ORDERABILITY OF SUBSPACES OF WELL-ORDERABLE TOPOLOGICAL SPACES

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Abstract. We will show that all subspaces of well-ordered spaces are orderable, and we will characterize the well-orderability of subspaces of well-ordered spaces.

1. Introduction

Let $\langle X, < \rangle$ be a linear ordered set, that is, a linear order $<$ is defined on X. For $a, b \in X$, set

- $(a, \rightarrow)_{\langle X,<\rangle} = \{x \in X : a < x\}$,
- $(\leftarrow, b)_{\langle X,<\rangle} = \{x \in X : x < b\}$,
- $(a, b)_{\langle X,<\rangle} = \{x \in X : a < x < b\}$.

Similarly one can define $[a, \rightarrow)_{\langle X,<\rangle}$, $(\leftarrow, a]_{\langle X,<\rangle}$, $[a, b]_{\langle X,<\rangle}$, $(a, b]_{\langle X,<\rangle}$, ...etc.. If contexts are clear, we often omit the suffix "$\langle X, < \rangle$" of the intervals, for instance $(a, b)_{\langle X,<\rangle}$ is written simply as (a, b). $\lambda(X, <)$ denotes the *order topology* on X generated by the collection $\{(a, \rightarrow)_{\langle X,<\rangle} : a \in X\} \cup \{(\leftarrow, a)_{\langle X,<\rangle} : a \in X\}$ as a subbase. Then the triple $\langle X, <, \lambda(X, <)\rangle$ is called an ordered space, and in this case, we simply say "X is an ordered space". Note that if $\prec$ is the reverse order of $<$, then $\lambda(X, <)$ and $\lambda(X, \prec)$ are the same topology. It is well-known that a non-empty ordered space X is compact iff every subset of X has a supremum (equivalently, an infimum), see [1, 3.12.3].

A linear order $<$ on a set X is said to be a *well-order* if every non-empty subset of X has a $<$-minimal element. It is well known that every well-ordered set $\langle X, < \rangle$ is order isomorphic to a unique ordinal with the usual order $<$, that is, the order $\in$, see [3, I, Theorem 7.6]. We call such a unique ordinal as the *order type* of $\langle X, < \rangle$ and it is denoted by $\mathrm{otp}\langle X, < \rangle$.

A topological space $\langle X, \tau \rangle$ is said to be *orderable* (*well-orderable*) if there is a linear order (well-order) $<$ on X with $\lambda(X, <) = \tau$. Therefore every well-orderable space is identified with an ordinal having the order topology. A topological space $\langle X, \tau \rangle$ is said to be *sub-orderable* if there is a linearly ordered set $\langle Y, < \rangle$ such that $X \subseteq Y$ and $\lambda(Y, <) \restriction X = \tau$, here $\lambda(Y, <) \restriction X$ means the subspace topology $\{U \cap X : U \in \lambda(Y, <)\}$ on X of the order topology on Y. Observe that if a linearly ordered set $\langle Y, < \rangle$ is given and $X \subseteq Y$, then $\lambda(X, <\restriction X) \subseteq \lambda(Y, <) \restriction X$ always holds, where $\lambda(X, <\restriction X)$ denotes the order topology on X induced by the restricted order $<\restriction X$ on X of $<$. We usually write "$<\restriction X$" simply by "$<$" if it is clear in the contexts on where the order $<$ is restricted.

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By the definition, subspaces of ordinals are sub-orderable. On the other hand, the Sorgenfley line and the Michael line are known to be sub-orderable but not orderable, see [2]. Even the subspace $(0,1) \cup \{2\}$ of the real line is not orderable, see [4]. In this line, the second author has conjectured that every stationary set X of ω_1 such that $\omega_1 \setminus X$ is also stationary is not orderable. In this paper, we will show that this conjecture is false. Also we will characterize the well-orderability of subspaces of ordinals.

For set theoretical and topological notions, the reader should refer [3] and [1], respectively. **Ord** denotes the class of all ordinals. While **Ord** is a proper class (= a class which is not a set), the order $\in$ is considered as a linear order on **Ord**, usually this order $\in$ is written as $<$. Note that for every sub*set* Z of **Ord**, the supremum $\sup Z$ exists. For notational conveniences, -1 is considered as the immediate predecessor of the minimal ordinal $0 = \emptyset$ and we consider as $\sup \emptyset = -1$. Throughout the paper, each ordinal μ is identified with the set $\{\alpha \in \mathbf{Ord} : \alpha < \mu\}$ and assumed to have the order topology induced by the usual order $<$, in other words, for every $\alpha \in \mu$, the collection $\{(\beta, \alpha] : -1 \leq \beta < \alpha\}$ is a neighborhood base at α. The cofinality of an ordinal α is denoted by $\mathrm{cf}\,\alpha$. ω and ω_1 denote the least infinite ordinal and the least uncountable ordinal respectively. For a set Z of ordinals, $\mathrm{Lim}(Z)$ denotes the set of all cluster points of Z in **Ord**, i.e.

$$\mathrm{Lim}(Z) = \{\alpha \in \mathbf{Ord} \setminus \{0\} : Z \cap (\gamma, \alpha) \neq \emptyset \text{ for every } \gamma \in \alpha\}.$$

In this paper, we use notations $\sup Z$ and $\mathrm{Lim}(Z)$ only for a set $Z \subseteq \mathbf{Ord}$ with the usual order $<$ on **Ord**.

2. Decomposition

In this section, we show some basic facts. And for an arbitrary subspace X of an ordinal, we give a decomposition $\mathcal{I}$ into well-orderable closed convex subsets of X. It is routine to check that the lemma below holds.

Lemma 2.1. *Let $<$ be a linear order on a set X, and $Z \subseteq X$. Assume that for every $c \in X \setminus Z$ with $Z \cap (\leftarrow, c) \neq \emptyset$ and $Z \cap (c, \rightarrow) \neq \emptyset$, $Z \cap (\leftarrow, c)$ has a maximal element if and only if $Z \cap (c, \rightarrow)$ has a minimal element. Then the subspace topology $\lambda(X, <) \restriction Z$ coincides with the order topology $\lambda(Z, <)$.*

Let $<$ be a linear order on a set X. We call a subset Z of X *convex* in $\langle X, < \rangle$ iff $(a, b) \subseteq Z$ for every $a, b \in Z$ with $a < b$. If Z is convex in $\langle X, < \rangle$ and c is a point in X with $Z \cap (\leftarrow, c) \neq \emptyset$ and $Z \cap (c, \rightarrow) \neq \emptyset$, then we have $c \in Z$. Therefore the following well-known lemma is easily seen from Lemma 2.1.

Lemma 2.2. *Let $<$ be a linear order on a set X, and Z a convex set of $\langle X, < \rangle$. Then $\lambda(X, <) \restriction Z$ coincides with $\lambda(Z, <)$.*

Lemma 2.3. *Let $<$ be the usual order on an ordinal μ, and $Z \subseteq \mu$. Assume that $\mathrm{Lim}(Z) \setminus Z \subseteq \{\sup Z\}$. Then $\lambda(\mu, <) \restriction Z$ coincides with $\lambda(Z, <)$.*

Proof. Assume $c \in \mu \setminus Z$, $Z \cap (\leftarrow, c) \neq \emptyset$, and $Z \cap (c, \rightarrow) \neq \emptyset$. Since $Z \cap (c, \rightarrow)$ has a minimal element, by Lemma 2.1, it suffices to show that $Z \cap (\leftarrow, c)$ has a maximal element. Assume that $Z \cap (\leftarrow, c)$ does not have a maximal element. Put $\alpha = \sup(Z \cap (\leftarrow, c))$, then we have $\alpha \in \mathrm{Lim}(Z)$, $\alpha \notin Z \cap (\leftarrow, c)$ and $\alpha \leq c$. It follows from $\alpha \notin Z \cap (\leftarrow, c)$ and $\alpha \leq c \notin Z$ that $\alpha \notin Z$. Hence, $\sup Z > c \geq \alpha \in \mathrm{Lim}(Z) \setminus Z$, a contradiction. $\square$

Lemma 2.4. *Let $<$ be the usual order on* **Ord**, *μ an ordinal and $X \subseteq \mu$, moreover τ denote the subspace topology $\lambda(\mu,<) \upharpoonright X$. Define*

$$\begin{array}{rcl} G & = & \mathrm{Lim}(X) \setminus X, \\ S & = & \begin{cases} (G \setminus \mathrm{Lim}(G)) \cup \{\mu\} & \text{if } X \text{ has a } <\text{-maximal element } \max X, \\ G \setminus \mathrm{Lim}(G) & \text{otherwise,} \end{cases} \\ \mathcal{I} & = & \langle I(\xi) \mid \xi \in S \rangle, \end{array}$$

where for each $\xi \in S$, $I(\xi) = X \cap [\sup(G \cap \xi), \xi)$ and $<_\xi$ denotes the restriction of $<$ on $I(\xi)$. Then the following hold:

1. *If $\xi \in S \setminus G$, then $\xi = \mu$, X has a $<$-maximal element, $G \cap \mu = G$, $I(\mu) = X \cap [\sup G, \max X]$, μ is a $<$-maximal element of S and $\max X$ is also the $<_\mu$-maximal element of $I(\mu)$. In particular, $\mathrm{cf}\,\mathrm{otp}\langle I(\mu), <_\mu \rangle = 1$.*
2. *If $\xi \in G \setminus \mathrm{Lim}(G)$, then $I(\xi)$ is a non-empty subset of X with no $<_\xi$-maximal element, also $I(\xi) = X \cap [\sup(G \cap \xi), \xi]$ and $\xi = \sup I(\xi)$ hold. In particular, $\mathrm{cf}\,\mathrm{otp}\langle I(\xi), <_\xi \rangle = \mathrm{cf}\,\xi \geq \omega$ holds.*
3. *$\mathcal{I}$ is a pairwise disjoint closed cover of the space $\langle X, \tau \rangle$ consisting of non-empty convex subsets of $\langle X, < \rangle$.*
4. *For each $\xi \in S$, $<_\xi$ is a well-order on $I(\xi)$, $I(\xi) \setminus \mathrm{int}_{\langle X,\tau \rangle}(I(\xi)) \subseteq \{\sup(G \cap \xi)\}$ holds, moreover the subspace topology $\tau \upharpoonright I(\xi)$ $(= \lambda(\mu,<) \upharpoonright I(\xi))$ coincides with $\lambda(I(\xi), <_\xi)$.*
5. *$\mathrm{Lim}(S) = \mathrm{Lim}(G)$ holds.*

Proof. Observe that G and S are subsets of $\mu + 1 = [0, \mu]$. (1) is trivial.

(2) Let $\xi \in G \setminus \mathrm{Lim}(G)$. It follows from $\xi \in G \subseteq \mathrm{Lim}(X)$ and $\xi \notin \mathrm{Lim}(G)$ that $\sup(X \cap \xi) = \xi$ and $\sup(G \cap \xi) < \xi$ respectively. Since $\xi \notin X$, all properties in (2) are easily verified.

(3) By (1) and (2), each member of $\mathcal{I}$ is non-empty convex in $\langle X, < \rangle$ and closed in $\langle X, \tau \rangle$.

To see that $\mathcal{I}$ is pairwise disjoint, let $\zeta, \xi \in S$ with $\zeta < \xi$. Then $\zeta \in G \setminus \mathrm{Lim}(G)$, and we have $\zeta \leq \sup(G \cap \xi)$. If $\alpha \in I(\zeta)$ and $\beta \in I(\xi)$, then $\alpha < \zeta \leq \sup(G \cap \xi) \leq \beta$ holds. Therefore we see $I(\zeta) \cap I(\xi) = \emptyset$.

To see that $\mathcal{I}$ covers X, let $\alpha \in X$. First assume that there is $\xi \in G$ with $\alpha < \xi$. Pick the such least ξ, then $\sup(G \cap \xi) \leq \alpha < \xi$. Thus we have $\xi \in G \setminus \mathrm{Lim}(G) \subseteq S$ and $\alpha \in I(\xi)$. Next assume that there is no $\xi \in G$ with $\alpha < \xi$. In this case, X has a $<$-maximal element, for otherwise, $\alpha < \sup X \in G$, a contradiction. Therefore we have $\sup G \leq \alpha \leq \max X$, so we see $\alpha \in I(\mu)$ with $\mu \in S$.

(4) Let $\xi \in S$. Obviously, $<_\xi$ is a well-order. It follows from $X \cap (\sup(G \cap \xi), \xi) \subseteq I(\xi)$ that $I(\xi) \setminus \mathrm{int}_{\langle X,\tau \rangle}(I(\xi)) \subseteq \{\sup(G \cap \xi)\}$.

We show the remaining property. By Lemma 2.3, it suffices to show $\mathrm{Lim}(I(\xi)) \setminus I(\xi) \subseteq \{\sup I(\xi)\}$. Let $\alpha \in \mathrm{Lim}(I(\xi)) \cap \xi$. By the definitions of $I(\xi)$ and $\mathrm{Lim}(I(\xi))$, we have $\alpha \in (\sup(G \cap \xi), \xi)$ therefore $\alpha \notin G$ holds. It follows from $\alpha \in \mathrm{Lim}(X) \setminus G \subseteq X$ that $\alpha \in X \cap [\sup(G \cap \xi), \xi) = I(\xi)$. This shows $\mathrm{Lim}(I(\xi)) \setminus I(\xi) \subseteq \{\xi\}$.

If $\xi \in G \setminus \mathrm{Lim}(G)$, then by (2), we have $\mathrm{Lim}(I(\xi)) \setminus I(\xi) \subseteq \{\xi\} = \{\sup I(\xi)\}$. If $\xi \in S \setminus G$, then by (1), X has a $<$-maximal element and $\xi = \mu > \max X$, so we have $\xi \notin \mathrm{Lim}(X) \supseteq \mathrm{Lim}(I(\xi))$. Therefore $\mathrm{Lim}(I(\xi)) \setminus I(\xi) = \emptyset \subseteq \{\sup I(\xi)\}$.

(5) It follows from $S \subseteq G \cup \{\mu\}$ that $\mathrm{Lim}(S) \subseteq \mathrm{Lim}(G \cup \{\mu\}) = \mathrm{Lim}(G)$. To see $\mathrm{Lim}(G) \subseteq \mathrm{Lim}(S)$, assume $\alpha \in \mathrm{Lim}(G) \setminus \mathrm{Lim}(S)$. Then $\sup(S \cap \alpha) < \alpha$ holds.

Now pick the $<$-minimal element β of G with $\sup(S \cap \alpha) < \beta < \alpha$. Then we have $\beta \in G \setminus \mathrm{Lim}(G) \subseteq S$, so $\beta \leq \sup(S \cap \alpha)$, a contradiction. □

3. Orderability

This section is devoted to prove:

Theorem 3.1. *Subspaces of ordinals are orderable.*

Before proving the theorem, we show some lemmas. Let $\langle\langle I(\xi), \prec_\xi\rangle \mid \xi \in S\rangle$ be a pairwise disjoint collection of non-empty linearly ordered sets, and $\prec_S$ a linear order on the index set S. Let $X = \bigcup_{\xi \in S} I(\xi)$ and for each $\alpha \in X$, let $\xi(\alpha)$ denotes the unique ξ with $\alpha \in I(\xi)$. For each $\alpha, \beta \in X$ define

$$\alpha \prec \beta \text{ by } \begin{cases} \xi(\alpha) \prec_S \xi(\beta) & \text{if } \xi(\alpha) \neq \xi(\beta), \\ \alpha \prec_{\xi(\alpha)} \beta & \text{if } \xi(\alpha) = \xi(\beta). \end{cases}$$

Then $\prec$ is a linear order on X. This linearly ordered set $\langle X, \prec\rangle$ is said to be the order sum of $\langle\langle I(\xi), \prec_\xi\rangle \mid \xi \in S\rangle$ with respect to $\prec_S$. Note that for each $\xi \in S$, $I(\xi)$ is a convex set in $\langle X, \prec\rangle$ and the whole order $\prec$ extends $\prec_\xi$.

Lemma 3.2. *Let $\langle\langle I(\xi), \prec_\xi\rangle \mid \xi \in S\rangle$ be a pairwise disjoint collection of non-empty linearly ordered sets and $\prec_S$ a well-order on S such that*

(*) $I(\zeta)$ has a $\prec_\zeta$-maximal element iff $I(\xi)$ has a $\prec_\xi$-minimal element for every pair $\zeta, \xi \in S$ with $\zeta \prec_S \xi$ and $(\zeta, \xi)_{\langle S, \prec_S\rangle} = \emptyset$.

Moreover, let $\langle X, \prec\rangle$ be the order sum of $\langle\langle I(\xi), \prec_\xi\rangle \mid \xi \in S\rangle$ with respect to $\prec_S$ and τ' the order topology $\lambda(X, \prec)$. Then for each $\xi \in S$, the following hold.

1. *The subspace topology $\tau' \restriction I(\xi)$ coincides with the order topology $\lambda(I(\xi), \prec_\xi)$.*
2. *For each $\alpha \in I(\xi)$, $\alpha \notin \mathrm{int}_{\langle X, \tau'\rangle}(I(\xi))$ holds if and only if the following three conditions hold:*
   - (2-1) *ξ is not a $\prec_S$-minimal element,*
   - (2-2) *$(\leftarrow, \xi)_{\langle S, \prec_S\rangle}$ does not have a $\prec_S$-maximal element,*
   - (2-3) *α is a $\prec_\xi$-minimal element of $I(\xi)$.*
3. *Assume that $\prec_\xi$ is a well-order on $I(\xi)$ and $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X, \tau'\rangle}(I(\xi))$. Then $V \subseteq X$ is a neighbourhood of α in $\langle X, \tau'\rangle$ if and only if $\{\alpha\} \cup \bigcup\{I(\eta) : \eta \in (\zeta, \xi)_{\langle S, \prec_S\rangle}\} \subseteq V$ for some $\zeta \in (\leftarrow, \xi)_{\langle S, \prec_S\rangle}$.*

Proof. Let $\xi \in S$. (1) follows from Lemma 2.2.

(2) Let us call $V \subseteq X$ an upper neighbourhood of $\alpha \in X$ if $[\alpha, \rightarrow)_{\langle X, \prec\rangle} \subseteq V$ or $[\alpha, \beta)_{\langle X, \prec\rangle} \subseteq V$ for some $\beta \in X$ with $\alpha \prec \beta$. And let us call V a lower neighbourhood of α if $(\leftarrow, \alpha]_{\langle X, \prec\rangle} \subseteq V$ or $(\gamma, \alpha]_{\langle X, \prec\rangle} \subseteq V$ for some $\gamma \in X$ with $\gamma \prec \alpha$. Obviously, $\alpha \in \mathrm{int}_{\langle X, \tau'\rangle}(V)$ iff V is a neighbourhood of α in $\langle X, \tau'\rangle$ iff V is both an upper neighbourhood and a lower neighbourhood of α.

Let $\alpha \in I(\xi)$. First we prove:

Claim 1. *$I(\xi)$ is an upper neighbourhood of α.*

Proof. In the case that ξ is a $\prec_S$-maximal element of S, $[\alpha, \rightarrow)_{\langle X, \prec\rangle} \subseteq I(\xi)$ holds. In the case that α is not a $\prec_\xi$-maximal element of $I(\xi)$, pick $\beta \in I(\xi)$ with $\alpha \prec_\xi \beta$, then $\alpha \prec \beta$ and $[\alpha, \beta)_{\langle X, \prec\rangle} \subseteq I(\xi)$ hold. The rest case is that α is a $\prec_\xi$-maximal element of $I(\xi)$ and ξ is not a $\prec_S$-maximal element of S. In this case, by the well-orderability of $\prec_S$, we can find $\zeta \in S$ with $\xi \prec_S \zeta$ and $(\xi, \zeta)_{\langle S, \prec_S\rangle} = \emptyset$. By (*),

$I(\zeta)$ has a $\prec_\zeta$-minimal element β. Then we have $[\alpha,\beta)_{\langle X,\prec\rangle} = \{\alpha\} \subseteq I(\xi)$ with $\alpha \prec \beta$. □

The "if" part of (2) is obvious. To see the "only if" part, we show $\alpha \in \mathrm{int}_{\langle X,\tau'\rangle}(I(\xi))$ by assuming that at least one of the conditions (2-1), (2-2), and (2-3) fails. By Claim 1, it suffices to show:

Claim 2. *$I(\xi)$ is a lower neighbourhood of α.*

Proof. As in the proof of Claim 1, we can see that $I(\xi)$ is a lower neighbourhood of α in case (2-1) or (2-3) fails. So we may assume that (2-1) and (2-3) hold but (2-2) fails, i.e. α is a $\prec_\xi$-minimal element of $I(\xi)$ and $(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ has a $\prec_S$-maximal element ζ. Then by (*), $I(\zeta)$ has a $\prec_\zeta$-maximal element γ. And we have $(\gamma,\alpha]_{\langle X,\prec\rangle} = \{\alpha\} \subseteq I(\xi)$ with $\gamma \prec \alpha$. □

(3) Assume that $\prec_\xi$ is a well-order on $I(\xi)$ and $\alpha \in I(\xi)\setminus\mathrm{int}_{\langle X,\tau'\rangle}(I(\xi))$. Then by (2), α is a $\prec_\xi$-minimal element of $I(\xi)$, $(\leftarrow,\xi)_{\langle S,\prec_S\rangle} \neq \emptyset$ and $(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ does not have a $\prec_S$-maximal element. Therefore the "only if" part is obvious. To see the "if" part, assume that $\{\alpha\} \cup \bigcup\{I(\eta) : \eta \in (\zeta,\xi)_{\langle S,\prec_S\rangle}\} \subseteq V$ for some $\zeta \in (\leftarrow,\xi)_{\langle S,\prec_S\rangle}$. Pick $\eta \in (\zeta,\xi)_{\langle S,\prec_S\rangle}$ and $\gamma \in I(\eta)$, then we have $(\gamma,\alpha]_{\langle X,\prec\rangle} \subseteq V$ with $\gamma \prec \alpha$, so V is a lower neighbourhood of α. Moreover since $\prec_\xi$ is a well-order, we can take $\beta \in X$ with $\alpha \prec \beta$ such that $[\alpha,\beta)_{\langle X,\prec\rangle} = \{\alpha\}$ (use the assumption (*) when α is a $\prec_\xi$-maximal element of $I(\xi)$ and $(\xi,\rightarrow)_{\langle S,\prec_S\rangle} \neq \emptyset$). Therefore V is also an upper neighbourhood of α. □

Lemma 3.3. *Let τ and τ' be topologies on a set X. Then the following properties hold.*

(1) *If $I \subseteq X$, $\tau \restriction I = \tau' \restriction I$ and $\alpha \in \mathrm{int}_{\langle X,\tau\rangle}(I) \cap \mathrm{int}_{\langle X,\tau'\rangle}(I)$, then for every subset V of X, V is a neighbourhood at α in $\langle X,\tau\rangle$ iff so is in $\langle X,\tau'\rangle$.*
(2) *If there is a cover $\langle I(\xi) \mid \xi \in S\rangle$ of X such that for each $\xi \in S$,*
   - *$\tau \restriction I(\xi) = \tau' \restriction I(\xi)$,*
   - *$\mathrm{int}_{\langle X,\tau\rangle}(I(\xi)) = \mathrm{int}_{\langle X,\tau'\rangle}(I(\xi))$,*
   - *for every $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$ and for every subset V of X, V is a neighbourhood at α in $\langle X,\tau\rangle$ iff so is in $\langle X,\tau'\rangle$,*

   then $\tau = \tau'$.

Proof. (1) Assume that V is a neighbourhood of α in $\langle X,\tau\rangle$. Let $I' = \mathrm{int}_{\langle X,\tau'\rangle}(I)$, then $\alpha \in I' \subseteq I$ and $V \cap I'$ is a neighbourhood of α in $\tau \restriction I'$. Since $\tau \restriction I' = (\tau \restriction I) \restriction I' = (\tau' \restriction I) \restriction I' = \tau' \restriction I'$ holds and I' is open in $\langle X,\tau'\rangle$, $V \cap I'$ is a neighbourhood of α in $\langle X,\tau'\rangle$. The proof of the reverse implication is similar.

(2) To see $\tau \subseteq \tau'$, let $\alpha \in V \in \tau$. Pick $\xi \in S$ with $\alpha \in I(\xi)$. By the first and the second assumptions and (1), we may assume $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$. Then by the third condition, V is a neighbourhood of α in $\langle X,\tau'\rangle$. Similarly we have $\tau' \subseteq \tau$. □

Lemma 3.4. *Let $\langle X,\tau\rangle$ be a space having a pairwise disjoint cover $\langle I(\xi) \mid \xi \in S\rangle$, where S is well-ordered by $\prec_S$, such that for each ξ, $I(\xi)$ is not empty and there is a well-order $<_\xi$ on $I(\xi)$ with $\tau \restriction I(\xi) = \lambda(I(\xi), <_\xi)$ satisfying*

(i) *if $I(\xi)$ has a $<_\xi$-maximal element and is open in $\langle X,\tau\rangle$, then ξ is a $\prec_S$-minimal element of S,*
(ii) *if $I(\xi)$ has a $<_\xi$-maximal element and is not open in $\langle X,\tau\rangle$, then ξ is a $\prec_S$-maximal element of S,*
(iii) *if $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$, then*

- *ξ is not a $\prec_S$-minimal element of S,*
- *$(\leftarrow, \xi)_{\langle S, \prec_S\rangle}$ does not have a $\prec_S$-maximal element,*
- *α is a $<_\xi$-minimal element of $I(\xi)$,*
- *$V \subseteq X$ is a neighbourhood of α in $\langle X, \tau\rangle$ if and only if $\{\alpha\} \cup \bigcup\{I(\eta) : \eta \in (\zeta, \xi)_{\langle S, \prec_S\rangle}\} \subseteq V$ for some $\zeta \in (\leftarrow, \xi)_{\langle S, \prec_S\rangle}$.*

Then, $\langle X, \tau\rangle$ is orderable.

Proof. By $\prec_S$-induction on S, define a function $d : S \to 2 = \{0, 1\}$ as follows. Let $\xi \in S$ and assume that $d(\zeta) \in 2$ is defined for every $\zeta \in S$ with $\zeta \prec_S \xi$. Put

$$d(\xi) = \begin{cases} 0 & \text{if } (\leftarrow, \xi)_{\langle S, \prec_S\rangle} \text{ does not have a } \prec_S\text{-maximal element,} \\ & \text{and } I(\xi) \text{ is not open in } \langle X, \tau\rangle, \\ 1 & \text{if } (\leftarrow, \xi)_{\langle S, \prec_S\rangle} \text{ does not have a } \prec_S\text{-maximal element,} \\ & \text{and } I(\xi) \text{ is open in } \langle X, \tau\rangle, \\ 1 - d(\zeta) & \text{if } (\leftarrow, \xi)_{\langle S, \prec_S\rangle} \text{ has a } \prec_S\text{-maximal element } \zeta. \end{cases}$$

In particular, $d(\min S) = 1$ iff $I(\min S)$ is open in $\langle X, \tau\rangle$ where $\min S$ is a $\prec_S$-minimal element of S.

For each $\xi \in S$, we define another linear order $\prec_\xi$ on $I(\xi)$ as follows. In case $d(\xi) = 0$, let $\prec_\xi$ be the same order with $<_\xi$. In case $d(\xi) = 1$, let $\prec_\xi$ be the reverse order of $<_\xi$. More precisely, for each pair $\alpha, \beta \in I(\xi)$, define

$$\alpha \prec_\xi \beta \text{ by } \begin{cases} \alpha <_\xi \beta & \text{if } d(\xi) = 0, \\ \alpha >_\xi \beta & \text{if } d(\xi) = 1, \end{cases}$$

Claim 1. *$\langle I(\xi) \mid \xi \in S\rangle$ satisfies the property (*) in Lemma 3.2.*

Proof. Let $\zeta, \xi \in S$, $\zeta \prec_S \xi$, and $(\zeta, \xi)_{\langle S, \prec_S\rangle} = \emptyset$. Then $d(\xi) = d(\zeta) - 1$.

First assume that $I(\zeta)$ has a $\prec_\zeta$-maximal element. If $I(\zeta)$ does not have a $<_\zeta$-maximal element, then $\prec_\zeta$ is different from $<_\zeta$, hence $d(\zeta) = 1$. If $I(\zeta)$ has a $<_\zeta$-maximal element, then it follows from the assumption (ii) that $I(\zeta)$ is open in $\langle X, \tau\rangle$ since ζ is not a $\prec_S$-maximal element. By the assumption (i), ζ is a $\prec_S$-minimal element of S. So $d(\zeta) = 1$ holds by the definition of d. We have $d(\zeta) = 1$ in both cases. So $d(\xi) = 0$ holds, therefore the order $\prec_\xi$ is the same as the well-order $<_\xi$, hence $I(\xi)$ has a $\prec_\xi$-minimal element.

Next assume that $I(\xi)$ has a $\prec_\xi$-minimal element. The assumption (iii) implies that $I(\xi)$ is open since $(\leftarrow, \xi)_{\langle S, \prec_S\rangle}$ has a $\prec_S$-maximal element ζ. By the assumption (i), we see that $I(\xi)$ does not have a $<_\xi$-maximal element since ξ is not a $\prec_S$-minimal element. Hence $\prec_\xi$ is different from the reverse order of $<_\xi$. So we have $d(\xi) = 0$ and $d(\zeta) = 1$. Therefore $\prec_\zeta$ is the reverse order of the well-order $<_\zeta$, hence $I(\zeta)$ has a $\prec_\zeta$-maximal element. □

Let $\langle X, \prec\rangle$ be the order sum of $\langle\langle I(\xi), \prec_\xi\rangle \mid \xi \in S\rangle$ with respect to the well order $\prec_S$ and τ' its order topology $\lambda(X, \prec)$. By Lemma 3.2 (1), we see that $I(\xi)$ is a convex set of $\langle X, \prec\rangle$, and

$$\tau \restriction I(\xi) = \lambda(I(\xi), <_\xi) = \lambda(I(\xi), \prec_\xi) = \tau' \restriction I(\xi)$$

holds for every $\xi \in S$. Obviously, $\langle X, \tau'\rangle$ is orderable. The goal of the proof of the lemma is to see that $\tau = \tau'$. By Lemma 3.3 (2), it suffices to show the claim below.

Claim 2. *For every $\xi \in S$, the following hold.*

- $\mathrm{int}_{\langle X,\tau\rangle}(I(\xi)) = \mathrm{int}_{\langle X,\tau'\rangle}(I(\xi))$,
- *for every* $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$ *and for every subset* V *of* X, V *is a neighbourhood at* α *in* $\langle X,\tau\rangle$ *iff so is in* $\langle X,\tau'\rangle$.

Proof. Assume that $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$. Then by the assumption (iii), we have:

- ξ is not a $\prec_S$-minimal element of S,
- $(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ does not have a $\prec_S$-maximal element,
- α is a $<_\xi$-minimal element of $I(\xi)$.

Since $I(\xi)$ is not open in $\langle X,\tau\rangle$, we have $d(\xi) = 0$ and so $\prec_\xi$ coincides with the original order $<_\xi$. In particular, α is a $\prec_\xi$-minimal element of $I(\xi)$, and $\prec_\xi$ is a well-order. By Claim 1 and Lemma 3.2, we have $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau'\rangle}(I(\xi))$. Moreover, the last condition of the assumption (iii) implies that for every subset V of X, V is a neighbourhood at α in $\langle X,\tau\rangle$ iff so is in $\langle X,\tau'\rangle$.

On the other hand, assume that $\alpha' \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau'\rangle}(I(\xi))$. By Lemma 3.2 (2), we have:

- ξ is not a $\prec_S$-minimal element of S,
- $(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ does not have a $\prec_S$-maximal element,
- α' is a $\prec_\xi$-minimal element of $I(\xi)$.

If $I(\xi)$ does not have a $<_\xi$-maximal element, then $\prec_\xi$ is different from the reverse order of $<_\xi$, so $d(\xi) = 0$ thus $I(\xi)$ is not open in $\langle X,\tau\rangle$. If $I(\xi)$ has a $<_\xi$-maximal element, then it follows from the assumption (i) that $I(\xi)$ is not open in $\langle X,\tau\rangle$, thus $d(\xi) = 0$. In any case, we have $d(\xi) = 0$ and $I(\xi)$ is not open in $\langle X,\tau\rangle$. Therefore $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$ exists. By the assumption (iii), α is a $<_\xi$-minimal element of $I(\xi)$. We have $\alpha' = \alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$ since $\prec_\xi$ is the same order with $<_\xi$. □

□

We prove Theorem 3.1.

Proof. Let $<$ be the usual order on **Ord**, μ an ordinal, $X \subseteq \mu$ and τ denote the subspace topology $\lambda(\mu,<) \restriction X$. Let G, S be subsets of $\mu+1$ and $\langle\langle I(\xi),<_\xi\rangle \mid \xi \in S\rangle$ a sequence described in Lemma 2.4. And let $<_\xi$ be the restriction of the order $<$ on $I(\xi)$. We define another well-order $\prec_S$ on S satisfying (i), (ii) and (iii) of Lemma 3.4. If X has a $<$-maximal element and $I(\mu)$ is open in X, then for each $\xi,\zeta \in S \setminus \{\mu\}$, let $\mu \prec_S \xi$ and let $\zeta \prec_S \xi$ iff $\zeta < \xi$. Otherwise, let $\prec_S = < \restriction S$. Obviously, $\prec_S$ is a well-order on S. If $\xi \in S$ and $I(\xi)$ has a $<_\xi$-maximal element, then by Lemma 2.4 (1) and (2), X has a $<$-maximal element and $\xi = \mu$ is a $<$-maximal element of S. Hence, $\prec_S$ satisfies (i) and (ii) of Lemma 3.4. To see (iii), let $\xi \in S$ and $\alpha \in I(\xi) \setminus \mathrm{int}_{\langle X,\tau\rangle}(I(\xi))$. In case $\prec_S = < \restriction S$, it is trivial that $(\leftarrow,\xi)_{\langle S,\prec_S\rangle} = S \cap \xi$. In case $\prec_S \neq < \restriction S$, we have $\xi \neq \mu$ since $I(\mu)$ is open and $I(\xi)$ is not open in $\langle X,\tau\rangle$, and μ is a $\prec_S$-minimal element of S, so $(\leftarrow,\xi)_{\langle S,\prec_S\rangle} = \{\mu\} \cup (S \cap \xi)$ holds.

Claim 1. *α is a $<_\xi$-minimal element of $I(\xi)$.*

Proof. By Lemma 2.4 (4), we have $\alpha = \sup(G \cap \xi)$. By the definition, $I(\xi) = X \cap [\sup(G\cap\xi),\xi)$ holds, so α is a $<_\xi$-minimal element of $I(\xi)$. (We have to distinguish intervals with respect to the order $\langle \mathbf{Ord},<\rangle$ and intervals with respect to the order $\langle S,\prec_S\rangle$. We omit the suffix only for the former. For instance, $[\sup(G\cap\xi),\xi)$ above is intended to mean an interval of $\langle \mathbf{Ord},<\rangle$.) □

Claim 2. *ξ is not a $\prec_S$-minimal element of S, and $(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ does not have a $\prec_S$-maximal element.*

Proof. To see the claim, it suffices to show that $S\cap\xi$ is non-empty and does not have a $<$-maximal element. If $G\cap\xi=\emptyset$, then $\sup(G\cap\xi)=\sup\emptyset=-1\notin X$. If $G\cap\xi$ has a $<$-maximal element, then $\sup(G\cap\xi)=\max(G\cap\xi)\in G=\mathrm{Lim}(X)\setminus X$. But either case does not happen since $\sup(G\cap\xi)=\alpha\in I(\xi)\subseteq X$. Hence, $G\cap\xi$ is non-empty and does not have a $<$-maximal element. So we have $\alpha=\sup(G\cap\xi)\in\mathrm{Lim}(G)=\mathrm{Lim}(S)$ by Lemma 2.4 (5). Therefore, $S\cap\alpha$ is non-empty and does not have a $<$-maximal element. Obviously, $(\sup(G\cap\xi),\xi)$ is disjoint from G, so $[\alpha,\xi)$ is disjoint from $S\cap\xi\subseteq G\setminus\mathrm{Lim}(G)$ since $\alpha=\sup(G\cap\xi)\in\mathrm{Lim}(G)$. We have $S\cap\xi=S\cap\alpha$. Hence, $S\cap\xi$ is non-empty and does not have a $<$-maximal element. □

Claim 3. *$V\subseteq X$ is a neighbourhood of α in $\langle X,\tau\rangle$ if and only if $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in(\zeta,\xi)_{\langle S,\prec_S\rangle}\}\subseteq V$ for some $\zeta\in(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$.*

Proof. By $S\cap\xi=S\cap\alpha$, $(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ coincides either $S\cap\alpha$ or $\{\mu\}\cup(S\cap\alpha)$ with $\prec_S$-minimal element μ. Hence, $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in(\zeta,\xi)_{\langle S,\prec_S\rangle}\}\subseteq V$ for some $\zeta\in(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ iff $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in(\zeta,\xi)_{\langle S,\prec_S\rangle}\}\subseteq V$ for some $\zeta\in S\cap\alpha$. And $(\zeta,\xi)_{\langle S,\prec_S\rangle}=S\cap(\zeta,\xi)=S\cap(\zeta,\alpha)$ holds for every $\zeta\in S\cap\alpha$. Therefore, $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in(\zeta,\xi)_{\langle S,\prec_S\rangle}\}\subseteq V$ for some $\zeta\in(\leftarrow,\xi)_{\langle S,\prec_S\rangle}$ iff $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in S\cap(\zeta,\alpha)\}\subseteq V$ for some $\zeta\in S\cap\alpha$.

First assume that $V\subseteq X$ is a neighbourhood of α in $\langle X,\tau\rangle$. Then there is $\gamma<\alpha$ such that $X\cap(\gamma,\alpha]\subseteq V$. By $\alpha\in\mathrm{Lim}(S)$, there is $\zeta\in S\cap\alpha\subseteq G$ such that $\gamma<\zeta$. If $\eta\in S\cap(\zeta,\alpha)$, then $\zeta\in G\cap\eta$ and so $\gamma<\zeta\le\sup(G\cap\eta)$, thus $I(\eta)=X\cap[\sup(G\cap\eta),\eta)\subseteq X\cap(\gamma,\alpha]\subseteq V$. Hence, $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in S\cap(\zeta,\alpha)\}\subseteq V$ for some $\zeta\in S\cap\alpha$.

Conversely, let $\{\alpha\}\cup\bigcup\{I(\eta):\eta\in S\cap(\zeta,\alpha)\}\subseteq V$ for some $\zeta\in S\cap\alpha$. To see V being a neighbourhood of α in $\langle X,\tau\rangle$, we show that $X\cap(\zeta,\alpha]\subseteq V$. By the assumption, $\alpha\in V$ holds, so it suffices to show that $X\cap(\zeta,\alpha)\subseteq V$. Let $\beta\in X\cap(\zeta,\alpha)$. Since $\alpha\in\mathrm{Lim}(G)$, there is $\eta\in G\cap(\beta,\alpha)$. Pick the least such η. Then we have $\eta\in(G\setminus\mathrm{Lim}(G))\cap(\beta,\alpha)\subseteq S\cap(\zeta,\alpha)$ and $\beta\in X\cap[\sup(G\cap\eta),\eta)=I(\eta)\subseteq V$. □

By Claim 1-Claim 3, $\prec_S$ satisfies the condition (iii) in Lemma 3.4. Hence, $\langle X,\tau\rangle$ is orderable. □

4. Well-orderability

In this section, we characterize the well-orderability of subspaces of ordinals. Throughout this section, $<$ denotes the usual order on the class **Ord** of all ordinals.

Theorem 4.1. *Let X be a subspace of an ordinal, and $G=\mathrm{Lim}(X)\setminus X$. Then X is well-orderable iff either* (i) *or* (ii) *below holds:*

(i) $X\cap\mathrm{Lim}(G)=\emptyset$ *and* $\mathrm{cf}\,\xi=\omega$ *for every* $\xi\in G$.
(ii) $|G|\le 1$.

We first characterize the well-orderability of spaces having a pairwise disjoint open cover by well-orderable subspaces.

Proposition 4.2. *Assume that X has a pairwise disjoint open cover $\langle I(\xi) \mid \xi \in S\rangle$ such that $I(\xi)$ is homeomorphic to a non-zero ordinal α_ξ for every $\xi \in S$. Then X is well-orderable if and only if either* (I) *or* (II) *below holds.*

(I) $|S| \leq \omega$ *and* $\operatorname{cf}\alpha_\xi \leq \omega$ *for every* $\xi \in S$.
(II) $|S| < \omega$ *and* $|\{\xi \in S : \operatorname{cf}\alpha_\xi \neq 1\}| \leq 1$.

Before proving the proposition, we show some lemmas.

Lemma 4.3. *Assume that X has a pairwise disjoint open cover $\langle I(\xi) \mid \xi \in S\rangle$ such that $I(\xi)$ is homeomorphic to a non-zero ordinal α_ξ for every $\xi \in S$.*

(1) *If* $|S| < \omega$, $\operatorname{cf}\alpha_{\xi_0} = \kappa$ *for some* $\xi_0 \in S$, *and* $\operatorname{cf}\alpha_\xi = 1$ *for every* $\xi \in S \setminus \{\xi_0\}$, *then X is homeomorphic to an ordinal of cofinality κ.*
(2) *If* $|S| = \omega$ *and* $\operatorname{cf}\alpha_\xi = 1$ *for every* $\xi \in S$, *then X is homeomorphic to an ordinal of cofinality ω.*

Proof. For (1), fix a well-order $\prec_S$ on S such that ξ_0 is a $\prec_S$-maximal element of S. For (2), put $\kappa = \omega$ and fix a well-order $\prec_S$ on S such that $\operatorname{otp}\langle S, \prec_S\rangle = \omega$. Pick a well-order $\prec_\xi$ on $I(\xi)$ for each $\xi \in S$ such that:

- the order topology $\lambda(I(\xi), \prec_\xi)$ coincides with the subspace topology of the original topology of X,
- $\operatorname{otp}\langle I(\xi), \prec_\xi\rangle = \alpha_\xi$.

Let $\langle X, \prec\rangle$ be the order sum of $\langle\langle I(\xi), \prec_\xi\rangle \mid \xi \in S\rangle$ with respect to $\prec_S$. Obviously in either cases, $\prec$ is a well-order on X and $\operatorname{cf}\operatorname{otp}\langle X, \prec\rangle = \kappa$. The original topology on X coincides with $\langle X, \lambda(X, \prec)\rangle$, thus X is homeomorphic to an ordinal of cofinality κ. □

Remark that for a non-zero ordinal β, $\operatorname{cf}\beta = 1$ ($\operatorname{cf}\beta = \omega$, $\operatorname{cf}\beta > \omega$) iff β is compact (non-compact Lindelöf, non-compact countably compact, respectively).

Lemma 4.4. *A topological space is homeomorphic to an ordinal of cofinality ω if and only if it can be represented as the free union of countably infinite many subspaces which are homeomorphic to successor ordinals, that is, it has a pairwise disjoint infinite countable open cover $\langle I(j) \mid j < \omega\rangle$ such that $I(j)$ is homeomorphic to a successor ordinal for each $j < \omega$.*

Proof. The 'if' part is immediately obtained from 4.3 (2). Conversely, let β be an ordinal with $\operatorname{cf}\beta = \omega$, and fix a strictly increasing sequence $\langle \beta_j \mid j < \omega\rangle$ of ordinals in β that is cofinal in β. Let $I(j) = (\beta_{j-1}, \beta_j]$ for each $j < \omega$, where $\beta_{-1} = -1$. Then $\langle I(j) \mid j < \omega\rangle$ is as desired. □

Lemma 4.5. *Let β be an ordinal.*

(1) *β is covered by* $\operatorname{cf}\beta$*-many compact clopen subsets but not covered by* $< \operatorname{cf}\beta$*-many compact subsets. In particular, β is locally compact.*
(2) *If* $\operatorname{cf}\beta > \omega$, *then there is not a disjoint pair of non-compact closed subsets of β.*
(3) *Assume that β has a pairwise disjoint open cover $\langle I(\xi) \mid \xi \in S\rangle$ where each $I(\xi)$ is non-empty. Then,* $|S| < \omega$ *holds in case* $\operatorname{cf}\beta \neq \omega$, *and* $|S| \leq \omega$ *holds in case* $\operatorname{cf}\beta = \omega$.

Proof. (1) is obvious. (2) is obtained from the well-known fact that closed unbounded sets generates a filter on β.

(3) We may assume $\operatorname{cf}\beta > \omega$. If S were infinite, then decompose S into infinite subsets S_0 and S_1. Then $\bigcup_{\xi\in S_0} I(\xi)$ and $\bigcup_{\xi\in S_1} I(\xi)$ are disjoint non-compact closed subsets, this contradicts (2). □

Lemma 4.6. *Assume that X has a pairwise disjoint open cover $\langle I(\xi) \mid \xi \in S\rangle$ such that $I(\xi)$ is homeomorphic to a non-zero ordinal α_ξ for every $\xi \in S$. Then the following hold.*

1. *X is homeomorphic to 0 if and only if $S = \emptyset$.*
2. *X is homeomorphic to a successor ordinal if and only if $0 < |S| < \omega$ and $\operatorname{cf}\alpha_\xi = 1$ for every $\xi \in S$.*
3. *X is homeomorphic to an ordinal of cofinality ω if and only if*
$$\max(\{\operatorname{cf}\alpha_\xi : \xi \in S\} \cup \{|S|\}) = \omega.$$
4. *X is homeomorphic to an ordinal of uncountable cofinality κ if and only if $|S| < \omega$, $\operatorname{cf}\alpha_{\xi_0} = \kappa$ for some $\xi_0 \in S$, and $\operatorname{cf}\alpha_\xi = 1$ for every $\xi \in S \setminus \{\xi_0\}$.*

Proof. (1) is trivial. (2) follows from Lemma 4.3 (1) and Lemma 4.5 (3).

(3) Assume that $\max(\{\operatorname{cf}\alpha_\xi : \xi \in S\} \cup \{|S|\}) = \omega$. For each $\xi \in S$, $\operatorname{cf}\alpha_\xi \leq \omega$ holds and by Lemma 4.4, we see that $I(\xi)$ can be represented as the free union of at most countably many subspaces which are homeomorphic to successor ordinals. By the assumption, X is also represented as the free union of countably infinite many subspaces which are homeomorphic to successor ordinals. By using Lemma 4.4 again, we see that X is homeomorphic to an ordinal of cofinality ω.

Conversely, assume that X is homeomorphic to an ordinal of cofinality ω. By (1) and (2), either $|S| \geq \omega$ or $\operatorname{cf}\alpha_\xi \geq \omega$ holds for some $\xi \in S$. By (3) of Lemma 4.5, we have $|S| \leq \omega$. Moreover since X is Lindelöf, we have $\operatorname{cf}\alpha_\xi \leq \omega$ for every $\xi \in S$. Therefore we have $\max(\{\operatorname{cf}\alpha_\xi : \xi \in S\} \cup \{|S|\}) = \omega$.

(4) The "if" part follows from Lemma 4.3 (1). Assume that X is homeomorphic to an ordinal of uncountable cofinality κ. By Lemma 4.5 (3), we have $|S| < \omega$. Since X is not Lindelöf, there is $\xi_0 \in S$ such that $\operatorname{cf}\alpha_{\xi_0} > \omega$. By Lemma 4.5 (2), ξ_0 is unique and $\operatorname{cf}\alpha_\xi = 1$ for every $\xi \in S \setminus \{\xi_0\}$. By Lemma 4.3 (1), we have $\operatorname{cf}\alpha_{\xi_0} = \kappa$. □

Now we prove Proposition 4.2.

Proof. Assume that X is well-orderable. If X is homeomorphic to an ordinal of cofinality $\leq \omega$ ($> \omega$), then by Lemma 4.6, (I) ((II), respectively) holds. Conversely, if (I) or (II) is true, then by Lemma 4.6 again, it is straightforward to see that X is well-orderable. □

Before proving Theorem 4.1, we show a lemma.

Lemma 4.7. *Let X be a subspace of an ordinal μ, and define G, S, and $\mathcal{I} = \langle I(\xi) \mid \xi \in S\rangle$ as in Lemma 2.4. Then the following conditions are equivalent.*

(a) *X is locally compact.*
(b) *$X \cap \operatorname{Lim}(G) = \emptyset$.*
(c) *$I(\xi)$ is open for every $\xi \in S$.*
(d) *X is represented as the free union of well-orderable subspaces.*

Proof. (a)→(b): Assume that $X \cap \operatorname{Lim}(G)$ has an element α. Let V be an arbitrary neighbourhood of α in X. Then there is $\gamma \in \alpha$ such that $X \cap (\gamma, \alpha] \subseteq V$. By $\alpha \in \operatorname{Lim}(G)$, we can pick $\xi_0, \xi_1 \in G$ such that $\gamma < \xi_0 < \xi_1 < \alpha$. Moreover, we may assume that $\xi_1 = \min(G \cap (\xi_0, \alpha))$. Then $\xi_1 \in G \setminus \mathrm{Lim}(G) \subseteq S$. By Lemma 2.4 (2), $I(\xi_1)$ is a closed subspace of X which is not compact. V is not compact since $I(\xi_1) = X \cap [\sup(G \cap \xi_1), \xi_1) = X \cap [\xi_0, \xi_1) \subseteq X \cap (\gamma, \alpha] \subseteq V$.

(b)→(c): Assume that $X \cap \mathrm{Lim}(G) = \emptyset$. It follows from $G \cap X = \emptyset$ that $\sup(G \cap \xi) \notin X$ holds for every $\xi \in S$. Thus $I(\xi) = X \cap [\sup(G \cap \xi), \xi) = X \cap (\sup(G \cap \xi), \xi)$ and it is open in X for every $\xi \in S$.

(c)→(d): Use Lemma 2.4 (4).

(d)→(a): Since an ordinal is locally compact, so is the free union of well-orderable subspaces. □

Now we prove Theorem 4.1.

Proof. Let X be a subspace of an ordinal μ, and define G, S, and $\mathcal{I} = \langle I(\xi) \mid \xi \in S \rangle$ as in Lemma 2.4. Put $\alpha_\xi = \mathrm{otp}\langle I(\xi), <_\xi \rangle$ for each $\xi \in S$, where $<_\xi$ is the restriction of $<$ on $I(\xi)$. By Lemma 2.4 (4), $I(\xi)$ is homeomorphic to α_ξ. By Lemma 2.4 (1) and (2), $\mathrm{cf}\, \alpha_\xi = 1$ holds in case $\xi \in S \setminus G$, and $\mathrm{cf}\, \alpha_\xi = \mathrm{cf}\, \xi \geq \omega$ holds in case $\xi \in G \setminus \mathrm{Lim}(G)$. Therefore we have $G \setminus \mathrm{Lim}(G) = \{\xi \in S : \mathrm{cf}\, \alpha_\xi \neq 1\}$.

Assume that X is well-orderable. Then X is locally compact, and by Lemma 4.7, we have $X \cap \mathrm{Lim}(G) = \emptyset$. Moreover, either the condition (I) or (II) in Proposition 4.2 holds. First assume that (II) holds. It follows from $|G \setminus \mathrm{Lim}(G)| = |\{\xi \in S : \mathrm{cf}\, \alpha_\xi \neq 1\}| \leq 1$ and $\mathrm{Lim}(G) = \mathrm{Lim}(S) = \emptyset$ that $|G| \leq 1$, so the condition (ii) in the theorem is true. Next assume that (I) holds. Let $\xi \in G$. If $\xi \notin \mathrm{Lim}(G)$, then it follows from $\xi \in G \setminus \mathrm{Lim}(G) \subseteq S$ that $\mathrm{cf}\, \xi = \mathrm{cf}\, \alpha_\xi \leq \omega$. If $\xi \in \mathrm{Lim}(G)$, then $G \cap \xi$ is a cofinal subset of ξ, and $G \setminus \{\sup G\} \ni \zeta \mapsto \min\{\eta \in G : \zeta < \eta\} \in G \setminus \mathrm{Lim}(G)$ is a 1-1 function. So we have $\mathrm{cf}\, \xi \leq |G \cap \xi| \leq |G| \leq \max(|G \setminus \mathrm{Lim}(G)|, \omega) \leq \max(|S|, \omega) = \omega$. We obtain $\mathrm{cf}\, \xi \leq \omega$ for every $\xi \in G$ in either cases. Therefore the condition (i) in the theorem holds.

Conversely, assume that the condition (i) in the theorem holds. Obviously, $\mathrm{cf}\, \alpha_\xi \leq \omega$ holds for every $\xi \in S$. Since $X \cap \mathrm{Lim}(G) = \emptyset$, we see that each $I(\xi)$ is open in X by Lemma 4.7. We show that $|G| \leq \omega$. For otherwise, there is a strictly increasing sequence $\langle \xi_j \mid j < \omega_1 \rangle$ of elements of G. Put $\xi = \sup\{\xi_j : j < \omega_1\}$. Then $\mathrm{cf}\, \xi = \omega_1$ and $\xi \in \mathrm{Lim}(G) \subseteq \mathrm{Lim}(X)$. It follows from $X \cap \mathrm{Lim}(G) = \emptyset$ that $\xi \in \mathrm{Lim}(X) \setminus X \subseteq G$. This contradicts the condition in (i). Thus $|S| \leq |G \cup \{\mu\}| \leq \omega$. Hence, the condition (I) in Proposition 4.2 holds, and so X is well-orderable.

Next assume that (ii) in the theorem holds. Then $X \cap \mathrm{Lim}(G) \subseteq \mathrm{Lim}(G) = \emptyset$. By Lemma 4.7, each $I(\xi)$ is open in X. Obviously, $|S| \leq |G \cup \{\mu\}| \leq 2 < \omega$ and $|\{\xi \in S : \mathrm{cf}\, \alpha_\xi \neq 1\}| = |G \setminus \mathrm{Lim}(G)| \leq |G| \leq 1$ hold. Hence, the condition (II) in Proposition 4.2 holds, and so X is well-orderable. □

The corollary below is obtained from Lemma 2.3.

Corollary 4.8. *If X is a closed subspace of an ordinal μ, then X is well-orderable.*

By applying Theorem 4.1, we obtain the corollary below.

Corollary 4.9. *Let $X \subseteq \omega_1$ and $G = \mathrm{Lim}(X) \setminus X$. Then X is well-orderable iff either* (i') *or* (ii') *holds:*

(i') *$X \cap \mathrm{Lim}(G) = \emptyset$ and X is not cofinal in ω_1.*
(ii') *X is closed in ω_1.*

Proof. If the condition (i') above holds, then $\sup G \le \sup X < \omega_1$. So the condition (i) in Theorem 4.1 holds, thus X is well-orderable. If (ii') holds, then by Corollary 4.8, X is well-orderable.

Conversely, assume that X is well-orderable. By Theorem 4.1, $X \cap \mathrm{Lim}(G) = \emptyset$ holds. If the condition (i') above fails, then X is cofinal in ω_1, so $\omega_1 \in G$. Hence, the condition (ii) in Theorem 4.1 holds. Therefore $G = \{\omega_1\}$ and so the condition (ii') holds. □

Remark that if X is a subspace of ω_1 such that both X and $\omega_1 \setminus X$ are stationary, then X is cofinal but not closed in ω_1, so X is not well-orderable by Corollary 4.9.

Corollary 4.10. *If X is a subspace of $\omega \cdot \omega$, then X is well-orderable.*

Proof. See [3, I, 7.19] for multiplication of ordinals and note $\omega \cdot \omega < \omega_1$. Let X be a subspace of $\omega \cdot \omega$. We verify the conditions (i') of Corollary 4.9.

It follows from $G \subseteq \mathrm{Lim}(X) \subseteq \{\omega \cdot n : n < \omega\} \cup \{\omega \cdot \omega\}$ that $\mathrm{Lim}(G) \subseteq \{\omega \cdot \omega\}$. By $\omega \cdot \omega \notin X$, we have $X \cap \mathrm{Lim}(G) = \emptyset$. □

On the other hand, using Corollary 4.9, we see that $X = [0, \omega \cdot \omega] \setminus \{\omega \cdot n : n \in \omega\}$ is a subspace of $\omega \cdot \omega + 1$ which is not well-orderable.

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