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NORMALITY AND COUNTABLE PARACOMPACTNESS OF HYPERSPACES OF ORDINALS

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Abstract. For an ordinal α, 2^α denotes the collection of all nonempty closed sets of α with the Vietoris topology and $\mathcal{K}(\alpha)$ denotes the collection of all nonempty compact sets of α with the subspace topology of 2^α. It is well-known that 2^α is normal iff $\mathrm{cf}\alpha = 1$. In this paper, we will prove that for every nonzero-ordinal α:
(1) 2^α is countably paracompact iff $\mathrm{cf}\alpha \neq \omega$.
(2) $\mathcal{K}(\alpha)$ is countably paracompact.
(3) $\mathcal{K}(\alpha)$ is normal iff, if $\mathrm{cf}\alpha$ is uncountable, then $\mathrm{cf}\alpha = \alpha$.
In (3), we use elementary submodel techniques.

Throughout the paper, spaces mean nonempty topological spaces and generally $\alpha, \beta, \gamma, ...(\kappa, \lambda, \mu, ..., k, l, m, ...)$ stand for ordinals (infinite cardinals, natural numbers). ω (ω_1) is the first infinite ordinal (the first uncountable ordinal, respectively) and $\mathrm{cf}\alpha$ denotes the cofinality of α. For notational convenience, we consider -1 as the immediate predeccesor of the ordinal 0. Ordinals are considered as spaces with the usual order topology, so $\mathrm{cf}\alpha = 1$ iff α is compact whenever α is a nonzero-ordinal.

For a space X, 2^X ($\mathcal{K}(X)$) denotes the collection of all nonempty closed (compact, respectively) subsets of X. For $n \in \omega$, $[X]^{\leq n}$ denotes the collection of all nonempty subsets of X of cardinality $\leq n$ and let $[X]^{<\omega} = \bigcup_{n\in\omega}[X]^{\leq n}$. Equip 2^X with the Vietoris topology τ_V and $\mathcal{K}(X)$ with its subspace topology. To describe τ_V, we need some notation. For every finite family $\mathcal{V}$ of subsets of X, let

$$\langle \mathcal{V} \rangle = \{F \in 2^X : F \subset \bigcup \mathcal{V}, \forall V \in \mathcal{V}(V \cap F \neq \emptyset)\}.$$

Then the collection of all subsets of 2^X of the form $\langle \mathcal{V} \rangle$, where $\mathcal{V}$ is a finite family of open sets of X, is a base for τ_V. For a subset U of X, let

$$U^- = \{F \in 2^X : F \cap U \neq \emptyset\}, U^+ = \{F \in 2^X : F \subset U\}.$$

Then it is well-known that τ_V has as a subbase all subsets of the form U^- and V^+, where U and V are open in X. Observe that $[X]^{\leq n}$ is closed in 2^X and $[X]^{<\omega}$ is dense in 2^X and contained in $\mathcal{K}(X)$.

The relations of separation axioms between the base space X and its hyperspace are interesting. For example, the following are shown in [5].

- If X is T_1 then 2^X is T_1.
- For a T_1-space X, X is normal iff 2^X is regular.
- For a T_1-space X, X is regular iff $\mathcal{K}(X)$ is regular.
- For a T_1-space X, X is compact iff 2^X is compact.

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One of the strong results proved by [7] is:

- For a T_1-space X, X is compact iff 2^X is normal.

Since an ordinal α is normal T_1, 2^α and $\mathcal{K}(\alpha)$ are at least regular T_1. Moreover by the results above, we have $\mathrm{cf}\alpha = 1$ iff 2^α is normal. An ordinal α is also known to be countably paracompact, that is, every countable open cover has a locally finite open refinement. In this paper, we characterize, as is listed in the abstract, normality and countable paracompactness of 2^α and $\mathcal{K}(\alpha)$ using the cofinality function $\mathrm{cf}\alpha$. From now on, spaces are assumed to be regular T_1.

It is well known that 2^ω is not normal [2, 3]. First we check the following:

Proposition 1. *2^ω is not countably paracompact.*

Proof. Decompose ω into two infinite subsets X_0 and X_1. Fix a 1-1 onto function $f_i : \omega \to X_i$ for each $i \in 2 = \{0, 1\}$ and for every subset $A \subset \omega$, define $F(A) = f_0(A) \cup f_1(\omega \setminus A)$. Keesling [3] proved that $\mathcal{F} = \{F(A) : A \subset \omega\}$ is closed discrete in 2^ω. Since $[\omega]^{<\omega}$ is dense in 2^ω, the following claim completes the proof. The author beleives the following claim have been already proved by someone, but the author could not find a reference for the following claim, so it is proved here for completeness.

Claim. In a separable countably paracompact space X, there does not exist a closed discrete subspace of cardinality c, here c denotes the cardinality of the set of all subsets of ω.

Proof. Let D be a countable dense subset of X. Assume that there is a discrete closed subset F of X with cardinality c. We may assume $F \cap D = \emptyset$ and identify $F = c$. Observe that the size of the collection of all countable sequences of subsets of D is at most $c^\omega = c$. So we can list all locally finite countable sequences as $\{\langle D_n^\alpha : n \in \omega\rangle : \alpha < c\}$, where some of these sequences can be repeated for different α's, if needed. For $\beta < c$, define $f_\beta : c \to \omega$ by for each $\alpha < c$,

$$f_\beta(\alpha) = \begin{cases} \max\{n \in \omega : \alpha \in \mathrm{Cl}_X D_n^\beta\} & \text{if } \alpha \in \bigcup_{n\in\omega} \mathrm{Cl}_X D_n^\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover define $g : c \to \omega$ by

$$g(\alpha) = f_\alpha(\alpha) + 1$$

for each $\alpha < c$. Since $\{g^{-1}(n) : n \in \omega\}$ is a discrete collection of closed sets in the countably paracompact space X, we can find a locally finite collection $\{G_n : n \in \omega\}$ of open sets in X with $g^{-1}(n) = G_n \cap F$. Take a $\beta < c$ satisfying $\langle D_n^\beta : n \in \omega\rangle = \langle G_n \cap D : n \in \omega\rangle$. Then for each $n \in \omega$, $g^{-1}(n) \subset G_n \subset \mathrm{Cl}_X G_n = \mathrm{Cl}_X(G_n \cap D) = \mathrm{Cl}_X D_n^\beta$. So for each $\alpha < c$, if $g(\alpha) = n$, then $\alpha \in \mathrm{Cl}_X D_n^\beta$ thus $f_\beta(\alpha) \geq n = g(\alpha)$. Therefore we have $f_\beta(\beta) \geq g(\beta) = f_\beta(\beta) + 1$, a contradiction. □

Remark. The referee of the present paper gave the following another proof of Proposition 1 : First note that the Sorgenfrey line S embeds into 2^ω as a closed subspace, see Example 5 of [6]. If N_0 and N_1 are disjoint pair of infinite subsets of ω, then $\langle A, B\rangle \to A \cup B$ embeds $2^{N_0} \times 2^{N_1}$ into 2^ω as a closed subspace. Thus the Sorgenfrey square $S \times S$ embeds into 2^ω as a closed subspace. Since it is known that $S \times S$ is not countably paracompact (this fact is also shown by the Claim above), 2^ω is not countably paracompact.

The author does not know whether the following is true.

Question A. Is 2^ω countably metacompact?

Immediately we have:

Corollary 2. *If 2^X is counatbly paracompact, then X is countably compact.*

Also note that if 2^X is normal, then it is countably paracompact (use the known results listed above).

Corollary 3. *For each nonzero-ordinal α, 2^α is countably paracompact iff $\mathrm{cf}\alpha \neq \omega$.*

Proof. The "only if" part follows from the Corollary above. Assume $\mathrm{cf}\alpha \neq \omega$. Then α is ω-bounded (= each countable subset has a compact closure). Therefore it follows from Theorem 5 of [3] that 2^α is countably compact. □

The author does not know the answer to:

Question B. Is X ω-bounded if 2^X is countably paracompact?

Now we discuss countable paracompactness of $\mathcal{K}(\alpha)$. The following is almost obvious:

Lemma 4. *If X is represented as the free union $X = \bigoplus_{n\in\omega} X_n$ of countably many non-empty clopen sets X_n, then $\mathcal{K}(X) = \bigcup_{n\in\omega} \mathcal{K}(\bigoplus_{i<n} X_i)$.*

Theorem 5. *$\mathcal{K}(\alpha)$ is countably paracompact for all nonzero-ordinal α.*

Proof. If $\mathrm{cf}\alpha = 1$, then $\mathcal{K}(\alpha) = 2^\alpha$ is compact. Next assume $\mathrm{cf}\alpha = \omega$. Take a strictly increasing sequence $\{\alpha_n : n \in \omega\}$ cofinal in α. By the lemma above, we have $\mathcal{K}(\alpha) = \bigcup_{n\in\omega} \mathcal{K}([0, \alpha_n]) = \bigcup_{n\in\omega} 2^{[0,\alpha_n]}$, which is σ-compact thus countably paracompact. Finally assume $\mathrm{cf}\alpha \geq \omega_1$. In this case, $\mathcal{K}(\alpha)$ is countably compact. Indeed, let $\{K_n : n \in \omega\}$ be a countable subset of $\mathcal{K}(\alpha)$. Since $\max K_n < \alpha$ for each $n \in \omega$ and $\mathrm{cf}\alpha \geq \omega_1$, we can find a $\gamma < \alpha$ with $\bigcup_{n\in\omega} K_n \subset [0, \gamma]$. Since $\{K_n : n \in \omega\}$ is a subset of the compact space $2^{[0,\gamma]}$, it has a cluster point in $2^{[0,\gamma]}$ and also in $\mathcal{K}(\alpha)$. Therefore $\mathcal{K}(\alpha)$ is countably compact. □

Now we focus on normality of $\mathcal{K}(\alpha)$.

Lemma 6. *Let X be a zero-dimensional space, F a nonempty compact subset of X and $\mathcal{V}$ a finite collection of open sets with $F \in \langle\mathcal{V}\rangle$. Then there is a pairwise disjoint finite collection $\mathcal{W}$ of clopen sets such that $F \in \langle\mathcal{W}\rangle \subset \langle\mathcal{V}\rangle$.*

Proof. For each $V \in \mathcal{V}$, fix $x(V) \in F \cap V$. Define an equivalence relation $V \sim V'$ on $\mathcal{V}$ by $x(V) = x(V')$. For each equivalence class $\mathcal{E} \in \mathcal{V}/\sim$, let $x_\mathcal{E} = x(V)$ for some (all) $V \in \mathcal{E}$.

Since $\{x_\mathcal{E} : \mathcal{E} \in \mathcal{V}/\sim\}$ is a finite subset of the zero-dimensional T_2 space X, one can find a pairwise disjoint finite collection $\{W_\mathcal{E} : \mathcal{E} \in \mathcal{V}/\sim\}$ of clopen sets with $x_\mathcal{E} \in W_\mathcal{E} \subset \bigcap\mathcal{E}$ for each $\mathcal{E} \in \mathcal{V}/\sim$.

If $F \setminus \bigcup_{\mathcal{E}\in\mathcal{V}/\sim} W_\mathcal{E} = \emptyset$, then $\mathcal{W} = \{W_\mathcal{E} : \mathcal{E} \in \mathcal{V}/\sim\}$ is as required. So assume $F \setminus \bigcup_{\mathcal{E}\in\mathcal{V}/\sim} W_\mathcal{E} \neq \emptyset$. Since $F \setminus \bigcup_{\mathcal{E}\in\mathcal{V}/\sim} W_\mathcal{E}$ is a compact subset of the zero-dimensional space X and it is covered by $\mathcal{V}$, there is a pairwise disjoint collection $\{W(V) : V \in \mathcal{V}\}$ of clopen sets covering it such that $W(V) \subset V$ and $(\bigcup_{\mathcal{E}\in\mathcal{V}/\sim} W_\mathcal{E}) \cap W(V) = \emptyset$ for each $V \in \mathcal{V}$. Then putting $\mathcal{V}' = \{V \in \mathcal{V} : W(V) \cap F \neq \emptyset\}$, $\mathcal{W} = \{W_\mathcal{E} : \mathcal{E} \in \mathcal{V}/\sim\} \cup \{W(V) : V \in \mathcal{V}'\}$ is the desired one. □

Observe that by a similar proof, "Let X be a zero-dimensional space, F nonempty compact subset of X..." in the above lemma can be replaced by "Let X be a normal strongly zero-dimensional space, F nonempty closed subset of X...".

Lemma 7. *Let γ be a nonzero-ordinal, $F \in \mathcal{K}(\gamma)$ and $\mathcal{V}$ a finite collection of open sets in γ with $F \in \langle \mathcal{V} \rangle$. Then there are $n_F \in \omega$ and decreasing sequences $\{\alpha_i : i < n_F\}$ and $\{\beta_i : i < n_F\}$ of ordinals in γ such that*

(1) $\alpha_0 = \max F$, $\{\alpha_i : i < n_F\} \subset F$.
(2) $\alpha_{i+1} \leq \beta_i < \alpha_i$ *for each* $i < n_F$, *where* $\alpha_{n_F} = -1$.
(3) $F \in \langle \{(\beta_i, \alpha_i] : i < n_F\} \rangle \subset \langle \mathcal{V} \rangle$.

Proof. By the Lemma above, we may assume that $\mathcal{V}$ is a finite pairwise disjoint collection of clopen sets in γ.

Let $\alpha \in F$. Fix the unique $V_\alpha \in \mathcal{V}$ with $\alpha \in V_\alpha$ and let $h_F(\alpha) = \min\{\beta \in F : [\beta, \alpha] \cap F \subset V_\alpha\}$. Since V_α is open and $h_F(\alpha) \in V_\alpha$, we can find $g_F(\alpha) < h_F(\alpha)$ with $-1 \leq g_F(\alpha)$ such that $(g_F(\alpha), h_F(\alpha)] \subset V_\alpha$. Then $g_F(\alpha) < \alpha$ and $\alpha \in (g_F(\alpha), \alpha] \cap F = [h_F(\alpha), \alpha] \cap F \subset V_\alpha$. Therefore if $F \cap [0, h_F(\alpha)) \neq \emptyset$, then by $F \cap [0, h_F(\alpha)) = F \cap [0, g_F(\alpha)]$, $\max(F \cap [0, h_F(\alpha)))$ exists and is $\leq g_F(\alpha)$.

Now we will define such sequences by downward induction. First let $\alpha_0 = \max F$ and $\beta_0 = g_F(\alpha_0)$. Assume that for each $i < n$, decreasing sequences $\{\alpha_i : i < n\}$ and $\{\beta_i : i < n\}$ are defined with $\beta_i = g_F(\alpha_i) < \alpha_i$. If $F \cap [0, h_F(\alpha_{n-1})) = \emptyset$, then stop the induction and let $n_F = n$. Otherwise let $\alpha_n = \max(F \cap [0, h_F(\alpha_{n-1})))$ and $\beta_n = g_F(\alpha_n)$.

Since such α_i's are strictly decreasing, this induction will be stopped in finite steps. Now it is straightforward to show that these sequences satisfy the required conditions. □

By $\mathrm{cf}\omega_1 \neq 1$, 2^{ω_1} is not normal. On the other hand:

Theorem 8. *If κ is a regular uncountable cardinal, then $\mathcal{K}(\kappa)$ is normal.*

Proof. Let $\mathcal{F}$ and $\mathcal{H}$ be disjoint closed sets in $\mathcal{K}(\kappa)$. Let M_0 be an elementary submodel of $H(\theta)$, where θ is large enough, such that $\mathcal{F}, \mathcal{H}, \kappa \in M_0$ and $|M_0| < \kappa$. For elementary submodels, the readers should refer to [1, 4]. Assume that elementary submodels $M_0, ..., M_{n-1}$ of $H(\theta)$ with $M_0 \subset ... \subset M_{n-1}$ and $|M_{n-1}| < \kappa$ are defined. Let M_n be an elementary submodel of $H(\theta)$ satisfying $M_{n-1} \cup \bigcup(M_{n-1} \cap \kappa) \subset M_n$ and $|M_n| < \kappa$. Then the union $M = \bigcup_{n \in \omega} M_n$ is also an elementary submodel of $H(\theta)$ and satisfies $\mathcal{F}, \mathcal{H}, \kappa \in M$, $|M| < \kappa$ and $\kappa \cap M$ is an ordinal. Let $\gamma = \kappa \cap M < \kappa$.

Claim 1. If $F \in \mathcal{K}(\kappa) \cap M$, then $\max F < \gamma$.

Proof. Since F is a compact subset of κ, $\max F$ exists and $< \kappa$. On the other hand $\max F$ is deteremined by F and $F \in M$, by elementarity, we have $\max F \in M$. Therefore $\max F \in \kappa \cap M = \gamma$.

To show that $\mathcal{F}$ and $\mathcal{H}$ are separated by disjoint open sets in $\mathcal{K}(\kappa)$, it suffices to show :

$(*)$: for each $F \in \mathcal{F} \cup \mathcal{H}$, there is a finite collection $\mathcal{V}_F$ of open sets of κ such that

$$F \in \langle \mathcal{V}_F \rangle,$$

$$(\bigcup_{F\in\mathcal{F}}\langle\mathcal{V}_F\rangle)\cap(\bigcup_{H\in\mathcal{H}}\langle\mathcal{V}_H\rangle)=\emptyset.$$

Therefore by elementarity, it suffices to show $M\models(*)$, that is, for each $F\in(\mathcal{F}\cup\mathcal{H})\cap M$, there is a finite collection $\mathcal{V}_F\in M$ of open sets of κ such that

$$F\in\langle\mathcal{V}_F\rangle,$$

$$\langle\mathcal{V}_F\rangle\cap\langle\mathcal{V}_H\rangle=\emptyset \text{ for each } F\in\mathcal{F}\cap M \text{ and } H\in\mathcal{H}\cap M.$$

Observe that by Claim 1, $\mathcal{F}\cap M$ and $\mathcal{H}\cap M$ are subsets of the compact space $\mathcal{K}([0,\gamma])=2^{[0,\gamma]}$.

Claim 2. $\mathrm{Cl}_{\mathcal{K}([0,\gamma])}(\mathcal{F}\cap M)\cap\mathrm{Cl}_{\mathcal{K}([0,\gamma])}(\mathcal{H}\cap M)=\emptyset$.

Proof. Assume that there is a $K\in\mathrm{Cl}_{\mathcal{K}([0,\gamma])}(\mathcal{F}\cap M)\cap\mathrm{Cl}_{\mathcal{K}([0,\gamma])}(\mathcal{H}\cap M)$. Let $\langle\mathcal{V}\rangle$ be an arbitrary neighborhood of K in $\mathcal{K}(\kappa)$, that is, $\mathcal{V}$ is a finite collection of open sets in κ with $K\in\langle\mathcal{V}\rangle$. Since $K\subset[0,\gamma]\subset\kappa$, we may assume that $\bigcup\mathcal{V}\subset[0,\gamma]$. Then $\langle\mathcal{V}\rangle$ is a neighborhood of K in $\mathcal{K}([0,\gamma])$. It follows from $K\in\mathrm{Cl}_{\mathcal{K}([0,\gamma])}(\mathcal{F}\cap M)$ that $\emptyset\neq\langle\mathcal{V}\rangle\cap(\mathcal{F}\cap M)\subset\langle\mathcal{V}\rangle\cap\mathcal{F}$. Since $\langle\mathcal{V}\rangle$ was an arbitrary neighborhood of K in $\mathcal{K}(\kappa)$, we have $K\in\mathrm{Cl}_{\mathcal{K}(\kappa)}\mathcal{F}=\mathcal{F}$. Similarly we have $K\in\mathcal{H}$, a contradiction.

By normality of $\mathcal{K}([0,\gamma])$ and Claim 2, for each $F\in(\mathcal{F}\cup\mathcal{H})\cap M$, there is a finite collection $\mathcal{V}_F$ of open sets of $[0,\gamma]$ such that

$$F\in\langle\mathcal{V}_F\rangle,$$

$$\langle\mathcal{V}_F\rangle\cap\langle\mathcal{V}_H\rangle=\emptyset \text{ for each } F\in\mathcal{F}\cap M \text{ and } H\in\mathcal{H}\cap M.$$

For each $F\in(\mathcal{F}\cup\mathcal{H})\cap M$, since $F\in\langle\mathcal{V}_F\rangle$, applying Lemma 7 to $\gamma+1=[0,\gamma]$, we can find two finite sequences $\{\alpha_i^F:i<n_F\}$ and $\{\beta_i^F:i<n_F\}$ of $[0,\gamma]$ satisfying (1)-(3).

By (3), we may assume $\mathcal{V}_F=\{(\beta_i^F,\alpha_i^F]:i<n_F\}$. By Claim 1, $\alpha_0^F=\max F<\gamma\subset M$, therefore these two sequences are subsets of M. So by elementarity, these two finite sequences are elements in M. $\mathcal{V}_F$ is determined by these two sequences, consequently we have $\mathcal{V}_F\in M$. This completes the proof. ☐

The following question seems to be strangely difficult:

Question C. Find a proof of the Theorem above without using elementary submodel techniques.

Lemma 9. *If $\omega_1\leq\mathrm{cf}\gamma<\gamma$, then $\mathcal{K}(\gamma)$ is not normal.*

Proof. Let $\kappa=\mathrm{cf}\gamma$. Choose a strictly incresing sequence $\{\gamma(\alpha):\alpha<\kappa\}$ such that

(1) $\mathrm{cf}\gamma<\gamma(0)$,
(2) $\gamma(\alpha)=\sup\{\gamma(\beta):\beta<\alpha\}$ if α is limit.
(3) $\gamma=\sup\{\gamma(\alpha):\alpha<\kappa\}$.

It is routine to check that the mapping $\langle\alpha,\beta\rangle\to\{\alpha,\gamma(\beta)\}$ embeds $(\kappa+1)\times\kappa$ into $[\gamma]^{\leq 2}$ and, hence, into $\mathcal{K}(\gamma)$ as a closed subspace. This concludes the proof, since it is well-known that $(\kappa+1)\times\kappa$ is not normal if κ is an uncountable regular cardinal. ☐

Corollary 10. *For every nonzero-ordinal α, $\mathcal{K}(\alpha)$ is normal iff, if $\mathrm{cf}\alpha$ is uncountable, then $\mathrm{cf}\alpha=\alpha$.*

Proof. The "only if" part follows from Lemma 9.

The "if" part: If $\mathrm{cf}\alpha = 1$, then $\mathcal{K}(\alpha)$ is compact so normal. If $\mathrm{cf}\alpha = \omega$, then as in the proof of Theorem 5, $\mathcal{K}(\alpha)$ is σ-compact so normal. If $\mathrm{cf}\alpha$ is uncountable, then $\mathrm{cf}\alpha = \alpha$ so this case follows from Theorem 8. □

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