TOPOLOGICAL PROPERTIES OF PRODUCTS OF ORDINALS

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Abstract. We study separation and covering properties of special subspaces of products of ordinals. In particular, it is proven that certain subspaces of Σ-products of ordinals are quasi-perfect preimages of Σ-products of copies of ω. We obtain as corollaries that products of ordinals are κ-normal and strongly zero-dimensional. Also, σ-products and Σ-products of ordinals are shown to be countably paracompact, κ-normal and strongly zero-dimensional. Normality in Σ-products and σ-products of ordinals is also characterized. It is also shown that any continuous real-valued function on a σ-product of ordinals has countable range.

0. Introduction

Products of ordinals provide a fairly comprehensive store of basic counterexamples delineating normality, countable paracompactness and closely related propertes. For example, it is well known that;

- $\omega_1 \times (\omega_1 + 1)$ is countably paracompact but not normal,
- if A and B are disjoint stationary sets in ω_1, then $A \times B$ is neither normal nor countably paracompact [KOT],
- ${}^{\omega}\omega_1$ is normal, but ${}^{\omega_1}\omega_1$ is not normal [Co],
- ${}^{\kappa}\omega_1$ is countably (para)compact for every κ,
- ${}^{\omega}\omega$ is metrizable, so normal and countably paracompact.

On the other hand,

- ${}^{\omega_1}\omega$ is neither normal nor countably paracompact [Na, Do, Co].

Also, Fleissner, Kemoto and Terasawa [FKT] proved that finite products of subspaces of ordinals are strongly zero-dimensional: every disjoint pair of zero-sets is separated by disjoint clopen sets.

In a different line Kalantan and Szeptycki [KS] proved that arbitrary products of ordinals are κ-normal: every disjoint pair of regular closed sets is separated by disjoint open sets. In contrast, if A, B and C are stationary subsets of ω_1 such that any two have stationary intersection but the intersection of all three is not stationary, then the product $A \times B \times C$ is not κ-normal [HK].

In Section 1, we prove a technical result characterizing Σ-products of certain subspaces of ordinals as quasi-perfect preimages of Σ-products of copies of the discrete space ω. Using this, we obtain as a corollary that products of ordinals (and

AMS subject Classification: 54D15, 54B10, 54C10, 54F45

Key words and phrases: infinite product, Σ-product, σ- product, elementary submodel, ordinal space, strongly zero-dimensional, countably paracompact, ω_1-compact, normal, κ-normal, quasi-perfect map

Acknowledgement. The second author acknowledges support from NSERC grant 238944

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certain dense subsets) are both κ-normal and strongly zero-dimensional. Further corollaries are obtained: in Section 2, ω_1-compactness, countable paracompactness and normality are characterized for Σ-products of ordinals.

In Section 3 we study σ-products of ordinals. We prove that many properties of a σ-product of ordinals holds if and only if it holds for every finite subproduct. Hence, such σ-products are countably paracompact, κ-normal, and strongly zero-dimensional. Also it is proven that a continuous real-valued function on a σ-product of ordinals has countable range. Elementary submodel techniques play a central role in the proofs of Section 3.

Notation. Throughout this paper, spaces are regular topological spaces. Let X_i be a space for each $i \in \kappa$ and κ a cardinal. $\prod_{i\in\kappa} X_i$ denotes the product space with the usual Tychonoff product topology. When $X_i = X$ for each $i \in \kappa$, we denote $\prod_{i\in\kappa} X_i$ by ${}^{\kappa}X$. For $x \in \prod_{i\in\kappa} X_i$, $x(i)$ denotes the i-th coordinate of x.

A Σ-product of the family of spaces $\{X_i : i \in \kappa\}$ with a base point $s \in \prod_{i\in\kappa} X_i$ is the subspace

$$\Sigma(\prod_{i\in\kappa} X_i, s) = \{x \in \prod_{i\in\kappa} X_i : |\{i \in \kappa : x(i) \neq s(i)\}| \leq \omega\}.$$

A σ-product of X_i's ($i \in \kappa$) with a base point $s \in \prod_{i\in\kappa} X_i$ means the subspace

$$\sigma(\prod_{i\in\kappa} X_i, s) = \{x \in \prod_{i\in\kappa} X_i : |\{i \in \kappa : x(i) \neq s(i)\}| < \omega\}.$$

For x in either the Σ-product or σ-product with a base point s, we let $\mathrm{supt}(x)$ denotes the set $\{i \in \kappa : x(i) \neq s(i)\}$.

A countable (finite) subproduct of $\prod_{i\in\kappa} X_i$ means a product $\prod_{i\in B} X_i$ for some countable (finite) $B \subset \kappa$.

For a subset $B \subset \kappa$, $\pi_B : \Sigma(\prod_{i\in\kappa} X_i, s) \to \Sigma(\prod_{i\in B} X_i, s \restriction B)$ denotes the canonical projection map.

For a basic open set U of a product space, $\mathrm{supt}(U)$ denotes the set $\{i \in \kappa : \pi_{\{i\}}(U) \neq X_i\}$. We will use a similar notation for basic open subsets of Σ and σ-products.

A quasi-perfect map is a closed, continuous and onto map whose point inverses are countably compact. The properties of quasi-perfect maps that we will use is that they inversely preserve expandability and ω_1-compactness. In addition countable paracompactness and other properties are inversely preserved by quasi-perfect mappings, but, in general, normality is not.

1. Strong zero-dimensionality and κ-normality

First we prepare some Lemmas.

Lemma 1.1. *For each $i \in \kappa$, let $Y_i = \bigoplus_{n\in\omega} Y_i(n)$ where each $Y_i(n)$ is sequentially compact. Define $g_i : Y_i \to \omega$ by $g_i(\beta) = n$ iff $\beta \in Y_i(n)$, moreover define $g = \prod_{i\in\kappa} g_i : \prod_{i\in\kappa} Y_i \to {}^{\kappa}\omega$ by $g(y)(i) = g_i(y(i))$. For each $s \in \prod_{i\in\kappa} Y_i$, the restriction*

$g' = g \restriction \Sigma(\prod_{i\in\kappa} Y_i, s) : \Sigma(\prod_{i\in\kappa} Y_i, s) \to \Sigma({}^{\kappa}\omega, g(s))$ *is a closed continuous onto map.*

Proof. Recall that a space is sequentially compact if every countably infinite subset has a convergent subsequence.

Obviuosly, g' is continuous and onto. Let $\Sigma_0 = \Sigma(\prod_{i\in\kappa} Y_i, s)$, $\Sigma_1 = \Sigma({}^{\kappa}\omega, g(s))$, E colsed in Σ_0 and $f \in \mathrm{Cl}_{\Sigma_1} g'(E) = \mathrm{Cl}_{\Sigma_1} g(E)$. Since by [KM, Corollary], Σ_1 is Fréchet, there is a subset $\{e_j : j \in \omega\} \subset E$ such that $\{g(e_j) : j \in \omega\}$ converges to f (we write $f = \lim_{j\in\omega} g(e_j)$). Then one can find an infinite countable subset $A \subset \kappa$ such that $\mathrm{supt}(f) \subset A$ and $\mathrm{supt}(e_j) \subset A$ for each $j \in \omega$. Well-order A as $A = \{i(n) : n \in \omega\}$ and set $U_n = \{u \in \Sigma_1 : f \restriction A_{n+1} = u \restriction A_{n+1}\}$, where $A_n = \{i(m) : m < n\}$. Then U_n is a neighborhood of f in Σ_1. Let $F_{-1} = \omega$. By induction on ω, we will define a decreasing sequence $\{F_n : n \in \omega\}$ of infinite subsets of ω and $y(i(n)) \in Y_{i(n)}(f(i(n)))$, $n \in \omega$, such that for each $n \in \omega$, $y(i(n)) = \lim_{j\in F_n} e_j(i(n))$. Assume that $F_0, ..., F_{n-1}$ and $y(i(0)), ..., y(i(n-1))$ have been already defined. Since U_n is a neighborhood of $f = \lim_{j\in F_{n-1}} g(e_j)$, $F'_n = \{j \in F_{n-1} : g(e_j) \in U_n\}$ is infinite. For $j \in F'_n$, by $i(n) \in A_{n+1}$ and $g(e_j) \in U_n$, we have $g(e_j)(i(n)) = f(i(n))$, thus $e_j(i(n)) \in Y_{i(n)}(f(i(n)))$, i. e., $\{e_j(i(n)) : j \in F'_n\} \subset Y_{i(n)}(f(i(n)))$. It follows from the sequential compactness of $Y_{i(n)}(f(i(n)))$ that there are an infinite subset F_n of F'_n and a point $y(i(n)) \in Y_{i(n)}(f(i(n)))$ such that $y(i(n)) = \lim_{j\in F_n} e_j(i(n))$.

Define $x \in \prod_{i\in\kappa} Y_i$ by

$$x(i) = \begin{cases} y(i(n)), & \text{if } i = i(n) \text{ for some } n \in \omega, \\ s(i), & \text{otherwise.} \end{cases}$$

Then $x \in \Sigma_0$. Moreover since $x(i(n)) = y(i(n)) \in Y_{i(n)}(f(i(n)))$ and $\mathrm{supt}(f) \subset A$, we have $g(x) = f$.

Fact. $x \in \mathrm{Cl}_{\Sigma_0}\{e_j : j \in \omega\}$.

Proof. Let U be a basic open neighborhood of x in Σ_0. It suffices to prove that $e_j \in U$ for infinitely many $j \in \omega$. Since $\mathrm{supt}(e_j) \subset A$ and $x(i) = s(i)$ for each $i \in \kappa\setminus A$, we may assume $\mathrm{supt}(U) = \{i(n) : n \le m\} \subset A$ for some $m \in \omega$. Since $\lim_{j\in F_n} e_j(i(n)) = y(i(n)) = x(i(n)) \in \pi_{\{i(n)\}}(U)$ for each $n \le m$ and $\{F_n : n \in \omega\}$ is decreasing, we have $e_j \in U$ for all but finitely many $j \in F_m$. This completes the proof of Fact. ⊣

Finally since $x \in \mathrm{Cl}_{\Sigma_0}\{e_j : j \in \omega\} \subset \mathrm{Cl}_{\Sigma_0} E = E$, $f = g(x) \in g(E)$, thus g is closed. ⊣

Lemma 1.2. *The Σ-product $\Sigma = \Sigma(\prod_{i\in\kappa} Y_i, s)$ of ω-bounded spaces Y_i's with a base point s is ω-bounded.*

Proof. Recall that a space is ω-bounded if each countable subset has a compact closure. Let H be a countable subset of Σ. Then $\prod_{i\in\kappa} \mathrm{Cl}_{Y_i}\{z(i) : z \in H\}$ is a compact subset of Σ which includes H. ⊣

Recall that a space has countable tightness if for each point x and a subset A with $x \in \mathrm{Cl}\, A$, there is a countable subset $A' \subset A$ such that $x \in \mathrm{Cl}\, A'$. It is well-known that a Σ-product space Σ has countable tightness if every finite subproduct of Σ has countable tightness [KM, Proposition 1].

Lemma 1.3. *Let Y be ω-bounded, $p_X : X \times Y \to X$ the natural projection, $g : X \to Z$ a closed continuous onto map, and Z have countable tightness. Then the composition $g \circ p_X : X \times Y \to Z$ is a closed continuous onto map.*

Proof. Let E be a closed subset in $X \times Y$ and $z \in \mathrm{Cl}_Z\, g \circ p_X(E)$. Since Z has countable tightness, there is a countable subset $\{e_j : j \in \omega\}$ of E such that $z \in \mathrm{Cl}_Z\{g \circ p_X(e_j) : j \in \omega\}$. Since g is closed, we can find a point $x \in g^{-1}(z) \cap \mathrm{Cl}_X\{p_X(e_j) : j \in \omega\}$. For each neighborhood U of x, set $H(U) = \{j \in \omega : p_X(e_j) \in U\}$. Since Y is ω-bounded, we can find a point $y \in \bigcap\{\mathrm{Cl}_Y\{p_Y(e_j) : j \in H(U)\} : U \text{ is a neighborhood of } x\}$, where p_Y denotes the natural projection to Y. Let $U \times V$ be a basic open neighborhood of $\langle x, y\rangle$ in $X \times Y$. It follows from $y \in \mathrm{Cl}_Y\{p_Y(e_j) : j \in H(U)\}$ that there is $j \in H(U)$ such that $p_Y(e_j) \in V$. Then $e_j = \langle p_X(e_j), p_Y(e_j)\rangle \in E \cap U \times V$. Thus $\langle x, y\rangle \in \mathrm{Cl}_{X\times Y} E = E$ and therefore $z = g(x) = g \circ p_X(\langle x, y\rangle) \in g \circ p_X(E)$. This shows that $g \circ p_X$ is closed. ⊣

Now we prove one of main results of this paper:

Theorem 1.4. *Let α_i be an ordinal and $Y_i = \{\beta < \alpha_i : \mathrm{cf}\,\beta \le \omega\}$ for each $i \in \kappa$. Then the Σ-product $\Sigma = \Sigma(\prod_{i\in\kappa} Y_i, s)$ with a base point s is a quasi-perfect preimage of a Σ-product of copies of ω.*

Proof. Set $A(0) = \{i \in \kappa : \mathrm{cf}\,\alpha_i = \omega\}$ and $A(1) = \kappa\backslash A(0)$. Moreover let $\Sigma_l = \Sigma(\prod_{i\in A(l)} Y_i, s \restriction A(l))$ for each $l \in 2 = \{0, 1\}$, then $\Sigma = \Sigma_0 \times \Sigma_1$. By Lemma 1.2, Σ_1 is ω-bounded.

For each $i \in A(0)$, fix a strictly increasing sequence $\langle \alpha_i(n) : n \in \omega\rangle$ cofinal in α_i with $s(i) < \alpha_i(0)$ such that $Y_i(n) = Y_i \cap (\alpha_i(n-1), \alpha_i(n)]$ is non-empty for each $n \in \omega$, where $\alpha_i(-1) = -1$. Then $Y_i = \bigoplus_{n\in\omega} Y_i(n)$ and each $Y_i(n)$ is sequentially compact and ω-bounded. Define for each $i \in A(0)$, $g_i : Y_i \to \omega$ by $g_i(\beta) = n$ iff $\beta \in Y_i(n)$. By Lemma 1.1, $g = (\prod_{i\in A(0)} g_i) \restriction \Sigma_0 : \Sigma_0 \to \Sigma_2 = \Sigma({}^{A(0)}\omega, 0)$ is a closed continuous onto map, where we let 0 denote the constant map having value 0. Since each $Y_i(n)$ is ω-bounded, each point inverse of g is ω-bounded. Since Σ_2 has countable tightness, by Lemma 1.3, the composition $g \circ p : \Sigma = \Sigma_0 \times \Sigma_1 \to \Sigma_2$ is a closed continuous onto, wher $p : \Sigma_0 \times \Sigma_1 \to \Sigma_0$ denotes the natural projection. Moreover for each $f \in \Sigma_2$, $(g \circ p)^{-1}(f) = g^{-1}(f) \times \Sigma_1$ is a product of two ω-bounded sets, so it is ω-bounded. Therefore $g \circ p$ is quasi-perfect. This shows Σ is a quasi-perfect preimage of $\Sigma({}^{A(0)}\omega, 0)$. ⊣

Recall that a space is ω_1-compact if there is no uncountable closed discrete subspace. The following two lemmas are due to [KM, Basic Lemma] and [En1, Theorem 1], see also [NU, Proposition 2.1] for the latter one.

Lemma 1.5. [KM] *Let Σ be a Σ-product of X_i's ($i \in \kappa$) with a base point s, moreover let F_0 and F_1 be disjoint closed sets in Σ. If Σ is ω_1-compact and has countable tightness, then there is a countable subset $B \subset \kappa$ such that $\mathrm{Cl}_{X(B)}\,\pi_B(F_0) \cap \mathrm{Cl}_{X(B)}\,\pi_B(F_1) = \emptyset$, where $X(B) = \prod_{i\in B} X_i$ and π_B is the canonical projection.*

Lemma 1.6. [En1] *Let Σ be a Σ-product space of X_i's ($i \in \kappa$) with a base point s, Z a T_2-space with G_δ-diagonal and $f : \Sigma \to Z$ a continuous map. If every finite subproduct of $\prod_{i\in\kappa} X_i$ is ω_1-compact, then there are a countable subset $B \subset \kappa$ and a continuous map $f' : X(B) = \prod_{i\in B} X_i \to Z$ such that $f = f' \circ \pi_B$.*

Lemma 1.7. *Every Σ-product of copies of ω is ω_1-compact.*

Proof. Let Σ be a Σ-product of copies of ω. Note that Σ-products of metric spaces are collectionwise normal [Le]. If Σ were not ω_1-compact, by the collectionwise normality of Σ, there would be a disjoint collection of uncountably many non-empty open sets in Σ. But Σ is a dense subspace of a product of separable spaces, hence ccc. This is a contradiction. $\dashv$

The following result is due to [KS, Lemma 2]. Note that the proof does not use elementary submodel techniques.

Lemma 1.8. [KS] *Let α_i be an ordinal and $Y_i = \{\beta < \alpha_i : \mathrm{cf}\,\beta \leq \omega\}$ for each $i \in \omega$. Then $Y = \prod_{i\in\omega} Y_i$ is a normal C*-embedded subspace of $\prod_{i\in\omega} \alpha_i$.*

Nagami [Na] proved that if every finite subproduct of $X = \prod_{i\in\omega} X_i$ has covering dimension $\leq n$ and X is normal, then X has also covering dimension $\leq n$. With the result of [FKT], these results yield:

Lemma 1.9. [KS, Na, FKT] *Let α_i be an ordinal and $Y_i = \{\beta < \alpha_i : \mathrm{cf}\,\beta \leq \omega\}$ for each $i \in \omega$. Then $Y = \prod_{i\in\omega} Y_i$ is a normal strongly zero-dimensional C*-embedded subspace of $\prod_{i\in\omega} \alpha_i$.*

Applying this lemma, we show:

Theorem 1.10. *Let α_i be an ordinal and $Y_i = \{\beta < \alpha_i : \mathrm{cf}\,\beta \leq \omega\}$ for each $i \in \kappa$. Then the Σ-product $\Sigma = \Sigma(\prod_{i\in\kappa} Y_i, s)$ with a base point s is a normal strongly zero-dimensional C*-embedded subspace of $\prod_{i\in\kappa} \alpha_i$.*

Proof. Since ω_1-compactness is inversely preserved by a quasi-perfect map, by Theorem 1.4 and Lemma 1.7, Σ is also ω_1-compact. Moreover since each Y_i is first countable, Σ has countable tightness, in fact it is Fréchet [KM]. It follows from Lemmas 1.9 and 1.5 that Σ is normal and strongly zero-dimensional. To show that Σ is C*-embedded in $\prod_{i\in\kappa} \alpha_i$, let $f : \Sigma \to \mathbb{I}$ be a coutinuous map, where $\mathbb{I}$ denotes the closed unit interval $[0, 1]$. Applying Lemma 1.6 to Σ and $\prod_{i\in\kappa} Y_i$, we can find a countable subset $B \subset \kappa$ and a continuous map $f' : Y(B) = \prod_{i\in B} Y_i \to \mathbb{I}$ such that $f = f' \circ \pi_B$, where $\pi_B : \Sigma \to Y(B)$ is the projection. Apply Lemma 1.8 to $Y(B)$, then we have a continuous map $h' : \prod_{i\in B} \alpha_i \to \mathbb{I}$ extending f'. Let $p_B : \prod_{i\in\kappa} \alpha_i \to \prod_{i\in B} \alpha_i$ be the projection map. Then $h = h' \circ p_B : \prod_{i\in\kappa} \alpha_i \to \mathbb{I}$ is a continuous extension of f to $\prod_{i\in\kappa} \alpha_i$. $\dashv$

The next corollary follows from the fact that if a space has a dense C*-embedded κ-normal (strongly zero-dimensional) subspace, then it is also κ-normal [KS, Theorem 1.3] (strongly zero-dimensional [En2, 7.1.17], respectively).

Corollary 1.11. *Let α_i be an ordinal, $Y_i = \{\beta < \alpha_i : \mathrm{cf}\,\beta \leq \omega\}$ for each $i \in \kappa$ and $\Sigma = \Sigma(\prod_{i\in\kappa} Y_i, s)$, where $s \in \prod_{i\in\kappa} Y_i$. If $\Sigma \subset Z \subset \prod_{i\in\kappa} \alpha_i$, then Z is κ-normal and strongly zero-dimensional. In particular, both $\prod_{i\in\kappa} \alpha_i$ and $\Sigma(\prod_{i\in\kappa} \alpha_i, s)$ are κ-normal and strongly zero-dimensional.*

Remark 1.12. Note that ω_1-compactness of $\Sigma(\prod_{i\in\kappa} Y_i, s)$ played an important role in the above proofs.

First observe that in general, ω_1-compactness is not productive (the Sorgenfrey line is hereditarily separable so ω_1-compact, but, as is well-known, the square is not ω_1-compact). On the other hand, it is known that if every finite subproduct of a product space is pseudo-ω_1-compact, then its full product and Σ-product are also pseudo-ω_1-compact [NU]. Recall that a space is pseudo-ω_1-compact if every locally finite collection of non-empty open sets is countable. This result cannot be extended to ω_1-compactness: Przymusinski [Pr] constructed a Lindelöf space X such that ${}^nX$ is Lindelöf for every $n \in \omega$, but ${}^\omega X$ is not normal. In fact, in his proof of non-normality of ${}^\omega X$, an uncountable closed discrete subspace of ${}^\omega X$ is constructed. Therefore each finite product of copies of X is ω_1-compact but the countable product ${}^\omega X$ is not ω_1-compact. Another examples is due to Mycielski [My]: he proved that ${}^{\omega_1}\omega$ is not ω_1-compact. In this case, countable subproducts or Σ-products of ${}^{\omega_1}\omega$ are ω_1-compact by Lemma 1.7.

2. Countable paracompactness and ω_1-compactness

Henceforth, a Σ-product (σ-product) of ordinals means a Σ-product (σ-product) $\Sigma(\prod_{i\in\kappa}\alpha_i, s)$ $(\sigma(\prod_{i\in\kappa}\alpha_i, s))$ of ordinals α_i's with some base point s. With minor changes to the proof of Theorem 1.4 (for example, set $Y_i(n) = (\alpha_i(n-1), \alpha_i(n)]$ in this case), one can prove:

Theorem 2.1. *Every Σ-product of ordinals is a quasi-perfect preimage of a Σ-product of copies of ω.*

Corollary 2.2. *Every Σ-product of ordinals is expandable (hence countably paracompact) and ω_1-compact.*

Proof. Recall a space is expandable if for each locally finite collecion $\mathcal{F}$ of closed sets, there exists a locally finite collection $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ of open sets with $F \subset U(F)$. Note that expandable spaces are countably paracompact and preserved by quasi-perfect preimages. Let Σ be a Σ-product of copies of ω. Since Σ-products of metric spaces are countably paracompact (in fact, shrinking) and collectionwise normal [Le], it is normal and expandable [Al]. By Theorem 2.1, Σ-product of ordinals are expandable. In a similar way, ω_1-compactness follows from Lemma 1.7. $\dashv$

We showed that products of ordinals are κ-normal and strongly zero-dimensional in Corollary 1.11. Conover characterized normality of products of ordinals in [Co, Theorem 3]. But the situation of countable paracompactness of such product spaces is somewhat different from these properties.

Corollary 2.3. *Let α_i be an ordinal for each $i \in \kappa$. Then $X = \prod_{i\in\kappa}\alpha_i$ is countably paracompact (equivalently, expandable) iff $A(0) = \{i \in \kappa : \operatorname{cf}\alpha_i = \omega\}$ is countable.*

Proof. Assume that $A(0)$ is not countable, then X contains a homeomorphic closed copy of ${}^{\omega_1}\omega$. So X is not countably paracompact.

Assume that $A(0)$ is countable. Then a similar proof of Theorem 1.4 or 2.1 works to show that $F : \prod_{i\in A(0)}\alpha_i \times \prod_{i\in A(1)}\alpha_i \to {}^{A(0)}\omega$ is quasi-perfect. Thus X is expandable. $\dashv$

Thus for example, ${}^\omega(\aleph_\omega) \times {}^{\omega_1}((\omega_1+1)\times\omega_1)$ is countably paracompact, but ${}^{\omega_1}(\aleph_\omega) \times {}^\omega((\omega_1+1)\times\omega_1)$ is not countably paracompact. In particular:

Corollary 2.4. [Ao] *Every countable product of ordinals is countably paracompact.*

Nagami [Na] proved that if a countable product space $X = \prod_{i\in\omega} X_i$ is countably paracompact and every finite subproduct of X is normal, then X is normal. So we have:

Corollary 2.5. *A countable product of ordinals is normal if and only if every finite subproduct of it is normal.*

It is known that $\Sigma({}^{\kappa}\omega_1, s)$, $s \in {}^{\kappa}\omega$, is normal for each cardinal κ [Ko2, Theorem 3 or 4]. On the other hand as is well known, since $\Sigma({}^{\omega_1}(\omega_1+1), s)$, $s \in {}^{\omega_1}(\omega_1+1)$, contains a closed copy of ω_1 and is homeomorphic to $(\omega_1+1) \times \Sigma({}^{\omega_1}(\omega_1+1), s)$ it contains a closed copy of $(\omega_1+1)\times\omega_1$, so it is not normal. The following result clarifies this situation.

Theorem 2.6. *Let α_i be an ordinal with $2 \le \alpha_i$ for each $i \in \kappa$ and $s \in \prod_{i\in\kappa}\alpha_i$, where κ is uncountable cardinal. Then $\Sigma = \Sigma(\prod_{i\in\kappa}\alpha_i, s)$ is normal iff $\alpha_i \le \omega_1$ for each $i \in \kappa$.*

Proof. Assume that $\omega_1 < \alpha_i$ for some $i \in \kappa$. Then as in the above argument, Σ contains a closed copy of $(\omega_1+1)\times\omega_1$.

To show the other direction, assume that $\alpha_i \le \omega_1$ for each $i \in \kappa$. By applying Theorem 3 of [Co] for finite products, we have that every finite subproduct of $\prod_{i\in\kappa}\alpha_i$ is normal. So by Corollary 2.5, every countable subproduct of $\prod_{i\in\kappa}\alpha_i$ is normal. Since each α_i is first countable, Σ has countable tightness. Moreover by Lemma 1.7 and Theorem 2.1, Σ is also ω_1-compact. Therefore by Lemma 1.5, Σ is normal. $\dashv$

Observe that countable paracompactness and ω_1-compactness are in general different topological properties (consider, for example, an uncountable discrete space). However, for some important classes of spaces they can coincide. Indeed using the fact that ${}^{\omega_1}\omega$ is not ω_1-compact [My], the following can be proved in a similar way to Corollary 2.3.

Corollary 2.7. *Let α_i be an ordinal for each $i \in \kappa$. Then $X = \prod_{i\in\kappa}\alpha_i$ is ω_1-compact iff $A(0) = \{i \in \kappa : \operatorname{cf}\alpha_i = \omega\}$ is countable. Thus countable paracompactness and ω_1-compactness are equivalent for products of ordinals.*

3. σ-PRODUCTS OF ORDINALS

In this section we study σ-products of ordinals. In particular we show that such spaces are countably paracompact and strongly zero-dimensional. Also, we prove that a σ-product of ordinals is normal if and only if every finite subproduct is normal. One should note that this characterization does not hold for arbitary spaces: If X is a Dowker space, then the product of X with $\sigma({}^{\omega}2, 0)$ is homeomorphic to a non-normal σ-product all of whose finite subproducts are normal. We should emphasize that it is open whether a general σ-product is countably paracompact if and only if every finite subproduct is countably paracompact.

First note that if every finite subproduct of a product space is ω_1-compact, then its σ-product is also ω_1-compact [NU]. This can be shown by assuming the existence of an uncountable closed discrete subset $\{x_\alpha : \alpha \in \omega_1\}$ and then by applying the

$\triangle$-system lemma to $\{\mathrm{supt}(x_\alpha) : \alpha \in \omega_1\}$. Therefore every σ-product of ordinals is also ω_1-compact.

The other results require more work. In particular, the approach of Sections 1 and 2 are not readily applicable to σ-products. Essential to proving that the map defined in Section 1 is quasi-perfect is that a Σ-product of a sequence ordinals of uncountable cofinality is countably compact. Clearly, this fails for σ-products. In this section, elementary submodel techniques play a crucial role in the proofs. For background on elementary submodels see [D].

Let X be the σ-product $\sigma(\prod_{i\in\kappa}\alpha_i, 0)$ of the sequence $\langle\alpha_i : i \in \kappa\rangle$ of ordinals where we let 0 denote the constant function with value 0. We will use an elementary submodel to define a metrizable space Y and a continuous surjection $p : X \to Y$. Although the map will not in general be closed, it will have additional properties allowing us to analyze basic properties of X.

Let M be a countable elementary submodel of H_θ where θ is large enough, so that

$$\{\alpha_i : i \in \kappa\} \in M.$$

For each ordinal $\beta < \sup(M \cap \mathrm{ON})$, where ON denotes the class of ordinal numbers, let $\tilde{p}(\beta) = \min(M \setminus \beta)$. Then for each $\beta < \sup(M \cap \mathrm{ON})$, "$\beta \leq \tilde{p}(\beta)$" and "$\beta \in M$ iff $\tilde{p}(\beta) = \beta$" hold.

For each $i \in M \cap \kappa$, define

$$Y_i = \{\tilde{p}(\beta) : \beta \in \alpha_i\}.$$

Then it follows that for each $i \in \kappa \cap M$,

(a) $Y_i = \alpha_i \cap M$ if $\mathrm{cf}(\alpha_i) \leq \omega$

(b) $Y_i = (\alpha_i + 1) \cap M$ if $\mathrm{cf}(\alpha_i) > \omega$

We give Y_i the order topology, equivalently, the elementary submodel topology determined by M. Therefore, each Y_i is homeomorphic to a countable ordinal. Let

$$Y = \sigma(\prod_{i\in\kappa\cap M} Y_i, 0)$$

Since every finite subset of M is an element of M we have $Y \subseteq M$. In the rest of this section the expression $\sup\{\gamma + 1 : \gamma \in M \cap \beta\}$ plays a crucial role. If $\beta \in M \cap ON$ and if we let $q(\beta) = \sup\{\gamma + 1 : \gamma \in M \cap \beta\}$ then $q(\beta) = \min\{\eta : \tilde{p}(\eta) = \beta\}$. Moreover, the preimage $\tilde{p}^{-1}(\beta)$ is the closed interval $[q(\beta), \beta]$.

Define a surjection $p : X \to Y$ by:

$$p(x)(i) = \tilde{p}(x(i)) \text{ for each } i \in \kappa \cap M.$$

Note that $x(i) \in M$ iff $p(x)(i) = x(i)$ for each $i \in \kappa \cap M$, and that $\mathrm{supt}(x) \cap M = \mathrm{supt}(p(x))$. It is not difficult to verify that p is continuous. Indeed, if U is the subbasic open set $\pi^{-1}_{\{i\}}((\beta_i, \gamma_i]\cap M)$ in Y with $\beta_i, \gamma_i \in M$, then $p^{-1}(U)$ is the subbasic open set $\pi^{-1}_{\{i\}}((\beta_i, \gamma_i])$ in X. Although, in general, p is not a closed mapping, it is the case that $p(D)$ is closed for any closed $D \subset X$ such that $D \in M$:

Lemma 3.1. *For each $D \in M$ if $D \subset X$ is closed in X then $p(D)$ is closed in Y.*

Proof. Suppose that $y \notin p(D)$. Let $A = \mathrm{supt}(y)$ and let

$$A_0 = \{i \in A \cap M : y(i) \neq \sup\{\gamma + 1 : \gamma \in M \cap y(i)\}\} \subset A,$$

Let $B_0 = \{i \in A_0 : y(i) = \alpha_i\}$. For each $i \in A_0$, let $\xi_i = \sup\{\gamma+1 : \gamma \in M \cap y(i)\}$ ($= \sup(M \cap y(i))$). For each $B \subset A_0$ such that $B_0 \subset B$ define $z_B \in X$ as follows

$$z_B(i) = \begin{cases} 0, & \text{if } i \notin M, \\ \xi_i, & \text{if } i \in B, \\ y(i), & \text{otherwise.} \end{cases}$$

Note that $p(z_B) = y$ for every such B. Thus if we let

$$C_B = \{x \in X : x(i) = y(i) \text{ for all } i \in A_0 \setminus B\}$$

where $B_0 \subset B \subset A_0$, then

$$z_B \notin \mathrm{cl}_X(C_B \cap D).$$

Otherwise, $z_B \in D$ and $y = p(z_B) \in p(D)$. So, for each such B, we may fix a basic clopen neighborhood U_B of z_B such that

$$U_B \cap C_B \cap D = \emptyset$$

U_B is determined by a finite family of clopen intervals restricted to M. Thus, we may fix

(a) a finite set F_B disjoint from $\mathrm{supt}(y)$
(b) a set of ordinals $\{\gamma_i^B : i \in A \setminus A_0\} \subset M$ such that $\gamma_i^B < y(i)$ for all $i \in A \setminus A_0$, and
(c) a set of ordinals $\{\gamma_i^B : i \in B\} \subset M$ such that $\gamma_i^B < y(i)$ for all $i \in B$,

such that $U_B =$

$$\{x \in X : \forall i \in F_B(x(i) = 0) \wedge \forall i \in A \setminus A_0(x(i) \in (\gamma_i^B, y(i)]) \wedge \forall i \in B(x(i) \in (\gamma_i^B, \xi_i])\}.$$

Note that the ordinals γ_i^B's in (b) and (c) may be chosen to be in M: in the case $i \in A \setminus A_0$, $y(i) = \sup\{\gamma + 1 : \gamma \in M \cap y(i)\}$, and in the case $i \in B$, $z_B(i) = \sup\{\gamma + 1 : \gamma \in M \cap y(i)\}$.

Set $\hat{F}_B = F_B \cap M$ and $\hat{U}_B =$

$$\{x \in X : \forall i \in \hat{F}_B(x(i) = 0) \wedge \forall i \in A \setminus A_0(x(i) \in (\gamma_i^B, y(i)]) \wedge$$

$$\forall i \in B_0(x(i) \in (\gamma_i^B, \alpha_i)) \wedge \forall i \in B \setminus B_0(x(i) \in (\gamma_i^B, y(i)])\}.$$

Obviously $C_B, \hat{F}_B \in M$. Since all parameteres in $\hat{U}_B$ are in M, we also have $\hat{U}_B \in M$. Moreover since $\mathrm{supt}(x) \subset M$ for each $x \in M$, we have $\hat{U}_B \cap C_B \cap D \cap M = \emptyset$. By elmentarity, we know that

(d) D is disjoint from $C_B \cap \hat{U}_B$.

Now define a basic open neighborhood V of y in Y as follows:

For each $i \in A_0$, let

$$\gamma_i = \max\{\gamma_i^B : B_0 \cup \{i\} \subset B \subset A_0\}.$$

For each $i \in A \setminus A_0$, let

$$\gamma_i = \max\{\gamma_i^B : B_0 \subset B \subset A_0\}.$$

Let $F = \bigcup\{\hat{F}_B : B_0 \subset B \subset A_0\}$.
Let V be the basic clopen neighborhood of y:

$$V = \{x \in Y : \forall i \in F(x(i) = 0) \wedge \forall i \in A(x(i) \in (\gamma_i, y(i)] \cap M)\}.$$

Claim. $V \cap p(D) = \emptyset$

Proof. If not, we may fix $x \in D$ such that $p(x) \in V$. Let

$$B = \{i \in A_0 : x(i) \neq y(i)\}$$

Note that $B_0 \subset B$. By definition of p and the fact that $p(x) \in V$, we have that
$x(i) = y(i)$ for all $i \in A_0 \setminus B$, and
$x(i) \in (\gamma_i^B, y(i)]$ for all $i \in B \setminus B_0$, and
$x(i) > \gamma_i^B$ for all $i \in B_0$, and
$x(i) \in (\gamma_i^B, y(i)]$ for all $i \in A \setminus A_0$, and
$x(i) = 0$ for all $i \in \hat{F}_B$
But this contradicts (d). Thus $V \cap p(D) = \emptyset$. This completes the proof that $p(D)$ is closed. ⊣

⊣

For X a σ-product of ordinals and for Y constructed as above, we will refer to Y as the quotient of X modulo M. Indeed, it is not hard to show that the map $p : X \to Y$ is a quotient map (this is discussed after the proof of Theorem 3.3 below).

To prove main result in this section, we will need another technical result:

Lemma 3.2. *Let $D \in M$ be a closed subset of X and $x \in D$ (but not necessarily in M). Then there is a $x' \in D$ such that* $\mathrm{supt}(x') = \mathrm{supt}(x) \cap M$*, $p(x) = p(x')$ and $x'(i) = \sup(M \cap x'(i))$ if $x'(i) \notin M$.*

Proof. Define $x' \in X$ by

$$x'(i) = \begin{cases} x(i), & \text{if } i \in \kappa \cap M, x(i) \in M, \\ \sup(M \cap x(i)), & \text{if } i \in \kappa \cap M, x(i) \notin M, \\ 0, & \text{otherwise.} \end{cases}$$

We show that $x' \in D$. The other properties are evident. Let

$$A = \mathrm{supt}(x) \cap M \text{ and } B = \{i \in A : x(i) \in M\}$$

Let $\{F_n : n \in \omega\}$ be an increasing sequence of finite subsets of $\kappa \cap M$ such that

$$\bigcup_{n\in\omega} F_n = (\kappa \cap M) \setminus A$$

Let $u_n \in \prod_{i\in A}(M \cap \alpha_i)$ be such that for $i \in B$, $u_n(i) = x(i)$ for all n, and for $i \in A \setminus B$, $\{u_n(i) : i \in \omega\}$ is an increasing sequence cofinal in $x'(i)$.

For each n, let $\Phi_n(z)$ be the following formula with one free variable z:

$$\forall i \in B(z(i) = x(i)) \wedge \forall i \in A \setminus B(u_n(i) < z(i) < p(x)(i)) \wedge \forall i \in F_n(z(i) = 0)$$

Note that all parameters of $\Phi_n(z)$ are in M ($p(x)(i) \in M$ even though $p \notin M$). Also, $\exists z \in D\Phi_n(z)$ is valid in H_θ (since x is a witness). Thus, by elementarity, there is an $x_n \in D \cap M$ such that $\Phi_n(x_n)$. Note that $A \subset \mathrm{supt}(x_n) \subset M$, and that by choice of the u_n's and the F_n's, we get that $\{x_n : n \in \omega\}$ is a sequence of elements of $M \cap D$ converging to x'. Therefore $x' \in D$.

$\dashv$

Now we come to the first application of this construction and of Lemmas 3.1 and 3.2:

Theorem 3.3. *Suppose that $\{\alpha_i : i \in \kappa\}$ is a family of ordinals and suppose that*

$$X = \sigma(\prod_{i\in\kappa} \alpha_i, 0).$$

Then X is countably paracompact.

Proof. Suppose that $\{D_n : n \in \omega\}$ is a decreasing family of closed subsets of X with empty intersection.

Let M be a countable elementary submodel of H_θ where θ is large enough, so that

$$\{\{\alpha_i : i \in \kappa\}\{D_n : n \in \omega\}\} \subset M.$$

Let Y be the quotient of X modulo M. Thus, $\{p(D_n) : n \in \omega\}$ is a decreasing sequence of closed subsets of Y.

Claim. $\bigcap\{p(D_n) : n \in \omega\} = \emptyset$

Proof. Suppose not. Let $y \in p(D_n)$ for every $n \in \omega$. Then for each n there is $x_n \in D_n$ such that $p(x_n) = y$. By Lemma 3.2, we may assume that for each n, $\mathrm{supt}(x_n) = \mathrm{supt}(y)$ and that for each i, if $x_n(i) \notin M$ then $x_n(i) = \sup(M \cap x_n(i))$. For each n let

$$A_n = \{i \in \kappa \cap M : x_n(i) \notin M\} = \{i \in \kappa \cap M : x_n(i) \neq y(i)\}$$

and

$$B_n = \{i \in \kappa \cap M : x_n(i) = y(i) \neq 0\}$$

Then $\mathrm{supt}(y) = A_n \cup B_n$ for each n. Since $\mathrm{supt}(y)$ is finite, there are sets A and B such that

$$\{n \in \omega : A = A_n \wedge B = B_n\} \text{ is infinite}$$

So, without loss of generality $A_n = A$ and $B_n = B$ for every $n \in \omega$. Therefore (since for each n $x_n(i) = \sup(M \cap y(i))$ for each $i \in A$, and $x_n(i) = y(i)$ for each $i \in B$), $x_n = x_m$ for all $n, m \in \omega$. Thus $\bigcap_{n\in\omega} D_n \neq \emptyset$. Contradiction. $\dashv$

We have now proven that $\{p(D_n) : n \in \omega\}$ is a decreasing sequence of closed subsets of Y with empty intersection. Since Y is metrizable, we may fix an open expansion $U_n \supset p(D_n)$ such that $\bigcap_{n\in\omega} \mathrm{Cl}_Y U_n = \emptyset$. Let $V_n = p^{-1}(U_n)$. Clearly $D_n \subset V_n$ and by continuity we have that V_n is open and $p(\mathrm{Cl}_X V_n) \subset \mathrm{Cl}_Y U_n$. Thus $\bigcap_{n\in\omega} \mathrm{Cl}_X V_n = \emptyset$. This completes the proof of Theorem 3.3. $\dashv$

Remark 3.4. Given X, a σ-product of ordinals (possibly only a finite product of ordinals), and given M a countable elementary submodel let Y be as above the quotient of X modulo M and let p be the associated surjection. We now show that $p : X \to Y$ is, in fact, a quotient map: suppose that $A \subset Y$ is such that $p^{-1}(A)$ is open. Let $y \in A$ be arbitrary. Let $x \in p^{-1}(y)$ be chosen so that $\mathrm{supt}(x) = \mathrm{supt}(y) \subset M$ and for all $i \in \mathrm{supt}(y)$, if $\sup\{\gamma + 1 : \gamma \in y(i) \cap M\} < y(i)$, then $x(i) = \sup\{\gamma + 1 : \gamma \in y(i) \cap M\}$. We may choose a basic open neighborhood $U(x) \subset p^{-1}(A)$ of x. We may suppose that for some finite set $F \subset \kappa$ disjoint from $\mathrm{supt}(x)$ and $\beta_i \in M \cap x(i)$ with $i \in \mathrm{supt}(x)$,

$$U(x) = \{z \in X : \forall i \in \mathrm{supt}(x)(z(i) \in (\beta_i, x(i)]) \wedge \forall i \in F(z(i) = 0)\}.$$

Thus, $p(U(x))$ is open in Y and clearly $y \in p(U(x)) \subset A$. Thus A is open and therefore p is a quotient map.

It is not hard to verify that the equivalence relation defining this quotient space is given by $x =_M y$ if and only if for all $i \in \kappa \cap M$,

$$(y(i) < x(i) \Rightarrow [y(i), x(i)) \cap M = \emptyset) \;\wedge\; (x(i) < y(i) \to [x(i), y(i)) \cap M = \emptyset).$$

Equivalently, $x =_M y$ if and only if $f(x) = f(y)$ for all continuous functions $f : X \to \mathbb{R}$ such that $f \in M$. This later definition was recently introduced and studied in the general case by T. Eisworth and A. Stanley. Since we need to use this fact in the proof of Theorem 3.9 below, we give a proof.

Lemma 3.5. *Suppose that X is a σ-product of ordinals and that M is a countable elementary submodel containing X. Let Y be the quotient of X modulo M and let p be the associated quotient map. Then for each $x, y \in X$, $p(x) = p(y)$ if and only if $f(x) = f(y)$ for all continuous functions $f : X \to \mathbb{R}$ such that $f \in M$.*

Proof. One direction is trivial. If $p(x) \neq p(y)$, then we may assume that there is $i \in \kappa \cap M$ and $\alpha \in M$ such that $y(i) \leq \alpha < x(i)$. So y and x can be separated by a clopen set in M, thus by a two-valued function in M.

Conversely, suppose that $p(x) = p(y)$ and that $f \in M$ is a real-valued continuous function on X. Without loss of generality we may assume that y has the property that $\mathrm{supt}(y) \subset M$ and that $y(i) = \sup\{\gamma + 1 : \gamma \in M \cap y(i)\}$ for all $i \in \mathrm{supt}(y)$. If $f(x) \neq f(y)$, let q be a rational number between $f(x)$ and $f(y)$. Without loss of generality we may assume that $f(x) < q < f(y)$. Let U be a basic open neighborhood of y such that $f(U) \subset (q, \infty)$. Let $r \in \prod_{i\in\mathrm{supt}(y)}(M \cap y(i))$ and let $F \subset \kappa$ be finite such that,

$$U = \{z \in X : \forall i \in \mathrm{supt}(y)(z(i) \in (r(i), y(i)]) \wedge \forall i \in F(z(i) = 0)\}$$

Let $\hat{F} = F \cap M$ and $A = \{i \in \operatorname{supt}(y) : p(y)(i) \neq y(i)\}$. Then A is the set of i such that $y(i) \notin M$. Define $\hat{U} =$

$$\{z \in X : \forall i \in \operatorname{supt}(y) \setminus A(z(i) \in (r(i), y(i)]) \wedge \forall i \in A(z(i) \in (r(i), p(y)(i)))$$

$$\wedge \forall i \in \hat{F}(z(i) = 0)\}.$$

Notice that the following holds:

(∗) For all $z \in \hat{U} \cap M$, $f(z) > q$ holds.

Thus, since all the parameters in the above statement are in M ($\operatorname{supt}(y)$ is in M even though y may not be), we may conclude that the same holds true for all $z \in \hat{U}$. Notice that x is in the closure of $\hat{U}$, so $f(x) \geq q$. Contradiction. Thus $f(x) = f(y)$. $\dashv$

The following lemma is a kind of pressing down lemma for finite products of ordinals. We will need it below but believe it should be of general interest.

Lemma 3.6. *Suppose that β_i are ordinals with $\omega < \operatorname{cf} \beta_i$ for $i < n$. Let $K \subset Z = \prod_{i<n} \beta_i$ be closed and cofinal in Z, i.e., K is (topologically) closed in Z and for every $x = \langle x(i) : i < n \rangle \in Z$, there is $k = \langle k(i) : i < n \rangle \in K$ such that $x(i) < k(i)$ for each $i < n$. Let $U \supset K$ be open in Z. Then there is $x \in Z$ such that $\prod_{i<n}(x(i), \beta_i) \subset U$.*

Proof. Fix an appropriate countable elementary submodel M containing everything relevant. For each $i < n$ let $s(i) = \sup(M \cap \beta_i)$. Let V be an arbitrary basic open set containing $s = \langle s(i) : i < n \rangle$. Since $M \cap s(i)$ is unbounded in each $s(i)$, we may fix $y \in M$ such that $y(i) < s(i)$ for all $i < n$ and such that

$$\prod_{i<n}(y(i), s(i)] \subset V$$

By elementarity, since K is cofinal, there is $z \in K \cap \prod_{i<n}(y(i), s(i)]$. Thus $V \cap K \neq \emptyset$. Thus $s \in K \subset U$. Since U is open, we can fix $x \in M$ with $x(i) < s(i)$ for all $i < n$ and

$$\prod_{i<n}(x(i), s(i)] \subset U.$$

Note that for all $z \in M \cap Z$ if $x(i) < z(i) < \beta_i$ for all $i < n$, then $z \in U$. Thus by elementarity this is true. Thus, $\prod_{i<n}(x(i), \beta_i) \subset U$, completing the proof of the lemma. $\dashv$

We now consider κ-normality and strong zero-dimensionality of σ-products of ordinals. We will use Lemmas 3.1 and 3.2 proven above and in addition we will need a characterization of normality for finite products of ordinals.

We will say that a pair of closed subsets H and K in a topological space can be strongly separated if there are open sets U and V containing H and K respectively such that $\operatorname{Cl} U \cap \operatorname{Cl} V = \emptyset$.

Theorem 3.7. *Suppose that H and K are disjoint closed subsets of a σ-product $X = \sigma(\prod_{i\in\kappa} \alpha_i, 0)$ of ordinals and that M is a countable elementary submodel such that $X, H, K \in M$. Let Y be the quotient of X modulo M and the surjection $p : X \to Y$ as before. Then H and K can be strongly separated by open subsets of X if and only if $p(H)$ and $p(K)$ are disjoint.*

Proof. Suppose that $p(H)$ and $p(K)$ are disjoint. By Lemma 3.1, they are both closed and since Y is a countable product of countable ordinals, Y is metrizable, hence normal. Let U_0 and U_1 be disjoint open sets in Y strongly separating $p(H)$ and $p(K)$. Since p is continuous, $p^{-1}(U_0)$ and $p^{-1}(U_1)$ strongly separate H and K.

Conversely, suppose that $x \in p(H) \cap p(K)$. For each $h' \in p^{-1}(x) \cap H$, set $B_{h'} = \{i \in \mathrm{supt}(x) : h'(i) \in M \wedge \sup\{\gamma + 1 : \gamma \in M \cap h'(i)\} < h'(i)\}$. Similarly, for each $k' \in p^{-1}(x) \cap K$, set $C_{k'} = \{i \in \mathrm{supt}(x) : k'(i) \in M \wedge \sup\{\gamma + 1 : \gamma \in M \cap k'(i)\} < k'(i)\}$. Choose $h \in p^{-1}(x) \cap H$ and $k \in p^{-1}(x) \cap K$ so that B_h is maximal with respect to all $B_{h'}$ for $h' \in p^{-1}(x) \cap H$ and C_k is maximal with respect to all $C_{k'}$ for $k' \in p^{-1}(x) \cap K$. Applying Lemma 3.2, we may moreover assume that $\mathrm{supt}(h) = \mathrm{supt}(k) = \mathrm{supt}(x)$, $h(i) = \sup(M \cap h(i))$ (equivalently, $h(i) = \sup\{\gamma + 1 : \gamma \in M \cap h(i)\}$) if $h(i) \notin M$ and $k(i) = \sup(M \cap k(i))$ if $k(i) \notin M$. Set $A = \{i \in \mathrm{supt}(x) : h(i) \neq k(i)\}$, $B = A \cap B_h$, $C = A \cap C_k$ and $D = \{i \in \mathrm{supt}(x) : h(i) = k(i) \notin M\}$.

Claim 1. *$A = B \cup C$, $B \cap C = \emptyset$ and $h(i) = k(i) = x(i)$ for each $i \in \kappa \cap M \setminus A \cup D$.*

Proof. Let $i \in A$. We may assume $k(i) < h(i)$. Since $p(k)(i) = p(h)(i) = x(i)$, we have $[k(i), h(i)) \cap M = \emptyset$, in particular $k(i) \notin M$. Therefore $\sup\{\gamma + 1 : \gamma \in M \cap h(i)\} = \sup\{\gamma + 1 : \gamma \in M \cap k(i)\} \leq k(i) < h(i)$. If we assume $h(i) \notin M$, then we have $h(i) = \sup(M \cap h(i)) = \sup(M \cap k(i)) = k(i)$, a contradiction. Thus $h(i) \in M$ and therefore $i \in B$.

To prove $B \cap C = \emptyset$, assume $i \in B \cap C$. We may assume $k(i) < h(i)$. Then as above $k(i) \notin M$, thus $i \notin C_k$, a contradiction.

To show the final property, let $i \in \kappa \cap M \setminus A \cup D$. We may assume $i \in \mathrm{supt}(x)$. It follows from $i \notin A$ that $h(i) = k(i)$. If $h(i) \neq x(i)$, then it follows from $p(h)(i) = x(i)$ that $h(i) \notin M$.

Thus $i \in D$, a contradiction. ⊣

Now we have:

(a) $x(i) = h(i)$ for all $i \in B$, and $x(i) = k(i)$ for all $i \in C$.

Since for each $i \in A \cup D$, $\sup\{\gamma + 1 : \gamma \in M \cap x(i)\} < x(i) \in M$, we have:

(b) $\mathrm{cf}\, x(i) > \omega$ for all $i \in A \cup D$.

And:

(c) $h(i) < x(i)$ for all $i \in C$, and $k(i) < x(i)$ for all $i \in B$.

By elementarity, H satisfies:

(d) For all $r \in \prod_{i\in C\cup D} x(i)$, there is $z \in H$ such that $z(i) = x(i)$ for all $i \in \kappa \cap M \setminus C \cup D$, $z(i) = 0$ for all $i \in \kappa \setminus M$ and $z(i) \in (r(i), x(i))$ for all $i \in C \cup D$.

Similarly K satisfies:

(e) For all $r \in \prod_{i \in B \cup D} x(i)$, there is $z \in K$ such that $z(i) = x(i)$ for all $i \in \kappa \cap M \backslash B \cup D$, $z(i) = 0$ for all $i \in \kappa \backslash M$ and $z(i) \in (r(i), x(i))$ for all $i \in B \cup D$.

Now we restrict to closed subsets of H and K. Let

$$H' = \{z \in H : \forall i \in \kappa \cap M \backslash C \cup D(z(i) = x(i)) \wedge \forall i \in \kappa \backslash M(z(i) = 0)$$
$$\wedge \forall i \in C \cup D(z(i) \in (h(i), x(i)))\}$$

and let

$$K' = \{z \in K : \forall i \in \kappa \cap M \backslash B \cup D(z(i) = x(i)) \wedge \forall i \in \kappa \backslash M(z(i) = 0)$$
$$\wedge \forall i \in B \cup D(z(i) \in (k(i), x(i)))\}$$

Claim 2. *Both H' and K' are closed in X.*

Proof. Suppose $z \in \mathrm{Cl}_X H' \backslash H'$. Then z has the property that $p(z) = x$, $z(i) = x(i)$ for all $i \in \kappa \cap M \backslash C \cup D$, $z(i) = 0$ for all $i \in \kappa \backslash M$ and $z \in H$. Since $z \notin H'$, $z(i) = x(i)$ for some $i \in C \cup D$. It follows from $B_h \cap (C \cup D) = \emptyset$ that B_z as defined above is a proper superset of B_h, contradicting the maximality of B_h. A similar argument shows that K' is closed. $\dashv$

Let

$$Z = \{z \in X : \forall i \in A \cup D(z(i) \leq x(i)) \wedge \forall i \in \kappa \cap M \backslash A \cup D(z(i) = x(i))$$
$$\wedge \forall i \in \kappa \backslash M(z(i) = 0)\}.$$

Since H' and K' are closed subsets of $Z \subset X$, it suffices to prove that H' and K' cannot be strongly separated in Z. Let U and V be open sets in Z with $H' \subset U$ and $K' \subset V$.

Note that by (d), H' is homeomorphic to a closed cofinal subset of $\prod_{i \in C \cup D} x(i)$, where by (b), each $x(i)$ has uncountable cofinality. Applying Lemma 3.6, there is a $g \in \prod_{i \in C \cup D} x(i)$ such that

$$\prod_{i \in C \cup D} (g(i), x(i)) \times \prod_{i \in \kappa \cap M \backslash C \cup D} \{x(i)\} \times \prod_{i \in \kappa \backslash M} \{0\} \subset U.$$

Similarly, considering the open sets V, we may extend the domain of g to include $B \cup D$ such that

$$\prod_{i \in B \cup D} (g(i), x(i)) \times \prod_{i \in \kappa \cap M \backslash B \cup D} \{x(i)\} \times \prod_{i \in \kappa \backslash M} \{0\} \subset V.$$

Let v be defined by

$$v(i) = \begin{cases} g(i) + 1, & \text{if } i \in D, \\ x(i), & \text{otherwise.} \end{cases}$$

By the definition of g, it follows that $v \in \mathrm{Cl}_Z U \cap \mathrm{Cl}_Z V$. This completes the proof of Theorem 3.7. $\dashv$

Corollary 3.8. *A σ-product of ordinals is normal if and only if each finite subproduct is normal.*

Proof. Since each finite subproduct is homeomorphic to a closed subset of the σ-product, one direction is trivial. For the other implication, suppose

$$X = \sigma(\prod_{i\in\kappa}\alpha_i, 0)$$

and suppose that each finite subproduct of $\{\alpha_i : i \in \kappa\}$ is normal. Fix H and K disjoint closed subsets of X. Let M be an appropriate countable elementary submodel containing κ, $\{\alpha_i : i \in \kappa\}$, H and K. Let Y be the restriction of X to M and let $p : X \to Y$ be the corresponding surjection. Then by Lemma 3.1 we have that $p(H)$ and $p(K)$ are closed subsets of Y. By Theorem 3.7, it suffices to show:

Claim. $p(H) \cap p(K) = \emptyset$.

Proof. Suppose not and let $h \in H$ and $k \in K$ be such that $p(h) = p(k)$. Then by Lemma 3.2, we may assume that $\mathrm{supt}(h) \subset M$ and $\mathrm{supt}(k) \subset M$. Thus $\mathrm{supt}(h) = \mathrm{supt}(k)$. Call this finite set A. Let

$$H' = \{z \in H : \mathrm{supt}(z) = A\} \text{ and } K' = \{z \in K : \mathrm{supt}(z) = A\}.$$

Then the projection of H' and K' to A are disjoint closed subsets of $\prod_{i\in A}\alpha_i$. However, since $p(H') \cap p(K') \neq \emptyset$, applying Theorem 3.7 to the finite product $\prod_{i\in A}\alpha_i = \sigma(\prod_{i\in A}\alpha_i, 0)$, we conclude that the projection of H' and K' cannot be strongly separated in $\prod_{i\in A}\alpha_i$. Thus, this finite subproduct is not normal. Contradiction. $\dashv$

$\dashv$

We now consider strong zero-dimensionality of σ-products.

Theorem 3.9. *For every real-valued continuous function f on a σ-product $X = \sigma(\prod_{i\in\kappa}\alpha_i, 0)$ of ordinals, the range of f is countable.*

Proof. Let $f : X \to \mathbb{R}$ be continuous. Let M be a countable elementary submodel such that κ, $\{\alpha_i : i \in \kappa\}$, $f \in M$. Let Y be the quotient of X modulo M and let $p : X \to Y$ be the corresponding quotient map. Since $f \in M$, it follows by Lemma 3.5 that f respects equivalence classes. Thus, f can be factored through Y. I.e., there is a continuous $g : Y \to \mathbb{R}$ such that $f = g \circ p$. But Y is countable, so the range of g is countable. Hence the range of f is countable. $\dashv$

Note that the irrationals $\mathbb{P}$ is identified with the product ${}^{\omega}\omega$. The inclusion map $f : \mathbb{P} \to \mathbb{R}$ defined by $f(x) = x$ does not have countable range. Therefore we cannot extend Theorem 3.9 for countable products or Σ-products. However we have the following analogous result of Corollary 1.11 for strong zero-dimensionality.

Corollary 3.10. *Every σ-product of ordinals is strongly zero-dimensional.*

Proof. Let H and K be disjoint zero-sets in X. Disjoint zero-sets can be functionally separated, so let $f : X \to [0,1]$ be continuous, such that $f(H) = 0$ and $f(K) = 1$. Since the range of f is countable, pick r in $[0,1]$ but not in the range of f. Then $f^{-1}([0,r))$ is a clopen set containing H and disjoint from K. Thus, X is strongly zero-dimensional. $\dashv$

Finally, we prove that σ-products of ordinals are κ-normal:

Theorem 3.11. *Every σ-product of ordinals is κ-normal.*

Proof. Suppose that H and K are disjoint regular closed sets in a σ-product $X = \sigma(\prod_{i\in\kappa}\alpha_i, 0)$ of ordinals. Let M be an appropriate countable elementary submodel containing κ, $\{\alpha_i : i \in \kappa\}$, H and K. Let Y be the restriction of X to M and let $p : X \to Y$ be the corresponding surjection. Then by Lemma 3.1 we have that $p(H)$ and $p(K)$ are closed subsets of Y. By Theorem 3.7, it suffices to show $p(H) \cap p(K) = \emptyset$. Write $H = \mathrm{Cl}_X(\bigcup\mathcal{U})$ and $K = \mathrm{Cl}_X(\bigcup\mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are collections of basic open subsets of X. We may assume that $\mathcal{U}$ and $\mathcal{V}$ are in M because of $H, K \in M$. For each $U \in \mathcal{U} \cap M$, set $\tilde{U} = \pi_{\kappa\cap M}(U) \cap Y$, where $\pi_{\kappa\cap M} : X \to \sigma(\prod_{i\in\kappa\cap M}\alpha_i, 0)$ is the canonical projection. Note that by elementarity, $\mathrm{supt}(U) \in M$ thus $\mathrm{supt}(U) \subset M$ for each $U \in \mathcal{U} \cap M$. Similarly define $\tilde{V}$ for each $V \in \mathcal{V} \cap M$.

Claim 1. *$\tilde{U} \subset p(U)$ for each $U \in \mathcal{U} \cap M$.*

Proof. Let $y \in \tilde{U}$ and pick $x \in U$ with $\pi_{\kappa\cap M}(x) = y$.

For each $i \in \kappa \cap M$, it follows from $x(i) = y(i) \in M$ that $p(x)(i) = x(i) = y(i)$. Thus $y = p(x) \in p(U)$.

Claim 2. *$p(H) = \mathrm{Cl}_Y(\bigcup\{\tilde{U} : U \in \mathcal{U} \cap M\})$.*

Proof. Since $p(H)$ is closed and $p(H) \supset p(U) \supset \tilde{U}$ for each $U \in \mathcal{U}\cap M$, one inclusion is obvious.

To show the remaining inclusion, let $x \in p(H)$ and W be a basic open neighborhood of x in Y. We may assume that

$$W = \{z \in Y : \forall i \in \mathrm{supt}(x)(z(i) \in (r(i), x(i)] \cap Y_i) \wedge \forall i \in F(z(i) = 0)\},$$

where F is a finite subset of $\kappa \cap M$ disjoint from $\mathrm{supt}(x)$ and $r(i) \in M \cap x(i)$ for each $i \in \mathrm{supt}(x)$. Note $F \in M$. Fix $h \in H$ with $x = p(h)$. By Lemma 3.2, we may assume that $\mathrm{supt}(h) = \mathrm{supt}(x)$ and

$$h(i) = \begin{cases} x(i), & \text{if } i \in \kappa \cap M, h(i) \in M, \\ \sup(M \cap h(i)), & \text{if } i \in \kappa \cap M, h(i) \notin M, \\ 0, & \text{otherwise.} \end{cases}$$

Set $A = \{i \in \kappa \cap M : h(i) < x(i)\}$. Note that $A \in M$ by $A \subset \mathrm{supt}(x) \subset M$ and that $r(i) < h(i)$ for each $i \in A$, by $r(i) \in x(i) \cap M$. Let

$$\hat{W} = \{t \in X : \forall i \in A(t(i) \in (r(i), x(i)))\wedge$$

$$\forall i \in \mathrm{supt}(x)\backslash A(t(i) \in (r(i), x(i)]) \wedge \forall i \in F(t(i) = 0)\}.$$

Since all parameters in the definion of $\hat{W}$ are in M, we have $\hat{W} \in M$. Since $\hat{W}$ is a neighborhood of $h \in H$ in X, by elementarity, there is $U \in \mathcal{U} \cap M$ such that $\hat{W} \cap U \neq \emptyset$. By elementarity, we can pick $t \in M$ with $t \in \hat{W} \cap U$. It follows from $t(i) \in M$ for each $i \in \kappa \cap M$ that $t(i) \in (r(i), x(i)] \cap Y_i$ for each $i \in \mathrm{supt}(x)$ and $t(i) = 0$ for each $i \in F$. Therefore $\pi_{\kappa\cap M}(t) \in W \cap \pi_{\kappa\cap M}(U) \subset W \cap \tilde{U}$. Thus $x \in \mathrm{Cl}_Y(\bigcup\{\tilde{U} : U \in \mathcal{U} \cap M\})$.

Similarly we have $p(K) = \mathrm{Cl}_Y(\bigcup\{\tilde{V} : V \in \mathcal{V} \cap M\})$. To show $p(H) \cap p(K) = \emptyset$, we assume $x \in p(H) \cap p(K)$. For each $i \in \kappa$, set

$$y(i) = \begin{cases} \sup\{\gamma + 1 : \gamma \in M \cap x(i)\}, & \text{if } i \in \mathrm{supt}(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then obviously $y \in X$, $\mathrm{supt}(y) = \mathrm{supt}(x)$ and $p(y) = x$. Moreover if $y(i) \neq 0$, then $\{(\gamma, y(i)] : \gamma \in M \cap y(i)\}$ is a neighborhood base at $y(i)$ in α_i. Let W be a basic open neighborhood of y in X. We may assume that

$$W = \{z \in X : \forall i \in \mathrm{supt}(y)(z(i) \in (r(i), y(i)]) \wedge \forall i \in F(z(i) = 0)\},$$

where F is a finite subset of κ disjoint from $\mathrm{supt}(y)$ and $r(i) \in M \cap y(i)$ for each $i \in \mathrm{supt}(y)$. Set $F' = F \cap M$ and

$$W' = \{t \in Y : \forall i \in \mathrm{supt}(y)(t(i) \in (r(i), x(i)] \cap Y_i) \wedge \forall i \in F'(t(i) = 0)\}.$$

Since W' is a neighborhood of x in Y, by Claim 2, there is $U \in \mathcal{U} \cap M$ such that $W' \cap \tilde{U} \neq \emptyset$. Pick $t \in W' \cap \tilde{U}$. It follows from $U \in M$ that $L = \mathrm{supt}(U) \in M$ thus $L \subset M$. Hence U can be represented as $U = \pi_L^{-1}(\prod_{i \in L} U_i)$, where U_i is open in α_i. We will show $W \cap U \neq \emptyset$. It suffices to show that $W_i \cap U_i \neq \emptyset$ for each $i \in (\mathrm{supt}(y) \cup F) \cap L$, where

$$W_i = \begin{cases} (r(i), y(i)], & \text{if } i \in \mathrm{supt}(y), \\ \{0\}, & \text{if } i \in F. \end{cases}$$

Let $i \in L$. Note that by $i \in L \subset M$, $U_i \in M$.

Case 1. $i \in \mathrm{supt}(y)$ and $y(i) < x(i)$.

In this case, if $x(i) \in U_i$, then by elementarity, there is $\gamma \in M \cap x(i)$ such that $(\gamma, x(i)] \subset U_i$. By the definition of $y(i)$, we have $\gamma < y(i)$. Thus $y(i) \in (r(i), y(i)] \cap (\gamma, x(i)] \subset W_i \cap U_i$. If $x(i) \notin U_i$, then $t(i) \in ((r(i), x(i)] \cap Y_i) \cap U_i = ((r(i), x(i)) \cap Y_i) \cap U_i \subset (r(i), y(i)) \cap U_i \subset W_i \cap U_i$.

Case 2. $i \in \mathrm{supt}(y)$ and $y(i) = x(i)$.

In this case, we obviously have $t(i) \in W_i \cap U_i$.

Case 3. $i \in F$.

Since $i \in L \subset M$, we have $i \in F' = F \cap M$ and thus obviously $t(i) = 0 \in W_i \cap U_i$.

These cases shows $W \cap U \neq \emptyset$, therefore $y \in H$. Similarly we have $y \in K$, a contradiction because of $H \cap K = \emptyset$. Thus $p(H) \cap p(K) = \emptyset$. $\dashv$

Remarks. It is natural to ask whether the results of this section extend to include σ-products of ordinals at base points other than $\mathbf{0}$. If we require that the base point has countable cofinality at all coordinates (or at all but finitely many coordinates) then all the results of Section 3 generalize, and the proofs are essentially the same.

However, if the base point has uncountable cofinality at infinitely many coordinates then the proofs of this section do not generalize.[1]

In another direction, it is also natural to ask if the results generalize to include σ-products of subspaces of ordinals. The proofs given in this section can be slightly modified to include σ-products of spaces X_i where for each i

$$\{\beta < \alpha_i : \mathrm{cof}(\beta) \leq \omega\} \subseteq X_i \subseteq \alpha_i.$$

In the proofs, all ordinals of countable cofinality must be included to assure that for a given countable elementary submodel M, if $\delta \in \alpha_i$ and $\delta = \sup(M \cap \delta)$ then $\delta \in X_i$. All the main results of this section hold for this general case (even with the other base points described above) and the proofs are essentially the same. For example, in the proof of Lemma 3.1, one must redefine the set B_0 to be $\{i \in A_0 : y(i) \notin X(i)\}$

Certainly, one cannot expect all the proofs to generalize to σ-products of arbitrary subspaces of ordinals. However, it is open whether σ-products of subspaces of ordinals are strongly zero-dimensional. In addition, it is open whether countable products of subspaces of ordinals are strongly zero-dimensional.

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[1] Examples to show that some of these theorems do not generalize in this direction will appear in a sequel to this paper.

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