a partial open refinement of $\mathcal{U}$ with $\bigcup \mathcal{V} = ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)$ because of $a \notin X$. To see that $\mathcal{V}$ is interior preserving, let $\langle x, y\rangle \in X \times Y$. We may assume $\langle x, y\rangle \in \bigcup \mathcal{V}$. Let $\alpha^* = \min\{\alpha \in S_0''' : \langle x, y\rangle \in V(\alpha)\}$. Then $V(\alpha^*)$ is a neighborhood of $\langle x, y\rangle$ contained in $\bigcap(\mathcal{V})_{\langle x,y\rangle}$, thus $\mathcal{V}$ is interior preserving. □

Lemma 2.4. *If $a \in lX \setminus X$, $b \in Y$, X is 0-stationary at a, $\mathcal{U}$ is a binary open cover of $X \times Y$ and $X \times Y$ is orthocompact, then there are $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and $U_0 \in \mathcal{U}$ such that*

$$((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U_0,$$

therefore there is a closed shrinking of $\mathcal{U}$ in

$$((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y).$$

Proof. The proof is parallel to that of the lemma above by using Lemma 2.1. Let $S_0 = \{\alpha \in \lambda_0 : a_0(\alpha) \in X\}$. For every $\alpha \in S_0 \cap \mathrm{Lim}(S_0)$, it follows from $\langle a_0(\alpha), b\rangle \in X \times Y$ that for some $U(\alpha) \in \mathcal{U}$, $f(\alpha) < \alpha$ and $g_i(\alpha) \in \mu_i \cup \{-1\}$ $(i \in 2)$,

$$((a_0(f(\alpha)), a_0(\alpha)] \cap X) \times ((b_0(g_0(\alpha)), b_1(g_1(\alpha))) \cap Y) \subset U(\alpha)$$

holds. This time using orthocompactness of $X \times Y$, we see:

Claim. $\lambda_0 \neq \mu_0$ and $\lambda_0 \neq \mu_1$ hold.

By PDL and $|\mathcal{U}| \le 2$, we find a stationary set $S_0' \subset S_0 \cap \mathrm{Lim}(S_0)$, $\alpha_0 \in \lambda_0$ and $U_0 \in \mathcal{U}$ such that $f(\alpha) \le \alpha_0$ and $U(\alpha) = U_0$ for every $\alpha \in S_0'$. Whenever $\lambda_0 < \mu_0$, take $\beta_0 \in \mu_0$ with $\sup\{g_0(\alpha) : \alpha \in S_0'\} \le \beta_0$ and set $S_0'' = S_0'$. Whenever $\lambda_0 > \mu_0$, also applying PDL, it follows that for some stationary set $S_0'' \subset S_0'$ and $\beta_0 \in \mu_0 \cup \{-1\}$, $g_0(\alpha) \le \beta_0$ holds for every $\alpha \in S_0''$. By $\lambda_0 \neq \mu_1$, similarly we can find a stationary set $S_0''' \subset S_0''$ and $\beta_1 \in \mu_1 \cup \{-1\}$ such that $g_1(\alpha) \le \beta_1$ holds for every $\alpha \in S_0'''$. Then $\alpha_0, \beta_0, \beta_1$ and U_0 are as desired and a desired closed shrinking can be easily constructed. □

The proof of the lemma below is analogous to Lemma 2.3.

Lemma 2.5. *If $a \in lX \setminus X$, $b \in Y$, X is 0-stationary at a, $\mathcal{U}$ is a countable open cover of $X \times Y$ and $X \times Y$ is normal, then there are $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and $U_0 \in \mathcal{U}$ such that*

$$((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U_0,$$

therefore there is a point finite partial open refinement $\mathcal{V}$ of $\mathcal{U}$ such that

$$\bigcup \mathcal{V} = ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y).$$

Lemma 2.6. *If $a \in lX \setminus X$, $b \in lY \setminus Y$, X is 0-stationary at a, Y is 0-stationary at b, $\mathcal{U}$ is an open cover of $X \times Y$ and $X \times Y$ is normal, then there are $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ such that*

$$\bigcup \mathcal{V} = ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y).$$

Proof. Let $S_0 = \{\alpha \in \lambda_0 : a_0(\alpha) \in X\}$ and $T_0 = \{\beta \in \mu_0 : b_0(\beta) \in Y\}$.

Case 1. $\lambda_0 < \mu_0$.

For every $\alpha \in S_0 \cap \mathrm{Lim}(S_0)$ and $\beta \in T_0 \cap \mathrm{Lim}(T_0)$, it follows from $\langle a_0(\alpha), b_0(\beta)\rangle \in X \times Y$ that for some $U(\alpha, \beta) \in \mathcal{U}$, $f(\alpha, \beta) < \alpha$ and $g(\alpha, \beta) < \beta$,

$$((a_0(f(\alpha, \beta)), a_0(\alpha)] \cap X) \times ((b_0(g_0(\alpha, \beta)), b_0(\beta)] \cap Y) \subset U(\alpha, \beta)$$

holds. First fix $\alpha \in S_0 \cap \mathrm{Lim}(S_0)$. Because of $|\{f(\alpha, \beta) : \beta \in T_0\}| \le |\alpha| < \lambda_0 < \mu_0$, applying PDL to g, we can find a stationary set $T_0(\alpha) \subset T_0 \cap \mathrm{Lim}(T_0)$, $f(\alpha) < \alpha$ and $g(\alpha) < \mu_0$ such that $f(\alpha, \beta) \le f(\alpha)$ and $g(\alpha, \beta) \le g(\alpha)$ for every $\beta \in T_0(\alpha)$. Next applying PDL to $S_0 \cap \mathrm{Lim}(S_0)$ and f, we can find a stationary set $S_0' \subset S_0 \cap \mathrm{Lim}(S_0)$ and $\alpha_0 \in \lambda_0$ such that $f(\alpha) \le \alpha_0$ for every $\alpha \in S_0'$. Because $g(\alpha) < \mu_0$ for every $\alpha \in S_0'$ and $|S_0'| = \lambda_0 < \mu_0$, we can take $\beta_0 \in \mu_0$ such that $g(\alpha) \le \beta_0$ for every $\alpha \in S_0'$. Now for every $\alpha \in S_0'$ and $\beta \in T_0(\alpha)$, we have

$$((a_0(\alpha_0)), a_0(\alpha)] \cap X) \times ((b_0(\beta_0), b_0(\beta)] \cap Y) \subset U(\alpha, \beta).$$

For every $\alpha \in (\alpha_0, \lambda_0)$ and $\beta \in (\beta_0, \mu_0)$, set

$$V(\alpha, \beta) = ((a_0(\alpha_0)), a_0(\alpha)) \cap X) \times ((b_0(\beta_0), b_0(\beta)) \cap Y).$$

And set

$$\mathcal{V} = \{V(\alpha, \beta) : \alpha \in (\alpha_0, \lambda_0), \beta \in (\beta_0, \mu_0)\}.$$

Then it is easy to see that $\mathcal{V}$ is a partial open refinement of $\mathcal{U}$ and $\bigcup \mathcal{V} = ((a_0(\alpha_0)), a] \cap X) \times ((b_0(\beta_0), b] \cap Y)$. To see that $\mathcal{V}$ is interior preserving, let $\langle x, y\rangle \in X \times Y$. We may assume $\langle x, y\rangle \in \bigcup \mathcal{V}$. First let $\alpha^* = \min\{\alpha \in \lambda_0 : x < a_0(\alpha)\}$ and next $\beta^* = \min\{\alpha \in \lambda_0 : y < b_0(\beta)\}$. Then $V(\alpha^*, \beta^*)$ is a neighborhood of $\langle x, y\rangle$ contained in $\bigcap(\mathcal{V})_{\langle x,y\rangle}$, thus $\mathcal{V}$ is interior preserving. Remark that this case does not require normality of $X \times Y$.

Case 2. $\lambda_0 > \mu_0$.

This case is similar to Case 1.

Case 3. $\lambda_0 = \mu_0$.

It follows from the normality of $X \times Y$ and Lemma 2.1 (1) that $S_0 \cap T_0$ is stationary in λ_0. For every $\alpha \in (S_0 \cap T_0) \cap \mathrm{Lim}(S_0 \cap T_0)$, take $U(\alpha) \in \mathcal{U}$, $f(\alpha) < \alpha$ such that

$$((a_0(f(\alpha)), a_0(\alpha)] \cap X) \times ((b_0(f(\alpha)), b_0(\alpha)] \cap Y) \subset U(\alpha)$$

holds. Applying PDL, we can find a stationary set $S_0' \subset (S_0 \cap T_0) \cap \mathrm{Lim}(S_0 \cap T_0)$ and $\alpha_0 \in \lambda_0$ such that $f(\alpha) \le \alpha_0$ for every $\alpha \in S_0'$. For every $\alpha \in S_0'$, set

$$V(\alpha) = ((a_0(\alpha_0)), a_0(\alpha)) \cap X) \times ((b_0(\alpha_0), b_0(\alpha)) \cap Y).$$

Then α_0 and $\mathcal{V} = \{V(\alpha) : \alpha \in S_0'\}$ with $\beta_0 = \alpha_0$ satisfy the required condition. □

Lemma 2.7. *If $a \in lX \setminus X$, $b \in lY \setminus Y$, X is 0-stationary at a, Y is 0-stationary at b, $\mathcal{U}$ is a binary open cover of $X \times Y$ and $X \times Y$ is orthocompact, then there are $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0$ and $U_0 \in \mathcal{U}$ such that*

$$((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y) \subset U_0,$$

therefore there is a closed shrinking of $\mathcal{U}$ in $((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y)$.

Proof. Let $S_0 = \{\alpha \in \lambda_0 : a_0(\alpha) \in X\}$ and $T_0 = \{\beta \in \mu_0 : b_0(\beta) \in Y\}$.

Case 1. $\lambda_0 < \mu_0$.

For every $\alpha \in S_0 \cap \mathrm{Lim}(S_0)$ and $\beta \in T_0 \cap \mathrm{Lim}(T_0)$, fix $U(\alpha, \beta) \in \mathcal{U}$, $f(\alpha, \beta) < \alpha$ and $g(\alpha, \beta) < \beta$ with

$$((a_0(f(\alpha, \beta)), a_0(\alpha)] \cap X) \times ((b_0(g_0(\alpha, \beta)), b_0(\beta)] \cap Y) \subset U(\alpha, \beta).$$

First fix $\alpha \in S_0 \cap \mathrm{Lim}(S_0)$. Applying PDL to g, we can find a stationary set $T_0(\alpha) \subset T_0 \cap \mathrm{Lim}(T_0)$, $f(\alpha) < \alpha$, $g(\alpha) < \mu_0$ and $U(\alpha) \in \mathcal{U}$ such that $f(\alpha, \beta) \leq f(\alpha)$, $g(\alpha, \beta) \leq g(\alpha)$ and $U(\alpha, \beta) = U(\alpha)$ for every $\beta \in T_0(\alpha)$. Next applying PDL to $S_0 \cap \mathrm{Lim}(S_0)$ and f, we can find a stationary set $S_0' \subset S_0 \cap \mathrm{Lim}(S_0)$, $\alpha_0 \in \lambda_0$ and $U_0 \in \mathcal{U}$ such that $f(\alpha) \leq \alpha_0$ and $U(\alpha) = U_0$ for every $\alpha \in S_0'$. By $\lambda_0 < \mu_0$, we can take $\beta_0 \in \mu_0$ such that $g(\alpha) \leq \beta_0$ for every $\alpha \in S_0'$. Then α_0, β_0 and U_0 are as required.

Case 2. $\lambda_0 > \mu_0$.

This case is similar to Case 1.

Case 3. $\lambda_0 = \mu_0$.

It follows from orthocompactness of $X \times Y$ and Lemma 2.1 (1) that $S_0 \cap T_0$ is stationary in λ_0. Then using PDL, we can find $\alpha_0 \in \lambda_0$ and $U_0 \in \mathcal{U}$ with $((a_0(\alpha_0)), a] \cap X) \times ((b_0(\alpha_0), b] \cap Y) \subset U_0$. $\square$

The proof of the following lemma is analogous to Lemma 2.6.

Lemma 2.8. *If $a \in lX \setminus X$, $b \in lY \setminus Y$, X is 0-stationary at a, Y is 0-stationary at b, $\mathcal{U}$ is a countable open cover of $X \times Y$ and $X \times Y$ is normal, then there are $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0$ and $U_0 \in \mathcal{U}$ such that*

$$((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y) \subset U_0,$$

therefore there is a point finite partial open refinement $\mathcal{V}$ of $\mathcal{U}$ such that

$$\bigcup \mathcal{V} = ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y).$$

3. Proof of "only if" part of Theorem 1.7

In this section we prove "only if" part of Theorem 1.7. Throughout this section, we assume that X and Y are GO-spaces satisfying:

(1) X is countably 1-compact,
(2) if there is $c \in lX$ with $1\text{-}\mathrm{cf}\, c \geq \omega$, then Y is countably compact.

Moreover we assume that $X \times Y$ is normal and $\mathcal{U}$ is an open cover of $X \times Y$. We will find an interior preserving open refinement $\mathcal{V}$ of $\mathcal{U}$. Generally in this section and the next section, for $a \in lX$ and $i \in 2$, we let $\lambda_i = i\text{-}\mathrm{cf}\, a$ and let $\{a_i(\alpha) : \alpha \in \lambda_i\}$ be the fixed i-normal sequence for a such that $\mathrm{Succ}(\lambda_i) \subset \{\alpha \in \lambda_i : a_i(\alpha) \in X\}$ whenever $\lambda_i \geq \omega$. Similarly for $b \in lY$ and $i \in 2$, we let $\mu_i = i\text{-}\mathrm{cf}\, b$ and let $\{b_i(\beta) : \beta \in \mu_i\}$ be the fixed i-normal sequence for b such that $\mathrm{Succ}(\mu_i) \subset \{\beta \in \mu_i : b_i(\beta) \in Y\}$ whenever $\mu_i \geq \omega$.

Definition 3.1. Let

$$A = \{a \in lX : \text{there is an interior preserving partial open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ with } \bigcup \mathcal{V} \supset ((\leftarrow, a] \cap X) \times Y.\ \},$$

$$B_a = \{b \in lY : \text{there are } \alpha_1 \in \lambda_1 \cup \{-1\} \text{ and an interior preserving partial open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ with } \bigcup \mathcal{V} \supset ((a, a_1(\alpha_1)] \cap X) \times ((\leftarrow, b]) \cap Y).\ \},$$

for every $a \in lX$.

Obviously A is an initial segment of lX, that is, if $a' \leq a \in A$ then $a' \in A$. Also for every $a \in lX$, B_a is an initial segment of lY.

Lemma 3.2. *If $a \in X$, then the following hold:*

(1) *there is an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup \mathcal{V} \supset \{a\} \times Y$,*
(2) *if $(\leftarrow, a) \subset A$, then $a \in A$.*

Proof. For $b, b' \in lY$, we define $b \cong b'$ by either one of the following:

(1) $b = b'$
(2) $b < b'$ and there are $\alpha_0 \in \lambda_0 \cup \{-1\}$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup \mathcal{V} \supset ((a_0(\alpha_0), a] \cap X) \times ([b, b'] \cap Y)$,
(3) $b' < b$ and there are $\alpha_0 \in \lambda_0 \cup \{-1\}$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup \mathcal{V} \supset ((a_0(\alpha_0), a] \cap X) \times ([b', b] \cap Y)$.

Obviously $\cong$ is an equivalence relation on lY and each equivalence class is convex in lY. Let $\mathcal{E}$ be the collection of all equivalence classes intersecting with Y, that is, $\mathcal{E} = \{E \in lY/_{\cong} : E \cap Y \neq \emptyset\}$.

Claim 1. $E \cap Y$ is open in Y for every $E \in \mathcal{E}$.

Proof. Let $b \in E \cap Y$, where $\lambda_i = i\text{-}\operatorname{cf} a$, $\mu_i = i\text{-}\operatorname{cf} b$, ..., etc as above. It follws from $\langle a, b\rangle \in X \times Y$ and Lemma 2.2 that there are $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and $U_0 \in \mathcal{U}$ such that

$$((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U_0.$$

If $y \in (b_0(\beta_0), b_1(\beta_1)) \cap Y$, then α_0 and $\mathcal{V} = \{U_0\}$ witnesse $y \cong b$ thus $y \in E \cap Y$. Hence $(b_0(\beta_0), b_1(\beta_1)) \cap Y$ is a neighborhood of b in Y contained in $E \cap Y$. □

Claim 1 shows $\{E \cap Y : E \in \mathcal{E}\}$ is a pairwise disjoint clopen cover of Y.

Claim 2. For every $E \in \mathcal{E}$, the following hold:

(1) there is an interior preserving partial open refinement $\mathcal{V}_E$ of $\mathcal{U}$ with $\bigcup \mathcal{V}_E \supset \{a\} \times (E \cap Y)$,
(2) if $(\leftarrow, a) \subset A$, then there is an interior preserving partial open refinement $\mathcal{V}_E$ of $\mathcal{U}$ with $\bigcup \mathcal{V}_E \supset ((\leftarrow, a] \cap X) \times (E \cap Y)$.

Proof. Because the proof of (1) is simpler than that of (2), we only show (2). Assume $(\leftarrow, a) \subset A$. For each $\alpha \in \lambda_0$, from $a_0(\alpha) \in A$, take an interior preserving partial open refinement $\mathcal{W}_\alpha$ of $\mathcal{U}$ with $\bigcup \mathcal{W}_\alpha \supset ((\leftarrow, a_0(\alpha)] \cap X) \times Y$. Moreover let $\mathcal{W}_{-1} = \emptyset$. Fix $y \in E \cap Y$ and let $b = \sup_{lY}(E \cap Y)$ (as stated above, reset $\mu_i = i\text{-}\operatorname{cf} b$, ..., etc). Obviously $y \leq b$. We show:

Subclaim 1. There is an interior preserving partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\bigcup \mathcal{V}' \supset ((\leftarrow, a] \cap X) \times ([y, b] \cap Y)$.

Proof. First assume $b \in Y$. Since $b \in \operatorname{Cl}_{lY}(E \cap Y)$ and $E \cap Y$ is clopen in Y, we have $b \in E \cap Y$ thus $b \cong y$. Take α_0 and $\mathcal{V}$ witnessing $b \cong y$, that is, $\mathcal{V}$ is an interior preserving partial open refinement of $\mathcal{U}$ with $\bigcup \mathcal{V} \supset ((a_0(\alpha_0), a] \cap X) \times ([y, b] \cap Y)$. Then $\mathcal{V}' = \mathcal{W}_{\alpha_0} \cup \mathcal{V}$ witnesses the subclaim.

Next assume $b \notin Y$, then $\mu_0 \geq \omega$. Take $\beta^* \in \mu_0$ with $y < b_0(\beta^*)$ and for every $\beta \in \mu_0$ with $\beta^* \leq \beta$, fix $\alpha(\beta) \in \lambda_0 \cup \{-1\}$ and an interior preserving partial open refinement $\mathcal{V}_\beta$ of $\mathcal{U}$ witnessing $y \cong b_0(\beta)$. When $\mu_0 = \omega$,

$$\mathcal{V}' = \big((\mathcal{V}_{\beta^*+1} \cup \mathcal{W}_{\alpha(\beta^*+1)}) \restriction X \times ((\leftarrow, b_0(\beta^* + 1)) \cap Y)\big) \cup \bigcup_{\beta^* < \beta \in \mu_0} (\mathcal{V}_{\beta+1} \cup \mathcal{W}_{\alpha(\beta+1)}) \restriction X \times ((b_0(\beta - 1), b_0(\beta + 1)) \cap Y)$$

witnesses the subclaim, because the collection of $X \times ((b_0(\beta-1), b_0(\beta+1)) \cap Y)$'s are point finite. Let $T_0 = \{\beta \in \mu_0 : b_0(\beta) \in Y\}$. When $\mu_0 > \omega$ and T_0 is not stationary in μ_0, take a club set $D \subset (\beta^*, \mu_0)$ in μ_0 disjoint from T_0. Then

$$\mathcal{V}' = \bigcup_{\beta \in \operatorname{Succ}(D)} (\mathcal{V}_\beta \cup \mathcal{W}_{\alpha(\beta)}) \restriction X \times ((b_0(p_D(\beta)), b_0(\beta)) \cap Y)$$

witnesses the subclaim, because the collection of $X \times ((b_0(p_D(\beta)), b_0(\beta)) \cap Y)$'s are point finite, in fact disjoint. When $\mu_0 > \omega$ and T_0 is stationary in μ_0 (i.e., Y is 0-stationary at b), by Lemma 2.3, we can take $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\beta_0 \in \mu_0$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup \mathcal{V} \supset ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y)$. We may assume $\beta^* \leq \beta_0$. Then

$$\mathcal{V}' = \mathcal{W}_{\alpha_0} \cup \mathcal{V}_{\beta_0} \cup \mathcal{V}$$

witnesses the subclaim. □

Let $b' = \inf_{lY}(E \cap Y)$, then similarly we have:

Subclaim 2. There is an interior preserving partial open refinement $\mathcal{V}''$ of $\mathcal{U}$ with $\bigcup \mathcal{V}'' \supset ((\leftarrow, a] \cap X) \times ([b', y] \cap Y)$.

Then $\mathcal{V}_E = \mathcal{V}' \cup \mathcal{V}''$ satisfies Claim 2. □

Now $\mathcal{V} = \bigcup_{E \in \mathcal{E}} \mathcal{V}_E \restriction X \times (E \cap Y)$ proves the lemma. □

Lemma 3.3. *If $a \in lX \setminus X$, X is 0-stationary at a and $(\leftarrow, a) \subset A$, then $a \in A$.*

Proof. For $b, b' \in lY$, define $b \cong b'$ as in the previous lemma. Then $\cong$ is an equivalence relation on lY. Set $\mathcal{E} = \{E \in lY/_{\cong} : E \cap Y \neq \emptyset\}$.

Claim 1. $E \cap Y$ is open in Y for every $E \in \mathcal{E}$.

Proof. Let $b \in E \cap Y$, where $\lambda_i = i\text{-}\mathrm{cf}\, a$, $\mu_i = i\text{-}\mathrm{cf}\, b$, ..., etc as above. By Lemma 2.3, we can take $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup \mathcal{V} \supset ((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)$. Then $(b_0(\beta_0), b_1(\beta_1)) \cap Y$ is a neighborhood of b in Y contained in $E \cap Y$. □

To see $a \in A$, it suffices to check the following claim (see the lemma above).

Claim 2. For every $E \in \mathcal{E}$, there is an interior preserving partial open refinement $\mathcal{V}_E$ of $\mathcal{U}$ with $\bigcup \mathcal{V}_E \supset ((\leftarrow, a] \cap X) \times (E \cap Y)$.

Proof. Let $E \in \mathcal{E}$, $y \in E \cap Y$ and $b = \sup_{lY}(E \cap Y)$ (as stated above, reset $\mu_i = i\text{-}\mathrm{cf}\, b$, ..., etc). For each $\alpha \in \lambda_0$, fix an interior preserving partial open refinement $\mathcal{W}_\alpha$ of $\mathcal{U}$ witnessing $a_0(\alpha) \in A$. Moreover let $\mathcal{W}_{-1} = \emptyset$.

Subclaim 1. There is an interior preserving partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\bigcup \mathcal{V}' \supset ((\leftarrow, a] \cap X) \times ([y, b] \cap Y)$.

Proof. The proof is similar to the corresponding proof of the lemma above by using Lemma 2.6 instead of Lemma 2.3, so we leave it to the reader. □

Let $b' = \inf_{lY}(E \cap Y)$, then similarly we have:

Subclaim 2. There is an interior preserving partial open refinement $\mathcal{V}''$ of $\mathcal{U}$ with $\bigcup \mathcal{V}'' \supset ((\leftarrow, a] \cap X) \times ([b', y] \cap Y)$.

Then $\mathcal{V}_E = \mathcal{V}' \cup \mathcal{V}''$ satisfies Claim 2. □

□

Lemma 3.4. *The following hold.*

(1) $\min lX \in A$,
(2) $\min lY \in B_a$ *for every* $a \in lX$,
(3) *if* $a \in A$ *and* $1\text{-}\operatorname{cf} a = 1$, *then* $a_1(0) \in A$,
(4) *if* $a \in lX$, $b \in B_a$ *and* $1\text{-}\operatorname{cf} b = 1$, *then* $b_1(0) \in B_a$.

Proof. (1) follows from Lemma 3.2 (1) (when $\min lX \notin X$, consider $\mathcal{V}$ as $\emptyset$ in the definition of A). (2) is similar. For (3), take a $\mathcal{V}$ witnessing $a \in A$. It follows from $1\text{-}\operatorname{cf} a = 1$ that $a_1(0)$ is the immediate successor of a in lX. When $a_1(0) \notin X$, $\mathcal{V}$ witnesses $a_1(0) \in A$. When $a_1(0) \in X$, by Lemma 3.2 (1) take an interior preserving partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\bigcup \mathcal{V}' \supset \{a_1(0)\} \times Y$. Then $\mathcal{V} \cup \mathcal{V}'$ witnesses $a_1(0) \in A$. (4) is similar to (3). □

Lemma 3.5. $\max A$ *exists.*

Proof. Let $a = \sup A$. It suffices to see $a \in A$. Note $(\leftarrow, a) \subset A$. It follows from (1) and (3) of Lemma 3.4 that we may assume $\lambda_0 = 0\text{-}\operatorname{cf} a \geq \omega$. By Lemma 3.2 (2), we may assume $a \notin X$. For each $\alpha \in \lambda_0$, fix an interior preserving partial open refinement $\mathcal{V}_\alpha$ of $\mathcal{U}$ witnessing $a_0(\alpha) \in A$. When $\lambda_0 = \omega$, $\bigcup_{\alpha\in\lambda_0} \mathcal{V}_\alpha \restriction ((a_0(\alpha - 1), a_0(\alpha + 1)) \cap X) \times Y$ witnesses $a \in A$. When $\lambda_0 > \omega$ and $S_0 = \{\alpha \in \lambda_0 : a_0(\alpha) \in X\}$ is not stationary in λ_0, taking a club set C disjoint from S_0,

$$\bigcup_{\alpha\in\operatorname{Succ}(C)} \mathcal{V}_\alpha \restriction ((a_0(p_C(\alpha)), a_0(\alpha)) \cap X) \times Y$$

witnesses $a \in A$. The remaining case (i.e., X is 0-stationary at a) follows from Lemma 3.3. □

Lemma 3.6. *If* Y *is countably compact, then for each* $a \in lX$ *with* $1\text{-}\operatorname{cf} a \geq \omega$, $\max lY \in B_a$ *holds.*

Proof. Let $a \in lX$ with $1\text{-}\operatorname{cf} a \geq \omega$, and reset $\lambda_i = i\text{-}\operatorname{cf} a$, ..., etc.

Claim 1. $\max B_a$ exists.

Proof. Let $b = \sup B_a$ and again reset $\mu_i = i\text{-}\operatorname{cf} b$, ..., etc. Because of (2) and (4) of Lemma 3.4, we may assume $\mu_0 \geq \omega$. For every $\beta \in \mu_0$, fix $\alpha(\beta) \in \lambda_1$ and an interior preserving partial open refinement $\mathcal{V}_\beta$ of $\mathcal{U}$ with $\bigcup \mathcal{V}_\beta \supset ((a, a_1(\alpha(\beta))] \cap X) \times ((\leftarrow, b_0(\beta)] \cap Y)$.

Case 1. $a \in X$ and $b \in Y$.

Take $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\alpha_1 \in \lambda_1$, $\beta_0 \in \mu_0$, $\beta_1 \in \mu_1 \cup \{-1\}$ and $U_0 \in \mathcal{U}$ such that

$$((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U_0.$$

Pick $\alpha^* \in \lambda_1$ with $\alpha^* > \max\{\alpha_1, \alpha(\beta_0)\}$. Then α^* and $\mathcal{V}_{\beta_0} \cup \mathcal{V}$ witnesses $b \in B_a$.

Case 2. $a \in X$ and $b \notin Y$.

Since Y is (0-)countably compact, Y is 0-stationary at b, see Lemma1.6. Using Lemma 2.3, take $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\alpha_1 \in \lambda_1$, $\beta_0 \in \mu_0$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup \mathcal{V} = ((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b] \cap Y)$. Pick $\alpha^* \in \lambda_1$ with $\alpha^* > \max\{\alpha_1, \alpha(\beta_0)\}$. Then α^* and $\mathcal{V}_{\beta_0} \cup \mathcal{V}$ witnesses $b \in B_a$.

Case 3. $a \notin X$ and $b \in Y$.

Similar to Case 2.

Case 4. $a \notin X$ and $b \notin Y$.

Since X is 1-stationary at a and Y is 0-stationary at b, use Lemma 2.6. □

The following completes the proof of Lemma 3.6.

Claim 2. $\max lY \in B_a$.

Proof. Let $b = \max B_a$ and again reset $\mu_i = i\text{-}\operatorname{cf} b$, ..., etc. Assume $b < \max lY$, then by Lemma 3.4 (4), we have $\mu_1 \geq \omega$. Take $\alpha_1^* \in \lambda_1$ and an interior preserving partial open refinement $\mathcal{V}$ of $\mathcal{U}$ witnessing $b \in B_a$.

Case 1. $a \in X$ and $b \in Y$.

Take $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\alpha_1 \in \lambda_1$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1$ and $U_0 \in \mathcal{U}$ such that

$$((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y) \subset U_0.$$

Pick $\alpha^* \in \lambda_1$ with $\alpha^* > \max\{\alpha_1, \alpha_1^*\}$. Then α^* and $\mathcal{V} \cup \{U_0\}$ witnesses $b_1(\beta_1 + 1) \in B_a$. This contradicts the maximality of b.

Case 2. $a \in X$ and $b \notin Y$.

Since Y is countably (1-)compact, Y is 1-stationary at b. Using Lemma 2.3, take $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\alpha_1 \in \lambda_1$, $\beta_1 \in \mu_1$ and an interior preserving partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\bigcup \mathcal{V}' = ((a_0(\alpha_0), a_1(\alpha_1)) \cap X) \times ([b, b_1(\beta_1)) \cap Y)$. Pick $\alpha^* \in \lambda_1$ with $\alpha^* > \max\{\alpha_1^*, \alpha_1\}$. Then α^* and $\mathcal{V} \cup \mathcal{V}'$ witnesses $b_1(\beta_1 + 1) \in B_a$, a contradiction.

Case 3. $a \notin X$ and $b \in Y$.

Since X is countably 1-compact, X is 1-stationary at a. Using Lemma 2.3, take $\alpha_1 \in \lambda_1$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1$ and an interior preserving partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\bigcup \mathcal{V}' = ([a, a_1(\alpha_1)) \cap X) \times$

$((b_0(\beta_0), b_1(\beta_1)) \cap Y)$. Pick $\alpha^* \in \lambda_1$ with $\alpha^* > \max\{\alpha_1^*, \alpha_1\}$. Then α^* and $\mathcal{V} \cup \mathcal{V}'$ witnesses $b_1(\beta_1 + 1) \in B_a$, a contradiction.

Case 4. $a \notin X$ and $b \notin Y$.

Since X and Y are 1-stationary at a and b respectively, using Lemma 2.6, take $\alpha_1 \in \lambda_1$, $\beta_1 \in \mu_1$ and an interior preserving partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\bigcup \mathcal{V}' = ([a, a_1(\alpha_1)) \cap X) \times ([b, b_1(\beta_1)) \cap Y)$. The remaining are similar. □

□

The following lemma completes the proof of "only if " part of the main theorem.

Lemma 3.7. $\max lX \in A$ *holds.*

Proof. Let $a = \max A$ and reset $\lambda_i = i\text{-}\operatorname{cf} a$, ..., etc. Assume $a < \max lX$, then $\lambda_1 \geq 1$.

Case 1. $1\text{-}\operatorname{cf} c \leq 1$ for every $c \in lX$.

In this case, it has to be $\lambda_1 = 1$. Then it follows from Lemma 3.4 (3) that $a_1(0) \in A$, this contradicts the maximality of a.

Case 2. Otherwise.

In this case, by the assumption (2) of the theorem, Y is countably compact. If $\lambda_1 = 1$, then the argument of Case 1 works. If $\lambda_1 \geq \omega$, then by Lemma 3.6, we have $\max lY \in B_a$. Take α_1 and $\mathcal{V}$ witnessing $\max lY \in B_a$. Also take $\mathcal{V}'$ witnessing $a \in A$. Then $\mathcal{V} \cup \mathcal{V}'$ witnesses $a_1(\alpha_1 + 1) \in A$, this contradicts the maximality of a. □

4. Proof of "if" part of Theorem 1.7

In this section we prove "if" part of Theorem 1.7. Throughout this section, we assume that X and Y are GO-spaces satisfying:

(1) X is countably 1-compact,
(2) if there is $c \in lX$ with $1\text{-}\operatorname{cf} c \geq \omega$, then Y is countably compact.

Moreover we assume that $X \times Y$ is orthocompact and $\mathcal{U}$ is a binary open cover of $X \times Y$. We will find a closed shrinking $\mathcal{F}$ of $\mathcal{U}$ in $X \times Y$. Except for some technical differences, the proof of this section is almost parallel to that of the previous section. So we will give their abstract proofs.

Definition 4.1. Let

$$A = \{a \in lX : \text{there is a closed shrinking } \mathcal{F} \text{ of } \mathcal{U} \text{ in } ((\leftarrow, a] \cap X) \times Y.\ \},$$

$$B_a = \{b \in lY : \text{there are } \alpha_1 \in \lambda_1 \cup \{-1\} \text{ and}$$
$$\text{a closed shrinking } \mathcal{F} \text{ of } \mathcal{U} \text{ in}$$
$$([a, a_1(\alpha_1)) \cap X) \times ((\leftarrow, b]) \cap Y).\ \},$$

for every $a \in lX$.

Since GO-spaces are normal, we have $\min lX \in A$ and for every $a \in lX$, $\min lY \in B_a$. Obviously A is an initial segment of lX and for every $a \in lX$, B_a is an initial segment of lY.

Lemma 4.2. *If $a \in X$ and $(\leftarrow, a) \subset A$, then $a \in A$.*

Proof. For $b, b' \in lY$, we define $b \cong b'$ by either one of the following:

(1) $b = b'$
(2) $b < b'$ and there are $\alpha_0 \in \lambda_0 \cup \{-1\}$ and a closed shrinking $\mathcal{F}$ of $\mathcal{U}$ in $((a_0(\alpha_0), a] \cap X) \times ([b, b'] \cap Y)$,
(3) $b' < b$ and there are $\alpha_0 \in \lambda_0 \cup \{-1\}$ and a closed shrinking $\mathcal{F}$ of $\mathcal{U}$ in $((a_0(\alpha_0), a] \cap X) \times ([b', b] \cap Y)$,

Obviously $\cong$ is an equivalence relation on lY. Let $\mathcal{E} = \{E \in lY/_{\cong} : E \cap Y \neq \emptyset\}$.

Claim 1. $E \cap Y$ is open in Y for every $E \in \mathcal{E}$.

Proof. Let $b \in E \cap Y$, where $\lambda_i = i\text{-}\operatorname{cf} a$, $\mu_i = i\text{-}\operatorname{cf} b$, ..., etc. It follws from $\langle a, b\rangle \in X \times Y$ and Lemma 2.2 that there are $\alpha_0 \in \lambda_0 \cup \{-1\}$, $\beta_0 \in \mu_0 \cup \{-1\}$, $\beta_1 \in \mu_1 \cup \{-1\}$ and a closed shrinking $\mathcal{F}$ of $\mathcal{U}$ in $((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b_1(\beta_1)) \cap Y)$. Let $y \in (b_0(\beta_0), b_1(\beta_1)) \cap Y$. We may assume $y \leq b$. Then α_0 and $\mathcal{F} \restriction ((a_0(\alpha_0), a] \cap X) \times ([y, b] \cap Y)$ witnesses $b \cong y$. Hence $(b_0(\beta_0), b_1(\beta_1)) \cap Y$ is a neighborhood of b in Y contained in $E \cap Y$. □

Claim 2. For every $E \in \mathcal{E}$, there is a closed shrinking $\mathcal{F}_E$ of $\mathcal{U}$ in $((\leftarrow, a] \cap X) \times (E \cap Y)$.

Proof. When $\lambda_0 = 0$ or 1, use normality of Y and $(\leftarrow, a) \subset A$. So we may assmue $\lambda_0 \geq \omega$. For each $\alpha \in \lambda_0$, from $a_0(\alpha) \in A$, take a closed shrinking $\mathcal{H}_\alpha$ of $\mathcal{U}$ in $((\leftarrow, a_0(\alpha)] \cap X) \times Y$. Fix $y \in E \cap Y$ and let $b = \sup_{lY}(E \cap Y)$ (as stated above, reset $\mu_i = i\text{-}\operatorname{cf} b$, ..., etc). We show:

Subclaim 1. There is a closed shrinking $\mathcal{F}'$ of $\mathcal{U}$ in $((\leftarrow, a] \cap X) \times ([y, b] \cap Y)$.

Proof. First assume $b \in Y$, then by Claim 1, we have $b \cong y$. Take α_0 and $\mathcal{F}_0$ witnessing $b \cong y$, that is, $\mathcal{F}_0$ is a closed shrinking of $\mathcal{U}$ in $((a_0(\alpha_0), a] \cap X) \times ([y, b] \cap Y)$. Then $\mathcal{F}' = \mathcal{H}_{\alpha_0+1} \restriction ((\leftarrow, a_0(\alpha_0+1)] \cap X) \times ([y, b] \cap Y) \vee \mathcal{F}_0 \restriction ([a_0(\alpha_0+1), a] \cap X) \times ([y, b] \cap Y)$ witnesses the subclaim.

Next assume $b \notin Y$, then $\mu_0 \geq \omega$ and $y < b$. Take $\beta^* \in \mu_0$ with $y < b_0(\beta^*)$ and for every $\beta \in \mu_0$ with $\beta^* \leq \beta$, fix $\alpha(\beta) \in \lambda_0 \cup \{-1\}$ and a closed shrinking $\mathcal{F}_\beta$ of $\mathcal{U}$ in $((a_0(\alpha(\beta)), a] \cap X) \times ([y, b_0(\beta)] \cap Y)$.

When $\mu_0 = \omega$,

$$\mathcal{F}' = \big(\mathcal{F}_{\beta^*} \restriction ([a_0(\alpha(\beta^*)+1), a] \cap X) \times ([y, b_0(\beta^*)] \cap Y) \vee$$

$$\mathcal{H}_{\alpha(\beta^*)+1)}) \restriction ((\leftarrow, a_0(\alpha(\beta^*)+1)] \cap X) \times ([y, b_0(\beta^*)] \cap Y)\big) \vee$$

$$\bigvee_{\beta^* < \beta \in \mu_0} \big(\mathcal{F}_\beta \restriction ([a_0(\alpha(\beta)+1), a] \cap X) \times ([b_0(\beta-1), b_0(\beta)] \cap Y) \vee$$

$$\mathcal{H}_{\alpha(\beta)+1)}) \restriction ((\leftarrow, a_0(\alpha(\beta)+1)] \cap X) \times ([b_0(\beta-1), b_0(\beta)] \cap Y)\big)$$

witnesses the subclaim, because the collection of $X \times ([b_0(\beta-1), b_0(\beta)] \cap Y)$'s are locally finite.

Let $T_0 = \{\beta \in \mu_0 : b_0(\beta) \in Y\}$. When $\mu_0 > \omega$ and T_0 is not stationary in μ_0, take a club set $D \subset (\beta^*, \mu_0)$ in μ_0 disjoint from T_0. Then

$$\mathcal{F}' = \big(\mathcal{F}_{\min D} \restriction ([a_0(\alpha(\min D)+1), a] \cap X) \times ([y, b_0(\min D)] \cap Y) \vee$$

$$\mathcal{H}_{\alpha(\min D)+1)}) \restriction ((\leftarrow, a_0(\alpha(\min D)+1)] \cap X) \times ([y, b_0(\min D)] \cap Y)\big) \vee$$

$$\bigvee_{\min D < \beta \in \mathrm{Succ}(D)} \big(\mathcal{F}_\beta \restriction ([a_0(\alpha(\beta)+1), a] \cap X) \times ([b_0(p_D(\beta)), b_0(\beta)] \cap Y) \vee$$

$$\mathcal{H}_{\alpha(\beta)+1)}) \restriction ((\leftarrow, a_0(\alpha(\beta)+1)] \cap X) \times ([b_0(p_D(\beta)), b_0(\beta)] \cap Y)\big)$$

witnesses the subclaim, because the collection of $X \times ((b_0(p_D(\beta)), b_0(\beta)) \cap Y)$'s are discrete.

When $\mu_0 > \omega$ and T_0 is stationary in μ_0, by Lemma 2.4, we can take $\alpha_0 \in \lambda_0$, $\beta_0 \in \mu_0$ and a closed shrinking $\mathcal{F}^*$ of $\mathcal{U}$ in $((a_0(\alpha_0), a] \cap X) \times ((b_0(\beta_0), b] \cap Y)$. We may assume $\beta^* \leq \beta_0$. Take $\alpha_0^* \in \lambda_0$ with $\max\{\alpha_0, \alpha(\beta_0+1)\} < \alpha_0^*$. Then

$$\mathcal{F}' = \mathcal{F}_{\beta_0+1} \restriction ([a_0(\alpha_0^*), a] \cap X) \times ([y, b_0(\beta_0+1)] \cap Y) \vee$$

$$\mathcal{F}^* \restriction ([a_0(\alpha_0^*), a] \cap X) \times ([b_0(\beta_0+1), b] \cap Y) \vee$$

$$\mathcal{H}_{\alpha_0^*} \restriction ((\leftarrow, a_0(\alpha_0^*)] \cap X) \times ([y, b] \cap Y)$$

witnesses the subclaim. This completes the proof of Subclaim 1. □

Let $b' = \inf_{lY}(E \cap Y)$, then similarly we have:

Subclaim 2. There is a closed shrinking $\mathcal{F}'$ of $\mathcal{U}$ in $((\leftarrow, a] \cap X) \times ([b', y] \cap Y)$.

Then $\mathcal{F}_E = \mathcal{F}' \vee \mathcal{F}''$ satisfies Claim 2. □

Finally $\bigvee_{E \in \mathcal{E}} \mathcal{F}_E$ is a closed shrinking of $\mathcal{U}$ in $((\leftarrow, a] \cap X) \times Y$, thus $a \in A$. □

Lemma 4.3. *If $a \in lX \setminus X$, X is 0-stationary at a and $(\leftarrow, a) \subset A$, then $a \in A$.*

Proof. For $b, b' \in lY$, define $b \cong b'$ as in the previous lemma. Then $\cong$ is an equivalence relation on lY. Set $\mathcal{E} = \{E \in lY/_{\cong} : E \cap Y \neq \emptyset\}$.

Using Lemma 2.3 this time, as in the proof of Claim 1 in the lemma above, we see:

Claim 1. $E \cap Y$ is open in Y for every $E \in \mathcal{E}$.

To see $a \in A$, it suffices to check the following claim. But the proof is similar to the corresponding one (when $\mu_0 > \omega$ and T_0 is stationary use Lemma 2.7 instead of Lemma 2.4) in the previous lemma.

Claim 2. For every $E \in \mathcal{E}$, there is a closed shrinking $\mathcal{F}_E$ of $\mathcal{U}$ in $((\leftarrow, a] \cap X) \times (E \cap Y)$. □

Using normality of GO-spaces, it is easy to see:

Lemma 4.4. *The following hold.*

(1) *if $a \in A$ and* 1- cf $a = 1$*, then $a_1(0) \in A$,*
(2) *if $a \in lX$, $b \in B_a$ and* 1- cf $b = 1$*, then $b_1(0) \in B_a$.*

The remaining arguments including the following lemma are also similar to the arguments in the previous section, so we leave it to the reader.

Lemma 4.5. max A *exists.*

Lemma 4.6. *If Y is countably compact, then for each $a \in lX$ with* 1- cf $a \geq \omega$, max $lY \in B_a$ *holds.*

The following lemma completes the proof of "if " part of the main theorem.

Lemma 4.7. max $lX \in A$ *holds.*

5. Proof of of Theorem 1.11

In this section we prove Theorem 1.11. Throughout this section, we assume that X and Y are GO-spaces satisfying:

(1) X is countably 1-compact,
(2) if there is $c \in lX$ with $1\text{-}\operatorname{cf} c \geq \omega$, then Y is countably compact.

Moreover we assume that $X \times Y$ is normal and $\mathcal{U}$ is a countable open cover of $X \times Y$. We will find a point finite open refinement $\mathcal{V}$ of $\mathcal{U}$.

Definition 5.1. Let

$$A = \{a \in lX : \text{there is a point finite partial open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ with } \bigcup \mathcal{V} \supset ((\leftarrow, a] \cap X) \times Y.\ \},$$

$$B_a = \{b \in lY : \text{there are } \alpha_1 \in \lambda_1 \cup \{-1\} \text{ and a point finite partial open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ with } \bigcup \mathcal{V} \supset ((a, a_1(\alpha_1)] \cap X) \times ((\leftarrow, b]) \cap Y).\ \},$$

for every $a \in lX$.

The remaining are similar to the proof of section 3, so we leave it to the reader.

Finally we ask the following:

Question 5.2. [14, Problem 3.1] Does there exist a pair of GO-spaces whose product is normal but not orthocompact?

Question 5.3. Does there exist a pair of GO-spaces whose product is normal but not countably paracompact?

Question 5.4. Are the two questions above equivalent?

The author has not tried yet, but he thinks that the arguments in the present paper work to see the following question.

Question 5.5. Assuming the same assumption in Theorem 1.7, are the problem lists of Problem 9.2 in [4] all affirmatively?

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