(1) Tel: +81-97-554-7569 (2) Fax: +81-97-554-7569
(3) e-mail: nkemoto@cc.oita-u.ac.jp

STRONG ZERO-DIMENSIONALITY OF HYPERSPACES

NOBUYUKI KEMOTO AND JUN TERASAWA

Abstract. For a space X, 2^X denotes the collection of all non-empty closed sets of X with the Vietoris topology, and $\mathcal{K}(X)$ denotes the collection of all non-empty compact sets of X with the subspace topology of 2^X. The following are known:

- 2^ω is not normal, where ω denotes the discrete space of countably infinite cardinality.
- For every non-zero ordinal γ with the usual order topology, $\mathcal{K}(\gamma)$ is normal iff $\mathrm{cf}\gamma = \gamma$ whenever $\mathrm{cf}\gamma$ is uncountable.

In this paper, we will prove:

(1) 2^ω is strongly zero-dimensional.
(2) $\mathcal{K}(\gamma)$ is strongly zero-dimensional, for every non-zero ordinal γ.

In (2), we use the technique of elementary submodels.

Throughout, spaces are Tychonoff spaces. And $\alpha, \beta, \gamma, ...$ stand for ordinals, while $k, l, m, ...$ for natural numbers. For the notational convenience, we consider -1 as the immediate predecessor of the ordinal 0. Ordinals are considered as spaces with the usual order topology.

For a space X, we let 2^X, resp. $\mathcal{K}(X)$, denote the collection of all non-empty closed, resp. compact, subsets of X.

We consider 2^X with the so-called Vietoris topology τ_V, and $\mathcal{K}(X)$ its subspace. X is called the base space, and 2^X and $\mathcal{K}(X)$ the hyperspaces or the exponential spaces of X.

To describe τ_V, we need some notation. For every finite family $\mathcal{V}$ of open subsets of X, let

$$\langle \mathcal{V} \rangle_{2^X} = \left\{ F \in 2^X : F \subset \bigcup \mathcal{V}, \forall V \in \mathcal{V} (V \cap F \neq \emptyset) \right\},$$
$$\langle \mathcal{V} \rangle_{\mathcal{K}(X)} = \left\{ F \in \mathcal{K}(X) : F \subset \bigcup \mathcal{V}, \forall V \in \mathcal{V} (V \cap F \neq \emptyset) \right\}.$$

Observe that $\langle \mathcal{V} \rangle_{2^X} \cap \mathcal{K}(X) = \langle \mathcal{V} \rangle_{\mathcal{K}(X)}$. Then the collection of all subsets of 2^X of the form $\langle \mathcal{V} \rangle_{2^X}$ is a base for τ_V. Obvioulsy, $\mathcal{K}(X)$ has the base of the form $\langle \mathcal{V} \rangle_{\mathcal{K}(X)}$ For the simplicity's sake, we will often write $\langle \mathcal{V} \rangle$ instead of $\langle \mathcal{V} \rangle_{2^X}$ or $\langle \mathcal{V} \rangle_{\mathcal{K}(X)}$, if the context is clear.

For an open subset U of X, let

$$U^- = \{F \in 2^X : F \cap U \neq \emptyset\}, \quad U^+ = \{F \in 2^X : F \subset U\}.$$

Then obviously, these sets form a subbase for τ_V.

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In the pioneering work [7], E.Michael established basic properties of the hyperspaces. In particular, 2^X is Tychonoff iff X is normal, and $\mathcal{K}(X)$ is Tychonoff iff X is Tychonoff. Hence, 2^γ and $\mathcal{K}(\gamma)$ are Tychonoff for a non-zero ordinal γ.

It is known that 2^ω is not normal [3, 4]. Previously [5], the first author showed that, for every non-zero ordinal γ, $\mathcal{K}(\gamma)$ is normal iff $\mathrm{cf}\gamma = \gamma$ whenever $\mathrm{cf}\gamma$ is uncountable.

We recall that a space X is *zero-dimensional* if it has a base consisting of clopen sets (that is, simultaneously-closed-and-open sets), and *strongly zero-dimensional* if its Stone-Čech compactification βX is zero-dimensional. It is well-known that X is strongly zero-dimensional iff its disjoint zero-sets are separated by a clopen set ([2, 6.2.4 and 6.2.12]). Obviously, every strongly zero-dimensional space is zero-dimensional, but not vice versa even for metrizable spaces. For Lindelöf spaces, it is known that zero-dimensionality implies strong zero-dimensionality ([2, 6.2.7])

In the literature it is often investigated whether disjoint closed sets of a certain space X are separated by clopen sets. This property is equivalent to "normality plus strong zero-dimensionality".

Therefore we need to investigate strong zero-dimensionality itself.

We note that 2^X is zero-dimensional if X is normal and strongly zero-dimensional [5, the comment after Lemma 6], and that $\mathcal{K}(X)$ is zero-dimensional if X is zero-dimensional [7, Proposition 4.13].

In this paper we will prove the following two theorems.

Theorem 1. 2^ω *is strongly zero-dimensional.*

Theorem 2. $\mathcal{K}(\gamma)$ *is strongly zero-dimensional for every non-zero ordinal* γ.

For the proof of the latter, we will use a countable elementary submodel of $H(\theta)$ for some suitably large regular cardinal θ.

1. Proof of Theorem 1

The following lemma was first shown by the second author [8] (see also [2, 6.2.C(b)]), and is useful for our purpose. Here a cozero set is the complement of a zero set.

Lemma 1. *A space is strongly zero-dimensional iff every cozero set can be represented as the union of countably many clopen sets.*

For the proof of Theorem 1, first, for every pair $F \in 2^\omega$ and $n \in \omega$, let

$$\mathcal{S}_n(F) = \{F' \in 2^\omega : F' \cap n = F \cap n, F' \subset F\}.$$

Observe that $\mathcal{S}_n(F) = \bigcap_{i \in F \cap n} \{i\}^- \cap F^+$ and hence, $\{\mathcal{S}_n(F) : n \in \omega\}$ is a decreasing neighborhood base at F in 2^ω. The following two claims are easy to prove.

Claim 1. If $F' \in \mathcal{S}_n(F)$, then $\mathcal{S}_n(F') \subset \mathcal{S}_n(F)$.

Claim 2. If $n \leq k$, $H \in \mathcal{S}_n(F)$, $K \in \mathcal{S}_k(F)$ and $H \cap k \neq \emptyset$, then $H \cap k \in \mathcal{S}_n(K)$.

Let $\mathcal{U}$ be a cozero set in 2^ω. We may assume $\mathcal{U} = f^{-1}[(0,1]]$ for a continuous map f on 2^ω into the unit interval $[0,1]$. Now let for each $n \in \omega$,

$$\mathcal{A}_n = \left\{ F \in 2^\omega : f[\mathcal{S}_n(F)] \subset \left[\frac{1}{n}, 1\right] \right\}.$$

Claim 3. $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{A}_n$.

Proof of Claim 3. $\mathcal{U} \supset \bigcup_{n\in\omega} \mathcal{A}_n$ is obvious. Let $F \in \mathcal{U}$ and take $n_0 \in \omega$ with $f(F) > \frac{1}{n_0}$. By the continuity of f one can take $n_1 \in \omega$ with $f[\mathcal{S}_{n_1}(F)] \subset \left(\frac{1}{n_0}, 1\right]$. Letting $n = \max\{n_0, n_1\}$, we have

$$f[\mathcal{S}_n(F)] \subset f[\mathcal{S}_{n_1}(F)] \subset \left(\frac{1}{n_0}, 1\right] \subset \left(\frac{1}{n}, 1\right] \subset \left[\frac{1}{n}, 1\right],$$

thus $F \in \mathcal{A}_n$. This shows $\mathcal{U} \subset \bigcup_{n\in\omega} \mathcal{A}_n$. □

Now let $n \in \omega$.

Claim 4. $\mathcal{A}_n$ is open in 2^ω.

Proof of Claim 4. Let $F \in \mathcal{A}_n$ and $F' \in \mathcal{S}_n(F)$. By Claim 1, $\mathcal{S}_n(F') \subset \mathcal{S}_n(F)$ holds. Now we have

$$f[\mathcal{S}_n(F')] \subset f[\mathcal{S}_n(F)] \subset \left[\frac{1}{n}, 1\right].$$

This shows $\mathcal{S}_n(F) \subset \mathcal{A}_n$, consequently $\mathcal{A}_n$ is open in 2^ω. □

Claim 5. $\mathcal{A}_n$ is closed in 2^ω.

Proof of Claim 5. Let $F \in \mathrm{Cl}_{2^\omega}\mathcal{A}_n$. We will show $F \in \mathcal{A}_n$, that is $f[\mathcal{S}_n(F)] \subset [\frac{1}{n}, 1]$. Let $H \in \mathcal{S}_n(F)$. For each $k \geq n$, since $\mathcal{S}_k(F)$ is a neighborhood of F, we can take $H_k \in \mathcal{S}_k(F) \cap \mathcal{A}_n$. Then by Claim 2, $H \cap k \in \mathcal{S}_n(H_k)$ holds for each $k \geq n$ with $H \cap k \neq \emptyset$. For such a k, by $H_k \in \mathcal{A}_n$, we have $f(H \cap k) \geq \frac{1}{n}$. Then since $\mathcal{H} = \{H \cap k : k \geq n, H \cap k \neq \emptyset\}$ converges to H (i.e., every neighborhood of H contains all but finitely many members of $\mathcal{H}$), we have $f(H) \geq \frac{1}{n}$. □

The last two claims complete the proof of Theorem 1.

2. Proof of Theorem 2

We use the following basic lemma about $\mathcal{K}(\gamma)$.

Lemma 2. [5] *Let γ be a non-zero ordinal, $F \in \mathcal{K}(\gamma)$ and $\mathcal{V}$ a finite collection of open sets in γ with $F \in \langle \mathcal{V} \rangle$. Then there are $n \in \omega$ and decreasing sequences $\{\alpha_i : i < n\}$ and $\{\beta_i : i < n\}$ of ordinals in γ such that*

(1) $\alpha_0 = \max F$, $\{\alpha_i : i < n\} \subset F$.
(2) $\alpha_{i+1} \leq \beta_i < \alpha_i$ *for each* $i < n$, *where* $\alpha_n = -1$.
(3) $F \in \langle \{(\beta_i, \alpha_i] : i < n\} \rangle \subset \langle \mathcal{V} \rangle$.

In this section, we use a *countable* elementary submodel of $H(\theta)$ for some large enough regular cardinal θ. Note that this approach is somewhat different from the use of elementary submodels in Theorem 8 of [5], where the cardinality of the elementary submodels are larger (in general not countable).

The proof of Theorem 2 is divided into six claims.

If γ is a successor ordinal, then it follows from the zero-dimensionality of γ and Proposition 4.13.1 and Theorem 4.2 in [7] that $2^\gamma = \mathcal{K}(\gamma)$ is zero-dimensional and compact therefore strongly zero-dimensional.

So we may assume that γ is a limit ordinal. To see that $X = \mathcal{K}(\gamma)$ is strongly zero-dimensional, let $f : X \to [0, 1]$ be a continuous map. We will show that the zero sets $f^{-1}[\{0\}]$ and $f^{-1}[\{1\}]$ are separated by a clopen set.

Let M be a countable elementary submodel of $H(\theta)$, where θ is large enough, such that $\gamma, f \in M$, see [1, 6] for basic facts about elementary submodels. For each $\beta < \gamma$, let

$$u(\beta) = \min([\beta, \gamma] \cap M).$$

Obviously we have:

(a) for each $\beta < \gamma$, $\beta \leq u(\beta) \in M$,
(b) for each $\beta < \gamma$, $\beta \in M$ iff $u(\beta) = \beta$,
(c) if $\beta' < \beta < \gamma$, then $u(\beta') \leq u(\beta)$.

Moreover let

$$Z = \{u(\beta) : \beta < \gamma\}.$$

Then $Z \subset [0, \gamma] \cap M$ and u can be considered as a function on γ onto Z, i.e., $u : \gamma \to Z$.

Claim 1. We have the following:

(1) If $\mathrm{cf}\gamma \geq \omega_1$, then $Z = [0, \gamma] \cap M$, $\gamma \in Z$ and $[0, \gamma) \cap M$ is bounded in γ.
(2) If $\mathrm{cf}\gamma = \omega$, then $Z = [0, \gamma) \cap M$, $\gamma \notin Z$ and $Z = [0, \gamma) \cap M$ is unbounded in γ.

Proof of Claim 1. It follows from (b) that $[0, \gamma) \cap M \subset Z$.

(1): Let $\mathrm{cf}\gamma \geq \omega_1$. Since M is countable, we can take $\beta < \gamma$ with $\sup(\gamma \cap M) < \beta$. Then by $\gamma \in M$, we have $\gamma = u(\beta) \in Z$. Other properties are almost obvious.

(2): Let $\mathrm{cf}\gamma = \omega$. There is a strictly increasing cofinal sequence $\{\gamma_n : n \in \omega\}$ in γ. By elementarity and $\gamma \in M$, we may assume $\{\gamma_n : n \in \omega\} \in M$. Since $\{\gamma_n : n \in \omega\}$ is countable and belongs to M, it is a subset of M, that is, $\{\gamma_n : n \in \omega\} \subset M$, see Theorem 1.6 of [1]. Therefore we see that $[0, \gamma) \cap M$ is unbounded in γ. Now let $\beta < \gamma$ and take $n \in \omega$ with $\beta < \gamma_n$. It follows from $\gamma_n \in M$ and the definition of $u(\beta)$ that $u(\beta) \leq \gamma_n$. This shows $\gamma \notin Z$ and $Z = [0, \gamma) \cap M$. □

Now we give Z the order topology. Note that this topology on Z is weaker than the subspace topology on Z of the ordinal $\gamma + 1 = [0, \gamma]$. Since Z is countable, it is homeomorphic to a countable ordinal. In particular by Claim 1, Z is homeomorphic to a successor ordinal $< \omega_1$ if $\mathrm{cf}\gamma \geq \omega_1$, and to a limit ordinal $< \omega_1$ if $\mathrm{cf}\gamma = \omega$.

We consider the hyperspace $Y = \mathcal{K}(Z)$. Since Z is second countable, by Proposition 4.5.2 of [7], $Y = \mathcal{K}(Z)$ is also second countable.

Now we investigate the relationship between $X = \mathcal{K}(\gamma)$ and $Y = \mathcal{K}(Z)$.

For each $\alpha \in Z$, let

$$d(\alpha) = \sup\{\delta + 1 : \delta \in \alpha \cap Z\}.$$

By Claim 1, $d(\alpha) = \sup\{\delta + 1 : \delta \in \alpha \cap M\}$ holds and d can be considered as a function on Z into γ, that is, $d : Z \to \gamma$. Obviously we have:

(d) for each $\alpha \in Z$, $d(\alpha) \leq \alpha$,
(e) if $\alpha', \alpha \in Z$ with $\alpha' < \alpha$, then $d(\alpha') \leq d(\alpha)$.

Claim 2. $u : \gamma \to Z$ and $d : Z \to \gamma$ are both continuous.

Proof of Claim 2. For u: Let $\beta < \gamma$ and V be a neighborhood of $u(\beta)$ in Z. By the definition of the topology of Z, we can find $\alpha \in Z$ with $\alpha < u(\beta)$ and $(\alpha, u(\beta)] \cap Z \subset V$. By $\alpha \in Z \subset M$, we have $\alpha < \beta$ and $u[(\alpha, \beta]] \subset (\alpha, u(\beta)] \cap Z \subset V$. We see that u is continuous.

For d: Let $\alpha \in Z$ and $\beta < d(\alpha)$. By the definition of $d(\alpha)$, we can find $\beta' \in \alpha \cap M$ with $\beta < \beta' + 1$. Then $\beta \leq \beta' \in M$ and $(\beta', \alpha] \cap Z$ is a neighborhood of α in Z. Now we have $d[(\beta', \alpha] \cap Z] \subset (\beta', d(\alpha)] \subset (\beta, d(\alpha)]$, so d is continuous. □

Claim 3. The functions u and d have the following properties:

(1) For every $\beta < \gamma$, $d(u(\beta)) = \sup\{\delta + 1 : \delta \in \beta \cap M\} \leq \beta$.
(2) For every $\alpha \in Z$, $u(d(\alpha)) = \alpha$ holds, that is, the composition $u \circ d$ is the identity map on Z.
(3) For every $\beta < \gamma$ and $\alpha \in Z$, if $\beta < d(\alpha)$, then $u(\beta) < d(\alpha) \leq \alpha$.
(4) If $\beta' < \beta < \gamma$, $\alpha \in Z$ and $d(\alpha) \in (\beta', \beta]$, then $\alpha \in (u(\beta'), u(\beta)]$.

Proof of Claim 3. (1): Let $\beta < \gamma$. When $\beta \in M$, by $u(\beta) = \beta$ we have $d(u(\beta)) = d(\beta) = \sup\{\delta + 1 : \delta \in \beta \cap M\}$. When $\beta \notin M$, by $[\beta, u(\beta)) \cap M = \emptyset$ we have $\beta \cap M = u(\beta) \cap M$. Therefore $d(u(\beta)) = \sup\{\delta + 1 : \delta \in \beta \cap M\} \leq \beta$.

(2): Let $\alpha \in Z$. Then by $d(\alpha) \leq \alpha \in Z \subset M$, clearly $u(d(\alpha)) \leq \alpha$ holds. Assume $u(d(\alpha)) < \alpha$. It follows from $u(d(\alpha)) \in \alpha \cap M$ and the definition of $d(\alpha)$ that $u(d(\alpha)) + 1 \leq d(\alpha)$. Then $d(\alpha) \leq u(d(\alpha)) < d(\alpha)$, a contradiction.

(3): Let $\beta < \gamma$, $\alpha \in Z$ and $\beta < d(\alpha)$. Then by the definition of $d(\alpha)$, there is $\delta \in \alpha \cap M$ with $\beta < \delta + 1$. Then we have $\beta \leq \delta < \delta + 1 \leq d(\alpha)$. It follows from $\delta \in M$ that $u(\beta) \leq \delta < d(\alpha) \leq \alpha$.

(4) easily follows from (2). □

Define $\tilde{u} : X \to Y$ and $\tilde{d} : Y \to X$ by

$$\tilde{u}(F) = u[F], \quad \tilde{d}(H) = d[H] \quad \text{for } F \in X \text{ and } H \in Y.$$

Then by the following general result, $\tilde{u}$ and $\tilde{d}$ are continuous.

Claim 4. For each continuous map $h : S \to T$, define $\tilde{h} : \mathcal{K}(S) \to \mathcal{K}(T)$ by $\tilde{h}(F) = h[F]$ for each $F \in \mathcal{K}(S)$. Then $\tilde{h}$ is continuous.

Claim 5. $\tilde{u} : X \to Y$ is quotient.

Proof of Claim 5. Let $\mathcal{U} \subset Y$ such that $\tilde{u}^{-1}[\mathcal{U}]$ is open in X. To see that $\mathcal{U}$ is open in Y, let $H \in \mathcal{U}$. By Claim 3(2) and $\tilde{u}(\tilde{d}(H)) = u[d[H]] = H \in \mathcal{U}$, we have $\tilde{d}(H) \in \tilde{u}^{-1}[\mathcal{U}]$. Since $\tilde{u}^{-1}[\mathcal{U}]$ is open in X, there is a finite collection $\mathcal{V}$ of open sets in γ such that $\tilde{d}(H) \in \langle \mathcal{V} \rangle \subset \tilde{u}^{-1}[\mathcal{U}]$. By Lemma 2, we may assume that $\mathcal{V} = \{(\beta_i, \alpha_i] : i < n\}$, where $n \in \omega$, $\{\alpha_i : i < n\}$ and $\{\beta_i : i < n\}$ are decreasing sequences in γ such that

(1) $\alpha_0 = \max \tilde{d}(H)$, $\{\alpha_i : i < n\} \subset \tilde{d}(H)$.
(2) $\alpha_{i+1} \leq \beta_i < \alpha_i$ for each $i < n$, where $\alpha_n = -1$.

Subclaim 1. $u(\beta_i) < u(\alpha_i)$ for each $i < n$.

Proof of Subclaim 1. Let $i < n$. It follows from $\alpha_i \in \tilde{d}(H) = d[H]$ that there is $\delta \in H$ with $d(\delta) = \alpha_i$. By Claim 3(2), we have $\delta = u(d(\delta)) = u(\alpha_i)$. Moreover by $\beta_i < \alpha_i = d(\delta)$ and Claim 3(3), $u(\beta_i) < d(\delta) \leq \delta$ holds. Therefore we have $u(\beta_i) < u(\alpha_i)$. □

Subclaim 2. $H \in \langle \{(u(\beta_i), u(\alpha_i)] \cap Z : i < n\} \rangle$.

Proof of Subclaim 2. First let $\delta \in H$. By $d(\delta) \in d[H] = \tilde{d}(H) \in \langle \mathcal{V} \rangle$, there is $i < n$ such that $d(\delta) \in (\beta_i, \alpha_i]$. It follows from Claim 3(4) that $\delta \in (u(\beta_i), u(\alpha_i)] \cap Z$. This shows $H \subset \bigcup_{i<n}((u(\beta_i), u(\alpha_i)] \cap Z)$.

Next let $i < n$. Then by $\tilde{d}(H) \in \langle\mathcal{V}\rangle$, we have $\emptyset \neq \tilde{d}(H) \cap (\beta_i, \alpha_i] = d[H] \cap (\beta_i, \alpha_i]$. Therefore we can take $\delta \in H$ with $d(\delta) \in (\beta_i, \alpha_i]$. Then as in the first paragraph above, we get $\delta \in H \cap ((u(\beta_i), u(\alpha_i)] \cap Z)$. Thus $H \cap ((u(\beta_i), u(\alpha_i)] \cap Z) \neq \emptyset$. □

Subclaim 3. $\langle\{(u(\beta_i), u(\alpha_i)] \cap Z : i < n\}\rangle \subset \mathcal{U}$.

Proof of Subclaim 3. Let $K \in \langle\{(u(\beta_i), u(\alpha_i)] \cap Z : i < n\}\rangle$. It suffices to see $d[K] \in \langle\mathcal{V}\rangle$, because this shows $K = u[d[K]] = \tilde{u}(d[K]) \in \tilde{u}[\langle\mathcal{V}\rangle] \subset \mathcal{U}$.

To see $d[K] \subset \bigcup_{i<n}(\beta_i, \alpha_i]$, let $\delta \in K$. Then there is an $i < n$ with $\delta \in (u(\beta_i), u(\alpha_i)] \cap Z$. If $\alpha_i < d(\delta)$ were true, then by Claim 3(3) we have $u(\alpha_i) < d(\delta) \leq \delta$, a contradiction. Therefore $d(\delta) \leq \alpha_i$ holds. Next if $d(\delta) \leq \beta_i$ were true, then $\delta = u(d(\delta)) \leq u(\beta_i)$ holds, a contradiction. Therefore $\beta_i < d(\delta)$ holds and we have $d(\delta) \in (\beta_i, \alpha_i]$.

To see $d[K] \cap (\beta_i, \alpha_i] \neq \emptyset$ for each $i < n$, let $i < n$. Then there is $\delta \in K$ with $\delta \in (u(\beta_i), u(\alpha_i)] \cap Z$ by $K \in \langle\{(u(\beta_i), u(\alpha_i)] \cap Z : i < n\}\rangle$. By a similar argument above, we have $d[K] \cap (\beta_i, \alpha_i] \neq \emptyset$. □

Obviously these Subclaims complete the proof of Claim 5. □

Claim 6. For every $F \in X$, $f(F) = f(\tilde{d}(\tilde{u}(F)))$.

Proof of Claim 6. Let $K = \tilde{d}(\tilde{u}(F)) = d[u[F]]$ and assume $f(F) \neq f(K)$. Let us consider the case $f(F) < f(K)$. (The proof for the case $f(F) > f(K)$ is quite similar.) Fix $r \in \mathbb{Q} \cap [0, 1]$ with $f(F) < r < f(K)$, where $\mathbb{Q}$ denotes the set of all rationals. By Lemma 2 and the continuity of f, we can find $n \in \omega$ and two decreasing sequences $\{\alpha_i : i < n\}$ and $\{\beta_i : i < n\}$ of ordinals in γ such that

(1) $\alpha_0 = \max K$, $\{\alpha_i : i < n\} \subset K$,
(2) $\alpha_{i+1} \leq \beta_i < \alpha_i$ for each $i < n$, where $\alpha_n = -1$,
(3) $K \in \langle\mathcal{V}\rangle$, where $\mathcal{V} = \{(\beta_i, \alpha_i] : i < n\}$,
(4) $f[\langle\mathcal{V}\rangle] \subset (r, 1]$.

Note that ω, $\mathbb{Q}$ and $\mathbb{R}$ (the set of all reals) are definable in $H(\theta)$. Therefore they are elements of the countable elementary submodel M. Since ω and $\mathbb{Q}$ are countable, we have $\omega \subset M$ and $\mathbb{Q} \subset M$. On the other hand, $\mathbb{R} \not\subset M$ holds because M is countable but not $\mathbb{R}$. Moreover the unit interval $[0, 1]$ belongs to M because it is definable from $0, 1 \in \mathbb{Q} \subset M$. Similarly we have $(r, 1] \in M$ whenever $r \in \mathbb{Q}$.

Note that $u(\beta_i) < u(\alpha_i)$ for every $i < n$ (use the same argument in Subclaim 1 of Claim 5). Now let for each $i < n$,

$$W_i = \begin{cases} (u(\beta_i), u(\alpha_i)) & \text{if } \alpha_i < u(\alpha_i), \text{ i.e., } \alpha_i \notin M, \\ (u(\beta_i), u(\alpha_i)] & \text{if } \alpha_i = u(\alpha_i), \text{ i.e., } \alpha_i \in M. \end{cases}$$

Then obviously $\mathcal{W} = \{W_i : i < n\}$ is a pairwise disjoint collection of open sets in γ. Since $(u(\beta_i), u(\alpha_i))$ and $(u(\beta_i), u(\alpha_i)]$ are definable from $u(\beta_i), u(\alpha_i) \in M$, W_i's are elements of M. Moreover, since $\mathcal{W}$ is finite, it also belongs to M.

Subclaim 1. For every $L \in \langle\mathcal{W}\rangle_X \cap M$, $f(L) > r$ holds.

Proof of Subclaim 1. Let $L \in \langle\mathcal{W}\rangle \cap M$, it suffices to see $L \in \langle\mathcal{V}\rangle$. For each $i < n$, set $L_i = L \cap W_i$. Note that by $L \in \langle\mathcal{W}\rangle$, each L_i is non-empty. Since $\mathcal{W}$ is a pairwise disjoint open cover of the compact set L, each L_i is compact. Since each L_i is determined by $L, W_i \in M$, it also belongs to M. By the compactness of L_i, the maximal element $\max L_i$ of L_i exists. Moreover by $L_i \in M$, both $\max L_i$ and $\min L_i$ are elements of M.

Let $i < n$, now we will show $L_i \subset (\beta_i, \alpha_i]$. It follows from $L_i \subset W_i$ that $\beta_i \le u(\beta_i) < \min L_i$. When $\alpha_i < u(\alpha_i)$, it follows from $\max L_i < u(\alpha_i)$ and $\max L_i \in M$ that $\max L_i + 1 \le d(u(\alpha_i)) \le \alpha_i$. When $\alpha_i = u(\alpha_i)$, we have $\max L_i \le u(\alpha_i) = \alpha_i$. In either cases, $L_i \subset (\beta_i, \alpha_i]$ holds.

Therefore we have $L = \bigcup_{i<n} L_i \in \langle \mathcal{V} \rangle$. □

Subclaim 1 says that

$$M \models \text{“For every } L \in \langle \mathcal{W} \rangle_X, f(L) > r \text{ holds.”}$$

Then by elementarity and $\mathcal{W}, f, r, \gamma \in M$,

$$(*)\ \text{“For every } L \in \langle \mathcal{W} \rangle_X, f(L) > r \text{ holds.”}$$

Subclaim 2. $F \in \mathrm{Cl}_X \langle \mathcal{W} \rangle$.

Proof of Subclaim 2. For each $i < n$, let

$$W'_i = \begin{cases} (u(\beta_i), u(\alpha_i)) & \text{if } u(\alpha_i) = \gamma, \\ (u(\beta_i), u(\alpha_i)] & \text{otherwise.} \end{cases}$$

Moreover let $\mathcal{W}' = \{W'_i : i < n\}$. Note that if $u(\alpha_i) = \gamma$, then $i = 0$ should hold because $\cdots < u(\alpha_1) \le u(\beta_0) < u(\alpha_0)$.

Further note that if $\alpha_i < u(\alpha_i)$, then $u(\alpha_i)$ is a limit ordinal. Otherwise, $u(\alpha_i) = \beta + 1$ for some ordinal β. Then β is the immediate predecessor of $u(\alpha_i) \in M$ (i.e., β is definable from $u(\alpha_i) \in M$), so by elementarity, we have $\beta \in M$ and $\alpha_i \le \beta < u(\alpha_i)$, which contradicts the definition of $u(\alpha_i)$.

Now by the definitions of W_i and W'_i, we have $\mathrm{Cl}_\gamma W_i = W'_i$ for each $i < n$. By a similar argument as in Proposition 2.3.2 of [7], we have

$$\mathrm{Cl}_X \langle \mathcal{W} \rangle = \langle \{\mathrm{Cl}_\gamma W_i : i < n\} \rangle = \langle \{W'_i : i < n\} \rangle = \langle \mathcal{W}' \rangle.$$

It suffices to see $F \in \langle \mathcal{W}' \rangle$.

First let $\delta \in F$. It follows from $K = d[u[F]] \in \langle \mathcal{V} \rangle$ that $d(u(\delta)) \in (\beta_i, \alpha_i]$ for some $i < n$. By $\beta_i < d(u(\delta))$ and Claim 3(3) we have $u(\beta_i) < d(u(\delta)) \le \delta$. On the other hand, by $d(u(\delta)) \le \alpha_i$ and Claim 3(2), $\delta \le u(\delta) = u(d(u(\delta))) \le u(\alpha_i)$ holds. In particular, when $u(\alpha_i) = \gamma$ (then $i = 0$ as above), by $\delta \in F \subset \gamma = u(\alpha_i)$, we have $\delta < u(\alpha_i)$. These arguments show $\delta \in W'_i$ therefore $F \subset \bigcup_{i<n} W'_i$. Next let $i < n$. Because of $K = d[u[F]] \in \langle \mathcal{V} \rangle$, we have $K \cap (\beta_i, \alpha_i] \ne \emptyset$. Take $\delta \in F$ with $d(u(\delta)) \in (\beta_i, \alpha_i]$. By a similar argument as above, we have $\delta \in W'_i$ thus $F \cap W'_i \ne \emptyset$. So we have $F \in \langle \mathcal{W}' \rangle$. □

Now $(*)$ and Subclaim 2 imply $f(F) \ge r$, which contradicts $f(F) < r$. Claim 6 is now established. □

Finally let us return to the proof of Theorem 2.

Let us define $g : Y = \mathcal{K}(Z) \to [0, 1]$ as follows:

$$g(H) = f(F), \text{ where } \tilde{u}(F) = H.$$

Note that $\tilde{u}$ is onto, by Claim 3(2).

To see that g is well-defined, let $\tilde{u}(F) = \tilde{u}(F') = H$. Then by $\tilde{d}(\tilde{u}(F)) = \tilde{d}(\tilde{u}(F'))$ and Claim 6 we have $f(F) = f(F')$. Therefore the value $g(H)$ does not depend on the choice of $F \in X$ with $\tilde{u}(F) = H$.

Since $\tilde{u}$ is quotient, f is continuous and $f = g \circ \tilde{u}$, we see g is continuous. Since Z is homeomorphic to a countable ordinal, Z is zero-dimensional and second countable. Then by Propositions 4.5.2 and 4.13.1 of [7], Y is also zero-dimensional and second countable. Moreover by Theorem 6.2.7 of [2], Y is strongly zero-dimensional. Therefore $g^{-1}[\{0\}]$ and $g^{-1}[\{1\}]$ are separated by a clopen set in Y. Since $\tilde{u}$ is continuous, $f^{-1}[\{0\}] = \tilde{u}^{-1}[g^{-1}[\{0\}]]$ and $f^{-1}[\{1\}] = \tilde{u}^{-1}[g^{-1}[\{1\}]]$ are separated by a clopen set. Therefore X is strongly zero-dimensional.

Thus the proof of Theorem 2 is complete.

3. Remarks

The authors do not know the answer to the following two questions, where $D(\omega_1)$ is the discrete space of cardinality ω_1:

Question A. Is $2^{D(\omega_1)}$ strongly zero-dimensional?

Question B. Is 2^{ω_1} strongly zero-dimensional?

Moreover we would like to ask:

Question C. Give a direct proof of Theorem 2 without using elementary submodels.

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Department of Mathematics, Faculty of Education, Oita University, Dannoharu, Oita, 870-1192, Japan
Department of Mathematics, The National Defense Academy, Yokosuka 239-8686, Japan

E-mail addresses:
nkemoto@cc.oita-u.ac.jp
jun.trswa@member.ams.org