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THE LEXICOGRAPHIC ORDERED PRODUCTS AND THE USUAL TYCHONOFF PRODUCTS

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Abstract. The usual Tychonoff product space of arbitrary many compact (ω-bounded) spaces is well-known to be also compact (ω-bounded). In this paper, we compare the lexicographic ordered topologies on some products of ordinals with the Tychonoff product topologies. We see:

- The lexicographic ordered space ω_1^ω is ω-bounded.
- The lexicographic ordered space $\omega_1^{\omega+1}$ is not ω-bounded.
- If α and β are ordinals with $\beta < \alpha$, then the lexicographic ordered space $[0,\beta]^\omega$ is a subspace of the lexicographic ordered space α^ω, thus the lexicographic ordered space 2^ω is a subspace of the lexicographic ordered space 3^ω.
- The lexicographic ordered space $2^{\omega+1}$ is not a subspace of the lexicographic ordered space $3^{\omega+1}$.
- For all $n < \omega$ with $2 \leq n$, the lexicographic ordered space n^ω is homeomorphic to the Cantor set.
- The lexicographic ordered space $2^{\omega+1}$ is not metrizable.
- The lexicographic ordered spaces $2^{\omega+1}$ and $3^{\omega+1}$ are not homeomorphic.
- The lexicographic ordered topology on $\omega \times 2^\omega$ coincides with its usual Tychonoff product topology.
- The lexicographic ordered topology on ω^ω is strictly weaker than its usual Tychonoff product topology.
- The lexicographic ordered topology on $\omega \times \omega \times \omega_1$ is strictly weaker than its usual Tychonoff product topology.
- The lexicographic ordered topology on $\omega \times 2 \times 3 \times 4 \times (\omega_1 + 1)$ coincide with its usual Tychonoff product topology.

1. Introduction

All spaces are regular T_1 and contain at least 2 elements without stated. The Greek letters α, β, γ,... denote the ordinal numbers. cfα denotes the cofinality of α. In particular ω denotes the first infinite ordinal and ω_1 denotes the first uncountable ordinal.

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Let $<$ be a linear order on a set X with $|X| \geq 2$. $\lambda(<)$ denotes the usual order topology, that is, the topology generated by

$$\{(a, \rightarrow) : a \in X\} \cup \{(\leftarrow, b) : b \in X\}$$

as a subbase, where $(a, \rightarrow) = \{x \in X : a < x\}$, $(a, b) = \{x \in X : a < x < b\}$,..., etc. If necessary, we write $<_X$ and $(a, b)_X$ instead of $<$ and (a, b) respectively. A *LOTS* X means the triple $\langle X, <, \lambda(<)\rangle$. LOTS is an abbreviation of "Linearly Ordered Topological Space". If $|X| = 1$, then X is considered as a trivial LOTS.

As usual, we consider an ordinal α as the set of smaller ordinals and as a LOTS with the order $\in$ (we identify it with $<$). $\mathbb{R}$ and $\mathbb{I}$ stand for the usual real line and the unit interval $[0, 1]$ of the real line respectively, with the usual order $<$. Then $\mathbb{R}$ is a Lindelöf LOTS and $\mathbb{I}$ is a compact LOTS. Similarly a *Generalized Ordered space* (*GO-space*) means the triple $\langle X, <, \tau\rangle$ where τ is a topology on X with $\lambda(<) \subset \tau$ which has a base consisting convex sets, where a subset A is *convex* if $(a, b) \subset A$ whenever $a, b \in A$ with $a < b$. It is known that the Sorgenfrey line is a GO-space but not a LOTS. The following are also well-known:

(1) If $\langle L, <_L, \lambda(<_L)\rangle$ is a LOTS and $X \subset L$, then $\langle X, <_L\restriction X, \lambda(<_L) \restriction X\rangle$ is a GO-space, where $<_L\restriction X$ is the restricted order of $<_L$ to X and $\lambda(<_L) \restriction X$ is the subspace topology on X with respect to the topology $\lambda(<_L)$ on L, that is $\{U \cap X : U \in \lambda(<_L)\}$. On the other hand:

(2) If $\langle X, <_X, \tau\rangle$ is a GO-space, then there is a LOTS $\langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ such that the space $\langle X, \tau\rangle$ is a dense subspace of $\langle L, \lambda(<_L)\rangle$ and $<_X = <_L\restriction X$. Moreover:

(3) If $\langle X, <_X, \lambda(<_X)\rangle$ is a LOTS, then there is a LOTS $\langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ and $<_X = <_L\restriction X$ such that the space $\langle L, \lambda(<_L)\rangle$ is compact and contains $\langle X, \lambda(<_X)\rangle$ as a dense subspace. Therefore by (2) and (3), we have:

(4) If $\langle X, <_X, \tau\rangle$ is a GO-space, then there is a compact LOTS $\langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ and $<_X = <_L\restriction X$ such that the compact space $\langle L, \lambda(<_L)\rangle$ contains $\langle X, \tau\rangle$ as a dense subspace. We say this situation as "a GO space $\langle X, <_X, \tau\rangle$ has a linearly ordered compactification $\langle L, <_L, \lambda(<_L)\rangle$"or more simply "a GO-space X has a linearly ordered compactification L". Usually, if there are no confusion, we do not distinguish the symbols $<_X$ and $<_L$, and simply write $<$.

Obviously a compact LOTS $\langle L, <_L, \lambda(<_L)\rangle$ has the largest element $\max L$ and the smallest element $\min L$. Remark that a LOTS $\langle L, <_L, \lambda(<_L)\rangle$ is compact iff every subset A of L has the least upper bound

$\sup_L A$ (equivalently, greatest lower bound $\inf_L A$), where we define $\sup_L \emptyset = \min L$ and $\inf_L \emptyset = \max L$, see [1, Problem 3.12.3(a)]. Also remark that if X is a convex subset of a LOTS $\langle L, <_L, \lambda(<_L)\rangle$, then the subspace topology $\lambda(<_L) \restriction X$ coincides with the order topology $\lambda(< \restriction X)$ on X. For more details, see [3, 4].

Definition 1.1. Let $\{X_\alpha : \alpha < \gamma\}$ be a sequence of LOTS X_α's with the order $<_\alpha$'s, where γ is an ordinal. The *lexicographic order* $<$ on the product $X = \Pi_{\alpha<\gamma} X_\alpha$ is defined by:

$$x < y \Leftrightarrow x \restriction \alpha = y \restriction \alpha \text{ and } x(\alpha) <_\alpha y(\alpha) \text{ for some } \alpha < \gamma,$$

for $x, y \in X$, where $x \restriction \alpha$ denotes the restriction $\langle x(\beta) : \beta < \alpha\rangle$ of $x = \langle x(\beta) : \beta < \gamma\rangle$. Remark that the lexicographic order is well-defined, because γ is well-ordered.

We say a LOTS $\Pi_{\alpha<\gamma} X_\alpha$ with the lexicographic order as a *lexicographic ordered space*. An order topology induced by a lexicographic order is called a *lexicographic ordered topology*. When $X_\alpha = Y$ for all $\alpha < \gamma$, we write $\Pi_{\alpha<\gamma} X_\alpha$ as Y^γ.

The usual Tychonoff product space of arbitrary many compact (ω-bounded) spaces is well-known to be also compact (ω-bounded), where a space is said to be ω-*bounded* if every countable subset has the compact closure. On the other hand, it is known that there are two countably compact spaces whose usual Tychonoff product is not countably compact, where a space is said to be *countably compact* if every countable subset has cluster points. Also it is well-known that the lexicographic ordered space $\mathbb{I}^2$ is compact [1, Problem 3.12.3(d)].

First we remark:

Lemma 1.2. *If X is a non-discrete LOTS, then the lexicographic ordered topology $\lambda = \lambda(<)$ on X^2 does not coincide with the usual Tychonoff product topology τ.*

Proof. Let $x_0 \in X$ be a non-isolated point of the LOTS X. We may assume $x_0 \in \mathrm{Cl}_X(\leftarrow, x_0)_X$. Then we have $\langle x_0, x_0\rangle \in \mathrm{Cl}_\tau(\leftarrow, x_0)_X \times \{x_0\}$, where Cl_τ denotes the closure with respect to the topology τ. On the other hand, pick $y \in X$ with $y < x_0$. Then $(\langle x_0, y\rangle, \rightarrow)_{X^2}$ is a neighborhood of $\langle x_0, x_0\rangle$ which is disjoint from $(\leftarrow, x_0)_X \times \{x_0\}$, thus we have $\langle x_0, x_0\rangle \notin \mathrm{Cl}_\lambda(\leftarrow, x_0)_X \times \{x_0\}$. □

Even if X is a discrete LOTS, in some order on X, the lexicographic ordered space X^2 can be non-discrete.

Lemma 1.3. *Let X be a discrete LOTS having the smallest element x_0 but not have a largest element, for instance, ω with the usual order* *is such an example. Then the lexicographic ordered space X^2 is not discrete.*

Proof. Since X is a discrete LOTS, x_0 has the immediate successor x_1 in X. Then $\langle x_1, x_0\rangle \in \mathrm{Cl}_{X^2}(\leftarrow, \langle x_1, x_0\rangle)_{X^2}$. □

It is easy to verify that if a discrete LOTS X has neither smallest nor largest elements (for instance, the LOTS $\mathbb{Z}$ of all integers is such an example), then the lexicographic ordered space X^2 is discrete. Therefore in this case, the lexicographic ordered topology coincides with the usual Tychonoff product topology. Further we remark the following.

Remark 1.4. $2 \times \mathbb{R}$ and $\mathbb{R} \times 2$ with the lexicographic orders are not homeomorphic, where $2 = \{0, 1\}$, because $2 \times \mathbb{R}$ is homeomorphic to the toplogical sum $\mathbb{R} \bigoplus \mathbb{R}$ of two $\mathbb{R}$'s, on the other hand $\mathbb{R} \times 2$ contains the subspace $\mathbb{R} \times \{0\}$ that is homeomorphic to the Sorgenfrey line. $2 \times \mathbb{I}$ and $\mathbb{I} \times 2$ with the lexicographic orders are both compat but by the the same reason, not homeomorphic. $\mathbb{I} \times 2$ is called a *Double Arrow space.*

Almost all topological properties cannot be preserved for lexicographic ordered products. Lemma 1.3 is such an example, moreover $\mathbb{R}$ is Lindelöf but $\mathbb{R}^2$ with the lexicographic order is not Lindelöf, because $\mathbb{R}^2$ contains the uncountable closed discrete subspace $\mathbb{R} \times \{0\}$. Also see Example 2.2.

Moreover remark that the lexicographic ordered space $\mathbb{I}^2$ is not homeomorphict to the subspace $\mathbb{I}^2$ of the lexicographic ordered space $\mathbb{R}^2$, because the lexicographic ordered space $\mathbb{I}^2$ is compact, but the subspace $\mathbb{I}^2$ of the lexicographic ordered space $\mathbb{R}^2$ is homeomorphic to the topological sum $\bigoplus_{x \in \mathbb{I}} \{x\} \times \mathbb{I}$ thus non-compact. Also note that the subspace $(0, 1) \times \{1\}$ of the lexicographic ordered space $\mathbb{I}^2$ is homemomorphic to the Sorgenfrey line. But the subspace $(0, 1) \times \{1\}$ of the lexicographic ordered space $\mathbb{R}^2$ is discrete closed.

Consider the usual Tyconoff product space $X \times Y$, then X can be identified with the closed subspace $X \times \{y\}$ for every $y \in Y$. However in general, this is not true for the lexicographic ordered spaces. To see this, consider the lexicographic ordered space ω_1^2. Then the subspace $\omega_1 \times \{1\}$ is discrete, because $\omega_1 \times \{1\} \cap (\langle \alpha, 0\rangle, \langle \alpha, 2\rangle)_{\omega_1^2} = \{\langle \alpha, 1\rangle\}$ holds for every $\alpha < \omega_1$. Moreover remark that $\omega_1 \times \{1\}$ is not closed in ω_1^2. But $\omega_1 \times \{0\}$ is closed in ω_1^2 and homeomorphic to ω_1.

2. Countable compactness

In this section, we consider countable compactness of lexicographic ordered spaces. First, we remark the following lemma which says that compactness is preserved by lexicographic ordered products. But, the author does not know whether it is known or not. Since it has important roles in the present paper, for reader's conveniences, we give the proof here.

Lemma 2.1. *Let $\{X_\alpha : \alpha < \gamma\}$ be a sequence of compact LOTS X_α's with the order $<_\alpha$'s, where γ is an ordinal. Then the lexicographic ordered space $X = \Pi_{\alpha<\gamma} X_\alpha$ is also compact.*

Proof. Let $<$ denote the lexicographic order on X. Put $0_\alpha = \min X_\alpha$ and $1_\alpha = \max X_\alpha$ for every $\alpha < \gamma$. Obviously $\langle 0_\alpha : \alpha < \gamma\rangle$ is the smallest element of X and $\langle 1_\alpha : \alpha < \gamma\rangle$ is the largest element of X. It suffices to see that every $A \subset X$ has the greatest lower bound $\inf_X A$ in X. By induction on $\alpha < \gamma$, we will define an element $u = \langle u(\alpha) : \alpha < \gamma\rangle$ in X and a decreasing sequence $\{A_\alpha : \alpha < \gamma\}$ of subsets of A as follows. First let $u(0) = \inf_{X_0}\{a(0) : a \in A\}$ and $A_0 = \{a \in A : a(0) = u(0)\}$. Note $u(0) = 1_0$ if $A = \emptyset$. Let $\alpha < \gamma$ and assume that $u \restriction \alpha$ and $\{A_\beta : \beta < \alpha\}$ are already defined. Set

$$u(\alpha) = \inf_{X_\alpha}\{a(\alpha) : a \in \bigcap_{\beta<\alpha} A_\beta\}$$

and

$$A_\alpha = \{a \in \bigcap_{\beta<\alpha} A_\beta : a(\alpha) = u(\alpha)\}.$$

Then $\{A_\alpha : \alpha < \gamma\}$ is decreasing sequence of subsets of A. We will see $u = \inf_X A$. Let

$$\alpha_0 = \begin{cases} \min\{\alpha < \gamma : \bigcap_{\beta<\alpha} A_\beta = \emptyset\} & \text{if exists,} \\ \gamma & \text{otherwise.} \end{cases}$$

Note that for every $\alpha < \gamma$ with $\alpha_0 \le \alpha$, $u(\alpha) = \inf_{X_\alpha} \emptyset = 1_\alpha$ holds.

Claim 1. u is a lower bound of A.

Proof. Assume indirectly that for some $a_0 \in A$, $a_0 < u$ holds. Then there is $\alpha < \gamma$ such that $a_0 \restriction \alpha = u \restriction \alpha$ and $a_0(\alpha) < u(\alpha)$. By the definition of $u(0)$, we have $0 < \alpha$ and therefore $a_0(0) = u(0)$. Thus we have $a_0 \in A_0$. Fix $\beta < \alpha$ and assume that $a_0 \in \bigcap_{\delta<\beta} A_\delta$ is proved. Now by $\beta < \alpha$, $u(\beta) = a_0(\beta)$ holds. Then we have $a_0 \in A_\beta$ because of $a_0 \in \bigcap_{\delta<\beta} A_\delta$. Thus by induction on β, we have $a_0 \in \bigcap_{\beta<\alpha} A_\beta$ and therefore $u(\alpha) \le a_0(\alpha)$, a contradiction. □

Claim 2. u is the greatest lower bound of A.

Proof. Let $u < v \in X$. It suffices to find $a \in A$ with $a < v$. Take $\alpha < \gamma$ with $u \restriction \alpha = v \restriction \alpha$ and $u(\alpha) < v(\alpha)$. By $u(\alpha) < v(\alpha) \leq 1_\alpha$, we have $\alpha < \alpha_0$ therefore $\emptyset \neq \bigcap_{\beta<\alpha} A_\beta$. Using the definition of $u(\alpha)$, take $a \in \bigcap_{\beta<\alpha} A_\beta$ with $a(\alpha) < v(\alpha)$. Note that for every $\beta < \alpha$, $a(\beta) = u(\beta)$ holds because of $a \in A_\beta$. Therefore we have $a \restriction \alpha = u \restriction \alpha = v \restriction \alpha$ and $a(\alpha) < v(\alpha)$ thus $a < v$. □

Now we have $u = \inf_X A$. □

But countable compactness is not preserved by the lexicograhic ordered products.

Example 2.2. There is a countably compact LOTS Z whose square Z^2 with the lexicograhic ordered topology is not countably compact.

To see this, let $\{x_\alpha : 0 < \alpha < \omega_1\}$ be a set of distinct points which is disjoint from ω_1. Our LOTS is $Z = \omega_1 \cup \{x_\alpha : 0 < \alpha < \omega_1\}$ with the following order $<_Z$. The order $<_Z$ on ω_1 coincides with the usual order on ω_1. The order $<_Z$ on $\{x_\alpha : 0 < \alpha < \omega_1\}$ is given by $x_\alpha <_Z x_\beta \Leftrightarrow \beta < \alpha$. Finally $x_\alpha <_Z 0 <_Z \beta$ holds for every $\alpha, \beta \in \omega_1 \setminus \{0\}$. Then obviously the LOTS Z is countably compact. Since the lexicograhic ordered space Z^2 contains the closed discrete subspace $Z \times \{0\}$, Z^2 is not countably compact.

Since countable compactness and ω-boundedness of GO-spaces coincide [2, Theorem 3], we remark:

Proposition 2.3. *The usual Tychonoff product space of arbitrary many countably compact GO-spaces is also countably compact.*

Theorem 2.4. *Let $\alpha(n)$ and γ_n be ordinals with $\alpha(n) < \gamma_n$ for every $n < \omega$ moreover let $X = \Pi_{n<\omega}\gamma_n$ be the lexicographic ordered space with the order $<$ and $Y = \Pi_{n<\omega}[0, \alpha(n)]$, where $[0, \alpha(n)] = \alpha(n) + 1$. Then:*

1. *the restriction $<\restriction Y$ of the order $<$ on Y coincides with the lexicographic order on Y,*
2. *the lexicographic ordered topology $\lambda(<\restriction Y)$ on Y coincides with the subspace topology $\tau = \lambda(<) \restriction Y$ on Y of the lexicographic ordered topology $\lambda(<)$ of X.*

Proof. (1) is obvious.

(2): We may assume $2 \leq |Y|$. $\lambda(<\restriction Y) \subset \tau$ is obvious, so for every $x \in X$, it suffices to see $(\leftarrow, x)_X \cap Y, (x, \rightarrow)_X \cap Y \in \lambda(<\restriction Y)$. Let $x \in X$. When $x \in Y$, we have $(\leftarrow, x)_X \cap Y = (\leftarrow, x)_Y \in \lambda(<\restriction Y)$. Similarly we have $(x, \rightarrow)_X \cap Y \in \lambda(<\restriction Y)$, so we may assume $x \notin Y$. Let

$$n_0 = \min\{n < \omega : x(n) > \alpha(n)\}.$$

Case 1. $x(n) = \alpha(n)$ for all $n < n_0$.

In this case, since $(\leftarrow, x)_X \cap Y \supset Y$ holds, we have $(\leftarrow, x)_X \cap Y = Y \in \lambda(<\restriction Y)$ and $(x, \rightarrow)_X \cap Y = \emptyset \in \lambda(<\restriction Y)$.

Case 2. Otherwise.

Set

$$n_1 = \max\{n < n_0 : x(n) < \alpha(n)\}.$$

Define $y \in Y$ by for each $n < \omega$,

$$y(n) = \begin{cases} x(n) & \text{if } n < n_1, \\ x(n) + 1 & \text{if } n = n_1, \\ 0 & \text{if } n > n_1, \end{cases}$$

Obviously, $x < y$ holds.

Claim 1. $(\leftarrow, x)_X \cap Y = (\leftarrow, y)_Y$.

Proof. "$\subset$" is obvious because of $x < y$.

$\supset$: Let $u \in (\leftarrow, y)_Y$. It follows from $u < y$ that for some $n_2 < \omega$, $u \restriction n_2 = y \restriction n_2$ and $u(n_2) < y(n_2)$ hold. Because of $y(n) = 0$ for every $n > n_1$, we have $n_2 \leq n_1$, therefore $x \restriction n_2 = y \restriction n_2 = u \restriction n_2$. When $n_2 < n_1$, it follows from $u(n_2) < y(n_2) = x(n_2)$ that $u < x$. Thus $u \in (\leftarrow, x)_X \cap Y$. Next we consider the case "$n_2 = n_1$". By $u(n_1) = u(n_2) < y(n_2) = y(n_1) = x(n_1) + 1$, we have $u(n_1) \leq x(n_1)$. The maximality of n_1 ensures that

$$(*)\ \alpha(n) = x(n) \text{ for every } n < n_0 \text{ with } n_1 < n.$$

Noting $u(n_0) \leq \alpha(n_0) < x(n_0)$, in the case "$u(n_1) < x(n_1)$", we evidently have $u < x$, also in the case "$u(n_1) = x(n_1)$, we have $u < x$ by (*). Therefore $u \in (\leftarrow, x)_X \cap Y$. □

Now let

$$z(n) = \begin{cases} x(n) & \text{if } n \leq n_1, \\ \alpha(n) & \text{if } n > n_1. \end{cases}$$

Then obviously $z \in Y$ and $z < x$.

Claim 2. $(x, \rightarrow)_X \cap Y = (z, \rightarrow)_Y$.

Proof. "$\subset$" is obvious because of $z < x$.

$\supset$: Let $u \in (z, \rightarrow)_Y$. It follows from $z < u$ that for some $n_2 < \omega$, $z \restriction n_2 = u \restriction n_2$ and $z(n_2) < u(n_2)$ hold. Because of $z(n) = \alpha(n)$ for every $n > n_1$ and $u \in Y$, we have $n_2 \leq n_1$, therefore $u \restriction n_2 = z \restriction n_2 = x \restriction n_2$. Also by $n_2 \leq n_1$, we have $u(n_2) > z(n_2) = x(n_2)$, we have $x < u$. Therefore $u \in (x, \rightarrow)_X \cap Y$. □

These claims show $(\leftarrow, x)_X \cap Y, (x, \rightarrow)_X \cap Y \in \lambda(<\restriction Y)$. □

Corollary 2.5. *If α and β are ordinals with $\beta < \alpha$, then the lexicographic ordered space $[0, \beta]^\omega$ is a subspace of the the lexicographic ordered space α^ω.*

Corollary 2.6. *Let γ_n be ordinals with $\mathrm{cf}\gamma_n \neq \omega$ for each $n < \omega$. Then the lexicographic ordered space $X = \Pi_{n<\omega}\gamma_n$ is countably compact.*

Proof. Let $\{x_m : m < \omega\} \subset X$. For each $n < \omega$, let $\alpha(n) = \sup\{x_m(n) : m < \omega\}$. Note $\alpha(n) < \gamma_n$ by $\mathrm{cf}\gamma_n \neq \omega$. Then $\{x_m : m < \omega\}$ is a subset of $Y = \Pi_{n<\omega}[0, \alpha(n)]$. By Lemma 2.1 and the theorem above, the subspace Y of X is compact. This argument shows that X is ω-bounded. □

Corollary 2.7. *Both lexicographic ordered spaces ω_1^ω and ω_1^2 are countably compact.*

For ω_1^2, set $\gamma_n = 1$ for every $n < \omega$ with $2 \leq n$.
The theorem above evidently shows:

Corollary 2.8. *The subspace topology on $Y = 2^\omega$ of the lexicographic ordered space $X = 3^\omega$ coincides with the lexicographic ordered topology of $Y = 2^\omega$, where $3 = \{0, 1, 2\}$.*

Example 2.9. The subspace topology on $Y = 2^{\omega+1}$ of the lexicographic ordered space $X = 3^{\omega+1}$ does not coincide with the lexicographic ordered topology of $Y = 2^{\omega+1}$.

To see this, define $x \in Y$ and $z \in X$ by:

$$x(n) = \begin{cases} 0 & \text{if } n < \omega, \\ 1 & \text{if } n = \omega, \end{cases}$$

$$z(n) = \begin{cases} 0 & \text{if } n < \omega, \\ 2 & \text{if } n = \omega. \end{cases}$$

Obviously, z is the immediate successor of x in X therefore $(\leftarrow, z)_X \cap Y = (\leftarrow, x]_Y$ is open with respect to the subspace topology on Y of the lexicographic ordered space space X.

On the other hand, x does not have an immediate successor in Y. To see this, let $x < u \in Y$. Take $n_0 \leq \omega$ with $x \restriction n_0 = u \restriction n_0$ and $x(n_0) < u(n_0)$. It follows from $1 = x(\omega)$ and $u(n_0) \leq 1$ that $n_0 < \omega$. Let $y \in Y$ by:

$$y(n) = \begin{cases} 0 & \text{if } n \leq n_0, \\ 1 & \text{if } n_0 < n \leq \omega. \end{cases}$$

Then we have $x < y < u$. Therefore x does not have an immediate successor in Y. This shows that $(\leftarrow, x]_Y$ is not open with respect to the lexicographic ordered topology of $Y = 2^{\omega+1}$.

This argument also shows that the closed subspace $(x, \rightarrow)_Y = [z, \rightarrow)_X \cap Y$ of Y, where Y is considered as the subspace of the lexicographic ordered space X, has the open cover $\{(u, \rightarrow)_Y : x < u \in Y\}$ which does not have a finite subcover. Thus the subspace Y of X is not compact.

Example 2.10. The lexicographic ordered space $X = \omega_1^{\omega+1}$ is not countably compact.

To see this, for each $m < \omega$, define $x_m \in X$ by:

$$x_m(n) = \begin{cases} n & \text{if } m \neq n < \omega, \\ n+1 & \text{if } m = n < \omega, \\ 0 & \text{if } n = \omega, \end{cases}$$

for every $n \leq \omega$. Note that the sequence $\{x_m : m < \omega\}$ is strictly decreasing. It suffices to see that $F = \{x_m : m < \omega\}$ is closed discrete in X. Let $x \in X$. If there is $m < \omega$ with $x_m < x$, then $(x_m, \rightarrow)$ is a neighborhood of x which meets F with at most finite members. So we may assume that $x \leq x_m$ for all $m < \omega$. In particular, by $x \leq x_1$, we have $0 \leq x(0) \leq x_1(0) = 0$ therefore we see $x(0) = 0$.

Case 1. $x(n) < n$ for some $n < \omega$.

In this case, take such a least $n_0 < \omega$. Then $x(n) \geq n$ for all $n < n_0$. Assuming $x(n) > n$ for some $n < n_0$, take such a least $n_1 < \omega$. Then $n_1 > 0$ and for all $n < n_1$, $x(n) = n$. Now we have $x_{n_1+1}(n) = x(n) = n$ for all $n < n_1$ and $x_{n_1+1}(n_1) = n_1 < x(n_1)$. Thus $x_{n_1+1} < x$ holds, a contradiction. Therefore for all $n < n_0$, we have $x(n) = n$. For each $n \leq \omega$, let

$$b(n) = \begin{cases} n & \text{if } n < n_0, \\ x(n)+1 & \text{if } n = n_0, \\ 0 & \text{if } n_0 < n \leq \omega. \end{cases}$$

Then $x < b$ and $b(n_0) = x(n_0) + 1 \leq n_0$. So we have $b(n) \leq n$ for all $n < \omega$. To see $(\leftarrow, b) \cap F = \emptyset$, assume on the contrary that $x_m < b$ for some $m < \omega$. Take $n_1 < \omega$ with $x_m \restriction n_1 = b \restriction n_1$ and $x_m(n_1) < b(n_1)$. We see $n_1 \leq n_0$ because $b(n) = 0$ whenever $n_0 < n$. Therefore we have $n_1 \leq x_m(n_1) < b(n_1) \leq n_1$, a contradiction.

Case 2. $x(n) \geq n$ for all $n < \omega$.

If there were $n < \omega$ with $x(n) > n$, then take such a least n_0. Then $x_{n_0+1} < x$ holds, a contradiction. Thus we have $x(n) = n$ for all $n < \omega$.

For each $n \leq \omega$, let

$$b(n) = \begin{cases} n & \text{if } n < \omega, \\ x(n) + 1 & \text{if } n = \omega. \end{cases}$$

Then $x < b$ and $(\leftarrow, b) \cap F = \emptyset$.

Remark 2.11. The referee of the present paper informed to the author that there is a simple way to see Example 2.10 as follows.

First we show "If $\{x_n : n \in \omega\}$ is a strictly decreasing sequence in a LOTS L, then in the lexicographic ordered space $L \times \omega_1$, $\{\langle x_n, 0\rangle : n \in \omega\}$ is closed discrete". Now let $X = \omega_1^{\omega+1}$ and $L = \omega_1^{\omega}$, then $X = L \times \omega_1$. Define $x_n \in L$ by

$$x_n(m) = \begin{cases} m & \text{if } m \neq n, \\ m + 1 & \text{if } m = n. \end{cases}$$

Then obviously $\{x_n : n \in \omega\}$ is a strictly decreasing sequence in L. Apply the above fact.

3. The lexicographic ordered topology versus the usual product topology

Remark that the lexicographic ordered topology λ on $X = (\omega+1)\times\omega$ cannot be compared with the usual Tychonoff product topology τ on X, because of, $\langle 1, 0\rangle \in \mathrm{Cl}_\lambda\{0\}\times\omega$, $\langle 1, 0\rangle \notin \mathrm{Cl}_\tau\{0\}\times\omega$, $\langle \omega, 1\rangle \notin \mathrm{Cl}_\lambda\omega\times\{1\}$ and $\langle \omega, 1\rangle \in \mathrm{Cl}_\tau\omega\times\{1\}$. On the other hand, the lexicographic ordered topology on $\omega \times (\omega + 1)$ coincides with the usual Tychonoff product topology. In this section, we discuss when these topologies on products of ordinals are comparable.

At first, we consider the lexicographic ordered products of infinite length.

Lemma 3.1. *Let X_n be a discrete LOTS (having any order) with $|X_n| \geq 1$ for every $n < \omega$. Then the lexicographic ordered topology λ on $X = \Pi_{n<\omega}X_n$ is weaker than the usual Tychonoff product topology τ on X, that is, $\lambda \subset \tau$.*

Proof. We may assume $|X| \geq 2$. It suffices to see that $(a, \rightarrow), (\leftarrow, a) \in \tau$ for every $a \in X$. Let $a \in X$ and $x \in (a, \rightarrow)$. Fix $n_0 < \omega$ with $a \restriction n_0 = x \restriction n_0$ and $a(n_0) < x(n_0)$. Let $U = \{y \in X : \forall n \leq n_0(y(n) = x(n))\}$. Then U is τ-open with $x \in U \subset (a, \rightarrow)$. Therefore $(a, \rightarrow)$ is τ-open. $(\leftarrow, a) \in \tau$ is similar. □

Corollary 3.2. *A LOTS X is discrete iff the lexicographic ordered topology on X^2 is weaker than the usual Tychonoff product topology on X^2.*

Proof. One direction follows from the lemma above. The other direction follows from the proof of Lemma 1.2. □

Theorem 3.3. *Let γ be an ordinal with $\gamma \geq \omega$ and for every $\alpha < \gamma$ let β_α be an ordinal with $2 \leq \beta_\alpha$. Then the lexicographic ordered topology λ on $X = \Pi_{\alpha<\gamma}\beta_\alpha$ is weaker than the usual Tychonoff product topology τ on X iff $\gamma = \omega$ and for every $\alpha < \gamma$, $\beta_\alpha \leq \omega$ holds.*

Proof. "if" part follows from Lemma 3.1. To see the other direction, assume $\lambda \subset \tau$.

Claim 1. $\gamma = \omega$.

Proof. Assume $\omega < \gamma$. Put

$$Y = \{x \in X : \forall\alpha \leq \omega(x(\alpha) \in 2), \forall\alpha < \gamma(\omega < \alpha \to x(\alpha) = 0)\}.$$

Let x be the smallest element of X, that is, $x(\alpha) = 0$ for every $\alpha < \gamma$. Then note $x \in Y$. For every $m < \omega$, define $x_m \in Y$ by

$$x_m(\alpha) = \begin{cases} 0 & \text{if } m \neq \alpha, \\ 1 & \text{if } m = \alpha, \end{cases}$$

for every $\alpha < \gamma$. Then obviously $x \in \mathrm{Cl}_\tau\{x_m : m < \omega\}$ holds. On the other hand, define $b \in Y$ by

$$b(\alpha) = \begin{cases} 0 & \text{if } \alpha \neq \omega, \\ 1 & \text{if } \alpha = \omega, \end{cases}$$

for every $\alpha < \gamma$. Then $x < b < x_m$ for every $m < \omega$, thus $x \notin \mathrm{Cl}_\lambda\{x_m : m < \omega\}$ holds. Therefore $\lambda \not\subset \tau$, a contradiction. □

Claim 2. $\beta_n \leq \omega$ for every $n < \gamma = \omega$.

Proof. Assume $\omega < \beta_{n_0}$ for some $n_0 < \gamma$. Define $x_m, x, a \in X$, where $m < \omega$, by:

$$x_m(n) = \begin{cases} 0 & \text{if } n < n_0, \\ m & \text{if } n = n_0, \\ 1 & \text{if } n_0 < n < \gamma, \end{cases}$$

$$x(n) = \begin{cases} 0 & \text{if } n < n_0, \\ \omega & \text{if } n = n_0, \\ 1 & \text{if } n_0 < n < \gamma, \end{cases}$$

$$a(n) = \begin{cases} 0 & \text{if } n < n_0, \\ \omega & \text{if } n = n_0, \\ 0 & \text{if } n_0 < n < \gamma, \end{cases}$$

for every $n < \gamma = \omega$. Then obviously $x \in \mathrm{Cl}_\tau\{x_m : m < \omega\}$ holds. And the element a witnesses $x \notin \mathrm{Cl}_\lambda\{x_m : m < \omega\}$. Therefore $\lambda \not\subset \tau$, a contradiction. □

□

Theorem 3.4. *Let γ be an ordinal with $\gamma \geq \omega$ and for every $\alpha < \gamma$ let β_α be an ordinal with $2 \leq \beta_\alpha$. Then the lexicographic ordered topology λ on $X = \Pi_{\alpha<\gamma}\beta_\alpha$ coincides with the usual Tychonoff product topology τ on X iff $\gamma = \omega$, $\beta_0 \leq \omega$ and for every $\alpha < \gamma$ with $1 \leq \alpha$, $\beta_\alpha < \omega$ holds.*

Proof. "only if" part: It follows from the theorem above that $\gamma = \omega$ and $\beta_n \leq \omega$ for every $n < \gamma = \omega$. Assume $\beta_{n_0} = \omega$ for some $n_0 < \omega$ with $1 \leq n_0$. Define $x_m, x \in X$, where $m < \omega$, by:

$$x_m(n) = \begin{cases} 0 & \text{if } n \neq n_0, \\ m & \text{if } n = n_0, \end{cases}$$

$$x(n) = \begin{cases} 0 & \text{if } n \neq n_0 - 1, \\ 1 & \text{if } n = n_0 - 1, \end{cases}$$

for every $n < \gamma = \omega$. Then $x_m < x$ holds for every $m < \omega$.

To see $x \in \mathrm{Cl}_\lambda\{x_m : m < \omega\}$, let $a < x$. Take $n_1 < \omega$ with $a \restriction n_1 = x \restriction n_1$ and $a(n_1) < x(n_1)$. By the definition of x, it has to be $n_1 = n_0 - 1$ and $a(n_1) < x(n_1) = x(n_0 - 1) = 1$. Therefore we have $a(n_1) = 0$. Take $m < \omega$ with $a(n_0) < m$, then $a < x_m$. This argument shows $x \in \mathrm{Cl}_\lambda\{x_m : m < \omega\}$.

Let $U = \{y \in X : y(n_0 - 1) = 1, y(n_0) = 0\}$. Then U is a τ-neighborhood of x which is disjoint from $\{x_m : m < \omega\}$, thus $x \notin \mathrm{Cl}_\tau\{x_m : m < \omega\}$. Hence $\tau \not\subset \lambda$, a contradiction.

"if" part: Assume that $2 \leq \beta_0 \leq \omega$ and for every $n < \omega$ with $1 \leq n$, $2 \leq \beta_n < \omega$. We shall show that the lexicographic ordered topology λ on $X = \Pi_{n<\omega}\beta_n$ coincides with the usual Tychonoff topology τ on X. $\lambda \subset \tau$ follows from Lemma 3.1. To see $\tau \subset \lambda$, let $n_0 < \omega$, $p \in \Pi_{n<n_0}\beta_n$ and $U = \{x \in X : p = x \restriction n_0\}$. It suffices to see:

Claim. $U \in \lambda$.

Proof. Since $p = \emptyset$ and $U = X \in \lambda$ when $n_0 = 0$, we may assume $n_0 > 0$. For every $n < \omega$, let 0_n and 1_n denote $\min \beta_n$ and $\max \beta_n$ respectively if exists, that is, $0_n = 0$ for every $n < \omega$, and $1_n = \beta_n - 1$ for every $n < \omega$ with $1 \leq n$. Moreover, $1_n = \beta_n - 1$ holds when $n = 0$ and $\beta_n < \omega$. In the case "$n = 0$ and $\beta_n = \omega$", for notational convenience, we let $1_n = \omega$.

Case 1. $p(n) = 0_n$ for every $n < n_0$.

Define $b \in X$ by

$$b(n) = \begin{cases} p(n) & \text{if } n < n_0 - 1, \\ p(n) + 1 & \text{if } n = n_0 - 1, \\ 0_n & \text{if } n > n_0 - 1, \end{cases}$$

for every $n < \omega$. Then it is straightforward to see $U = (\leftarrow, b) \in \lambda$.

Case 2. $p(n) = 1_n$ for every $n < n_0$.

Define $a \in X$ by

$$a(n) = \begin{cases} p(n) & \text{if } n < n_0 - 1, \\ p(n) - 1 & \text{if } n = n_0 - 1, \\ 1_n & \text{if } n > n_0 - 1, \end{cases}$$

for every $n < \omega$. Then $U = (a, \rightarrow) \in \lambda$ holds.

Case 3. Otherwise.

We consider 3 subcases.

Subcase 1. $p(n_0 - 1) = 0_{n_0 - 1}$.

In this case, let $n_1 = \max\{n < n_0 - 1 : 0_n < p(n)\}$. Define $a, b \in X$ by

$$a(n) = \begin{cases} p(n) & \text{if } n < n_1, \\ p(n) - 1 & \text{if } n = n_1, \\ 1_n & \text{if } n > n_1, \end{cases}$$

$$b(n) = \begin{cases} p(n) & \text{if } n < n_0 - 1, \\ p(n) + 1 & \text{if } n = n_0 - 1, \\ 0_n & \text{if } n > n_0 - 1, \end{cases}$$

for every $n < \omega$.

We shall show $U = (a, b) \in \lambda$. "$U \subset (a, b)$" is obvious. To see "$(a, b) \subset U$", let $x \in (a, b)$. By $a < x$, take $n_2 < \omega$ with $a \restriction n_2 = x \restriction n_2$ and $a(n_2) < x(n_2)$. By the definition of a, we have $n_2 \leq n_1$. Then we have $n_2 = n_1$, otherwise $b < x$, a contradiction. Therefore $x \restriction n_1 = p \restriction n_1$. It follows from $x(n_1) = x(n_2) > a(n_2) = a(n_1) = p(n_1) - 1$ that $x(n_1) \geq p(n_1)$. Now we have $x(n_1) = p(n_1)$, otherwise $b < x$, a contradiction. Moreover we have $x(n) = 0_n$ for every $n < \omega$ with $n_1 < n < n_0 - 1$, otherwise $b < x$, a contradiction. Thus we have $x \restriction (n_0 - 1) = p \restriction (n_0 - 1) = b \restriction (n_0 - 1)$. Finally we have $x(n_0 - 1) = 0$, otherwise $x \geq b$, a contradiction. Therefore we have $x \restriction n_0 = p$ thus $x \in U$.

Subcase 2. $p(n_0 - 1) = 1_{n_0-1}$.

In this case, let $n_1 = \max\{n < n_0 - 1 : p(n) < 1_n\}$. Define $a, b \in X$ by

$$a(n) = \begin{cases} p(n) & \text{if } n < n_0 - 1, \\ p(n) - 1 & \text{if } n = n_0 - 1, \\ 1_n & \text{if } n > n_0 - 1, \end{cases}$$

$$b(n) = \begin{cases} p(n) & \text{if } n < n_1, \\ p(n) + 1 & \text{if } n = n_1, \\ 0_n & \text{if } n > n_1, \end{cases}$$

for every $n < \omega$. Then similarly we have $U = (a, b) \in \lambda$.

Subcase 3. $0_{n_0-1} < p(n_0 - 1) < 1_{n_0-1}$.

In this case, define $a, b \in X$ by

$$a(n) = \begin{cases} p(n) & \text{if } n < n_0 - 1, \\ p(n) - 1 & \text{if } n = n_0 - 1, \\ 1_n & \text{if } n > n_0 - 1, \end{cases}$$

$$b(n) = \begin{cases} p(n) & \text{if } n < n_0 - 1, \\ p(n) + 1 & \text{if } n = n_0 - 1, \\ 0_n & \text{if } n > n_0 - 1, \end{cases}$$

for every $n < \omega$. Then it is easy to see $U = (a, b) \in \lambda$. □

□

These theorems imply:

Corollary 3.5. *The lexicographic ordered topology on $\omega \times 2^\omega = \omega \times 2 \times 2 \times \cdots$ coincides with its usual Tychonoff product topology. The lexicographic ordered topology on ω^ω strictly weaker than its usual Tychonoff product topology.*

Corollary 3.6. *Let β_n be an ordinal with $2 \leq \beta_n < \omega$ for every $n < \omega$. Then the lexicographic ordered space $X = \Pi_{n<\omega}\beta_n$ is homeomorphic to the Cantor set $\mathbb{C}$, that is, $\mathbb{C} = 2^\omega$ with the usual Tychonoff product topology. In particular, the lexicographic ordered spaces 2^ω and 3^ω are homeomorphic each other and metrizable.*

Proof. It is well-known that every compact zero-dimensional second countable space without isolated points is homeomorphic to the Cantor set, for instance see [5, Theorem 1.5.5]. Here a space is said to be *zero-dimensional* if it has a base consisting clopen sets. Using this characterization, we easily show that the usual Tychonoff product space

$X = \Pi_{n<\omega}\beta_n$ is homeomorphic to the Cantor set. Now apply Theorem 3.4. □

Example 3.7. The lexicographic ordered spaces $2^{\omega+1}$ and $3^{\omega+1}$ are not homeomorphic.

To see this, first note that the lexicographic ordered space $2^{\omega+1}$ has only two isolated points, that is, the smallest and the largest elements. On the other hand, all elements of $A = \{x \in 3^{\omega+1} : x(\omega) = 1\}$ are isolated in the lexicographic ordered space $3^{\omega+1}$. Because for each $x \in A$, let $x^-, x^+ \in 3^{\omega+1}$ by

$$x^-(n) = \begin{cases} x(n) & \text{if } n < \omega, \\ 0 & \text{if } n = \omega, \end{cases}$$

$$x^+(n) = \begin{cases} x(n) & \text{if } n < \omega, \\ 2 & \text{if } n = \omega, \end{cases}$$

for every $n \leq \omega$. then $\{x\} = (x^-, x^+)_{3^{\omega+1}}$ for each $x \in A$.

Example 3.8. The lexicographic ordered space $X = 2^{\omega+1}$ is not metrizable.

To see this, assume that X is metrizable. Since it is compact, it has a countable base $\{B_n : n < \omega\}$ for X. Let $A = \{x \in X : x(\omega) = 0\}$. Note $|A| = 2^\omega$, here 2^ω means the cardinality of 2^ω. Remark that every $x \in A$ has the immediate successor x^+ in X defined by

$$x^+(n) = \begin{cases} x(n) & \text{if } n < \omega, \\ 1 & \text{if } n = \omega, \end{cases}$$

for every $n \leq \omega$. Therefore $(\leftarrow, x]_X$ is open in X for every $x \in A$. So for every $x \in A$, one can fix $n(x) < \omega$ with $x \in B_{n(x)} \subset (\leftarrow, x]_X$. By $\mathrm{cf}2^\omega > \omega$, for some $A' \subset A$ with $|A'| = 2^\omega$ and $n < \omega$, $n(x) = n$ holds for every $x \in A'$. Pick any $x, y \in A'$ with $x < y$. Then $x \in B_n \subset (\leftarrow, x]_X \not\ni y$, a contradiction. Therefore X is not metrizable.

The situation of products of finite sequences of ordinals is somewhat different from that of infinite sequences.

Theorem 3.9. *Let $2 \leq n_0 < \omega$ and for every $n \leq n_0$, β_n be an ordinal with $2 \leq \beta_n$. Then the lexicographic ordered topology λ on $X = \Pi_{n \leq n_0}\beta_n$ is weaker than the usual Tychonoff product topology τ on X iff for every $n < n_0$, $\beta_n \leq \omega$ holds.*

Proof. "only if" part: Assume $\lambda \subset \tau$ and $\beta_{n_1} > \omega$ for some $n_1 < n_0$. Define $x_m, x \in X$, where $m < \omega$, by:

$$x_m(n) = \begin{cases} 0 & \text{if } n \notin \{n_1, n_1+1\}, \\ m & \text{if } n = n_1, \\ 1 & \text{if } n = n_1 + 1, \end{cases}$$

$$x(n) = \begin{cases} 0 & \text{if } n \notin \{n_1, n_1+1\}, \\ \omega & \text{if } n = n_1, \\ 1 & \text{if } n = n_1 + 1, \end{cases}$$

for every $n \leq n_0$. Then we have $x \in \mathrm{Cl}_\tau\{x_m : m < \omega\}$. But $a \in X$ defined by

$$a(n) = \begin{cases} 0 & \text{if } n \neq n_1, \\ \omega & \text{if } n = n_1, \end{cases}$$

for every $n \leq n_0$ witnesses $x \notin \mathrm{Cl}_\lambda\{x_m : m < \omega\}$, a contradiction.

"if" part: Assume $\beta_n \leq \omega$ for every $n < n_0$. It suffices to see $(a, \rightarrow), (\leftarrow, a) \in \tau$ for every $a \in X$. Let $a \in X$ and $x \in (a, \rightarrow)$. Take $n_1 \leq n_0$ with $a \restriction n_1 = x \restriction n_1$ and $a(n_1) < x(n_1)$. In the case "$n_1 < n_0$", let $U = \{y \in X : \forall n \leq n_1(y(n) = x(n))\}$. Then U is a τ-neighborhood of x contained in $(a, \rightarrow)$. In the case "$n_1 = n_0$", let $U = \{y \in X : \forall n < n_1(y(n) = x(n)), a(n_0) < y(n_0)\}$. Then U is a τ-neighborhood of x contained in $(a, \rightarrow)$. Therefore $(a, \rightarrow) \in \tau$. $(\leftarrow, a) \in \tau$ is similar. □

Theorem 3.10. *Let $2 \leq n_0 < \omega$ and for every $n \leq n_0$, β_n be an ordinal with $2 \leq \beta_n$. Then the lexicographic ordered topology λ on $X = \Pi_{n \leq n_0}\beta_n$ coincides with the usual Tychonoff product topology τ on X iff $\beta_0 \leq \omega$, for every $n < n_0$ with $1 \leq n$, $\beta_n < \omega$ holds moreover β_{n_0} is a successor ordinal, that is,* $\mathrm{cf}\beta_{n_0} = 1$.

Proof. "only if" part: Assume $\lambda = \tau$. We have $\beta_n \leq \omega$ for every $n < n_0$ by the previous theorem, in particular $\beta_0 \leq \omega$.

Assume that for some $n_1 < n_0$ with $1 \leq n$, $\beta_{n_1} = \omega$ holds. Define $x_m, x \in X$, where $m < \omega$, by:

$$x_m(n) = \begin{cases} 0 & \text{if } n \neq n_1, \\ m & \text{if } n = n_1, \end{cases}$$

$$x(n) = \begin{cases} 0 & \text{if } n \neq n_1 - 1, \\ 1 & \text{if } n = n_1 - 1, \end{cases}$$

for every $n \leq n_0$. Then as we have seen somewhere above, we have $x \in \mathrm{Cl}_\lambda\{x_m : m < \omega\}$ and $x \notin \mathrm{Cl}_\tau\{x_m : m < \omega\}$, a contradiction.

Assume that β_{n_0} is limit. Define $x_\alpha, x \in X$, where $\alpha < \beta_{n_0}$, by:

$$x_\alpha(n) = \begin{cases} 0 & \text{if } n \neq n_0, \\ \alpha & \text{if } n = n_0, \end{cases}$$

$$x(n) = \begin{cases} 0 & \text{if } n \neq n_0 - 1, \\ 1 & \text{if } n = n_0 - 1, \end{cases}$$

for every $n \leq n_0$. Then we have $x \in \mathrm{Cl}_\lambda\{x_\alpha : \alpha < \beta_{n_0}\}$ and $x \notin \mathrm{Cl}_\tau\{x_\alpha : \alpha < \beta_{n_0}\}$, a contradiction. Therefore β_{n_0} is successor.

"if" part: Assume that $\beta_0 \leq \omega$, $\beta_n < \omega$ $(1 \leq n < n_0)$ and β_{n_0} is successor.

To see "$\lambda \subset \tau$", let $a \in X$. It suffices to see $(a, \rightarrow), (\leftarrow, a) \in \tau$. Let $x \in (a, \rightarrow)$ and take $n_1 \leq n_0$ with $a \restriction n_1 = x \restriction n_1$ and $a(n_1) < x(n_1)$. In the case "$n_1 < n_0$", $U = \{y \in X : y \restriction (n_1 + 1) = x \restriction (n_1 + 1)\}$ is a τ-neighborhood of x contained in $(a, \rightarrow)$.

In the case "$n_1 = n_0$", $U = \{y \in X : y \restriction n_0 = x \restriction n_0, a(n_0) < y(n_0)\}$ is a τ-neighborhood of x contained in $(a, \rightarrow)$. Thus we see $(a, \rightarrow) \in \tau$. $(\leftarrow, a) \in \tau$ is similar.

To see "$\tau \subset \lambda$", for every $p \in \Pi_{n<n_0}\beta_n$, let $X_p = \{y \in X : y \restriction n_0 = p\}$. Note that X_p is a convex set with respect to the lexicographic order $<$ on X, therefore we have $\lambda(<\restriction X_p) = \lambda(<) \restriction X_p$ $(= \lambda \restriction X_p)$. Also note that X_p is τ-open, because β_n is discrete for every $n < n_0$. Since $\lambda(<\restriction X_p)$ coincides with $\tau \restriction X_p$, it suffices to see:

Claim. $X_p \in \lambda$ for every $p \in \Pi_{n<n_0}\beta_n$.

But by letting $U = X_p$, the proof of this claim is very similar to that of the Claim in Theorem 3.4. So we leave it to the readers. □

Remark 3.11. The referee also gives the following easy proof of the "if" part of the theorem above.

Assume that $\beta_0 \leq \omega$, $\beta_n < \omega$ $(1 \leq n < n_0)$ and β_{n_0} is successor. Then it is easy to see that $\Pi_{n<n_0}\beta_n$ is finite or homeomorphic to ω in both lexicographic and usual Tychonoff product topology. Thus we may regard $\Pi_{n\leq n_0}\beta_n$ as $k \times \beta_{n_0}$ for some $k \leq \omega$. It is obvious that whenever $k \leq \omega$, $k \times \beta_{n_0}$ is homeomorphic to the topological sum of k many copies of β_{n_0} in both lexicographic and usual Tychonoff product topology since β_{n_0} is successor.

Remark 3.12. Using the theorems above, for instance, we see the following, where as above λ and τ denote the lexicographic ordered topology and the usual Tychonoff topology on X respectively.

- If X is one of $\omega_1 \times 2$ and $\omega \times \omega_1 \times \omega$, then $\lambda \not\subset \tau$.
- If X is one of $\omega \times \omega \times \omega_1$ and $\omega \times \omega \times (\omega_1 + 1)$, then $\lambda \subsetneq \tau$

- If X is one of $\omega\times(\omega+1)$, $\omega\times(\omega_1+1)$ and $\omega\times 2\times 3\times 4\times(\omega_1+1)$, then $\lambda=\tau$

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