ORTHOCOMPACTNESS VERSUS NORMALITY IN HYPERSPACES

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Abstract. For a regular space X, 2^X denotes the collection of all non-empty closed sets of X with the Vietoris topology and $\mathcal{K}(X)$ denotes the collection of all non-empty compact sets of X with the subspace topology of 2^X. In this paper, we will prove:

- $\mathcal{K}(\gamma)$ is orthocompact iff either $\mathrm{cf}\gamma \leq \omega$ or γ is a regular uncountable cardinal, as a corollary normality and orthocompactness of $\mathcal{K}(\gamma)$ are equivalent for every non-zero ordinal γ.

We present its two proofs, one proof uses the elementary submodel techniques and another does not. This also answers Question C of [4]. Moreover we discuss the natural question whether 2^ω is orthocompact or not. We prove that

- 2^ω is orthocompact iff it is countably metacompact,
- The hyperspace $\mathcal{K}(\mathbb{S})$ of the Sorgenfrey line $\mathbb{S}$ *is* orthocompact therefore so is the Sorgenfrey plane $\mathbb{S}^2$.

1. Introduction

Throughout spaces are assumed to be regular. $\alpha, \beta, \gamma, ...$ stand for ordinals, while $i, j, k, ...$ for natural numbers. For the notational convenience, we consider -1 as the immediate predecessor of the ordinal 0. Ordinals are considered as spaces with the usual order topology. For an ordinal γ, $\mathrm{cf}\gamma$ denotes the cofinality of γ and $\mathrm{Lim}(\gamma)$ denotes the set of all limit ordinals in γ. $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the set of all reals, rationals and integers respectively.

For a space X, we let 2^X, or $\mathcal{K}(X)$ denote the collection of all non-empty closed sets, or of all non-empty compact sets, respectively, of X.

We consider 2^X with the so-called Vietoris topology τ_V, and $\mathcal{K}(X)$ its subspace. X is called the base space, and 2^X and $\mathcal{K}(X)$ the hyperspaces or the exponential spaces of X.

To describe τ_V, we need some notation. For every finite collection $\mathcal{V}$ of open subsets of X, let

$$\langle \mathcal{V} \rangle_{2^X} = \left\{ F \in 2^X : F \subset \bigcup \mathcal{V}, \forall V \in \mathcal{V} (V \cap F \neq \emptyset) \right\},$$

$$\langle \mathcal{V} \rangle_{\mathcal{K}(X)} = \left\{ F \in \mathcal{K}(X) : F \subset \bigcup \mathcal{V}, \forall V \in \mathcal{V} (V \cap F \neq \emptyset) \right\}.$$

Observe that $\langle \mathcal{V} \rangle_{2^X} \cap \mathcal{K}(X) = \langle \mathcal{V} \rangle_{\mathcal{K}(X)}$. Then the collection of all subsets of 2^X of the form $\langle \mathcal{V} \rangle_{2^X}$ is a base for τ_V. Obviously, $\mathcal{K}(X)$ has the base of the form $\langle \mathcal{V} \rangle_{\mathcal{K}(X)}$. For the simplicity's sake, we will often write $\langle \mathcal{V} \rangle$ instead of $\langle \mathcal{V} \rangle_{2^X}$ or $\langle \mathcal{V} \rangle_{\mathcal{K}(X)}$, if the context is clear. E. Michael [6] established basic properties of hyperspaces.

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Observe that whenever $\mathcal{B}$ is a base for a space X, $\{\langle\mathcal{V}\rangle_{\mathcal{K}(X)} : \mathcal{V} \in [\mathcal{B}]^{<\omega}\}$ forms a base for $\mathcal{K}(X)$, where $[\mathcal{B}]^{<\omega}$ denotes the set of all finite subsets of $\mathcal{B}$. So we call such a $\langle\mathcal{V}\rangle_{\mathcal{K}(X)}$ *basic open* if $\mathcal{V} \in [\mathcal{B}]^{<\omega}$. We always fix the base $\mathcal{B} = \{(\alpha, \beta] : -1 \leq \alpha < \beta < \gamma\}$ for an ordinal γ. Therefore open sets of form $\langle\{(\alpha_i, \beta_i] : i < n\}\rangle_{\mathcal{K}(\gamma)}$, where $1 \leq n \in \omega$ and $-1 \leq \alpha_i < \beta_i < \gamma$, are basic open in $\mathcal{K}(\gamma)$. In particular, when $\delta < \gamma$, basic open sets of form $\langle\{(\alpha_i, \beta_i] : i < n\}\rangle_{\mathcal{K}(\gamma)}$ is said to be $< \delta$-*basic open* if $\beta_i < \delta$ for each $i < n$. $\leq \delta$-basic open sets are similarly defined. Observe that even if $\mathcal{B}$ is a base for a space X, $\{\langle\mathcal{V}\rangle_{2^X} : \mathcal{V} \in [\mathcal{B}]^{<\omega}\}$ need not be a base for 2^X in general, e.g., $X = \omega$ and $\mathcal{B} = \{\{n\} : n \in \omega\}$ is such an example.

For an open subset U of X, let

$$U^- = \{F \in \mathcal{K}(X) : F \cap U \neq \emptyset\}, \quad U^+ = \{F \in \mathcal{K}(X) : F \subset U\}.$$

Then obviously, these sets form a subbase for $\mathcal{K}(X)$. Observe that $\langle\mathcal{V}\rangle_{\mathcal{K}(X)} = (\bigcap_{V \in \mathcal{V}} V^-) \cap (\bigcup \mathcal{V})^+$ whenever $\mathcal{V}$ is a finite collection of open sets, and that U^- and U^+ are clopen in $\mathcal{K}(X)$ if U is clopen in X.

It is well-known that there are deep relations between normality and countable paracompactness in the product theory. The most famous one due to [1] is:

(1) for every space X, $X \times \mathbb{I}$ is normal iff X is normal and countably paracompact, where $\mathbb{I}$ denotes the unit interval $[0, 1] \subset \mathbb{R}$.

It is also known that orthocompactness (see the definition below) versus countable metacompactness behaves like normality versus countable paracompactness in the product theory, e.g.,

(2) for every space X, $X \times \mathbb{I}$ is orthocompact iff X is countably metacompact [7].

In comparing with (1) and (2), orthocompactness seems to be weaker than normality in the product theory. However it has been known in some part of the product theory, the opposite can be occur, e.g.,

(3) for every paracompact space X and every regular uncountable cardinal κ, if $X \times \kappa$ is orthocompact, then it is normal but not vice versa [5].

We list related topological properties on hyperspaces. One of powerful results is:

(4) for every space X, 2^X is normal iff X is compact [9].

This shows that for every ordinal $\gamma \neq 0$, 2^γ is normal iff $\mathrm{cf}\gamma = 1$, in particular that the hyperspace 2^ω is not normal. The following is also worth noting:

(5) the hyperspace 2^ω contains the Sorgenfrey line $\mathbb{S}$ as a closed subspace [8].

Since ω can be decomposed into two infinite sets N_0 and N_1, $2^{N_0} \times 2^{N_1}$ is embed into 2^ω as a closed subspace. Therefore 2^ω has a closed copy of the Sorgenfrey plane $\mathbb{S}^2$. This also shows that 2^ω is neither normal nor countably paracompact, while the problem whether 2^ω is countably metacompact remains open [4, Question A].

Moreover on hyperspaces of ordinals, the following are also known in [4], for every ordinal $\gamma \neq 0$,

(6) 2^γ is countably paracompact iff $\mathrm{cf}\gamma \neq \omega$,

(7) $\mathcal{K}(\gamma)$ is countably paracompact.

(8) $\mathcal{K}(\gamma)$ is normal iff either $\mathrm{cf}\gamma \leq \omega$ or γ is a regular uncountable cardinal.

In this paper, we will prove:

- $\mathcal{K}(\gamma)$ is orthocompact (shrinking, collectionwise normal) iff either $\mathrm{cf}\gamma \le \omega$ or γ is a regular uncountable cardinal, therefore normality and orthocompactness of $\mathcal{K}(\gamma)$ are equivalent for every non-zero ordinal γ.

We present two proofs, one proof uses the elementary submodel techniques and another does not. This also answers Question C of [4]. Moreover we discuss the natural question whether 2^ω is orthocompact or not. We prove that

- 2^ω is (countably) orthocompact iff it is countably metacompact,
- $\mathcal{K}(\mathbb{S})$ *is* orthocompact therefore so is the Sorgenfrey plane.

2. Covering properties

In this section, we give definitions and facts about topological properties. In particular, we present an auxiliary covering property so called property (P) for later use.

Let $\mathcal{U}$ be an open cover of a space X. A collection $\mathcal{W}$ of subsets of X is a *partial refinement* (*partial regular refinement*) of $\mathcal{U}$ if for every $W \in \mathcal{W}$, there is $U \in \mathcal{U}$ such that $W \subset U$ ($\mathrm{Cl}_X W \subset U$, respectively), where $\mathrm{Cl}_X W$ denotes the closure of W in X. In particular, simply we call $\mathcal{W}$ a *refinement* (*regular refinement*) of $\mathcal{U}$ if $\mathcal{W}$ covers X. A refinement $\mathcal{W}$ of an open cover of $\mathcal{U}$ is a *shrinking* if $\mathcal{W}$ is represented as $\{W(U) : U \in \mathcal{U}\}$ with $\mathrm{Cl}_X W(U) \subset U$ for every $U \in \mathcal{U}$. An open (closed) refinement is a refinement whose elements are open (closed), similarly an open (closed) shrinking is defined.

Recall that a space is *normal* if every pair of disjoint closed sets are separated by disjoint open sets, equivalently every binary open cover has a closed shrinking. We are concerned with two generalizations of normality. One is collectionwise normality, where a space X is *collectionwise normal* if for every discrete collection $\mathcal{F}$ of closed sets of X, there is a pairwise disjoint collection $\{W(F) : F \in \mathcal{F}\}$ of open sets with $F \subset W(F)$ for every $F \in \mathcal{F}$. Another one is the shrinking property, where a space X is *shrinking* if every open cover has a closed shrinking. Also recall that a space X is *countably paracompact* (*countably metacompact*) if every countable open cover has a locally finite (point finite, respectively) open refinement.

A collection $\mathcal{W}$ of open sets in a space X is *interior preserving* if for every subcollection $\mathcal{W}' \subset \mathcal{W}$, $\bigcap \mathcal{W}'$ is open, where we put $\bigcap \mathcal{W}' = X$ whenever $\mathcal{W}' = \emptyset$. Obviously point finite collections of open sets are interior preserving. Observe that a collection $\mathcal{W}$ of open sets is interior preserving iff for every $x \in X$, $\bigcap (\mathcal{W})_x$ is a neighborhood of x, where $(\mathcal{W})_x = \{W \in \mathcal{W} : x \in W\}$. Moreover observe that if $\mathcal{W}_\lambda$ is an interior preserving collection of open sets for every $\lambda \in \Lambda$ and $\{\bigcup \mathcal{W}_\lambda : \lambda \in \Lambda\}$ is point finite, then $\bigcup_{\lambda \in \Lambda} \mathcal{W}_\lambda$ is also interior preserving. Therefore in a countably metacompact space, every σ-interior preserving open cover (i.e., an open cover which is the countable sum of interior preserving open collections) has an interior preserving open refinement. A space is (*countably*) *orthocompact* if every (countable) open cover has an interior preserving open refinement. Note that (countably) paracompact spaces are (countably) metacompact and (countably) metacompact spaces are (countably) orthocompact. A collection $\mathcal{W}$ of subsets is *well-monotone* if $\mathcal{W}$ is represented as $\mathcal{W} = \{W(\alpha) : \alpha \in A\}$ for some well-ordered set A with the order $<$ so that $W(\alpha') \subset W(\alpha)$ whenever $\alpha', \alpha \in A$ with $\alpha' < \alpha$. In this definition, we may assume that A is an ordinal with the usual order. It is easy to see that every well-monotone collection of open sets is interior preserving.

We consider the following property:

(P): Every open cover $\mathcal{U}$ has an open regular refinement $\mathcal{W}$ represented as $\mathcal{W} = \bigcup_{j\in J}\mathcal{W}_j$ such that

(1) $\{\bigcup\mathcal{W}_j : j\in J\}$ is locally finite and has a closed shrinking,
(2) $\mathcal{W}_j$ is well-monotone for every $j\in J$.

In this definition, by taking cofinal subsequences, we may assume for each $j\in J$, $\mathcal{W}_j = \{W_j(\alpha) : \alpha < \delta_j\}$ for some ordinal δ_j, where $\delta_j = 1$ or δ_j is an infinite regular cardinal. Obviously compact spaces have property (P), more generally we have:

Lemma 2.1. *Paracompact spaces have property* (P).

Proof. Let $\mathcal{U}$ be an open cover of a paracompact space X. By regularity of X, one can find a locally finite open regular refinement $\mathcal{W}$ of $\mathcal{U}$. Put for each $W\in\mathcal{W}$, $\mathcal{W}_W = \{W\}$, then this is obviously well-monotone. Since X is normal and $\mathcal{W}$ is point finite, by [2, 1.5.18] it has a closed shrinking. □

Lemma 2.2. *Ordinals have property* (P).

Proof. Let γ be an ordinal. If $\mathrm{cf}\gamma\le\omega$, then γ is Lindelöf so apply the lemma above. Assume $\mathrm{cf}\gamma>\omega$ and fix a normal (= strictly increasing continuous cofinal) sequence $\{\gamma(\beta) : \beta<\mathrm{cf}\gamma\}$ in γ. Let $\mathcal{U}$ be an open cover of γ. For every $\beta\in\mathrm{Lim}(\mathrm{cf}\gamma)$, fix $f(\beta)<\beta$ and $U_\beta\in\mathcal{U}$ with $(\gamma(f(\beta)),\gamma(\beta)]\subset U_\beta$. By the Pressing Down Lemma, we can find $\beta_0<\mathrm{cf}\gamma$ and a stationary set $S\subset\mathrm{Lim}(\mathrm{cf}\gamma)$ such that $f(\beta)=\beta_0$ for each $\beta\in S$. Then the collection $\{(\gamma(\beta_0),\gamma(\beta)] : \beta_0<\beta\in S\}$ is a well-monotone partial refinement of $\mathcal{U}$ whose union is $(\gamma(\beta_0),\gamma)$. Since $[0,\gamma(\beta_0)]$ is a compact clopen subset of γ, one can easily construct an open regular refinement of $\mathcal{U}$ satisfying property (P). □

Lemma 2.3. *Spaces having property* (P) *are orthocompact, shrinking and collectionwise normal.*

Proof. Orthocompactness is trivial.

To see shrinking, let $\mathcal{U}$ be an open cover of a space X. Take an open regular refinement $\mathcal{W}=\bigcup_{j\in J}\mathcal{W}_j$ with (1) and (2) in property (P), and let $\mathcal{F}=\{F_j : j\in J\}$ be a closed shrinking of $\{\bigcup\mathcal{W}_j : j\in J\}$. We may assume, for every $j\in J$, $\mathcal{W}_j=\{W_j(\alpha) : \alpha<\delta_j\}$ for some ordinal δ_j, where $\delta_j=1$ or δ_j is an infinite regular cardinal. It suffices to find a closed shrinking $\mathcal{F}_j=\{F_j(U) : U\in\mathcal{U}\}$ of $\{U\cap F_j : U\in\mathcal{U}\}$ in F_j for each $j\in J$, because by the local finiteness of $\mathcal{F}$, $\{\bigcup_{j\in J}F_j(U) : U\in\mathcal{U}\}$ is a closed shrinking of $\mathcal{U}$. If $\delta_j=1$, then take $U_0\in\mathcal{U}$ with $\mathrm{Cl}_X W_j(0)\subset U_0$. Set for each $U\in\mathcal{U}$,

$$F_j(U) = \begin{cases} F_j & \text{if } U = U_0,\\ \emptyset & \text{otherwise.}\end{cases}$$

Then $\mathcal{F}_j=\{F_j(U) : U\in\mathcal{U}\}$ is a closed shrinking of $\{U\cap F_j : U\in\mathcal{U}\}$ in F_j.

Assume that δ_j is an infinite regular cardinal. For each $U\in\mathcal{U}$, let $K(U)=\{\alpha<\delta_j : \mathrm{Cl}_X W_j(\alpha)\subset U\}$. If there is $U_0\in\mathcal{U}$ such that $K(U_0)$ is cofinal in δ_j, then set for each $U\in\mathcal{U}$,

$$F_j(U) = \begin{cases} F_j & \text{if } U = U_0,\\ \emptyset & \text{otherwise.}\end{cases}$$

Then $\mathcal{F}_j=\{F_j(U) : U\in\mathcal{U}\}$ is a closed shrinking of $\{U\cap F_j : U\in\mathcal{U}\}$ in F_j. If $K(U)$ is bounded in δ_j for each $U\in\mathcal{U}$, then we define a strictly increasing cofinal sequence $\{\alpha(\beta) : \beta < \delta_j\}$ in δ_j and $\{U_\beta : \beta < \delta_j\} \subset \mathcal{U}$ as follows. First let $\alpha(0) = 0$ and pick $U_0 \in \mathcal{U}$ with $\mathrm{Cl}_X W_j(\alpha(0)) \subset U_0$. Assume that $\{\alpha(\gamma) : \gamma < \beta\}$ and $\{U_\gamma : \gamma < \beta\}$ have been defined for some $\beta < \delta_j$. Pick $\alpha(\beta) < \delta_j$ and $U_\beta \in \mathcal{U}$ with $\sup(\bigcup\{K(U_\gamma) : \gamma < \beta\} \cup \{\alpha(\gamma) : \gamma < \beta\}) < \alpha(\beta)$ and $\mathrm{Cl}_X W_j(\alpha(\beta)) \subset U_\beta$. Obviously $U_\gamma \neq U_\beta$ holds for every $\gamma < \beta$. Now let for each $U \in \mathcal{U}$,

$$F_j(U) = \begin{cases} F_j \cap \mathrm{Cl}_X W_j(\alpha(\beta)) & \text{if } U = U_\beta \text{ for some } \beta < \delta_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}_j = \{F_j(U) : U \in \mathcal{U}\}$ is a closed shrinking of $\{U \cap F_j : U \in \mathcal{U}\}$ in F_j. Now the space is normal.

To see collectionwise normality of X, let $\mathcal{F}$ be a discrete collection of closed sets and $U(F) = X \setminus \bigcup(\mathcal{F} \setminus \{F\})$ for each $F \in \mathcal{F}$. Take an open regular refinement $\mathcal{W} = \bigcup_{j \in J} \mathcal{W}_j$ of the open cover $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ satisfying (1) and (2) in property (P). Note that for each $j \in J$, $(\bigcup \mathcal{W}_j) \cap F \neq \emptyset$ holds for at most one member $F \in \mathcal{F}$. For each $F \in \mathcal{F}$, let $W(F) = \bigcup\{\bigcup \mathcal{W}_j : F \cap (\bigcup \mathcal{W}_j) \neq \emptyset\}$. Then $\{W(F) : F \in \mathcal{F}\}$ is locally finite and $F \subset W(F) \subset X \setminus \bigcup(\mathcal{F} \setminus \{F\})$ holds for each $F \in \mathcal{F}$. Since X is normal, take an open set $V(F)$ in X such that $F \subset V(F) \subset \mathrm{Cl}_X V(F) \subset W(F)$ for each $F \in \mathcal{F}$. Then $\{V(F) \setminus \bigcup_{H \in \mathcal{F} \setminus \{F\}} \mathrm{Cl}_X V(H) : F \in \mathcal{F}\}$ separates $\mathcal{F}$. □

3. Orthocompactness and normality in $\mathcal{K}(\gamma)$

According to the result (7) in the Introduction, we know that $\mathcal{K}(\gamma)$ is countably orthocompact for every ordinal γ. In this section, we will see that $\mathcal{K}(\gamma)$ is orthocompact (shrinking, collectionwise normal) iff $\mathrm{cf}\gamma = \gamma$ whenever $\mathrm{cf}\gamma$ is uncountable, also answers Question C of [4].

For every $\delta < \gamma$, define $p_\delta : \gamma \to [0, \delta]$ by $p_\delta(\alpha) = \min\{\delta, \alpha\}$ for every $\alpha < \gamma$, also we can define $\tilde{p}_\delta : \mathcal{K}(\gamma) \to \mathcal{K}([0, \delta]) = 2^{[0,\delta]}$ by $\tilde{p}_\delta(K) = p_\delta[K]$ for every $K \in \mathcal{K}(\gamma)$. It is easy to see that both functions p_δ and $\tilde{p}_\delta$ are continuous.

The following is a main result of this section.

Lemma 3.1. *Let κ be a regular uncountable cardinal and $\mathbb{U}$ a basic open cover of $\mathcal{K}(\kappa)$, that is, an open cover by basic open sets. Then there is $\delta < \kappa$ such that for every $\mathcal{U} \in \mathbb{U}$,*

$$\{[0, \alpha]^+ \cap \tilde{p}_\delta^{-1}[\mathcal{U} \cap [0, \delta]^+] : \alpha < \kappa\}$$

is a partial refinement of $\mathbb{U}$.

First, we give a proof using elementary submodels.

Proof. Let M be an elementary submodel of $H(\theta)$, where θ is large enough, such that $\mathbb{U}, \kappa \in M$, $|M| < \kappa$ and $\kappa \cap M$ is an ordinal, see the beginning of the proof of Theorem 8 in [4]. We will show that $\delta = \kappa \cap M$ is as desired. Let $\mathcal{U} \in \mathbb{U}$, say $\mathcal{U} = \langle\{(\alpha_i, \beta_i] : i < n\}\rangle_{\mathcal{K}(\kappa)}$ where $1 \leq n \in \omega$ and $-1 \leq \alpha_i < \beta_i < \kappa$. We may assume $\mathcal{U} \cap [0, \delta]^+ \neq \emptyset$ (otherwise obvious), then we have $\alpha_i < \delta$ for every $i < n$. Note $\alpha_i < p_\delta(\beta_i)$ for each $i < n$.

Claim 1. $\mathcal{U} \cap [0, \delta]^+ = \langle\{(\alpha_i, p_\delta(\beta_i)] : i < n\}\rangle$ holds.

Proof. By $p_\delta(\beta_i) \leq \beta_i$ and $p_\delta(\beta_i) \leq \delta$, "$\supset$" is almost obvious. For "$\subset$", let $K \in \mathcal{U} \cap [0, \delta]^+$. First to see $K \subset \bigcup_{i<n}(\alpha_i, p_\delta(\beta_i)]$, let $\gamma \in K$. Since $\gamma \leq \delta$ and for some $i < n$, $\gamma \in (\alpha_i, \beta_i]$ holds, we have $\gamma \in (\alpha_i, p_\delta(\beta_i)]$. Next let $i < n$. It follows from $K \in \mathcal{U}$ that $\gamma \in K \cap (\alpha_i, \beta_i]$ for some γ. Then as above we have $\gamma \in K \cap (\alpha_i, p_\delta(\beta_i)]$. □

Now let for each $i < n$,

$$W_i = \begin{cases} (\alpha_i, \beta_i] & \text{if } \beta_i < \delta, \\ (\alpha_i, \kappa) & \text{if } \beta_i \geq \delta. \end{cases}$$

Then we have $(\alpha_i, p_\delta(\beta_i)] \subset (\alpha_i, \beta_i] \subset W_i$ for each $i < n$. It follows from $\{\alpha_i : i < n\} \cup \{\kappa\} \subset M$ and $\beta_i \in M$ for $\beta_i < \delta$ that W_i belongs to M and is clopen in κ for each $i < n$. Therefore the finite set $\{W_i : i < n\}$ belongs to M and $\mathcal{W}(\mathcal{U}) = \langle \{W_i : i < n\} \rangle_{\mathcal{K}(\kappa)}$ also belongs to M and clopen in $\mathcal{K}(\kappa)$. The following two claims can be similarly shown as in Claim 1.

Claim 2. $\mathcal{U} \cap [0, \delta]^+ = \mathcal{W}(\mathcal{U}) \cap [0, \delta]^+$ holds.

Claim 3. $\tilde{p_\delta}^{-1}[\mathcal{U} \cap [0, \delta]^+] = \mathcal{W}(\mathcal{U})$ holds.

The function $f = \{[0, \alpha]^+ : \alpha < \kappa\} : \kappa \to \mathcal{K}(\kappa)$ is definable from $\kappa \in H(\theta)$ and θ was taken large enough so that $f \in H(\theta)$. Therefore we may consider that the definable function f from $\kappa \in M$ also belongs to M. Now the function $\mathbb{U}(\mathcal{U}) = \{[0, \alpha]^+ \cap \mathcal{W}(\mathcal{U}) : \alpha < \kappa\} : \kappa \to \mathcal{P}(\mathcal{K}(\kappa))$, which is definable from $f, \mathcal{W}(\mathcal{U}) \in M$, also belongs to M. By Claim 3, the following claim completes the proof of the lemma.

Claim 4. $\mathbb{U}(\mathcal{U})$ is a partial refinement of $\mathbb{U}$.

Proof. By $\mathbb{U}(\mathcal{U}), \mathbb{U} \in M$, it suffices to see:

$$M \models \mathbb{U}(\mathcal{U}) \text{ is a partial refinement of } \mathbb{U},$$

that is,

$$M \models \forall \alpha < \kappa \exists \mathcal{V} \in \mathbb{U}(\mathbb{U}(\mathcal{U})(\alpha) \subset \mathcal{V}),$$

where $\mathbb{U}(\mathcal{U})(\alpha) = [0, \alpha]^+ \cap \mathcal{W}(\mathcal{U})$. Therefore it suffices to see that for every $\alpha \in \kappa \cap M = \delta$,

$$(*) \qquad \exists \mathcal{V} \in \mathbb{U}\Big([0, \alpha]^+ \cap \mathcal{W}(\mathcal{U}) \subset \mathcal{V}\Big).$$

Because by $\alpha < \delta$ and Claim 2, we have

$$[0, \alpha]^+ \cap \mathcal{W}(\mathcal{U}) = [0, \alpha]^+ \cap \mathcal{U} \subset \mathcal{U},$$

so $\mathcal{V} = \mathcal{U}$ witnesses $(*)$. □

□

Next we give a proof of the lemma without using elementary submodels.

Proof. For each $\xi < \kappa$, let

$$\mathbb{A}(\xi) = \{\langle \mathcal{W}, s \rangle : \mathcal{W} \text{ is a finite collection of } < \xi\text{-basic intervals, } s \in [\xi]^{<\omega}\},$$

$$\mathbb{A}_0(\xi) = \{\langle \mathcal{W}, s \rangle \in \mathbb{A}(\xi) : \{\tilde{\mathcal{W}}(s, \alpha) : \alpha < \kappa\} \text{ is not a partial refinement of } \mathbb{U}\},$$

where $\tilde{\mathcal{W}}(s, \alpha) = \langle \mathcal{W} \cup \{(\gamma, \alpha] : \gamma \in s\} \rangle_{\mathcal{K}(\kappa)}$, and

$$\alpha(\mathcal{W}, s) = \min\{\alpha < \kappa : \tilde{\mathcal{W}}(s, \alpha) \not\subset \mathcal{U} \text{ for any } \mathcal{U} \in \mathbb{U}\}$$

for each $\langle \mathcal{W}, s \rangle \in \mathbb{A}_0(\xi)$. Note that $|\mathbb{A}(\xi)| < \kappa$ holds for every $\xi < \kappa$. Take a strictly increasing sequence $\{\delta_n : n < \omega\}$ of ordinals $< \kappa$ such that $\alpha(\mathcal{W}, s) < \delta_{n+1}$ holds for each $n \in \omega$ and $\langle \mathcal{W}, s \rangle \in \mathbb{A}_0(\delta_n)$. And let $\delta = \sup\{\delta_n : n < \omega\}$. Then we have $\delta < \kappa$ and $\mathbb{A}_0(\delta) = \bigcup_{n<\omega} \mathbb{A}_0(\delta_n)$. So $\alpha(\mathcal{W}, s) < \delta$ holds for each $\langle \mathcal{W}, s \rangle \in \mathbb{A}_0(\delta)$.

We will show that this δ is as desired. Let $\mathcal{U} = \langle \{(\alpha_i, \beta_i] : i < n\} \rangle_{\mathcal{K}(\kappa)} \in \mathbb{U}$. The proof is parallel to the above proof except for Claim 4, so we only give the proof of it.

Claim 4. $\mathbb{U}(\mathcal{U})$ is a partial refinement of $\mathbb{U}$.

Proof. Let $A = \{i < n : \beta_i < \delta\}$. Then $s = \{\alpha_i : i \in n \setminus A\}$ belongs to $[\delta]^{<\omega}$ and $\mathcal{W} = \{(\alpha_i, \beta_i] : i \in A\}$ is a finite collection of $<\delta$-basic intervals. So $\langle \mathcal{W}, s \rangle \in \mathbb{A}(\delta)$. As in the proof of Claim 1, we can show $\tilde{\mathcal{W}}(s, \alpha) = [0, \alpha]^+ \cap \mathcal{W}(\mathcal{U})$ for each $\alpha < \kappa$. If $\mathbb{U}(\mathcal{U}) = \{\tilde{\mathcal{W}}(s, \alpha) : \alpha < \kappa\}$ were not a partial refinement of $\mathbb{U}$, then $\langle \mathcal{W}, s \rangle \in \mathbb{A}_0(\delta)$, so by putting $\alpha_0 = \alpha(\mathcal{W}, s) < \delta$, we have $\tilde{\mathcal{W}}(s, \alpha_0) \not\subset \mathcal{U}$. On the other hand by Claim 2, we have

$$\tilde{\mathcal{W}}(s, \alpha_0) = [0, \alpha_0]^+ \cap \mathcal{W}(\mathcal{U}) = [0, \alpha_0]^+ \cap \mathcal{U} \subset \mathcal{U},$$

a contradiction. □

□

Now we consider the following property (P_0) which is stronger than (P):

(P_0): Every open cover $\mathcal{U}$ has a clopen refinement $\mathcal{W}$ represented as $\mathcal{W} = \bigcup_{j \in J} \mathcal{W}_j$ such that

(1) J is finite and $\{\bigcup \mathcal{W}_j : j \in J\}$ is pairwise disjoint,
(2) $\mathcal{W}_j$ is well-monotone for every $j \in J$.

Lemma 3.2. *Let κ be a regular uncountable cardinal. Then $\mathcal{K}(\kappa)$ has property (P_0), therefore it is orthocompact, shrinking and collectionwise normal.*

Proof. Let $\mathbb{U}$ be an open cover of $\mathcal{K}(\kappa)$. By taking a refinement, we may assume that $\mathbb{U}$ is a basic open cover. By Lemma 3.1, there is $\delta < \kappa$ such that for every $\mathcal{U} \in \mathbb{U}$, $\{[0, \alpha]^+ \cap \tilde{p}_\delta^{-1}[\mathcal{U} \cap [0, \delta]^+] : \alpha < \kappa\}$ is a well-monotone partial refinement of $\mathbb{U}$. Note that the ordinal space $[0, \delta]$ is compact and zero-dimensional, therefore its hyperspace $2^{[0,\delta]} = \mathcal{K}([0, \delta]) = [0, \delta]^+$ is also compact and zero-dimensional. Since $\mathbb{U}$ covers $[0, \delta]^+$, there is a pairwise disjoint finite partial clopen refinement $\{\mathcal{V}_j : j \in J\}$ of $\mathbb{U}$ with $[0, \delta]^+ = \bigcup_{j \in J} \mathcal{V}_j$. Set for each $j \in J$, $\mathcal{W}_j = \{[0, \alpha]^+ \cap \tilde{p}_\delta^{-1}[\mathcal{V}_j] : \alpha < \kappa\}$. Then $\mathcal{W} = \bigcup_{j \in J} \mathcal{W}_j$ is as desired. □

Lemma 3.3. *If γ is an ordinal with $\mathrm{cf}\gamma \le \omega$, then $\mathcal{K}(\gamma)$ is Lindelöf therefore it is orthocompact, shrinking and collectionwise normal.*

Proof. Whenever $\mathrm{cf}\gamma = 1$, $\mathcal{K}(\gamma)$ is compact. Assume $\mathrm{cf}\gamma = \omega$. Take a cofinal subset $\{\gamma_n : n \in \omega\}$ of γ such that $\mathrm{cf}\,\gamma_n = 1$ for every $n \in \omega$. Then $\mathcal{K}(\gamma_n)$ is compact for every $n < \omega$. Since $\mathcal{K}(\gamma) = \bigcup_{n<\omega} \mathcal{K}(\gamma_n)$ is a countable union of compact subspaces, it is Lindelöf. □

Remark that according to the result (7) in Introduction, $\mathcal{K}(\gamma)$ is countably orthocompact for every ordinal γ. We now characterize the orthocompactness of $\mathcal{K}(\gamma)$. Although the equivalence (4) $\leftrightarrow$ (5) in the next theorem is shown in [4], for the readers' convenience, we prove it simultaneously.

Lemma 3.4. *For every non-zero ordinal γ, the following are equivalent:*

(1) *$\mathcal{K}(\gamma)$ is orthocompact,*

(2) *$\mathcal{K}(\gamma)$ is shrinking,*
(3) *$\mathcal{K}(\gamma)$ is collectionwise normal,*
(4) *$\mathcal{K}(\gamma)$ is normal,*
(5) *either* $\mathrm{cf}\gamma \leq \omega$ *or γ is a regular uncountable cardinal.*

Proof. (2) $\rightarrow$ (4) and (3) $\rightarrow$ (4) are obvious.

(1) $\rightarrow$ (5) and (4) $\rightarrow$ (5) can be proved simultaneously as in Lemma 9 of [4]. To see this, assume $\omega < \mathrm{cf}\gamma < \gamma$ and let $\kappa = \mathrm{cf}\gamma$. By fixing a normal sequence $\{\gamma(\alpha) : \alpha < \kappa\}$ in γ with $\kappa < \gamma(0)$, $(\kappa + 1) \times \kappa$ can be embedded into $\mathcal{K}(\gamma)$ as a closed subspace with the map $\langle \alpha, \beta \rangle \rightarrow \{\alpha, \gamma(\beta)\}$. Note that whenever ξ and η are ordinals, normality of $\xi \times \eta$ is equivalent to its orthocompactness, see [7, Theorem 3.3]. Since κ is regular uncountable, $(\kappa + 1) \times \kappa$ is not normal. Therefore $\mathcal{K}(\gamma)$ is neither normal nor orthocompact.

(5) $\rightarrow$ (1), (5) $\rightarrow$ (2) and (5) $\rightarrow$ (3): The case $\mathrm{cf}\gamma \leq \omega$ follows from Lemma 3.3. The other case follows from Lemma 3.2. $\square$

4. Orthocompactness of 2^ω

As noted in Introduction, the Sorgenfrey plane $\mathbb{S}^2$ is neither normal nor countably paracompact and can be embedded in 2^ω as a closed subspace. Also observe that $\mathbb{S}^\omega$ is perfect (= closed sets are G_δ) [3], therefore it is countably metacompact. It is natural to ask:

Question 4.1. Is 2^ω orthocompact?

The second author also asked in [4, Question A] whether 2^ω is countably metacompact. Although we know the answers of neither, in this section, we discuss these questions. First we prove that these questions are equivalent.

Like the proof of the fact that every σ-interior preserving open cover of a countably metacompact space has an interior preserving open refinement, the following lemma can be similarly proved.

Lemma 4.2. *Countably metacompact spaces having a σ-interior preserving base are orthocompact, where a σ-interior preserving base is a base which is represented as the countable sum of interior preserving collections of open sets.*

The proofs of the following are routine or well-known.

Lemma 4.3. *The following hold.*

(1) *If a space has a σ-interior preserving base, then so does every subspace.*
(2) *If X_n has a σ-interior preserving base for every $n \in \omega$, then so does $X = \prod_{n\in\omega} X_n$.*
(3) *If a space is perfect, then so is every subspace.*
(4) *If every finite subproduct of $X = \prod_{n\in\omega} X_n$ is perfect, then so is $X = \prod_{n\in\omega} X_n$.*

Lemma 4.4. *If a countably orthocompact space X has an open F_σ dense countably metacompact subspace Y, then X is countably metacompact.*

Proof. Let $\mathcal{U} = \{U_n : n \in \omega\}$ be a well-monotone open cover of X. It suffices to find its point finite open refinement. Let $\{W_n : n \in \omega\}$ be a decreasing sequence of open sets with $X \setminus Y = \bigcap_{n\in\omega} W_n$. Set $V_n = U_n \cap W_n$ for each $n \in \omega$. Then $\{V_n : n \in \omega\} \cup \{Y\}$ is a countable open cover of X. By the countable orthocompactness of X, one can take an interior preserving open cover $\{V'_n : n \in \omega\} \cup \{Y'\}$ of X with $V'_n \subset V_n$ for each $n \in \omega$ and $Y' \subset Y$. Since Y is countably metacompact, it suffices to see $\mathcal{V}' = \{V'_n : n \in \omega\}$ is point finite. To see this, let $x \in X$. If $x \in Y$, then there is $n \in \omega$ with $x \notin W_n$. Then for every $m \in \omega$ with $n \leq m$, $x \notin W_n \supset W_m \supset V_m \supset V'_m$, thus $\mathcal{V}'$ is point finite at x. Now assume that $x \in X \setminus Y$ and $M = \{n \in \omega : x \in V'_n\}$ is infinite. Since $\mathcal{V}'$ is interior preserving, there is a neighborhood U of x with $U \subset \bigcap_{n\in M} V'_n \subset \bigcap_{n\in M} V_n \subset \bigcap_{n\in M} W_n = X \setminus Y$, this contradicts that Y is dense in X. □

Lemma 4.5. 2^ω *has a σ-interior preserving base.*

Proof. For each $s \in [\omega]^{<\omega}$ and $C \in 2^\omega$ with $s \subset C$, let

$$\mathcal{B}_s(C) = \{D \in 2^\omega : s \subset D \subset C\}.$$

Then $\{\mathcal{B}_s(C) : s \in [C]^{<\omega}\}$ is a neighborhood base at C in 2^ω. Now for each $s \in [\omega]^{<\omega}$, set

$$\mathcal{B}_s = \{\mathcal{B}_s(C) : s \subset C \in 2^\omega\}.$$

For every pair $s \in [\omega]^{<\omega}$ and $D \in 2^\omega$, $\mathcal{B}_s(D) \subset \mathcal{B}_s(C)$ holds whenever $D \in \mathcal{B}_s(C)$. Therefore we have $\mathcal{B}_s(D) \subset \bigcap\{\mathcal{B}_s(C) : D \in \mathcal{B}_s(C)\}$, this shows that $\mathcal{B}_s$ is interior preserving. Then obviously $\mathcal{B} = \bigcup_{s\in[\omega]^{<\omega}} \mathcal{B}_s$ is the desired base. □

Although the question whether 2^ω is orthocompact remains open, we have:

Proposition 4.6. *The following are equivalent.*

(1) 2^ω *is orthocompact.*
(2) 2^ω *is countably orthocompact.*
(3) 2^ω *is countably metacompact.*

Proof. (1) → (2) is obvious and (3) → (1) follows from Lemma 4.2 and Proposition 4.5.

(2) → (3): Assume that 2^ω is countably orthocompact. Since $[\omega]^{<\omega} \setminus \{\emptyset\}$ is a countable dense subset of 2^ω consisting of isolated points, by Lemma 4.4, 2^ω is countably metacompact. □

Finally improving the proof of [3], we will show that $\mathcal{K}(\mathbb{S})$ is orthocompact. Recall that the Sorgenfrey line $\mathbb{S}$ is the space whose underlying set is $\mathbb{R}$ and whose topology is generated by the collection $\{(a, b] : a, b \in \mathbb{R}, a < b\}$, where $(a, b]$ denotes the usual interval in $\mathbb{R}$. For the notational convenience, $(a, \infty]$ denotes the interval $\{x \in \mathbb{S} : a < x\}$.

Theorem 4.7. $\mathcal{K}(\mathbb{S})$ *is perfect and has a σ-interior preserving base, therefore it is orthocompact.*

Proof. The following two claims are easy to prove.

Claim 1. Every $K \in \mathcal{K}(\mathbb{S})$ has the minimal element $\min K$ and the maximal element $\max K$.

Claim 2. $\{(a, p] : a \in \mathbb{Q}, a < p\}$ is a neighborhood base at $p \in \mathbb{S}$.

For each $n \in \omega$ with $1 \leq n$, let

$$A_n = \{a \in \mathbb{Q}^n : a(0) < a(1) < \cdots < a(n-1)\}.$$

For each $a \in A = \bigcup_{1\leq n\in\omega} A_n$, the length $\mathrm{lh}(a)$ of a denotes the n such that $a \in A_n$.

Let $a \in A$. Put

$$\mathcal{B}_a = \langle \{(a(i), a(i+1)] : i < \mathrm{lh}(a)\} \rangle_{\mathcal{K}(\mathbb{S})},$$

$$Q_a = \bigcup_{r \subseteq \mathrm{lh}(a)} \prod_{i \in r} \Big((a(i), a(i+1)] \cap \mathbb{Q} \Big),$$

where $a(\mathrm{lh}(a)) = \infty$. Note that $|A| = \omega$, $\emptyset \in Q_a$ and $|\bigcup_{a \in A} Q_a| = \omega$ hold. Now let $q = \langle q(i) : i \in \mathrm{dom}(q) \rangle \in Q_a$. Set

$$\mathcal{B}_a(q) = \langle \{(a(i), q(i)] : i \in \mathrm{dom}(q)\} \cup \{(a(i), a(i+1)] : i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q)\} \rangle_{\mathcal{K}(\mathbb{S})},$$

$$P_a(q) = \prod_{i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q)} (a(i), a(i+1)].$$

Moreover for each $K \in \mathcal{B}_a(q)$, define $p_{a,q,K} \in P_a(q)$ by

$$p_{a,q,K}(i) = \max \Big(K \cap (a(i), a(i+1)] \Big)$$

for each $i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q)$. Define binary relations $\preceq$ and $\prec$ on $P_a(q)$ by

$$p \preceq p' \Leftrightarrow p(i) \leq p'(i) \text{ for every } i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q),$$

$$p \prec p' \Leftrightarrow p \preceq p' \text{ and } p \neq p'.$$

For each $p \in P_a(q)$, let

$$\mathcal{B}_a(q,p) = \langle \{(a(i), q(i)] : i \in \mathrm{dom}(q)\} \cup \{(a(i), p(i)] : i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q)\} \rangle_{\mathcal{K}(\mathbb{S})},$$

$$\mathcal{B}_a^*(q,p) = \mathcal{B}_a(q,p) \setminus \bigcup \{\mathcal{B}_a(q,p') : p' \in P_a(q), p' \prec p\}.$$

And let

$$\mathbb{B}_a(q) = \{\mathcal{B}_a(q,p) : p \in P_a(q)\}.$$

Claim 3. For each $a \in A$ and $q \in Q_a$, the following hold:

(1) $\mathcal{B}_a$ and $\mathcal{B}_a(q)$ are clopen sets of $\mathcal{K}(\mathbb{S})$ with $\mathcal{B}_a(q) \subset \mathcal{B}_a$, moreover for each $p \in P_a(q)$, $\mathcal{B}_a(q,p)$ is a clopen subset of $\mathcal{B}_a(q)$.
(2) $\mathcal{B}_a(q,p) \subseteq \mathcal{B}_a(q,p')$ holds for every $p, p' \in P_a(q)$ with $p \preceq p'$.
(3) For each $K \in \mathcal{B}_a(q)$ and $p \in P_a(q)$, $K \in \mathcal{B}_a(q,p)$ holds iff $p_{a,q,K} \preceq p$.
(4) $\mathbb{B}_a(q)$ is an interior preserving collection of clopen sets of $\mathcal{K}(\mathbb{S})$.
(5) For each $p \in P_a(q)$, $\mathcal{B}_a^*(q,p) = \{K \in \mathcal{B}_a(q,p) : p_{a,q,K} = p\}$ holds.
(6) If $p \in P_a(q)$, $K \in \mathcal{B}_a(q,p) \setminus \mathcal{B}_a^*(q,p)$, then there are $q' \in Q_a$ and $p' \in P_a(q')$ such that $q \subsetneq q'$ and $K \in \mathcal{B}_a(q',p') \subseteq \mathcal{B}_a(q,p)$.

Proof. (1), (2), (3) and (5) are obvious.
(4): Let $K \in \mathcal{K}(\mathbb{S})$ and set $P' = \{p \in P_a(q) : K \in \mathcal{B}_a(q,p)\}$. If $p \in P'$, then by (3) we have $p_{a,q,K} \preceq p$ therefore $\mathcal{B}_a(q, p_{a,q,K}) \subset \mathcal{B}_a(q,p)$. This shows $\mathcal{B}_a(q, p_{a,q,K}) \subset \bigcap \{\mathcal{B}_a(q,p) : p \in P'\} = \bigcap (\mathbb{B}_a(q))_K$.
(6): Let $s = \{i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q) : p_{a,q,K}(i) < p(i)\}$. Then $s \neq \emptyset$. Define q' with $\mathrm{dom}(q') = \mathrm{dom}(q) \cup s$ as follows: for each $i \in s$, fix $q'(i) \in \mathbb{Q}$ with $p_{a,q,K}(i) < q'(i) < p(i)$ and for each $i \in \mathrm{dom}(q)$, let $q'(i) = q(i)$. Moreover define $p' \in P_a(q')$ by $p'(i) = p(i)$ for each $i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q')$. Then q' and p' are as desired. $\square$

Claim 4. $\mathbb{B} = \bigcup_{a \in A} \mathbb{B}_a(\emptyset)$ is a σ-interior preserving base of $\mathcal{K}(\mathbb{S})$ by clopen sets.

Proof. By Claim 3 (4), it suffices to show that $\mathbb{B}$ is a base for $\mathcal{K}(\mathbb{S})$, i.e. each $K \in \mathcal{K}(\mathbb{S})$ satisfies that

$(*K)$: for each finite family $\mathcal{V}$ of open sets of $\mathbb{S}$ with $K \in \langle \mathcal{V} \rangle$, there are $a \in A$ and $p \in P_a(\emptyset)$ such that $K \in \mathcal{B}_a(\emptyset, p) \subset \langle \mathcal{V} \rangle$.

Let $K \in \mathcal{K}(\mathbb{S})$. Then K is well-ordered by the usual order $<$. Actually, if there is a strictly decreasing sequence $\{x(j) : j < \omega\}$ of elements of K, then by letting $x(\omega) = \inf\{x(j) : j \in \omega\}$, we obtain an open cover

$$\{(x(j+1), x(j)] : j \in \omega\} \cup \{(-\infty, x(\omega)], (x(0), \infty)\}$$

of $\mathbb{S}$ which does not have a finite subfamily covering K. This contradicts that K is compact.

For each $c \in \mathbb{S}$, let $K_c = \{x \in K : x \leq c\}$ and $K_{<c} = \{x \in K : x < c\}$. We will show $(*K_u)$ by induction on $u \in K$. After finishing induction, we see that $(*K)$ holds since $K_{\tilde{u}} = K$ for $\tilde{u} = \max K$.

Let $u \in K$ and assume that $(*K_{u'})$ holds for every $u' \in K$ with $u' < u$. And let $\mathcal{V}$ be a finite family of open sets of $\mathbb{S}$ with $K_u \in \langle \mathcal{V} \rangle$. We would like to find $a \in A$ and $p \in P_a(\emptyset)$ such that $K \in \mathcal{B}_a(\emptyset, p) \subset \langle \mathcal{V} \rangle$. Put $\mathcal{V}' = \{V \in \mathcal{V} : V \cap K_{<u} \neq \emptyset\}$. Take $c \in \mathbb{Q}$ with $c < u$ such that $(c, u] \subset \bigcap(\mathcal{V})_u$, and $V' \cap K_c \neq \emptyset$ for every $V' \in \mathcal{V}'$. In case u is a minimal element of K, we have $K_u = \{u\}$ and $u \in V$ holds for every $V \in \mathcal{V}$, so by taking $a \in A_1$ and $p \in P_a(\emptyset)$ such that $a(0) = c$ and $p(0) = u$, we have $K_u \in \langle \{(c, u]\} \rangle = \mathcal{B}_a(\emptyset, p) \subset \langle \mathcal{V} \rangle$. In case u is not a minimal element of K, we have $\mathcal{V}' \neq \emptyset$, so $K_c \neq \emptyset$. Let $u' = \max K_c$. Then $u' \in K$, $u' \leq c < u$, and $K_{u'} = K_c \in \langle \mathcal{V}' \rangle$ hold. By inductive hypothesis, there are $a' \in A$ and $p' \in P_{a'}(\emptyset)$ such that $K_{u'} \in \mathcal{B}_{a'}(\emptyset, p') \subset \langle \mathcal{V}' \rangle$. We may assume $p'(\mathrm{lh}(a') - 1) \leq u' \leq c$. Define $a \in A$ and $p \in P_a(\emptyset)$ by $\mathrm{lh}(a) = \mathrm{lh}(a') + 1$, $a(i) = a'(i)$ and $p(i) = p'(i)$ for each $i < \mathrm{lh}(a')$, $a(\mathrm{lh}(a')) = c$, and $p(\mathrm{lh}(a')) = u$. Then $K_u = K_{u'} \cup (K \cap (c, u]) \in \mathcal{B}_a(\emptyset, p) \subset \langle \mathcal{V} \rangle$. □

To see that $\mathcal{K}(\mathbb{S})$ is perfect, let $\mathcal{U}$ be an open set. For each $a \in A$ and $q \in Q_a$, put

$$P'_a(q) = \{p \in P_a(q) : \mathcal{B}_a(q, p) \subseteq \mathcal{U}\},$$
$$P^*_a(q) = \{p \in P'_a(q) : \neg\exists \tilde{p} \in P'_a(q)\ (p \prec \tilde{p})\},$$
$$\mathcal{B}^*_a(q) = \bigcup\{\mathcal{B}^*_a(q, p) : p \in P^*_a(q)\}.$$

If $p \in P^*_a(q)$, then by $p \in P'_a(q)$, $\mathcal{B}^*_a(q, p) \subseteq \mathcal{B}_a(q, p) \subseteq \mathcal{U} \cap \mathcal{B}_a(q)$ holds. Therefore $\mathcal{B}^*_a(q)$ is a subset of $\mathcal{U} \cap \mathcal{B}_a(q)$.

Claim 5. $\mathcal{B}^*_a(q)$ is a closed set of $\mathcal{K}(\mathbb{S})$.

Proof. Let $K \notin \mathcal{B}^*_a(q)$. It suffices to find a neighborhood $\mathcal{V}$ of K in $\mathcal{K}(\mathbb{S})$ which is disjoint from $\mathcal{B}^*_a(q)$. Since $\mathcal{B}_a(q)$ is clopen and contains $\mathcal{B}^*_a(q)$, we may assume $K \in \mathcal{B}_a(q) \setminus \mathcal{B}^*_a(q)$. Put

$$r(p) = \{i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q) : p(i) < p_{a,q,K}(i)\} \text{ for each } p \in P^*_a(q) \text{ with } p \preceq p_{a,q,K},$$
$$\mathcal{R} = \{r(p) : p \in P^*_a(q), p \preceq p_{a,q,K}\}.$$

Then $\mathcal{R}$ is finite.

Since $K \in \mathcal{B}^*_a(q, p_{a,q,K}) \setminus \mathcal{B}^*_a(q)$, we have $p_{a,q,K} \notin P^*_a(q)$ therefore $\emptyset \notin \mathcal{R}$. For each $r \in \mathcal{R}$, fix $p_r \in P^*_a(q)$ with $p_r \preceq p_{a,q,K}$ and $r(p_r) = r$. Let

$$\mathcal{V} = \{K' \in \mathcal{B}_a(q, p_{a,q,K}) : \forall r \in \mathcal{R} \forall i \in r (K' \cap (p_r(i), p_{a,q,K}(i)] \neq \emptyset)\}.$$

Then $\mathcal{V} = \mathcal{B}_a(q, p_{a,q,K}) \cap \bigcap_{r \in \mathcal{R}, i \in r} (p_r(i), p_{a,q,K}(i)]^-$ is a neighborhood of K in $\mathcal{K}(\mathbb{S})$. We show $\mathcal{V} \cap \mathcal{B}^*_a(q) = \emptyset$. To the contrary, assume that there is $K' \in \mathcal{V} \cap \mathcal{B}^*_a(q)$.

Then $K' \in \mathcal{B}^*_a(q,p)$ for some $p \in P^*_a(q)$. By Claim 3 (5) and $K' \in \mathcal{B}_a(q, p_{a,q,K})$, we have $p = p_{a,q,K'} \preceq p_{a,q,K}$. Therefore $r = r(p) \in \mathcal{R}$ and $p_r \in P^*_a(q)$ are defined. Let $i \in \mathrm{lh}(a) \setminus \mathrm{dom}(q)$. Whenever $i \notin r = r(p) = r(p_r)$, we have $p_r(i) = p(i) = p_{a,q,K}(i)$. Whenever $i \in r = r(p) = r(p_r)$, by $K' \in (p_r(i), p_{a,q,K}(i)]^-$, we have $p_r(i) < p_{a,q,K'}(i) = p(i)$. Therefore we have $p_r \prec p$. This contradicts $p_r \in P^*_a(q)$. □

The following claim completes the proof of the theorem.

Claim 6. $\mathcal{U} = \bigcup_{a\in A, q\in Q_a} \mathcal{B}^*_a(q)$.

Proof. "$\supset$" is evident. To see "$\subset$", let $K \in \mathcal{U}$. Since $\mathbb{B}$ is a base for $\mathcal{K}(\mathbb{S})$, there are $a \in A$ and $p \in P_a(\emptyset)$ with $K \in \mathcal{B}_a(\emptyset, p) \subset \mathcal{U}$. Then note $p \in P'_a(\emptyset)$. Take such an $a \in A$. Then $q = \emptyset$ witnesses the sentence that there are $q \in Q_a$ and $p \in P'_a(q)$ with $K \in \mathcal{B}_a(q,p)$. Take a maximal element $q \in Q_a$ with respect to the inclusion "$\subset$" such that there is $p \in P'_a(q)$ with $K \in \mathcal{B}_a(q,p)$. Moreover fix such a $p \in P'_a(q)$ with $K \in \mathcal{B}_a(q,p)$, then note $p_{a,q,K} \preceq p$. It suffices to see $K \in \mathcal{B}^*_a(q)$, that is, $p \in P^*_a(q)$ and $K \in \mathcal{B}^*_a(q,p)$.

First assume $p \notin P^*_a(q)$, then by the definition, there is $\tilde{p} \in P'_a(q)$ with $p \prec \tilde{p}$. It follows from $K \in \mathcal{B}_a(q,p) \subseteq \mathcal{B}_a(q,\tilde{p})$ and $p \prec \tilde{p}$ that $K \in \mathcal{B}_a(q,\tilde{p}) \setminus \mathcal{B}^*_a(q,\tilde{p})$. By Claim 3 (6), there are $q' \in Q_a$ and $p' \in P_a(q')$ such that $q \subsetneq q'$ and $K \in \mathcal{B}_a(q',p') \subseteq \mathcal{B}_a(q,\tilde{p}) \subseteq \mathcal{U}$, this contradicts the maximality of q. Therefore we have $p \in P^*_a(q)$.

Next assume $K \notin \mathcal{B}^*_a(q,p)$. By $K \in \mathcal{B}_a(q,p)$, similarly applying Claim 3(6), there are $q' \in Q_a$ and $p' \in P_a(q')$ such that $q \subsetneq q'$ and $K \in \mathcal{B}_a(q',p') \subseteq \mathcal{B}_a(q,p) \subseteq \mathcal{U}$, also we have a contradiction. We see $K \in \mathcal{B}^*_a(q,p)$. □

□

Corollary 4.8. *The product $\mathbb{S}^\omega$ of countably many copies of the Sorgenfrey line $\mathbb{S}$ is perfect and has a σ-interior preserving base, therefore $\mathbb{S}^\omega$ and $\mathbb{S}^2$ are orthocompact.*

Proof. Remark that $\mathbb{S}$ is homeomorphic to its subspace $(0,1]$. Because, let

$$I_m = (m, m+1] \text{ for every } m \in \mathbb{Z},$$

$$J_n = \Big(\frac{1}{n+2}, \frac{1}{n+1}\Big] \text{ for every } n \in \omega.$$

Then obviously I_m and J_n are homeomorphic, therefore $\mathbb{S} = \bigoplus_{m\in\mathbb{Z}} I_m$ and $(0,1] = \bigoplus_{n\in\omega} J_n$ are homeomorphic.

Now let $n \in \omega$, then $\prod_{0\le m<n} I_m$ can be embedded into $\mathcal{K}(\mathbb{S})$ as a closed subspace with the map $x = \langle x(m) : 0 \le m < n\rangle \to \{x(m) : 0 \le m < n\}$. Since $\mathbb{S}^n$ is homeomorphic to $\prod_{0\le m<n} I_m$, it can be embedded into $\mathcal{K}(\mathbb{S})$ as a closed subspace. Therefore $\mathbb{S}^n$ is perfect and has a σ-interior preserving base for every $n \in \omega$. By applying Lemma 4.3, we see that $\mathbb{S}^\omega$ is perfect and has a σ-interior preserving base, therefore it is orthocompact. □

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