ORDERABILITY OF PRODUCTS

NOBUYUKI KEMOTO

Abstract. We prove that for non-discrete spaces X and Y,

(1) If the product space $X \times Y$ is suborderable, then both X and Y are hereditarily paracompact and there is a unique regular infinite cardinal κ such that for every $z \in X \cup Y$, the cofinality from left (right) of z (this notion will be defined precisely below) is either $0, 1$ or κ.

(2) If X and Y are subspaces of an ordinal, then the converse implication of (1) is also true.

Recently, a kind of orderability of X^2 is known to be related to selection theory, see [5, 3]. In this paper, we see the results in the abstract.

Spaces mean regular topological spaces. Let $<$ be a linear order on a set X. $\lambda(<)$ denotes the usual order topology, that is, the topology generated by

$$\{(a, \rightarrow) : a \in X\} \cup \{(\leftarrow, b) : b \in X\}$$

as a subbase, where $(a, \rightarrow) = \{x \in X : a < x\}$, $(a, b) = \{x \in X : a < x < b\}$,..., etc. If necessary, we write $<_X$ and $(a, b)_X$ instead of $<$ and (a, b) respectively. A *LOTS* X means the triple $\langle X, <, \lambda(<)\rangle$. LOTS is an abbreviation of "Linearly Ordered Topological Space". As usual, we consider an ordinal α as the set of smaller ordinals and as a LOTS with the order $\in$ (we identify it with $<$). Similarly a *Generalized Ordered space* (*GO-space*) means the triple $\langle X, <, \tau\rangle$ where τ is a topology on X with $\lambda(<) \subset \tau$ which has a base consisting convex sets, where a subset A is *convex* if $(a, b) \subset A$ whenever $a, b \in A$ with $a < b$.

A topological space $\langle X, \tau\rangle$, where τ is a topology on X, is said to be *orderable* if $\tau = \lambda(<)$ for some linear order $<$ on X. Also a topological space $\langle X, \tau\rangle$ is said to be *suborderable* if it is a subspace of some orderable space. It is well-known that orderable spaces are hereditarily normal. Also it is well-known that:

(1) If $\langle L, <_L, \lambda(<_L)\rangle$ is a LOTS and $X \subset L$, then $\langle X, <_L\restriction X, \lambda(<_L)\restriction X\rangle$ is a GO-space, where $<_L\restriction X$ is the restricted order of $<_L$ to

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X and $\lambda(<_L)\restriction X$ is the subspace topology of $\lambda(<_L)$ on X, that is $\{U \cap X : U \in \lambda(<_L)\}$. On the other hand:

(2) If $\langle X, <_X, \tau\rangle$ is a GO-space, then there is a LOTS $\langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ such that the space $\langle X, \tau\rangle$ is a dense subspace of $\langle L, \lambda(<_L)\rangle$ and $<_X = <_L\restriction X$, therefore $\langle X, \tau\rangle$ is suborderable. Obviously a suborderable space is a GO-space with some linear order. Moreover:

(3) If $\langle X, <_X, \lambda(<_X)\rangle$ is a LOTS, then there is a LOTS $\langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ and $<_X = <_L\restriction X$ such that the space $\langle L, \lambda(<_L)\rangle$ is compact and contains $\langle X, \lambda(<_X)\rangle$ as a dense subspace. Therefore by (2) and (3), we have:

(4) If $\langle X, <_X, \tau\rangle$ is a GO-space, then there is a compact LOTS $\langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ and $<_X = <_L\restriction X$ such that the compact space $\langle L, \lambda(<_L)\rangle$ contains $\langle X, \tau\rangle$ as a dense subspace. We say this situation as "a GO space $\langle X, <_X, \tau\rangle$ has a linearly ordered compactification $\langle L, <_L, \lambda(<_L)\rangle$"or more simply "a GO-space X has a linearly ordered compactification L".

Remark that a compact LOTS $\langle L, <_L, \lambda(<_L)\rangle$ has the largest element $\max L$ and the smallest element $\min L$. Also remark that if X is a convex subset of a LOTS $\langle L, <_L, \lambda(<_L)\rangle$, then the subspace topology $\lambda(<_L)\restriction X$ coincides with the order topology $\lambda(<\restriction X)$ on X. For more details, see [10] and [8]. Usually, if there are no confusion, we do not distinguish the symbols $<_X$ and $<_L$, and simply write $<$.

In general, a GO-space can have many linearly ordered compactifications. But it is known that a GO-space X has a linearly ordered compactification lX such that for every linearly ordered compactification cX of X, there is a continuous function $f : cX \to lX$ with $f(x) = x$ for every $x \in X$, see [9]. Observe that by the definition, lX is unique up to order isomorphisms. lX is said to be the minimal linearly ordered compactification of X and it is characterized as follows:

Lemma 1. [9, Lemma 2.1] *A linearly ordered compactification cX of a GO-space X is minimal if and only if $(a, b)_{cX} \neq \emptyset$ for every $a, b \in cX \setminus X$ with $a < b$.*

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a pairwise disjoint collection of spaces X_α's. $\bigoplus_{\alpha\in\Lambda} X_\alpha$ denotes the topological sum of X_α's, that is, the space $\bigcup_{\alpha\in\Lambda} X_\alpha$ with the topology generated by $\bigcup_{\alpha\in\Lambda} \tau_\alpha$ as a base, where τ_α is the topology on X_α. Remark that the subspace $\{0\} \cup (1, 2)$ of the real line is suborderable but not orderable. This means that the topological sum of orderable spaces need not be orderable. On the other hand, the infinite discrete space $D(\kappa)$ of cardinality κ is orderable, because the

LOTS $\kappa \times \mathbb{Z}$ with the lexicographic order is homeomorphic to $D(\kappa)$, where $\mathbb{Z}$ is the set of integers.

Let S be a subset of an ordinal α. $\mathrm{Lim}_\alpha(S)$ denotes the set $\{\beta \in \alpha : \sup(S \cap \beta) = \beta\}$, that is, the set of all cluster points of S in α. If the contexts are clear, we simply write as $\mathrm{Lim}(S)$. Obviously if S is closed in α, then $\mathrm{Lim}(S) \subset S$. $\mathrm{Succ}(S)$ denotes the set $S \setminus \mathrm{Lim}(S)$, that is the set of all isolated points of S.

A subset S of a regular uncountable cardinal κ is *stationary* if it intersects with all closed unbounded (club) set C of κ, where a subset C of κ is unbounded if for every $\alpha < \kappa$, there is $\beta \in C$ with $\alpha \leq \beta$. Note that if S is unbounded in κ, then $\mathrm{Lim}(S)$ is club in κ.

Lemma 2. *Let S be a stationary set in a regular uncountable cardinal κ and X a non-discrete space of cardinality $< \kappa$. Then the subspace $X \times S$ of $X \times \kappa$ is not hereditarily normal.*

Proof. Let x be a non-isolated point of X and $Y = (X \setminus \{x\}) \times S \cup \{x\} \times \mathrm{Succ}(S)$. Then it is routine to check that $F_0 = \{x\} \times \mathrm{Succ}(S)$ and $F_1 = (X \setminus \{x\}) \times (S \cap \mathrm{Lim}(S))$ are disjoint closed sets in Y which cannot be separated by disjoint open sets. □

Lemma 3. *Let κ and λ be regular infinite cardinals with $\kappa \neq \lambda$. Then the subspace $(\mathrm{Succ}(\kappa) \cup \{\kappa\}) \times (\mathrm{Succ}(\lambda) \cup \{\lambda\})$ of $(\kappa + 1) \times (\lambda + 1)$ is not suborderable.*

Proof. Let $X = \mathrm{Succ}(\kappa) \cup \{\kappa\}$ and $Y = \mathrm{Succ}(\lambda) \cup \{\lambda\}$ and assume that $X \times Y$ is suborderable. Denote the product topology of $X \times Y$ by τ. Fix a linearly ordered set $\langle L, <_L \rangle$ such that $X \times Y \subset L$ and $\lambda(<_L) \restriction X \times Y = \tau$, where $\lambda(<_L)$ denotes the order topology on L. Denote the restricted order $<_L\restriction X \times Y$ on $X \times Y$ by $<$. We may assume $\omega \leq \kappa < \lambda$. Let $F_0 = \{\kappa\} \times \mathrm{Succ}(\lambda)$ and $F_1 = \mathrm{Succ}(\kappa) \times \{\lambda\}$. Put

$$\begin{aligned} F_0^- &= \{\beta \in \mathrm{Succ}(\lambda) : \langle \kappa, \beta \rangle < \langle \kappa, \lambda \rangle\}, \\ F_0^+ &= \{\beta \in \mathrm{Succ}(\lambda) : \langle \kappa, \lambda \rangle < \langle \kappa, \beta \rangle\}, \\ F_1^- &= \{\alpha \in \mathrm{Succ}(\kappa) : \langle \alpha, \lambda \rangle < \langle \kappa, \lambda \rangle\}, \\ F_1^+ &= \{\alpha \in \mathrm{Succ}(\kappa) : \langle \kappa, \lambda \rangle < \langle \alpha, \lambda \rangle\}. \end{aligned}$$

Note $F_0 = \{\kappa\} \times (F_0^- \cup F_0^+)$ and $F_1 = (F_1^- \cup F_1^+) \times \{\lambda\}$.

Claim 1. $|F_1^-| < \kappa$ or $|F_1^+| < \kappa$.

Proof. Assume that both F_1^- and F_1^+ have cardinality κ. For every $\alpha \in F_1^-$, since $(\leftarrow, \langle \kappa, \lambda \rangle)_L \cap X \times Y$ is a τ-neighborhood of $\langle \alpha, \lambda \rangle$ in $X \times Y$, there is $g(\alpha) < \lambda$ such that $\{\alpha\} \times (g(\alpha), \lambda] \cap X \times Y \subset (\leftarrow, \langle \kappa, \lambda \rangle)_L \cap X \times Y$, where $(\leftarrow, \langle \kappa, \lambda \rangle)_L$ denotes the interval in L and

$(g(\alpha),\lambda]$ denotes the usual interval in $\lambda+1$. Similarly for every $\alpha\in F_1^+$, we can find $g(\alpha)<\lambda$ such that $\{\alpha\}\times(g(\alpha),\lambda]\cap X\times Y\subset(\langle\kappa,\lambda\rangle,\to)_L\cap X\times Y$.

Put $\beta_0=\sup\{g(\alpha):\alpha\in F_1^-\cup F_1^+\}$. Then by $\kappa<\lambda$, we have $\beta_0<\lambda$. Pick $\beta\in(\beta_0,\lambda)\cap\mathrm{Succ}(\lambda)$. We may assume $\beta\in F_0^-$, then $\langle\kappa,\beta\rangle<_L\langle\kappa,\lambda\rangle$. On the other hand, by $|F_1^+|=\kappa$ and $F_1^+\times\{\beta\}\subset(\langle\kappa,\lambda\rangle,\to)_L$, we have $\langle\kappa,\beta\rangle\in\mathrm{Cl}_\tau F_1^+\times\{\beta\}\subset[\langle\kappa,\lambda\rangle,\to)_L$. Therefore $\langle\kappa,\lambda\rangle\leq_L\langle\kappa,\beta\rangle$, a contradiction. □

Now we may assume $|F_1^+|<\kappa$, then $|F_1^-|=\kappa$ and $\langle\kappa,\lambda\rangle\in\mathrm{Cl}_\tau F_1^-\times\{\lambda\}\subset(\leftarrow,\langle\kappa,\lambda\rangle]_L$

Claim 2. $|F_0^+|=\lambda$.

Proof. Assume $|F_0^+|<\lambda$, then $|F_0^-|=\lambda$. Therefore we have $\langle\kappa,\lambda\rangle\in\mathrm{Cl}_\tau\{\kappa\}\times F_0^-\subset(\leftarrow,\langle\kappa,\lambda\rangle]_L$. For every $\beta\in F_0^-$, since $(\langle\kappa,\beta\rangle,\to)_L\cap X\times Y$ is a τ-neighborhood of $\langle\kappa,\lambda\rangle$ and $\langle\kappa,\lambda\rangle\in\mathrm{Cl}_\tau F_1^-\times\{\lambda\}$, there is $\alpha(\beta)\in F_1^-$ such that $\langle\kappa,\beta\rangle<_L\langle\alpha(\beta),\lambda\rangle$. Since $\kappa<\lambda$, there are $\alpha_0\in F_1^-$ and $F\subset F_0^-$ of size λ such that $\alpha(\beta)=\alpha_0$ for each $\beta\in F$. Note $\langle\alpha_0,\lambda\rangle<_L\langle\kappa,\lambda\rangle$. Then $\{\kappa\}\times F\subset(\leftarrow,\langle\alpha_0,\lambda\rangle)_L$, therefore $\mathrm{Cl}_\tau\{\kappa\}\times F\subset(\leftarrow,\langle\alpha_0,\lambda\rangle]_L$. On the other hand, it follows from $|F|=\lambda$ that $\langle\kappa,\lambda\rangle\in\mathrm{Cl}_\tau\{\kappa\}\times F$, thus $\langle\kappa,\lambda\rangle\leq_L\langle\alpha_0,\lambda\rangle$, a contradiction. □

Now for each $\beta\in F_0^+$, it follows from $\langle\kappa,\lambda\rangle<_L\langle\kappa,\beta\rangle$ that there is $f(\beta)<\kappa$ such that

$$(*)\ ((\mathrm{Succ}(\kappa)\cup\{\kappa\}\cap(f(\beta),\kappa])\times\{\beta\}\subset(\langle\kappa,\lambda\rangle,\to)_L.$$

By $\kappa<\lambda$, there are $\alpha_0<\kappa$ and $F\subset F_0^+$ of cardinality λ such that $f(\beta)=\alpha_0$ for every $\beta\in F$.

Since $|F_1^-|=\kappa$, one can pick $\alpha\in F_1^-$ with $\alpha_0<\alpha$. Then $\langle\alpha,\lambda\rangle<_L\langle\kappa,\lambda\rangle$. On the other hand by $(*)$, we have $\{\alpha\}\times F\subset(\langle\kappa,\lambda\rangle,\to)_L$, therefore $\langle\alpha,\lambda\rangle\in\mathrm{Cl}_\tau\{\alpha\}\times F\subset[\langle\kappa,\lambda\rangle,\to)_L$, a contradiction. □

Definition 4. Let κ be a regular infinite cardinal, $\mathcal{X}=\{X_\alpha:\alpha\in\Lambda\}$ a pairwise disjoint collection of non-empty spaces and x_0 a point with $x_0\notin\bigcup_{\alpha\in\Lambda}X_\alpha$, where $\Lambda\subset\kappa$. Put $X=(\bigcup_{\alpha\in\Lambda}X_\alpha)\cup\{x_0\}$ and equip the topology τ generated by

$$(\bigcup_{\alpha\in\Lambda}\tau_\alpha)\cup\{(\bigcup_{\alpha\in\Lambda\cap(\gamma,\kappa)}X_\alpha)\cup\{x_0\}:\gamma<\kappa\}$$

as a base, where τ_α is the topology on X_α. We call this topological space $\langle X,\tau\rangle$ as 1-*point extension of the topological sum* $\bigoplus_{\alpha\in\Lambda}X_\alpha$ *with the* κ*-limit point* x_0 and denoted by $X(\mathcal{X},x_0)$.

In the definition above, remark:

- For every $\alpha \in \Lambda$, X_α is clopen in X. Thus the topological sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ is a subspace of X.
- x_0 has a neighborhood base of cardinality $\leq \kappa$.
- Λ is unbounded in κ iff x_0 is a non-isolated point of X.

Now let C be a club set in a regular infinite cardinal κ and $\alpha < \kappa$. Let

$$\alpha_C^- = \sup(C \cap \alpha),\ \alpha_C^+ = \min\{\beta \in C : \alpha < \beta\},$$

where $\sup \emptyset = -1$. If contexts are clear, then we usually write simply α^- and α^+. Remark that $\alpha \in \mathrm{Succ}(C)$ iff $\alpha^- < \alpha$ and that $\alpha < \alpha^+$ for every $\alpha \in C$.

Lemma 5. *Let κ be a regular infinite cardinal, $\mathcal{X} = \{X_\alpha : \alpha \in \Lambda\}$ a pairwise disjoint collection of non-empty suborderable spaces with $\Lambda \subset \mathrm{Succ}(C)$ for some club set C of κ and $x_0 \notin \bigcup_{\alpha\in\Lambda} X_\alpha$. Then the 1-point extension $X(\mathcal{X}, x_0)$ of $\bigoplus_{\alpha\in\Lambda} X_\alpha$ with the κ-limit point x_0 is suborderable.*

Proof. For every $\alpha \in \Lambda$, pick a compact LOTS $\langle L_\alpha, <_\alpha, \lambda(<_\alpha)\rangle$ such that $\langle L_\alpha, \lambda(<_\alpha)\rangle$ contains $\langle X, \tau_\alpha\rangle$ as a dense subspace, where τ_α denotes the topology on X_α. For every $\alpha \in C \setminus \Lambda$, let $L_\alpha = \{l_\alpha\}$ be a one point set with the trivial order $<_\alpha$. By taking isomorphic compact LOTS', we may assume that $\{L_\alpha : \alpha \in C\}$ is pairwise disjoint with $x_0 \notin \bigcup_{\alpha\in C} L_\alpha$. Let $L = (\bigcup_{\alpha\in C} L_\alpha) \cup \{x_0\}$ and define a linear order $<_L$ on L as follows:

- for every $x \in \bigcup_{\alpha\in C} L_\alpha$, $x <_L x_0$, that is, $x_0 = \max L$,
- if $x, y \in L_\alpha$ for some $\alpha \in C$, then $x <_L y$ is defined by $x <_\alpha y$,
- if $x \in L_\alpha$ and $y \in L_\alpha$ with $\alpha, \beta \in C$ and $\alpha \neq \beta$, then $x <_L y$ is defined by $\alpha < \beta$.

Then obviously $<_L\restriction L_\alpha$ coincides with $<_\alpha$ for every $\alpha \in C$.

Claim 1. For every $\alpha \in \mathrm{Succ}(C)$, L_α is open in $\langle L, \lambda(<_L)\rangle$.

Proof. It follows from $L_\alpha = (\max L_{\alpha^-}, \min L_{\alpha^+})_L$ that L_α is open in $\langle L, \lambda(<_L)\rangle$. □

Claim 2. For every $\alpha \in C$, $\langle L_\alpha, \lambda(<_\alpha)\rangle$ is a convex closed subspace of $\langle L, \lambda(<_L)\rangle$.

Proof. Since L_α is represented as $L_\alpha = [\min L_\alpha, \max L_\alpha]_L$, it is closed and convex. Therefore $\lambda(<_L) \restriction L_\alpha = \lambda(<_L\restriction L_\alpha) = \lambda(<_\alpha)$. □

Since $\lambda(<_\alpha) \restriction X_\alpha = \tau_\alpha$ for each $\alpha \in \Lambda$, by the claim above, we have:

Claim 3. For every $\alpha \in \Lambda$, $\langle X_\alpha, \tau_\alpha\rangle$ is a subspace of $\langle L, \lambda(<_L)\rangle$.

To finish the proof, it suffices to see:

Claim 4. $\tau = \lambda(<_L) \restriction X$, where τ denotes the topology of $X = X(\mathcal{X}, x_0)$.

Proof. First we prove $\tau \subset \lambda(<_L) \restriction X$. Let $\mathcal{B}$ be the base $(\bigcup_{\alpha\in\Lambda} \tau_\alpha) \cup \{(\bigcup_{\alpha\in\Lambda\cap(\gamma,\kappa)} X_\alpha) \cup \{x_0\} : \gamma < \kappa\}$ of τ. It suffices to see $\mathcal{B} \subset \lambda(<_L) \restriction X$. Let $U \in \mathcal{B}$.

Case 1. $U \in \tau_\alpha$ for some $\alpha \in \Lambda$.

In this case by Claim 3, there is $V \in \lambda(<_L)$ with $V \cap X_\alpha = U$. By Claim 1, we have $X_\alpha = X \cap L_\alpha \in \lambda(<_L) \restriction X$. Therefore $U = V \cap X_\alpha = (V \cap X) \cap X_\alpha \in \lambda(<_L) \restriction X$ holds.

Case 2. $U = (\bigcup_{\alpha\in\Lambda\cap(\gamma,\kappa)} X_\alpha) \cup \{x_0\}$ for some $\gamma < \kappa$.

In this case, let $\alpha_0 = \min(\Lambda \cap (\gamma, \kappa))$. Then we have $\alpha_0 \in \Lambda \subset \mathrm{Succ}(C)$ and $U = ((\bigcup_{\alpha\in(\alpha_0^-,\kappa)\cap C} L_\alpha) \cup \{x_0\}) \cap X = (\max L_{\alpha_0^-}, x_0]_L \cap X \in \lambda(<_L) \restriction X$.

Next we show $\tau \supset \lambda(<_L) \restriction X$. Let $z \in L$. It suffices to see the following two facts.

Fact 1. $(\leftarrow, z)_L \cap X \in \tau$.

In the case $z = x_0$, $(\leftarrow, z)_L \cap X = \bigcup_{\alpha\in\Lambda} X_\alpha \in \tau$ holds. So we may assume $z \neq x_0$. Take $\alpha \in C$ with $z \in L_\alpha$. If $\alpha \notin \Lambda$, then $(\leftarrow, z)_L \cap X = \bigcup_{\beta\in\Lambda\cap\alpha} X_\beta \in \tau$. If $\alpha \in \Lambda$, then by Claim 3, we have $(\leftarrow, z)_L \cap X_\alpha \in \tau_\alpha \subset \tau$, therefore $(\leftarrow, z)_L \cap X = (\bigcup_{\beta\in\Lambda\cap\alpha} X_\beta) \cup ((\leftarrow, z)_L \cap X_\alpha) \in \tau$.

Fact 2. $(z, \rightarrow)_L \cap X \in \tau$.

In the case $z = x_0$, $(z, \rightarrow)_L \cap X = \emptyset \in \tau$. So we may assume $z \neq x_0$. Take $\alpha \in C$ with $z \in L_\alpha$. If $\alpha \notin \Lambda$, then $(z, \rightarrow)_L \cap X = (\bigcup_{\beta\in\Lambda\cap(\alpha,\kappa)} X_\beta) \cup \{x_0\} \in \tau$. If $\alpha \in \Lambda$, then by Claim 3, we have $(z, \rightarrow)_L \cap X_\alpha \in \tau_\alpha \subset \tau$, therefore $(z, \rightarrow)_L \cap X = (\bigcup_{\beta\in\Lambda\cap(\alpha,\kappa)} X_\beta) \cup ((z, \rightarrow)_L \cap X_\alpha) \in \tau$. □

□

The following corollary is well-known by different approaches.

Corollary 6. *If $\mathcal{X} = \{X_\alpha : \alpha \in \Lambda\}$ is a pairwise disjoint collection of non-empty suborderable spaces, then the topological sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ is also suborderable.*

Proof. We may assume that all X_α's are non-empty. Take a suitably large regular infinite cardinal κ with $|\Lambda| \leq \kappa$ and we may assume $\Lambda \subset \mathrm{Succ}(\kappa)$. By the lemma above, $X(\mathcal{X}, x_0)$ is suborderable for some x_0. Therefore the subspace $\bigoplus_{\alpha\in\Lambda} X_\alpha$ of $X(\mathcal{X}, x_0)$ is suborderable. □

This corollary shows:

Corollary 7. *If X is a suborderable space and Y is a discrete space, then $X \times Y$ is suborderable.*

Therefore when we discuss suborderability of $X \times Y$, we may assume that both X and Y are non-discrete. Additionally remark that, if X is an orderable space and Y is a discrete space, then $X \times Y$ is orderable.

Corollary 8. *Let κ be a regular infinite cardinal. Then $X = (\mathrm{Succ}(\kappa) \cup \{\kappa\})^2$ is suborderable.*

Proof. For every $\alpha \in \mathrm{Succ}(\kappa)$. let

$$X_\alpha = (\{\alpha\} \times [\alpha, \kappa] \cap X) \bigoplus ((\alpha, \kappa] \times \{\alpha\} \cap X),$$

moreover let

$$\mathcal{X} = \{X_\alpha : \alpha \in \mathrm{Succ}(\kappa)\}.$$

Then obviously $\mathcal{X}$ is a pairwise disjoint collection of suborderable spaces. One can check that the both topologies of X and $X(\mathcal{X}, \langle \kappa, \kappa \rangle)$ coincide by carefully comparing the both neighborhood bases at $\langle \kappa, \kappa \rangle$. The lemma above shows that X is suborderable. □

In particular, $(\omega + 1)^2$ is suborderable ([7]).
Remark:

Lemma 9. [1, Problem 3.12.3(a)] *Let $\langle L, <, \lambda(<) \rangle$ be a LOTS. Then the following are equivalent:*

1. *The space $\langle L, \lambda(<) \rangle$ is compact.*
2. *For every subset A of L, A has the least upper bound $\sup_L A$ in $\langle L, < \rangle$.*
3. *For every subset A of L, A has the greatest lower bound $\inf_L A$ in $\langle L, < \rangle$.*

Note that $\sup \emptyset = \min L$ and $\inf \emptyset = \max L$ whenever L is a compact LOTS.

Definition 10. Let L be a compact LOTS and $x \in L$. A subset $A \subset (\leftarrow, x)_L$ is said to be 0-*unbounded* for x in L if for every $y < x$, there is $a \in A$ with $y \leq a$. Similarly for a subset $A \subset (x, \rightarrow)_L$, "1-unbounded for x" is defined. Now 0-cofinality $0\text{-}\mathrm{cf}_L\, x$ of x in L is defined by:

$$0\text{-}\mathrm{cf}_L\, x = \min\{|A| : A \text{ is 0-unbounded for } x \text{ in } L.\}.$$

Also $1\text{-}\mathrm{cf}_L\, x$ is defined. If there are no confusion, we write simply $0\text{-}\mathrm{cf}\, x$ and $1\text{-}\mathrm{cf}\, x$. Observe that

- if x is the smallest element of L, then $0\text{-}\mathrm{cf}\, x = 0$,

- if x has the immediate predecessor in L, then $0\text{-}\operatorname{cf} x = 1$,
- otherwise, then $0\text{-}\operatorname{cf} x$ is a regular infinite cardinal.

Moreover, remark:

- $\omega \leq 0\text{-}\operatorname{cf} x$ iff $\sup_L(\leftarrow, x)_L = x$ iff $x \in \operatorname{Cl}_L(\leftarrow, x)_L$.

If $0\text{-}\operatorname{cf} x = \kappa$, then we can define a strictly increasing function $c : \kappa \to L$ which is continuous with its range $c[\kappa]$ 0-unbounded for x. We call such a function c as a 0-normal function for x in L. The reader should remark that these methods in compact LOTS' extend the usual methods in ordinal numbers.

Observe that in the notation above, for every closed set F of κ, $c[F]$ is also closed in $(\leftarrow, x)_L$. Therefore c is an embedding such that $c[\kappa]$ is closed in $(\leftarrow, x)$ and 0-unbounded for x. Note that there can be many 0-normal functions for x in L.

Also note that if cX and $c'X$ are two linearly ordered compactifications of a GO-space X, then $i\text{-}\operatorname{cf}_{cX} x$ coincides with $i\text{-}\operatorname{cf}_{c'X} x$ for every $x \in X$ and $i \in 2 = \{0, 1\}$. In our discussion, we apply these methods for $L = lX$ with a GO-space X and consider $0\text{-}\operatorname{cf}_{lX} x$ or $1\text{-}\operatorname{cf}_{lX} x$ for $x \in lX$. In particular, if X is a subspace of an ordinal, say $X \subset [0, \gamma]$, with the usual order, then we can check using Lemma 1 $lX = \operatorname{Cl}_{[0,\gamma]} X$. Moreover in this case, for every $x \in lX$, obviously $1\text{-}\operatorname{cf} x$ is 0 or 1, furthermore we can easily check that $0\text{-}\operatorname{cf} x$ is equal to $\operatorname{cf} x$ in the usual sense whenever $x \in \operatorname{Lim}(X)$. Let X be a GO-space, $x \in X$, $\kappa = 0\text{-}\operatorname{cf} x \geq \omega$ and fix a 0-normal function $c : \kappa \to lX$. Inductively one can take a strictly increasing sequence $\{x(\alpha) : \alpha < \kappa\} \subset (\leftarrow, x)_{lX} \cap X$ with $\sup(\{c(\beta) : \beta \leq \alpha\} \cup \{x(\beta) : \beta < \alpha\}) < x(\alpha)$. Then obviously $\{x(\alpha) : \alpha < \kappa\} \cup \{x\}$ is homeomorphic to $\operatorname{Succ}(\kappa) \cup \{\kappa\}$. Similarly whenever X is a subspace of an ordinal and $\alpha \in X \cap \operatorname{Lim}(X)$, one can fix a strictly increasing sequence $\{\alpha(\gamma) : \gamma < \kappa\} \subset X$ which is cofinal in α such that $\{\alpha(\gamma) : \gamma < \kappa\} \cup \{\alpha\}$ is homeomorphic to $\operatorname{Succ}(\kappa) \cup \{\kappa\}$, where $\kappa = \operatorname{cf} \alpha$.

Engelking and Lutzer [2] proved that a suborderable space is paracompact iff it does not have a closed subspace which is homeomorphic to a stationary set in a regular uncountable cardinal. Therefore:

Lemma 11 ([2]). *A suborderable space is hereditarily paracompact iff it does not have a subspace which is homeomorphic to a stationary set in a regular uncountable cardinal.*

Now we have prepared to find properties implied by suborderability of product spaces. Remark that if the product space $X \times Y$ is suborderable, then both X and Y are suborderable. Therfore we may assume that X and Y are GO-spaces under the assumption that $X \times Y$ is suborderable.

Theorem 12. *Let X and Y be non-discrete GO-spaces. If the product space $X \times Y$ is suborderable, then*

(1) *X and Y are hereditarily paracompact,*
(2) *there is a unique regular infinite cardinal κ such that for every $z \in X \cup Y$ and $i \in 2$, $i\text{-}\operatorname{cf} z$ is $0, 1$ or κ, where $i\text{-}\operatorname{cf} z$ means $i\text{-}\operatorname{cf}_{lX} z$ ($i\text{-}\operatorname{cf}_{lY} z$) whenever $z \in X$ ($z \in Y$ respectively).*
(3) *X or Y are hereditarily disconnected.*

Proof. Assume that $X \times Y$ is suborderable. Fix a linearly ordered set $\langle L, <_L\rangle$ such that $X \times Y$ is a subspace of $\langle L, \lambda(<_L)\rangle$.

(1): We will see that Y is hereditarily paracompact (the case for X is similar). Assume not, then by Lemma 11, there is a subspace which is homeomorphic to a stationary set S in a regular uncountable cardinal in κ. Since X is non-discrete, there is $i \in 2$ and $x \in X$ with $\lambda = i\text{-}\operatorname{cf}_{lX} x \geq \omega$. As mentioned above, X has a subspace which is homeomorphic to $\operatorname{Succ}(\lambda) \cup \{\lambda\}$.

Case 1. $\lambda < \kappa$.

In this case, by Lemma 2, the hereditarily normal space $X \times Y$ has a non-hereditarily normal subspace, a contradiction.

Case 2. $\kappa \leq \lambda$.

In this case, since S is stationary, we can take $\alpha \in S \cap \operatorname{Lim}(S)$. Set $\mu = \operatorname{cf} \alpha$, then $\mu < \lambda$. As mentioned above, S has a subspace which is homeomorphic to $\operatorname{Succ}(\mu) \cup \{\mu\}$. Then the suborderable space $X \times Y$ contains a subspace which is homeomorphic to $(\operatorname{Succ}(\lambda) \cup \{\lambda\}) \times (\operatorname{Succ}(\mu) \cup \{\mu\})$. This contradicts Lemma 3.

(2): Assume that (2) does not hold. Since both X and Y are non-discrete, there are $x \in X$, $y \in Y$ and $i, j \in 2$ with $i\text{-}\operatorname{cf} x \geq \omega$, $j\text{-}\operatorname{cf} y \geq \omega$ and $i\text{-}\operatorname{cf} x \neq j\text{-}\operatorname{cf} y$. Set $\kappa = i\text{-}\operatorname{cf} x$ and $\lambda = j\text{-}\operatorname{cf} y$. Then the suborderable space $X \times Y$ contains a subspace which is homeomorphic to $(\operatorname{Succ}(\kappa) \cup \{\kappa\}) \times (\operatorname{Succ}(\lambda) \cup \{\lambda\})$. This contradicts Lemma 3.

(3): Recall that a space is *hereditarily disconnected* if every non-empty connected subset is a one-point set. Assume neither X nor Y is hereditarily disconnected. Then there are connected subsets C and D of X and Y respectively with $2 \leq |C|$ and $2 \leq |D|$. Fix $x_0, x_1 \in C$ and $y_0, y_1 \in D$ with $x_0 \neq x_1$ and $y_0 \neq y_1$. We may assume $\langle x_0, y_0\rangle <_L \langle x_0, y_1\rangle <_L \langle x_1, y_1\rangle$, otherwise change the indeces. Then $\langle x_1, y_0\rangle \in C \times \{y_0\} \cap \{x_1\} \times D$, moreover both $C \times \{y_0\}$ and $\{x_1\} \times D$ are connected. Therefore $C \times \{y_0\} \cup \{x_1\} \times D$ is a connected subset of $X \times Y \setminus \{\langle x_0, y_1\rangle\}$

containing the points $\langle x_0, y_0\rangle$ and $\langle x_1, y_1\rangle$. On the other hand, the disjoint open sets $(\leftarrow, \langle x_0, y_1\rangle)_L \cap X \times Y$ and $(\langle x_0, y_1\rangle, \rightarrow)_L \cap X \times Y$ separate the connected set $C \times \{y_0\} \cup \{x_1\} \times D$, a contradiction. $\square$

Whenever X and Y are subspaces of an ordinal, then the converse implication of the theorem above is also true:

Theorem 13. *Let X and Y be non-discrete subspaces of an ordinal. Then the product space $X \times Y$ is suborderable, if*

(1) *X and Y are hereditarily paracompact,*
(2) *there is a unique regular infinite cardinal κ such that for every $z \in X \cup Y$ and $i \in 2$, $\mathrm{cf}\, z$ is either $0, 1$ or κ, equivalently for every $z \in (X \cap \mathrm{Lim}(X)) \cup (Y \cap \mathrm{Lim}(Y))$, $\mathrm{cf}\, z = \kappa$.*

Proof. Note that every subspace of an ordinal is hereditarily disconnected. We may assume $X \cup Y \subset [0, \gamma]$ for some ordinal γ. It suffices to see that by induction on $\alpha \le \gamma$, $(X \cap [0, \alpha]) \times Y$ is suborderable (because $\alpha = \gamma$ finishes the proof). Assume that $\alpha \le \gamma$ and for every $\alpha' < \alpha$, $(X \cap [0, \alpha']) \times Y$ is suborderable.

Case 1. $\alpha \notin \mathrm{Lim}(X)$.

In this case, let $\alpha' = \sup(X \cap \alpha)$. By $\alpha' < \alpha$, since $(X \cap [0, \alpha]) \times Y$ is homeomorphic to $(X \cap [0, \alpha']) \times Y \bigoplus (X \cap \{\alpha\}) \times Y$, it is suborderable by the assumption.

Case 2. $\alpha \in \mathrm{Lim}(X)$.

Set $\lambda = \mathrm{cf}\, \alpha$ and fix a normal function $c : \lambda \to \alpha$ for α, that is, it is a strictly increasing continuous cofinal function into α, where $c(-1) = -1$. Since λ is homeomorphic to $c[\lambda]$, by Lemma 11, $c^{-1}[X]$ is non-stationary in λ whenever λ is uncountable.

Subcase 1. $\alpha \notin X$.

When $\lambda = \omega$, $(X \cap [0, \alpha]) \times Y$ is homeomorphic to $\bigoplus_{n \in \omega}(X \cap (c(n-1), c(n)]) \times Y$. When $\omega < \lambda$, taking a club set C in λ with $C \cap c^{-1}[X] = \emptyset$, $(X \cap [0, \alpha]) \times Y$ is homeomorphic to $\bigoplus_{\delta \in \mathrm{Succ}(C)}(X \cap (c(\delta^-), c(\delta)]) \times Y$. In either cases, $(X \cap [0, \alpha]) \times Y$ is suborderable by the inductive assumption.

Subcase 2. $\alpha \in X$.

By the assumption (2), we have $\lambda = \kappa$. We will see by induction $\beta \le \gamma$ that $(X \cap [0, \alpha]) \times (Y \cap [0, \beta])$ is suborderable (then $\beta = \gamma$ finishes this subcase). Assume that $\beta \le \gamma$ and for every $\beta' < \beta$, $(X \cap [0, \alpha]) \times (Y \cap [0, \beta'])$ is suborderable. It suffices to check the case $\beta \in Y \cap \mathrm{Lim}(Y)$, because other cases are similar to Case 1 and Subcase

1 of Case 2. By the assumption (2), we have $\operatorname{cf}\beta = \kappa$. Let $d : \kappa \to \beta$ be a normal function for β. When $\kappa = \omega$, let $C = \omega$. When $\kappa > \omega$, by Lemma 11, take a club set C of κ with $C \cap (c^{-1}[X] \cup d^{-1}[Y]) = \emptyset$. For every $\delta \in \operatorname{Succ}(C)$, let $Z_\delta =$

$$(X\cap(c(\delta^-), \alpha])\times(Y\cap(d(\delta^-), d(\delta)]) \bigoplus (X\cap(c(\delta^-), c(\delta)])\times(Y\cap(d(\delta), \beta]).$$

By the inductive assumption, Z_δ is suborderable. Put $\Lambda = \{\delta \in \operatorname{Succ}(C) : Z_\delta \neq \emptyset\}$ and $\mathcal{Z} = \{Z_\delta : \delta \in \Lambda\}$. Note that $\mathcal{Z}$ is pairwise disjoint. It is easy to see that $(X \cap [0, \alpha]) \times (Y \cap [0, \beta]) = (\bigcup_{\delta\in\Lambda} Z_\delta) \cup \{\langle\alpha, \beta\rangle\}$ and whose product topology coincides with topology of the 1-point extension of $\bigoplus_{\delta\in\Lambda} Z_\delta$ with the κ-limit point $\langle\alpha, \beta\rangle$. It follows from Lemma 5 that $(X\cap[0, \alpha])\times(Y\cap[0, \beta])$ is suborderable. $\square$

Note that the product of two subspaces of an ordinal is scattered (= every subspace has an isolated point), and that scattered suborderable spaces are orderable ([11]). Thus in Theorem 13, "suborderable" is replaced by "orderable".

Example 14. The square $\mathbb{S}^2$ of the Sorgenfrey line $\mathbb{S}$ with the usual order satisfies (1),(2) and (3) with $X = Y = \mathbb{S}$ in Theorem 12. But $\mathbb{S}^2$ is not suborderable.

Because, it is well-known that $\mathbb{S}$ is hereditarily paracompact and hereditarily disconnected. Since $\mathbb{S}^2$ is not normal, it is not suborderable. We check (2). We may assume $\mathbb{S} = (0, 1)$ with the usual order and the topology induced by $\{(a, \to) : a \in (0, 1)\} \cup \{(\leftarrow, b] : b \in (0, 1)\}$, where $(0, 1)$ denotes the unit open interval. Then using Lemma 1 and 9, it is easy to check $l\mathbb{S} = [0, 1] \times \{0\} \cup (0, 1) \times \{1\}$ with the lexicographic order identifying $\mathbb{S}$ with $(0, 1) \times \{0\}$. Then for every $x \in l\mathbb{S}$ and $i \in 2$, i-$\operatorname{cf} x$ is either $0, 1$ or ω.

Question 15. For non-discrete suborderable spaces X and Y, characterize suborderability of $X \times Y$.

Concerning monotonical normality, the following are known:

- If $X \times Y$ is monotonically normal and if Y contains a countable set with a limit point, then X is stratifiable ([6]).
- If X^2 is monotonically normal, then X is hereditarily paracompact and X^n is monotonically normal for each finite n ([4]).

So we also ask:

Question 16. Characterize suborderable spaces X and Y for which $X \times Y$ is monotonically normal.

References

[1] R. Engelking, *General Topology.* Herdermann Verlag, Berlin (1989).

[2] R. Engelking and D. Lutzer, *Paracompactness in ordered spaces* Fund. Math., **94** (1977), 49–58.

[3] S. Garcia-Ferreira, K. Miyazaki and T. Nogura, *Continuous weak selections for products* , Top. Appl., **160** (2013), 2465–2472.

[4] P. Gartside, *Monotone normality in products* , Top. Appl., **191** (1999), 621–640.

[5] V.Gutev and T. Nogura, *Weak orderability of topological spaces* , Top. Appl., **157** (2010), 1249–1274.

[6] R. Heath, D. Lutzer, P. Zenor, *Monotonically normal spaces*, Trans. Amer. Math. Soc., **178** (1973), 481–493.

[7] H. Herrich, *Ordnungsfähigkeit topologischer Räume*, Inaugural-Dissertation zur Erlangung der Doktorwürde der Mathematisch-Naturwissenschaftlichen Fakultät der Freien Universität Berlin (1962), 72 pp.

[8] R. Kaufman, *Ordered sets and compact spaces*, Colloq. Math., **17** (1967), 35–39.

[9] N. Kemoto, *Normality of products of GO-spaces and cardinals*, Topology Proc., **18** (1993), 133–142.

[10] D.J. Lutzer, *On generalized ordered spaces*, Dissertationes Math. Rozprawy Mat., **89** (1971).

[11] S. Purisch, *Scattered compactifications and the orderability of scattered spaces*, Proc. Amer. Math. Soc., **95** (1985), 636–640.

Faculty of Education, Oita University, Dannoharu, Oita, 870-1192, Japan

E-mail addresses:
nkemoto@cc.oita-u.ac.jp