NORMALITY AND PARANORMALITY IN PRODUCT SPACES

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Abstract. Let $\beta\omega$ denote the Stone-Čech compactification of the countable discrete space ω. We show that if p is a point of $\beta\omega\backslash\omega$, then all subspaces of $(\omega \cup \{p\}) \times \omega_1$ are paranormal, where $(\omega \cup \{p\})$ is considered as a subspace of $\beta\omega$. This answers a van Douwen's question. Moreover we show that the existence of a paranormal non-normal subspace of $(\omega + 1) \times \omega_1$ is independet of ZFC, where $\omega + 1$ is the ordinal space $\{0, 1, 2, ..., \omega\}$ with the usual order toplogy.

1. Introduction

Throughout this paper, we assume that all spaces are regular and T_1. As usual, an ordinal is equal to the set of smaller ordinals, for example, $j = \{0, 1, 2, 3, ..., j-1\}$ for each natural number j. The symbols ω and ω_1 stand for the set of all finite and respectively all countable ordinals. If an ordinal is considered as a topological space, then its topology is induced by the usual order. It is well known that all subspaces of ordinals are normal and countably paracompact. A space X is *paranormal* ([vD]) if for every countable discrete collection $\{F(n) : n \in \omega\}$ of closed sets, there is a collection $\{U(n,k) : n, k \in \omega\}$ of open sets such that $F(n) \subset U(n,k)$ for each $n, k \in \omega$, and $\bigcap_{n,k\in\omega} \mathrm{Cl}_X\, U(n,k) = \emptyset$. Let us recall that $\{F(n) : n \in \omega\}$ is *discrete* in X if each point in X has a neighborhood U with $|\{n \in \omega : F(n) \cap U \neq \emptyset\}| \leq 1$. Obviously, the normal spaces as well as the countably paracompact ones are paranormal. Van Douwen and Katětov, respectively, proved the following.

Proposition 1.1. [vD, Theorem 5.2] *If a space Y has a closed subset K which is not regular G_δ, then the subspace $Z = \omega \times Y \cup \{\omega\} \times (Y\backslash K)$ of $(\omega + 1) \times Y$ is not paranormal.*

Here a subset K is a *regular G_δ* subset if it is an intersection of a countable collection of closed neighborhoods.

Proposition 1.2. [Ka, Corollary 1] *If a space Y has a closed subset K which is not regular G_δ and C is a countable space with a non-isolated point p, then the subspace $Z = (C\backslash\{p\}) \times Y \cup \{p\} \times (Y\backslash K)$ of $C \times Y$ is not normal.*

Van Douwen [vD, the remark after Theorem 5.2] also pointed out that Zenor essentially proved the following.

AMS subject Classification: 54B10, 54D15, 54A35, 03E35.

Key words and phrases: normal, paranormal, product space, ZFC, Continuum Hypothesis, Martin's Axiom

Proposition 1.3. [Ze, Theorem 8] *If a space Y has a closed subset K which is not regular G_δ, and C is a countable space with a non-isolated point p, then the subspace $Z = (C\setminus\{p\}) \times Y \cup \{p\} \times (Y\setminus K)$ of $C \times Y$ is not countably paracompact.*

Van Douwen asked:

Van Douwen's question. *If a space Y has a closed subset K which is not regular G_δ and C is a countable space with a non-isolated point, then does $C \times Y$ have a non-paranormal subspace?*

Observe that in the ordinal ω_1, the set Lim consisting of all limit ordinals in ω_1 is not a (regular) G_δ subset in ω_1.

In the present paper, we will show that if p is a point of $\beta\omega\setminus\omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the countable discrete space ω, then all subspaces of $(\omega \cup \{p\}) \times \omega_1$ are paranormal. Moreover, we will show that the existence of a paranormal non-normal subspace of $(\omega + 1) \times \omega_1$ is independet of ZFC.

Now we introduce two known notions lying between normality and paranormality. A space X is said to be *discretely ω-expandable* ([SK]) if for every countable discrete collection $\{F(n) : n \in \omega\}$ of closed sets, there is a locally finite collection $\{U(n) : n \in \omega\}$ of open sets with $F(n) \subset U(n)$ for each $n \in \omega$. A space X is said to have the *weak $D(\omega)$-property* ([KOT]) if for every countable discrete collection $\{F(n) : n \in \omega\}$ of closed sets, there is a collection $\{U(n) : n \in \omega\}$ of open sets such that $F(n) \subset U(n)$ for each $n \in \omega$ and $\bigcap_{n\in\omega} \mathrm{Cl}_X U(n) = \emptyset$. It is easy to see that discrete ω-expandability is a common generalization of both normality and countable paracompactness. Moreover, the discrete ω-expandability implies the weak $D(\omega)$-property and the weak $D(\omega)$-property implies paranormality. Here we introduce a new notion, which will play an important role in our discussion, and which lies between discrete ω-expandability and the weak $D(\omega)$-property. A space X is said to have the *weak $C(\omega)$-property* if for every countable discrete collection $\{F(n) : n \in \omega\}$ of closed sets, there is a collection $\{U(n) : n \in \omega\}$ of open sets such that $F(n) \subset U(n)$ for each $n \in \omega$ and $\bigcap_{n\in J} \mathrm{Cl}_X U(n) = \emptyset$ for each $J \in [\omega]^\omega$, where $[A]^\kappa$ denotes the set $\{B \subset A : |B| = \kappa\}$ for a set A and a cardinal κ.

Observe that the remark after [vD, Theorem 5.2] works to prove the following:

Proposition 1.4. *If a space Y has a closed subset K which is not regular G_δ, and C is a countable space with a non-isolated point p, then the subspace $Z = (C\setminus\{p\}) \times Y \cup \{p\} \times (Y\setminus K)$ of $C \times Y$ is not discretely ω-expandable.*

So relating to the van Douwen's question, it is natural to ask:

Question I. *If a space Y has a closed subset K which is not regular G_δ and C is a countable space with a non-isolated point, then does $C \times Y$ have a subspace without the weak $C(\omega)$-property?*

It is known that if A and B are subspaces of ordinals, then the countable paracompactness of $A \times B$ is equivalent to the weak $D(\omega)$-property of $A \times B$ ([KOT]). Another related question is :

Question II. [KOT, problem (ii)] *For every subspace X of a product space of two ordinals, is the countable paracompactness of X equivalent to the weak $D(\omega)$-property?*

We will also answer these questions.

Throughout the present paper, we use the following specific notation. Let X be a subspace of a product space $S \times T$, and let $s \in S$, $A \subset S$ and $B \subset T$. Set

$$V_s(X) = \{t \in T : \langle s, t\rangle \in X\},$$

$$X_A = X \cap A \times T, X^B = X \cap S \times B, X_A^B = X \cap A \times B.$$

For each $A \subset \omega_1$, we denote by $\mathrm{Lim}(A)$ the set $\{\alpha < \omega_1 : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of A in ω_1. For technical reasons only, we consider "-1" is the immediate predecessor of the ordinal 0, and $\sup \emptyset = -1$. Note that $\mathrm{Lim}(A)$ is closed unbounded (cub) in ω_1 whenever A is unbounded in ω_1. In particular, assume that C is a cub set in ω_1, then $\mathrm{Lim}(C) \subset C$. In this case, we set $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}(C)$, and $p_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Note that, for each $\alpha \in C$, $p_C(\alpha) \in C \cup \{-1\}$, and $p_C(\alpha) < \alpha$ iff $\alpha \in \mathrm{Succ}(C)$. So, $p_C(\alpha)$ is the immediate predecessor of α in $C \cup \{-1\}$ whenever $\alpha \in \mathrm{Succ}(C)$. Moreover observe that $\omega_1 \setminus C$ is the union of the pairwise disjoint collection $\{(p_C(\alpha), \alpha) : \alpha \in \mathrm{Succ}(C)\}$ of open intervals of ω_1. So if Y is a subspace of ω_1 which is disjoint from some cub set C of ω_1, then Y can be represented as the free union

$$Y = \bigoplus_{\alpha \in \mathrm{Succ}(C)} (Y \cap (p_C(\alpha), \alpha]).$$

For simplicity, we use Lim and Succ instead of $\mathrm{Lim}(\omega_1)$ and $\mathrm{Succ}(\omega_1)$, respectively.

2. On paranormality

According to Proposition 1.4, the subspace $Z = \omega \times \omega_1 \cup \{p\} \times (\omega_1 \setminus \mathrm{Lim})$ of $(\omega \cup \{p\}) \times \omega_1$ is not discrete ω-expandable whenever p is a point of $\beta\omega \setminus \omega$. In this section, we will show that all subspaces of $(\omega \cup \{p\}) \times \omega_1$ have the weak $C(\omega)$-property.

We omit the proof of the following easy lemmas.

Lemma 2.1. *If X is a hereditarily normal space and $Y \subset X$, then for every countable discrete (in Y) collection $\{F(n) : n \in \omega\}$ of closed sets in Y, there is a pairwise disjoint collection $\{U(n) : n \in \omega\}$ of open sets in X with $F(n) \subset U(n)$ for each $n \in \omega$, and such that $\{\mathrm{Cl}_X U(n) \cap Y : n \in \omega\}$ is also pairwise disjoint.*

Lemma 2.2. *If I is a finite set and $\{W_i(n) : n \in \omega\}$ is a collection of subsets of X such that $\bigcap_{n\in J} \mathrm{Cl}_X W_i(n) = \emptyset$ for every $i \in I$ and $J \in [\omega]^\omega$, then $\bigcap_{n\in J} \mathrm{Cl}_X(\bigcup_{i\in I} W_i(n)) = \emptyset$ for every $J \in [\omega]^\omega$.*

Recall that a subset X of ω_1 is said to be *stationary* in ω_1 if $X \cap C \neq \emptyset$ for any cub set C in ω_1.

Lemma 2.3. *Assume that X is stationary in ω_1 and $\mathcal{F} = \{F(n) : n \in \omega\}$ is a countable discrete collection of closed sets of X. Then there is an $\alpha < \omega_1$ such that $|\{n \in \omega : F(n) \cap (\alpha, \omega_1) \neq \emptyset| \leq 1$.*

Proof. Since $\mathcal{F}$ is discrete in X, for each $\beta \in X$, fix an $f(\beta) < \beta$ with $|\{n \in \omega : F(n) \cap (f(\beta), \beta] \neq \emptyset| \leq 1$. Then, by Pressing Down Lemma (PDL), we find a stationary set $S \subset X$ and an $\alpha < \omega_1$ such that $f(\beta) = \alpha$ for every $\beta \in S$. It is straightforward to see this α is as desired. $\square$

Let C be a countable space with a unique non-isolated point p, say $C = \omega \cup \{p\}$. Then we can identify the point p as the filter $\{U \cap \omega : U$ is a neighborhood of $p\}$ on ω. In this case, this filter p is free, that is, $\bigcap p = \emptyset$. If p is a free ultrafilter on ω, then $C = \omega \cup \{p\}$ is a subspace of $\beta\omega$. Moreover if $p = \{U \subset \omega : \omega \backslash U$ is finite $\}$, then $C = \omega \cup \{p\}$ is considered as the ordinal space $\omega + 1$.

Theorem 2.4. *Let $C = \omega \cup \{p\}$ be a countable space with a unique non-isolated point p. Then for every pairwise disjoint collection $\{A(n) : n \in \omega\}$ of ω, $\{n \in \omega : p \in \mathrm{Cl}_{\omega\cup\{p\}} A(n)\}$ is finite if and only if all subspaces of $(\omega \cup \{p\}) \times \omega_1$ have the weak $C(\omega)$-property.*

Proof. "if" part: Assume there is a pairwise disjoint collection $\{A(n) : n \in \omega\}$ of ω such that $J_0 = \{n \in \omega : p \in \mathrm{Cl}_{\omega\cup\{p\}} A(n)\}$ is infinite. Let $X = \omega \times \omega_1 \cup \{p\} \times \mathrm{Succ}$ and $F(n) = A(n) \times \mathrm{Lim}$ for each $n \in \omega$. Then $\mathcal{F} = \{F(n) : n \in \omega\}$ is a discrete collection of closed sets in X. Let $\{U(n) : n \in \omega\}$ be a collection of open sets such that $F(n) \subset U(n)$ for each $n \in \omega$. Let $j \in A = \bigcup_{n\in J_0} A(n)$, say $j \in A(n)$. Since $V_j(F(n)) = \mathrm{Lim}$ is stationary, there is an $\alpha_j < \omega_1$ such that $X_{\{j\}}^{(\alpha_j,\omega_1)} \subset U(n)$. Pick a $\beta \in \mathrm{Succ}$ with $\sup\{\alpha_j : j \in A\} < \beta$. Then $\langle p, \beta\rangle \in \mathrm{Cl}_X U(n)$ for each $n \in J_0$. Hence X does not have the weak $C(\omega)$-property.

"only if" part: Let $X \subset (\omega \cup \{p\}) \times \omega_1$. We consider two cases.

Case 1. $V_p(X)$ is stationary in ω_1.

We will show the normality of X, which implies the weak $C(\omega)$-property of X. Let $F(0)$ and $F(1)$ be disjoint closed sets in X. By Lemma 2.3 one of $F(0)$ and $F(1)$, say $F(0)$, has the empty intersection with $X_{\{p\}}^{(\alpha_0,\omega_1)}$ for some $\alpha_0 < \omega_1$. Since $\langle p, \alpha\rangle \notin F(0)$, for each $\alpha \in V_p(X) \cap (\alpha_0, \omega_1)$, we can find an open neighborhood of $N(\alpha)$ of p and an $f(\alpha) < \alpha$ in such a way that $\alpha_0 \leq f(\alpha)$ and $X_{N(\alpha)}^{(f(\alpha),\alpha]} \cap F(0) = \emptyset$. Applying PDL, we find an $\alpha_1 < \omega_1$ and a stationary set $S \subset V_p(X) \cap (\alpha_0, \omega_1)$ such that $f(\alpha) = \alpha_1$ for each $\alpha \in S$. Define $W(\alpha) = \bigcup\{W \subset \omega \cup \{p\} : X_W^{(\alpha_1,\alpha]} \cap F(0) = \emptyset$ and W is open in $\omega \cup \{p\}\}$ for each $\alpha \in S$. Then $W(\alpha)$ is an open neighborhood of p in $\omega \cup \{p\}$ and $N(\alpha) \subset W(\alpha)$. Since the collection $\mathcal{W} = \{W(\alpha) : \alpha \in S\}$ is decreasing and $\omega \cup \{p\}$ is countable, we find a $\gamma \in S$ such that $W(\gamma) = W(\alpha)$ for each $\alpha \in S\backslash\gamma$. Let $W = W(\gamma)$. Then we have $X_W^{(\alpha_1,\omega_1)} \cap F(0) = \emptyset$. Since both W and (α_1, ω_1) are closed and open (clopen) in $\omega \cup \{p\}$ and ω_1 respectively, so is $X_W^{(\alpha_1,\omega_1)}$ in X. So X can be represented as the free union

$$X = X^{[0,\alpha_1]} \bigoplus X_{\omega\backslash W}^{(\alpha_1,\omega_1)} \bigoplus X_W^{(\alpha_1,\omega_1)}.$$

The subspace $X^{[0,\alpha_1]}$ is normal because it is countable, and also $X_{\omega\backslash W}^{(\alpha_1,\omega_1)}$ is normal, because it can be represented as the free union

$$X_{\omega\backslash W}^{(\alpha_1,\omega_1)} = \bigoplus_{j\in\omega\backslash W} X_{\{j\}}^{(\alpha_1,\omega_1)}$$

of subspaces of X which are homeomorphic to some subspaces of ω_1. Moreover $X_W^{(\alpha_1,\omega_1)}$ does not meet $F(0)$. These considerations yield disjoint open sets $U(0)$ and $U(1)$ containing $F(0)$ and $F(1)$, respectively. Therefore X is normal.

Case 2. $V_p(X)$ is not stationary in ω_1.

Let $\mathcal{F} = \{F(n) : n \in \omega\}$ be a discrete collection of closed sets in X. Since $\{V_p(F(n)) : n \in \omega\}$ is a discrete collection of closed sets in $V_p(X) \subset \omega_1$, by Lemma 2.1, there is a pairwise disjoint collection $\{U(n) : n \in \omega\}$ of open sets in ω_1 such that $V_p(F(n)) \subset U(n)$ for each $n \in \omega$ and $\{\mathrm{Cl}_{\omega_1} U(n) \cap V_p(X) : n \in \omega\}$ remains pairwise disjoint. Set $W_0(n) = X^{U(n)}_{(n,\omega)\cup\{p\}}$ for each $n \in \omega$. Then $W_0(n)$ is open and $F(n) \cap X_{\{p\}} \subset W_0(n)$ for each $n \in \omega$. Since $\{\mathrm{Cl}_{\omega_1} U(n) \cap V_p(X) : n \in \omega\}$ is pairwise disjoint, we have $\bigcap_{n\in J} \mathrm{Cl}_X W_0(n) = \emptyset$ for each $J \in [\omega]^\omega$.

Next let $A = \{j \in \omega : V_j(X)$ is stationary in $\omega_1\}$. It follows from Lemma 2.3 that, for each $j \in A$, there is an $\alpha_j < \omega_1$ with $|\{n \in \omega : F(n) \cap X^{(\alpha_j,\omega_1)}_{\{j\}} \neq \emptyset\}| \leq 1$. Let $\gamma_0 = \sup\{\alpha_j : j \in A\}$. Then we have $|\{n \in \omega : F(n) \cap X^{(\gamma_0,\omega_1)}_{\{j\}} \neq \emptyset\}| \leq 1$ for each $j \in A$. Let $A(n) = \{j \in A : F(n) \cap X^{(\gamma_0,\omega_1)}_{\{j\}} \neq \emptyset\}$. Since the collection $\{A(n) : n \in \omega\}$ is pairwise disjoint, the set $\{n \in \omega : p \in \mathrm{Cl}_{\omega\cup\{p\}} A(n)\}$ is finite. Set $W_1(n) = X^{(\gamma_0,\omega_1)}_{A(n)}$ for each $n \in \omega$. Then $W_1(n)$ is open and $F(n) \cap X^{(\gamma_0,\omega_1)}_{A} \subset W_1(n)$ for each $n \in \omega$. Since $\{n \in \omega : p \in \mathrm{Cl}_{\omega\cup\{p\}} A(n)\}$ is finite, we have $\bigcap_{n\in J} \mathrm{Cl}_X W_1(n) = \emptyset$ for each $J \in [\omega]^\omega$.

Since $X^{[0,\gamma_0]}$ is countable so normal, so there is a collection $W_2(n) : n \in \omega\}$ of open sets such that $F(n) \cap X^{[0,\gamma_0]} \subset W_2(n)$ for every $n \in \omega$ and $\bigcap_{n\in J} \mathrm{Cl}_X W_2(n) = \emptyset$ for each $J \in [\omega]^\omega$.

Next for each $j \in (\omega \setminus A) \cup \{p\}$, we can find a cub set D_j disjoint from $V_j(X)$ because $V_j(X)$ is not stationary. Since $D = \bigcap_{j\in(\omega\setminus A)\cup\{p\}} D_j$ is cub in ω_1 and $X^D_{(\omega\setminus A)\cup\{p\}} = \emptyset$, we represent

$$X_{(\omega\setminus A)\cup\{p\}} = \bigoplus_{\alpha\in \mathrm{Succ}(D)} X^{(p_D(\alpha),\alpha]}_{(\omega\setminus A)\cup\{p\}}$$

as it is mentioned in the introduction. Therefore $Y = X_{(\omega\setminus A)\cup\{p\}}$ is normal as a free union of countable spaces. So there is a collection $\{U(n) : n \in \omega\}$ of open sets in Y such that $F(n) \cap Y \subset U(n)$ for each $n \in \omega$ and $\{\mathrm{Cl}_Y U(n) : n \in \omega\}$ is pairwise disjoint. Since $X_{\omega\setminus A}$ is open in X, $W_3(n) = U(n) \cap X_{\omega\setminus A}$ is also open in X and $F(n) \cap X_{\omega\setminus A} \subset W_3(n)$. Since $\{\mathrm{Cl}_Y U(n) : n \in \omega\}$ is pairwise disjoint, so is $\{\mathrm{Cl}_X W_3(n) : n \in \omega\}$. Therefore $\bigcap_{n\in J} \mathrm{Cl}_X W_3(n) = \emptyset$ for each $J \in [\omega]^\omega$.

Finally set $W(n) = \bigcup_{0\leq i\leq 3} W_i(n)$ for each $n \in \omega$. Then $F(n) \subset W(n)$ for each $n \in \omega$ and, by Lemma 2.2, $\bigcap_{n\in J} \mathrm{Cl}_X W(n) = \emptyset$ for each $J \in [\omega]^\omega$. □

If p is a free ultrafilter on ω, then for every pairwise disjoint collection $\{A(n) : n \in \omega\}$ of ω, $p \in \mathrm{Cl}_{\omega\cup\{p\}} A(n)$ for at most one $n \in \omega$. So we have:

Corollary 2.5. *Let $p \in \beta\omega\setminus\omega$. Then all subspaces of $(\omega \cup \{p\}) \times \omega_1$ have the weak $C(\omega)$-property, therefore they are paranormal.*

Corollary 2.5 answers Question I and the van Douwen's question.

3. On normality

In this section, we will show that the discretely ω-expandable subspaces of $C \times \omega_1$ are normal whenever C is a countable space with a unique non-isolated point.

Theorem 3.1. *Let $C = \omega \cup \{p\}$ be a countable space with a unique non-isolated point p. Then for every subspace X of $C \times \omega_1$, the following are equivalent:*

(1) *X is normal.*
(2) *X is countably paracompact.*
(3) *X is discretely ω-expandable.*

Proof. (1) → (2): Assume that X is a normal subspace of $C \times \omega_1$. Since X is a countable union of countably metacompact closed subspaces $X_{\{j\}}$'s ($j \in C$), X is also countably metacompact. By the normality, X is countably paracompact.

(2) → (3): Obvious.

(3) → (1): Let $X \subset (\omega \cup \{p\}) \times \omega_1$ be discretely ω-expandable. Let $A = \{j \in \omega : V_j(X) \text{ is stationary in } \omega_1\}$. We may assume that $V_p(X)$ is not stationary in ω_1, otherwise, by the proof of Case 1 of Theorem 2.4, X is normal. For each $j \in (\omega \setminus A) \cup \{p\}$, take a cub set D_j disjoint from $V_j(X)$. Then $D = \bigcap_{j \in (\omega \setminus A) \cup \{p\}} D_j$ is a cub set in ω_1. Clearly $X_{\{p\}}$ and X^D are disjoint closed sets in X and $X^D = X^D_A$.

Claim. *$X_{\{p\}}$ and X^D are separated by disjoint open sets in X.*

Proof. For each $j \in A$, set $F(j) = X^D_{\{j\}}$. It is easy to see that $\mathcal{F} = \{F(j) : j \in A\}$ is a discrete collection of closed sets in X. By the discrete ω-expandability of X, there is a locally finite collection $\mathcal{U} = \{U(j) : j \in A\}$ of open sets of X with $F(j) \subset U(j)$ for each $j \in A$. Since for each $j \in A$, $V_j(F(j)) = V_j(X) \cap D$ is stationary in ω_1, by PDL, we can find an $\alpha_j < \omega_1$ such that $X^{(\alpha_j, \omega_1)}_{\{j\}} \subset U(j)$. Let $\gamma_0 = \sup\{\alpha_j : j \in A\}$. Then we have $X^{(\gamma_0, \omega_1)}_{\{j\}} \subset U(j)$ for each $j \in A$. Since $\mathcal{U}$ is locally finite, so is $\{X^{(\gamma_0, \omega_1)}_{\{j\}} : j \in A\}$. Therefore for each $\alpha \in V_p(X)$, there is a neighborhood $N(\alpha)'$ of p and an $f(\alpha) < \alpha$ such that $H(\alpha) = \{j \in A : X^{(f(\alpha), \alpha]}_{N(\alpha)'} \cap X^{(\gamma_0, \omega_1)}_{\{j\}} \neq \emptyset\}$ is finite. Then $N(\alpha) = N(\alpha)' \setminus H(\alpha)$ is also a neighborhood of p for each $\alpha \in V_p(X)$. Since $X^{(f(\alpha), \alpha]}_{N(\alpha)} \cap X^{(\gamma_0, \omega_1)}_{\{j\}} = \emptyset$ for each $j \in A$ and $\alpha \in V_p(X)$, $\bigcup_{\alpha \in V_p(X)} X^{(f(\alpha), \alpha]}_{N(\alpha)}$ and $X^{(\gamma_0, \omega_1)}_A$ are disjoint open sets which separate $X^{(\gamma_0, \omega_1)}_{\{p\}}$ and $X^{D \cap (\gamma_0, \omega_1)}_A$ respectively. Since $X^{[0, \gamma_0]}$ is normal and $X^D = X^D_A$, X^D and $X_{\{p\}}$ are also separated by disjoint open sets in X. This completes the proof of Claim.

Using this claim, take disjoint open sets U and V containing $X_{\{p\}}$ and X^D respectively. Observe that $X \setminus U \subset X \setminus X_{\{p\}} = X_\omega = \bigoplus_{j \in \omega} X_{\{j\}}$ and

$$X \setminus V \subset X \setminus X^D = X^{\omega_1 \setminus D} = \bigoplus_{\alpha \in \mathrm{Succ}(D)} X^{(p_D(\alpha), \alpha]}.$$

Since $\bigoplus_{j \in \omega} X_{\{j\}}$ and $\bigoplus_{\alpha \in \mathrm{Succ}(D)} X^{(p_D(\alpha), \alpha]}$ are normal, $X \setminus U$ and $X \setminus V$ are also normal closed subspaces. Hence $X = (X \setminus U) \cup (X \setminus V)$ is normal. □

Problem 3.2. *Let C be a countable space and $X \subset C \times \omega_1$. Are normality, countable paracompactness and discrete ω-expandability of X equivalent?*

4. Paranormal non-normal subspaces of $(\omega + 1) \times \omega_1$

In this section, we will find equivalent combinatorial conditions of the existence of a subspace of $(\omega + 1) \times \omega_1$ with the weak $C(\omega)$-property which is not discretely ω-expandable as well as the existence of a paranormal non-normal subspace of $(\omega + 1) \times \omega_1$.

Theorem 4.1. *The following are equivalent:*

(1) *There is a non-normal subspace of* $(\omega+1)\times\omega_1$ *which has the weak* $C(\omega)$-*property.*

(2) *The following property* (A) *holds:*

(A)*: There is a collection* $\{N(\alpha):\alpha<\omega_1\}$ *of infinite subsets of* ω *such that, for every pairwise disjoint collection* $\{A(n):n\in\omega\}$ *of subsets of* ω *and* $K\in[\omega_1]^{\omega_1}$*, there is an* $\alpha\in K$ *such that for every* $J\in[\omega]^\omega$, $A(n)\cap N(\alpha)$ *is finite for some* $n\in J$.

Proof. (1) → (2): Assume that X is a non-normal subspace of $(\omega+1)\times\omega_1$ having the weak $C(\omega)$-property. Since X is not normal, similarily to Case 1 of the proof of Theorem 2.4, $V_\omega(X)$ is not stationary in ω_1. Let

$$A=\{j\in\omega : V_j(X)\text{ is stationary in }\omega_1\},$$

and take a cub set D_j disjoint from $V_j(X)$ for each $j\in(\omega\backslash A)\cup\{\omega\}$. Let $D=\bigcap_{j\in(\omega\backslash A)\cup\{\omega\}}D_j$. Then it is straightforward to show that X^D and $X_{\{\omega\}}$ are disjoint closed sets in X, $X^D_A=X^D$, and X_A is an open set including $X^D_A=X^D$.

Claim 1. $V_\omega(\mathrm{Cl}_X X_A)$ *is uncountable.*

Proof. Assume that $V_\omega(\mathrm{Cl}_X X_A)$ is countable. Take an ordinal $\gamma_0<\omega_1$ with $V_\omega(\mathrm{Cl}_X X_A)\subset\gamma_0$. Then $X=X^{[0,\gamma_0]}\bigoplus X^{(\gamma_0,\omega_1)}$ and $X^{[0,\gamma_0]}$ is normal. Let $Y=X^{(\gamma_0,\omega_1)}$. Since $\mathrm{Cl}\,Y_A\cap Y_{\{\omega\}}=\emptyset$, $U=Y\backslash\mathrm{Cl}\,Y_A$ and $V=Y_A$ are disjoint open sets including $Y_{\{\omega\}}$ and Y^D, respectively. Then, as in the proof of Theorem 3.1, $Y=(Y\backslash U)\cup(Y\backslash V)$ is normal. Hence X is normal, a contradiciton. This completes the proof of Claim 1.

Enumerate $V_\omega(\mathrm{Cl}_X X_A)=\{\alpha(\gamma):\gamma<\omega_1\}$ with the increasing order. For each $\gamma<\omega_1$, since $\langle\omega,\alpha(\gamma)\rangle\in\mathrm{Cl}_X X_A$ and X is first countable, we can take two increasing functions $f_\gamma:\omega\to A$ and $h_\gamma:\omega\to[0,\alpha(\gamma)]$ in such a way that the sequence $S(\gamma)=\{\langle f_\gamma(l),h_\gamma(l)\rangle : l\in\omega\}$ converges to $\langle\omega,\alpha(\gamma)\rangle$. Since the sequence $S(\gamma)$ is contained in X_A and converges to $\langle\omega,\alpha(\gamma)\rangle$, we may assume that the function f_γ is strictly increasing, i.e., $f_\gamma(n')<f_\gamma(n)$ whenever $n'<n$. Therefore $N(\gamma)=\mathrm{ran}(f_\gamma)$ $(=\{f_\gamma(l):l\in\omega\})$ is an infinite subset of A for each $\gamma<\omega_1$. We will show that the collection $\{N(\gamma):\gamma<\omega_1\}$ is as desired. Let $\{A(n):n\in\omega\}$ be a pairwise disjoint collection of ω and $K\in[\omega_1]^{\omega_1}$. Define $F(n)=X^D_{A(n)\cap A}$ for each $n\in\omega$. Then $\mathcal{F}=\{F(n):n\in\omega\}$ is a discrete collection of closed sets in X. By the weak $C(\omega)$-property, there is a collection $\{U(n):n\subset\omega\}$ of open sets in X such that $F(n)\subset U(n)$ for each $n\in\omega$ and $\bigcap_{n\in J}\mathrm{Cl}_X U(n)=\emptyset$ for each $J\in[\omega]^\omega$. Let $j\in A(n)\cap A$ and $\alpha\in V_j(F(n))$. Since $\langle j,\alpha\rangle\in F(n)\subset U(n)$ and $U(n)$ is open, there is a $g_j(\alpha)<\alpha$ satisfying $X^{(g_j(\alpha),\alpha]}_{\{j\}}\subset U(n)$. Moreover since $V_j(F(n))=V_j(X)\cap D$ is stationary, by PDL, we can find an $\alpha_j<\omega_1$ such that $X^{(\alpha_j,\omega_1)}_{\{j\}}\subset U(n)$. Let $\gamma_0=\sup\{\alpha_j : j\in(\bigcup_{n\in\omega}A(n))\cap A\}$. Then clearly we have $X^{(\gamma_0,\omega_1)}_{A(n)\cap A}\subset U(n)$ for each $n\in\omega$. Since K is uncountable, take a $\gamma\in K$ with $\gamma_0<\alpha(\gamma)$. Then $\langle\omega,\alpha(\gamma)\rangle\in X$. Let $J\in[\omega]^\omega$. It follows from $\bigcap_{n\in J}\mathrm{Cl}_X X^{(\gamma_0,\omega_1)}_{A(n)\cap A}\subset\bigcap_{n\in J}\mathrm{Cl}_X U(n)=\emptyset$ that there is an $n\in J$ such that $\langle\omega,\alpha(\gamma)\rangle\notin\mathrm{Cl}_X X^{(\gamma_0,\omega_1)}_{A(n)\cap A}$. If $A(n)\cap N(\gamma)$ is infinite, then the subsequence $\{\langle f_\gamma(l),h_\gamma(l)\rangle : l\in A(n)\cap N(\gamma), h_\gamma(l)>\gamma_0\}$ of $S(\gamma)$ is a subset of $X^{(\gamma_0,\omega_1)}_{A(n)\cap A}$ and converges to $\langle\omega,\alpha(\gamma)\rangle$. So we have $\langle\omega,\alpha(\gamma)\rangle \in \mathrm{Cl}_X X^{(\gamma_0,\omega_1)}_{A(n)\cap A}$, a contradiction. Therefore $A(n)\cap N(\gamma)$ is finite.

(2) $\to$ (1): Let $\{N(\alpha) : \alpha < \omega_1\}$ be a collection satisfying the property (A). Let X be the subspace

$$X = \omega \times \mathrm{Lim} \cup \bigcup_{\alpha<\omega_1} (N(\alpha)\cup\{\omega\})\times\{\alpha+1\}$$

of $(\omega+1)\times\omega_1$.

Claim 2. *X is not normal.*

Proof. Obviously X^{Lim} and $X_{\{\omega\}}$ are disjoint closed sets in X. Let W be an open set containing X^{Lim}. For each $j\in\omega$ and $\alpha \in V_j(X^{\mathrm{Lim}}) = \mathrm{Lim}$, since $\langle j,\alpha\rangle \in X^{\mathrm{Lim}} \subset W$ and W is open, we can find an $f_j(\alpha) < \alpha$ such that $X^{(f_j(\alpha),\alpha]}_{\{j\}} \subset W$. Applying PDL, for each $j\in\omega$, there is an $\alpha_j < \omega_1$ satisfying $X^{(\alpha_j,\omega_1)}_{\{j\}} \subset W$. Setting $\gamma_0 = \sup\{\alpha_j : j\in\omega\}$, we have $X^{(\gamma_0,\omega_1)}_{\omega} \subset W$. For any $\alpha \ge \gamma_0$, since $N(\alpha)\times\{\alpha+1\} \subset X^{(\gamma_0,\omega_1)}_{\omega} \subset W$, we have $\langle\omega,\alpha+1\rangle \in \mathrm{Cl}_X W \cap X_{\{\omega\}}$. Thus X^{Lim} and $X_{\{\omega\}}$ cannot be separated by disjoint open sets.

Claim 3. *X has the weak $C(\omega)$-property.*

Proof. Let $\mathcal{F} = \{F(n) : n\in\omega\}$ be a countable discrete collection of closed sets in X. As in Case 2 of Theorem 2.4, we can find a collection $\{W_0(n) : n\in\omega\}$ of open sets in X such that $F(n)\cap X_{\{\omega\}} \subset W_0(n)$ for each $n\in\omega$ and $\bigcap_{n\in J}\mathrm{Cl}_X W_0(n) = \emptyset$ for each $J\in[\omega]^\omega$.

Next, for every $j\in\omega$, applying Lemma 2.3, fix an $\alpha_j<\omega_1$ with $|\{n\in\omega : F(n)\cap X^{(\alpha_j,\omega_1)}_{\{j\}} \ne \emptyset\}| \le 1$. Let $\gamma_0 = \sup\{\alpha_j : j\in\omega\}$. Then $|\{n\in\omega : F(n)\cap X^{(\gamma_0,\omega_1)}_{\{j\}} \ne \emptyset\}| \le 1$ for every $j\in\omega$. So setting $A(n) = \{j\in\omega : F(n)\cap X^{(\gamma_0,\omega_1)}_{\{j\}} \ne \emptyset\}$ for each $n\in\omega$, we have that $\{A(n) : n\in\omega\}$ is a pairwise disjoint collection of subsets of ω. Let

$$K = \bigcup_{J\in[\omega]^\omega}\bigcap_{n\in J}\{\alpha<\omega_1 : A(n)\cap N(\alpha) \text{ is infinite }\}.$$

Then it follows from the property (A) that K is countable. So pick a $\gamma_1<\omega_1$ with $\gamma_0\le\gamma_1$ and $\alpha<\gamma_1$ for every $\alpha\in K$. Define $W_1(n) = X^{(\gamma_1,\omega_1)}_{A(n)}$ for each $n\in\omega$. Note that, for each $n\in\omega$, $F(n)\cap X^{(\gamma_1,\omega_1)}_{\omega} \subset W_1(n)$, $W_1(n)$ is open in X, $\mathrm{Cl}_X W_1(n)\backslash W_1(n) \subset X^{(\gamma_1,\omega_1)}_{\{\omega\}}$ and the collection $\{W_1(n) : n\in\omega\}$ is pairwise disjoint. Assume that there is a $J\in[\omega]^\omega$ with $\langle j,\beta\rangle \in \bigcap_{n\in J}\mathrm{Cl}_X W(n)$ for some j and β. Since $\{W_1(n) : n\in\omega\}$ is pairwise disjoint and $\mathrm{Cl}_X W_1(n)\backslash W_1(n) \subset X^{(\gamma_1,\omega_1)}_{\{\omega\}}$, we have $j=\omega$ and $\gamma_1<\beta$. It follows from $\langle j,\beta\rangle = \langle\omega,\beta\rangle \in X$ that $\beta = \alpha+1$ for some $\alpha<\omega_1$. Therefore $\gamma_1\le\alpha$, thus $\alpha\notin K$. So it follows from $\alpha \notin \bigcap_{n\in J}\{\alpha<\omega_1 : A(n)\cap N(\alpha) \text{ is infinite }\}$ that $F = A(n)\cap N(\alpha)$ is finite for some $n\in J$. Let $N(\alpha)' = N(\alpha)\backslash F$. Then $(N(\alpha)'\cup\{\omega\})\times\{\alpha+1\}$ is a neighborhood of $\langle\omega,\alpha+1\rangle = \langle j,\beta\rangle$ disjoint from $W_1(n) = X^{(\gamma_1,\omega_1)}_{A(n)}$. Now we have $\langle j,\beta\rangle \notin \mathrm{Cl}_X W_1(n)$, a contradiction. Therefore $\bigcap_{n\in J}\mathrm{Cl}_X W_1(n) = \emptyset$ for each $J\in[\omega]^\omega$.

Since $X^{[0,\gamma_1]}$ is countable so normal, so there is a collection $\{W_2(n) : n\in\omega\}$ of open sets such that $F(n)\cap X^{[0,\gamma_1]} \subset W_2(n)$ for every $n\in\omega$ and $\bigcap_{n\in J}\mathrm{Cl}_X W_2(n) = \emptyset$ for each $J\in[\omega]^\omega$.

Finally set $W(n) = \bigcup_{0 \le i \le 2} W_i(n)$ for each $n \in \omega$. Then $F(n) \subset W(n)$ for each $n \in \omega$ and, by Lemma 2.2, $\bigcap_{n \in J} \mathrm{Cl}_X W(n) = \emptyset$ for each $J \in [\omega]^\omega$. This completes the proof of Claim 3. □

A similar proof of Theorem 4.1 works to show the following theorem, so we leave its proof to the reader.

Theorem 4.2. *The following are equivalent:*

(1) *There is a non-normal subspace of $(\omega+1) \times \omega_1$ which is paranormal.*
(2) *There is a non-normal subspace of $(\omega+1) \times \omega_1$ which has the weak $D(\omega)$-property.*
(3) *The following property (B) holds:*
(B): There is a collection $\{N(\alpha) : \alpha < \omega_1\}$ of infinite subsets of ω such that, for every pairwise disjoint collection $\{A(n) : n \in \omega\}$ of subsets of ω and $K \in [\omega_1]^{\omega_1}$, $A(n) \cap N(\alpha)$ is finite for some $\alpha \in K$ and $n \in \omega$.

Note that the property (A) implies the property (B). Here we ask:

Problem 4.3. *In ZFC, are the properties (A) and (B) equivalent?*

5. Independence of the existence of a paranormal non-normal subspace of $(\omega+1) \times \omega_1$

In this section, we see that the Continuum Hypothesis (CH) implies the property (A) and that the Martin's Axiom at ω_1 $(MA(\omega_1))$ implies the negation of the property (B). For undefined notions in this section, see [Ku].

Theorem 5.1. *If there is a P-point p in $\beta\omega\backslash\omega$ which has a neighborhood base of cardinality ω_1, then the property (A) holds.*

Proof. Recall that a point p in a space X is a *P-point* in X if the intersection of countably many neighborhoods of p is also a neighborhood of p.

Assume that p is a P-point in $\beta\omega\backslash\omega$ with a neighborhood base $\{B(\alpha) : \alpha < \omega_1\}$ at p in $\beta\omega\backslash\omega$. Inductively defining $N(\alpha)$'s, we can find a collection $\{N(\alpha) : \alpha < \omega_1\}$ of infinite subsets of ω such that $\{\mathrm{Cl}_{\beta\omega} N(\alpha)\backslash\omega : \alpha < \omega_1\}$ forms a neighborhood base at p in $\beta\omega\backslash\omega$ and $N(\alpha) \subset^* N(\beta)$, that is, $N(\alpha)\backslash N(\beta)$ is finite, whenever $\beta < \alpha$. To show that the collection $\{N(\alpha) : \alpha < \omega_1\}$ satisfies the property (A), let $\{A(n) : n \in \omega\}$ be a pairwise disjoint collection of subsets of ω and $K \in [\omega_1]^{\omega_1}$. Then $F = \{n \in \omega : p \in \mathrm{Cl}_{\beta\omega} A(n)\}$ has at most one member. So for each $n \in \omega\backslash F$, we can take an $\alpha(n) < \omega_1$ such that $\mathrm{Cl}_{\beta\omega} A(n) \cap (\mathrm{Cl}_{\beta\omega} N(\alpha(n))\backslash\omega) = \emptyset$, that is, $A(n) \cap N(\alpha(n))$ is finite. Let $\alpha = \sup\{\alpha(n) : n \in \omega\backslash F\}$. Then $N(\alpha) \subset^* N(\alpha(n))$ for each $n \in \omega\backslash F$. Let $J \in [\omega]^\omega$. Then $A(n) \cap N(\alpha)$ is finite for $n \in J\backslash F$. □

If CH is assumed, then there is a P-point in $\beta\omega\backslash\omega$ ([Ru]) and every point of $\beta\omega\backslash\omega$ has a neighborhood base of cardinality ω_1, so we get:

Corollary 5.2. *If CH is assumed, then there is a non-normal subspace of $(\omega+1) \times \omega_1$ which has the weak $C(\omega)$-property.*

Moreover by Corollary 3.1:

Corollary 5.3. *If CH is assumed, then there is a subspace of $(\omega+1) \times \omega_1$ which has the weak $D(\omega)$-property but is not countably paracompact.*

This corollary gives a consistently negative answer to Question II.

Lemma 5.4. *If $MA(\omega_1)$ is assumed, then for every collection $\{N(\alpha) : \alpha < \omega_1\}$ of infinite subsets of ω, there are disjoint subsets $A(0)$ and $A(1)$ of ω such that $A(i) \cap N(\alpha)$ is infinite for every $\alpha < \omega_1$ and $i \in 2$.*

Proof. Let

$$P = \{p : p \subset \omega \times 2, |p| < \omega \text{ and } p \text{ is a partial function }\}.$$

Define $p \leq q$ iff $q \subset p$. Then the partially ordered set $\langle P, \leq \rangle$ satisfies the countable chain condition (ccc)(see [Ku, p.54]). For each $n \in \omega$ and $\alpha < \omega_1$, define

$$D_{n\alpha} = \{p \in P : n \in \text{dom}(p) \text{ and there are } m_0, m_1 \in \text{dom}(p) \text{ such that}$$

$$n < m_0, n < m_1, m_0 \in N(\alpha), m_1 \in N(\alpha), p(m_0) = 0 \text{ and } p(m_1) = 1 \}.$$

Then it is straightforward to show that each $D_{n\alpha}$ is dense in P. Applying $MA(\omega_1)$, find a generic filter G in P such that $G \cap D_{n\alpha} \neq \emptyset$ for each $n \in \omega$ and $\alpha < \omega_1$. Observe that $\bigcup G$ is a function from ω to 2. Let $A(i) = (\bigcup G)^{-1}(i)$ for each $i \in 2$. Assume that $A(i) \cap N(\alpha)$ is finite for some $\alpha < \omega_1$ and $i \in 2$. Take an $n \in \omega$ with $A(i) \cap N(\alpha) \subset n$ and a $p \in G$ with $p \in D_{n\alpha}$. Then there is $m_i \in \text{dom}(p)$ such that $n < m_i, m_i \in N(\alpha)$ and $p(m_i) = i$. It follows from $(\bigcup G)(m_i) = p(m_i) = i$ that $m_i \in A(i)$. But by $m_i \in N(\alpha)$, we have $m_i \in A(i) \cap N(\alpha) \subset n$. So $m_i < n < m_i$, a contradiction. Hence $A(i) \cap N(\alpha)$ is infinite for every $\alpha < \omega_1$ and $i \in 2$. □

Theorem 5.5. *If $MA(\omega_1)$ is assumed, then for every collection $\{N(\alpha) : \alpha < \omega_1\}$ of infinite subsets of ω, there is a pairwise disjoint collection $\{A(n) : n \in \omega\}$ of susets of ω such that $A(n) \cap N(\alpha)$ is infinite for every $\alpha < \omega_1$ and $n \in \omega$.*

Proof. Let $\{N(\alpha) : \alpha < \omega_1\}$ be a collection of infinite subsets of ω. We will define, by induction on $n \in \omega$, disjoint subsets $A_n(0)$ and $A_n(1)$ of ω such that $A_n(i) \cap N(\alpha)$ is infinite for each $\alpha < \omega_1$ and $i \in 2$. The existence of $A_0(0)$ and $A_0(1)$ follows from Lemma 5.4. Assume $A_n(0)$ and $A_n(1)$ have been already defined. By the inductive assumption, $A_n(1) \cap N(\alpha)$ is infinite for every $\alpha < \omega_1$, so applying Lemma 5.4 to the collection $\{N(\alpha) \cap A_n(1) : \alpha < \omega_1\}$, we get disjoint subsets $A_{n+1}(0)$ and $A_{n+1}(1)$ of $A_n(1)$ such that $A_{n+1}(i) \cap N(\alpha)$ is infinite for every $\alpha < \omega_1$ and $i \in 2$. This completes the inductive construction. Finally put $A(n) = A_n(0)$ for each $n \in \omega$ Then $\{A(n) : n \in \omega\}$ is as desired. □

Corollary 5.6. *If $MA(\omega_1)$ is assumed, then the property (B) does not hold.*

It follows Theorem 4.2 and Corollary 3.3 that:

Corollary 5.7. *If $MA(\omega_1)$ is assumed, then the paranormal subspaces of $(\omega+1) \times \omega_1$ are normal.*

Corollary 5.8. *If $MA(\omega_1)$ is assumed, then the subspaces of $(\omega + 1) \times \omega_1$ which have the weak $D(\omega)$-property are countably paracompact.*

Finally we ask:

Problem 5.9. *In ZFC, does there exist a subspace of ω_1^2 which has the weak $D(\omega)$-property but is not countably paracompact?*

Acknowledgement. The authors thank professor V. Gutev for his careful reading of our manuscript and many suggestions. Moreover they also thank the referee for many helpful comments and suggestions.

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