PRODUCTS OF MONOTONICALLY NORMAL SPACES WITH VARIOUS SPECIAL FACTORS

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Abstract. We give several characterizations of normality, orthocompactness and rectangularity for products of monotonically normal spaces and various special factors in terms of some neighborhood properties of the factors. Such a special factor is a compact factor, a $\mathbb{DC}$-like factor, an almost discrete factor or an ordinal factor. Moreover, we deal with the same properties for products of GO-spaces with ordinal factors.

Contents

1. Introduction

This paper is a continuation of several papers [9, 10, 17, 18]. The study for products of monotonically normal spaces with special factors was actually begun with the following result.

Theorem 1.1 ([17]). *Let X be a monotonically normal space and K a compact space. If $X \times K$ is orthocompact, then it is normal.*

Subsequently, in [18], this result was extended for the products $X \times Y$ of monotonically normal spaces X with $\mathbb{DC}$-like factors Y defined by topological games of Telgársky. Moreover, it was proved in there that if such a product space $X \times Y$ is normal and rectangular, then it is collectionwise normal and has the shrinking property.

In Section 2, as our preliminaries, we explain monotone normality, normal covers and rectangular products, which play important roles in this paper.

In Section 3, we define three neighborhood properties for spaces. This section might be somewhat boring for the reader without the background. However, these new concepts are a key of our several characterizations.

From Section 4, we begin our theorems. In this section, we give a characterization of normality of the product $X \times K$ of a monotonically normal space X and a compact space K, in terms of two neighborhood properties stated in the previous section.

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In Section 5, we consider the product $X \times Y$ of a monotonically normal space X and a paracompact $\mathbb{DC}$-like space Y. We give characterizations of normality and orthocompactness of $X \times Y$, respectively.

In Section 6, an almost discrete space means a space with only one non-isolated point. We give a characterization of normality of the product $X \times Y$ of a monotonically normal space X and an almost discrete space Y. Moreover, we show an unexpected result in the sense that rectangularity of $X \times Y$ implies its normality.

Scott [14] proved that for any two ordinals λ and μ, the product $\lambda \times \mu$ is orthocompact iff it is normal. This result was extended to the products of two subspaces of an ordinal as follows.

Theorem 1.2 ([7, 8, 10])**.** *Let A and B be subspaces of an ordinal. Then the following are equivalent.*

(a) *$A \times B$ is orthocompact.*
(b) *$A \times B$ is normal and rectangular.*
(c) *$A \times B$ is normal.*
(d) *$A \times B$ is collectionwise normal.*
(e) *$A \times B$ has the shrinking property.*

In Section 7, to extend some implications above, we consider the product $X \times B$ of a monotonically normal space X and a subspace B of an ordinal. We show that (a)⇒(b) in Theorem 1.2 holds for the product $X \times B$ and that (c)⇔(d)⇔(e) in there holds under rectangularity of $X \times B$. On the other hand, examples of $X \times B$ refuting (b)⇒(a) and (c)⇒(b) were found in [17] and [12], respectively, where X is a certain almost discrete space and $B = \kappa$ is a regular uncountable cardinal.

In Section 8, we show that (a)⇔(b)⇔(c) in Theorem 1.2 holds for the product $X \times B$ of a GO-space X and a subspace B of an ordinal. Moreover, we prove the equivalence of rectangularity and countable paracompactness for such a product $X \times B$.

Throughout this paper, we will try to put our theorems at the beginning of each subsection, and put the proof at the last. Thus we hope the reader will easily understand the purpose of each subsection. Moreover, since we have to prepare several new concepts and tools for our results and their proofs, we will try to explain them as just before we use as possible. All spaces are assumed to be Hausdorff. The letters κ and τ always mean infinite cardinals. We follow the books [3, 11] for notation and terminology which are not explained here.

2. Some preliminaries

In this section, as our preliminaries, we explain normal covers, rectangular products and monotone normality which play important roles in this paper. When we discuss monotone normality, the notation $\mathcal{S}(X), \mathcal{S}(X, \kappa)$ and $\mathcal{S}^*(X)$ are necessary, because we often make use of Balogh-Rudin's Theorem stated below. Moreover, we sometimes use a characterization of normal covers of monotonically normal spaces given here.

Normal covers and rectangular products. Recall that a subset U of a space X is *cozero* (or a *cozero-set*) in X if there is a continuous function $f : X \to \mathbb{I}$ such that $U = \{x \in X : f(x) > 0\}$, where $\mathbb{I} = [0, 1]$ is the unit interval in the real line. It is well-known (as a part of Stone-Michael-Morita's Theorem) that an open cover of a space has a locally finite cozero refinement iff it has a σ-locally finite cozero refinement iff it has a σ-discrete cozero refinement. An open cover having such a refinement is said to be *normal*. It is also well-known that an open cover of a normal (and countably paracompact) space is normal iff it has a locally finite (σ-disjoint) open refinement.

Let $X \times Y$ be a product of two spaces. A subset of the form $U \times V$ in $X \times Y$ is called a *rectangle*. A rectangle $U \times V$ is called a *cozero (open, closed) rectangle* in $X \times Y$ if U and V are cozero (open, closed) in X and Y, respectively. A cover $\mathcal{G}$ of $X \times Y$ is *rectangular* if each member of $\mathcal{G}$ is a rectangle in $X \times Y$. We say that a product space $X \times Y$ is *rectangular* [13] if every finite (equivalently, binary) cozero cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement. It is well-known that $X \times Y$ is rectangular iff every cozero-set in $X \times Y$ is the union of a σ-locally finite collection by cozero rectangles. It is often used the fact that $X \times Y$ is normal and rectangular iff every binary open cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement. In particular, if $X \times Y$ is normal and rectangular, then each closed rectangle $E \times F$ in $X \times Y$ is also normal and rectangular.

As stated in [13, Proposition 1], there are many kinds of rectangular products. In particular, we will use the following typical rectangular products, which was proved in [2].

Lemma 2.1 (Terasawa). *If X is a space and K is a compact space, then $X \times K$ is rectangular.*

Monotone normality. A subset S in an ordinal λ with $\mathrm{cf}(\lambda) > \omega$ is *stationary* if every club (= closed unbounded) set in λ meets S. We denote by $\mathrm{Lim}(S)$ the subset of λ consisting of all limit points of S, that is, $\mathrm{Lim}(S) = \{\alpha \in \lambda : \sup(S \cap \alpha) = \alpha\}$, where $\sup \emptyset = -1$. For stationary sets, we will frequently use a well-known lemma called the Pressing Down Lemma (see [11, Lemma 6.15]), which is abbreviated by PDL.

Let X be a space. For each regular uncountable cardinal κ, we let

$$\mathcal{S}(X, \kappa) = \{E : E \text{ is a closed set in } X \text{ such that it is homeomorphic to a stationary subset in } \kappa\}.$$

For each $E \in \mathcal{S}(X, \kappa)$, we assign a stationary subset S_E in κ and a homeomorphism $e_E : S_E \to E$ onto E, and fix them. We say a subset E of X is *almost contained in* a subset U of X if $|E \setminus U| < |E|$. When $E \in \mathcal{S}(X, \kappa)$ and $U \subset X$, remark that E is almost contained in U iff $e_E(S_E \cap (\gamma, \kappa)) \subset U$ for some $\gamma < \kappa$. Moreover, the following fact witnesses that the choices of S_E and e_E will have no influence on later arguments (see Lemma 3.2 and Lemma 8.6 etc. below).

Fact 2.2 (folklore). *Let S and T be subspaces of a regular uncountable cardinal κ which are homeomorphic to each other. Then there is a club set C in κ such that $S \cap C = T \cap C$. Moreover, if both $e : S \to E$ and $f : T \to E$ are homeomorphism onto a space E, then we can take C as $e \restriction (S \cap C) = f \restriction (T \cap C)$.*

In fact, it is not difficult to show the following: Let S be a subset in a regular uncountable cardinal κ. If $f : S \to \kappa$ is a continuous map such that $f(S)$ is unbounded in κ, then there is a club set C in κ such that $f(\alpha) = \alpha$ for each $\alpha \in S \cap C$.

The following well-known fact is easily obtained by PDL, and we frequently use it.

Fact 2.3 (folklore). *Let X be a space with $E \in \mathcal{S}(X, \kappa)$. If $\mathcal{U}$ is a point-countable family of open sets in X with $E \subset \bigcup \mathcal{U}$, then E is almost contained in some member of $\mathcal{U}$.*

For a space X and a regular uncountable cardinal κ, using $\mathcal{S}(X, \kappa)$, we let

$$\mathcal{S}^*(X) = \{\kappa : \kappa \text{ is a regular uncountable cardinal with } \mathcal{S}(X, \kappa) \neq \emptyset\} \text{ and}$$
$$\mathcal{S}(X) = \bigcup \{\mathcal{S}(X, \kappa) : \kappa \in \mathcal{S}^*(X)\}.$$

Definition 1. A space X is said to be *monotonically normal* if for any two disjoint closed sets E and F in X, one can assign an open set $M(E, F)$, satisfying that

(i) $E \subset M(E, F) \subset \overline{M(E, F)} \subset X \setminus F$,
(ii) if $E \subset E'$ and $F \supset F'$, then $M(E, F) \subset M(E', F')$.

Lemma 2.4 ([5]). *A space X is monotonically normal if and only if for each open set U in X and for each $x \in U$, one can assign an open set $H(x, U)$ in X, satisfying that*

(i) *$x \in H(x, U) \subset U$,*
(ii) *$H(x, U) \cap H(y, V) \neq \emptyset$ implies that $x \in V$ or $y \in U$.*

The function M in Definition 1 is called a *monotone normality operator,* and we call the function H in Lemma 2.4 a *monotone normality assignment* for X. We will use them a couple of times. Instead of them, we will frequently make use of the following powerful results.

Theorem 2.5 (Balogh and Rudin [1])**.** *Let X be a monotonically normal space. For every open cover $\mathcal{U}$ of X, there are a σ-disjoint partial open refinement $\mathcal{V}$ of $\mathcal{U}$ and a discrete collection $\mathcal{F} \subset \mathcal{S}(X)$ such that $X \setminus \bigcup \mathcal{V} = \bigcup \mathcal{F}$.*

Through this theorem, the notation $\mathcal{S}(X, \kappa)$, $\mathcal{S}^*(X)$ and $\mathcal{S}(X)$ defined above are very useful to observe many topological properties of a monotonically normal space X. For example, the following lemma was proved by applying this theorem.

Lemma 2.6 ([18, Lemma 8.2])**.** *Let X be a monotonically normal space and K a compact space. Let $\mathcal{G}$ be an open cover of $X \times K$, satisfying that for each $E \in \mathcal{S}(X)$ and each $y \in K$, there is an open rectangle $P \times Q$ in $X \times K$ such that E is almost contained in P, $y \in Q$ and $P \times Q$ is contained in some member of $\mathcal{G}$. Then there is a locally finite open (cozero) cover $\mathcal{U}$ of X and a family $\{\mathcal{V}_U : U \in \mathcal{U}\}$ of finite open (cozero) covers of K such that $\{U \times V : U \in \mathcal{U}$ and $V \in \mathcal{V}_U\}$ refines $\mathcal{G}$.*

Considering $K = \{q\}$ as one-point space, this lemma immediately yields

Corollary 2.7. *Let X be a monotonically normal space. An open cover $\mathcal{U}$ of X is normal if and only if each $E \in \mathcal{S}(X)$ is almost contained in some member of $\mathcal{U}$.*

Conversely, Lemma 2.6 can be easily derived from this corollary, which is so useful that we will sometimes use it in later proofs.

3. Three neighborhood properties and related products

In this section, we state three neighborhood properties, which are necessary to characterize normality and orthocompactness of products of monotonically normal spaces and several special factors in later sections. Here, we state their definitions, and discuss their implications and the properties of related products.

Definitions for three neighborhood properties. Let us begin with the definitions of three neighborhood properties.

Definition 2 ([9])**.** Let κ be an infinite cardinal. A space Y has *orthocaliber κ at $q \in Y$* if for each collection $\mathcal{V}$ of open neighborhoods of q in X with $|\mathcal{V}| = \kappa$, there is a subcollection $\mathcal{W}$ of $\mathcal{V}$ such that $|\mathcal{W}| = \kappa$ and $q \in \mathrm{Int}(\bigcap \mathcal{W})$. We say that a space Y has *orthocaliber κ* if it has orthocaliber κ at each point of Y.

For a set S of ordinals, we say that a sequence $\{V_\alpha : \alpha \in S\}$ of subsets in a set Y is *descending (increasing)* if $V_\beta \supset V_\alpha$ ($V_\beta \subset V_\alpha$) for each $\alpha, \beta \in S$ with $\beta < \alpha$.

Definition 3. Let κ be a regular cardinal. We say that a space Y has the *κ-descending open preserving property at $q \in Y$* (the *κ-dop property at $q \in Y$* for short) if for every descending sequence $\{V_\alpha : \alpha \in \kappa\}$ of open neighborhoods of q in Y, $q \in \mathrm{Int}(\bigcap_{\alpha \in \kappa} V_\alpha)$ holds. We say that a space Y has the *κ-dop property* if it has the κ-dop property at each point of Y.

It is obvious that a space Y has the κ-dop property iff for every descending sequence $\{V_\alpha : \alpha \in \kappa\}$ of open sets in Y, $\bigcap_{\alpha \in \kappa} V_\alpha$ is open in Y.

A sequence $\{V_\alpha : \alpha \in S\}$ of subsets in a space Y, where S is a set of ordinals, is *continuously descending* if it is descending and $V_\alpha = \bigcap \{V_\beta : \beta \in S \cap \alpha\}$ for each $\alpha \in S \cap \mathrm{Lim}(S)$.

Definition 4. Let Y be a space and S a set of ordinals. We say that Y has the *S-descending open continuously shrinking property* (the *S-docs property* for short) at $q \in Y$ if for each descending sequence $\mathcal{V} = \{V_\alpha : \alpha \in S\}$ of open neighborhoods of q in Y, there is a continuously descending sequence $\mathcal{F} = \{F_\alpha : \alpha \in S\}$ of closed neighborhoods of q in Y such that $F_\alpha \subset V_\alpha$ for each $\alpha \in S$. We call such $\mathcal{F}$ a *continuous shrinking* of $\mathcal{V}$ by closed neighborhoods of q. We say that a space Y has the *S-docs property* if it has the S-docs property at each point of Y.

Obviously, a regular space Y has the ω-docs property, and the S-docs property if $\max S$ exists. The following is easily seen.

Fact 3.1. *Let S and T be unbounded subsets in a limit ordinal λ with $T \subset S$. If a space Y has the S-docs property at $q \in Y$, then it has the T-docs property at q.*

Lemma 3.2. *Let S be a stationary subset in a regular uncountable cardinal κ, and Y a space with $q \in Y$.*

(1) *A descending sequence $\{V_\alpha : \alpha \in S\}$ of open neighborhoods of q in Y has a continuous shrinking by closed neighborhoods of q if $\{V_\alpha : \alpha \in S \cap C\}$ has such a shrinking for some club set C in κ.*

(2) *Y has the S-docs property at q if and only if it has the $(S \cap C)$-docs property at q for some (any) club set C in κ.*

Proof. (1): Take a descending sequence $\{F_\alpha : \alpha \in S \cap C\}$ of closed neighborhoods of q with $F_\alpha \subset V_\alpha$ for each $\alpha \in S \cap C$. For each $\alpha \in S \setminus C$, let $\alpha^+ = \min\{\alpha' \in S \cap C : \alpha < \alpha'\}$ and let $F_\alpha = F_{\alpha^+}$. Then $\mathcal{F} := \{F_\alpha : \alpha \in S\}$ is a descending sequence of closed neighborhoods of q with $F_\alpha \subset V_\alpha$ for each $\alpha \in S$. Pick $\alpha \in S \cap \mathrm{Lim}(S)$. In case $\alpha \in \mathrm{Lim}(S \cap C)$: By $\alpha \in S \cap C \cap \mathrm{Lim}(S \cap C)$, we have $F_\alpha = \bigcap_{\beta \in S \cap C \cap \alpha} F_\beta = \bigcap_{\beta \in S \cap \alpha} F_\beta$. In case $\alpha \notin \mathrm{Lim}(S \cap C)$: Let $\delta = \sup(S \cap C \cap \alpha)$. Then $\delta < \alpha$ and $S \cap C \cap (\delta, \alpha) = \emptyset$. For each $\beta \in S \cap (\delta, \alpha)$, by $\beta \notin C$, we have $\beta^+ \leq \alpha^+$ and $\alpha \leq \beta^+$. Hence we see $\beta^+ = \alpha^+$, that is, $F_\alpha = F_{\alpha^+} = F_{\beta^+} = F_\beta$. Thus $F_\alpha = \bigcap_{\beta \in S \cap (\delta,\alpha)} F_\beta = \bigcap_{\beta \in S \cap \alpha} F_\beta$. This means that $\mathcal{F}$ is continuously descending.

(2): The "only if" part follows from the Fact 3.1. And the "if" part is obvious from (1). □

Remark 3.3. As shown in Example 6.20 below, the S-docs property for a stationary subset S in a regular uncountable cardinal κ depends on the choice of S. On the other hand, Lemma 3.2 shows that for a fixed $E \in \mathcal{S}(X, \kappa)$, the S_E-docs property does not depend on the choice of a stationary set S_E in κ which is homeomorphic to E (by Fact 2.2).

Three neighborhood properties defined above are all different with each other, and several examples dividing them are given later in this section. Here we state the following lemma which is easily seen.

Lemma 3.4. *Let κ be a regular cardinal.*

(1) *If a space Y has orthocaliber κ at $q \in Y$, then it has the κ-dop property at q.*

(2) *If a regular space Y has the κ-dop property at $q \in Y$, then it has the S-docs property at q for each unbounded subset S in κ.*

The relations to products with stationary sets. Let us recall that a space X is *orthocompact* if every open cover of X has an interior-preserving open refinement, where a collection $\mathcal{V}$ of open sets in a space X is *interior-preserving* if $\bigcap\{W : W \in \mathcal{W}\}$ is open in X for any $\mathcal{W} \subset \mathcal{V}$.

Lemma 3.5 ([9, 10])**.** *Let S be a stationary subset in a regular uncountable cardinal κ and Y a space. If $S \times Y$ is orthocompact, then Y has orthocaliber κ.*

We can give analogous results for the κ-dop and S-docs properties to the above.

Lemma 3.6. *Let S be a stationary subset in a regular uncountable cardinal κ and Y a space. If $S \times Y$ is normal and rectangular, then Y has the κ-dop property.*

Proof. Pick any $q \in Y$. Let $\{V_\alpha : \alpha \in \kappa\}$ be a descending sequence of open neighborhoods of q in Y. Then $G := \bigcup\{(S \cap [0, \alpha]) \times V_\alpha : \alpha \in \kappa\}$ is an open set in $S \times Y$ with $S \times \{q\} \subset G$. Since $S \times Y$ is normal and rectangular, there is a σ-locally finite collection $\mathcal{H}$ of cozero rectangles in $S \times Y$ such that $S \times \{q\} \subset \bigcup \mathcal{H} \subset G$. Applying Fact 2.3 for a σ-locally finite open cover $\{U : U \times W \in \mathcal{H}, q \in W\}$ of S, we obtain $U \times W \in \mathcal{H}$ and $\gamma < \kappa$ with $S \cap (\gamma, \kappa) \subset U$ and $q \in W$. Pick any $y \in W$ and $\alpha \in S \cap (\gamma, \kappa)$. Since $\langle \alpha, y \rangle \in U \times W \subset \bigcup \mathcal{H} \subset G$, there is an $\alpha_0 \in S$ with $\langle \alpha, y \rangle \in (S \cap [0, \alpha_0]) \times V_{\alpha_0}$. By $\alpha \leq \alpha_0$, we have $y \in V_{\alpha_0} \subset V_\alpha$. Hence $W \subset \bigcap_{\alpha \in S \cap (\gamma,\kappa)} V_\alpha = \bigcap_{\alpha \in \kappa} V_\alpha$ holds. This means that $q \in \mathrm{Int}(\bigcap_{\alpha \in \kappa} V_\alpha)$. □

Lemma 3.7. *Let S be a stationary subset in a regular uncountable cardinal κ and Y a space. If $S \times Y$ is normal, then Y has the S-docs property.*

Proof. Pick any $q \in Y$. Let $\mathcal{V} = \{V_\alpha : \alpha \in S\}$ be a descending sequence of open neighborhoods of q in Y. Then $G := \bigcup\{(S \cap [0,\alpha]) \times V_\alpha : \alpha \in S\}$ is an open set in $S \times Y$ with $S \times \{q\} \subset G$. There is an open set H in $S \times Y$ such that $S \times \{q\} \subset H \subset \overline{H} \subset G$. For each $\alpha \in S$, there are a $\gamma(\alpha) < \alpha$ and an open neighborhood Q_α of q in Y such that $\big(S \cap (\gamma(\alpha), \alpha]\big) \times Q_\alpha \subset H$. By PDL, there are $T \subset S$ and $\gamma \in \kappa$ such that T is stationary in κ and $\gamma(\alpha) = \gamma$ for each $\alpha \in T$. To see that $\mathcal{V}$ has a continuous shrinking by closed neighborhoods of q, it suffices to find a continuous shrinking $\{F_\alpha : \alpha \in S \cap (\gamma, \kappa)\}$ of $\{V_\alpha : \alpha \in S \cap (\gamma,\kappa)\}$ by closed neighborhoods of q. For each $\alpha \in S \cap (\gamma,\kappa) \setminus \mathrm{Lim}(S)$, let $F_\alpha = \overline{\{y \in Y : \big(S \cap (\gamma,\alpha]\big) \times \{y\} \subset H\}}$. Taking a $\delta \in T$ with $\alpha \leq \delta$, we have $\big(S \cap (\gamma,\alpha]\big) \times Q_\delta \subset \big(S \cap (\gamma(\delta),\delta]\big) \times Q_\delta \subset H$, so $q \in Q_\delta \subset F_\alpha$ holds. Hence F_α is a closed neighborhood of q in Y. For each $\alpha \in S \cap (\gamma,\kappa) \cap \mathrm{Lim}(S)$, let $F_\alpha = \bigcap\{F_\xi : \xi \in S \cap (\gamma,\alpha) \setminus \mathrm{Lim}(S)\}$. Then $\{F_\alpha : \alpha \in S \cap (\gamma,\kappa)\}$ is a continuously descending sequence of closed neighborhoods of q in Y.

It suffices to show that each F_α is contained in V_α. Pick any $\alpha \in S \cap (\gamma,\kappa)$ and any $z \in F_\alpha$. Take any open rectangle $U \times W$ in $S \times Y$ containing $\langle \alpha, z\rangle$. We can pick $\beta \in U \cap (S \setminus \mathrm{Lim}(S))$ with $\gamma < \beta \leq \alpha$. Since $z \in F_\alpha \subset F_\beta$, by the choice of F_β, we can pick $y \in W$ such that $\big(S \cap (\gamma,\beta]\big) \times \{y\} \subset H$. We obtain $\langle \beta, y\rangle \in (U \times W) \cap H$. This means that $\langle \alpha, z\rangle \in \overline{H} \subset G$. By the choice of G, there is $\eta \in S$ with $\langle \alpha, z\rangle \in (S \cap [0,\eta]) \times V_\eta$. By $\alpha \leq \eta$, we obtain that $z \in V_\eta \subset V_\alpha$. Hence $F_\alpha \subset V_\alpha$ holds. □

Pulling together lemmas stated above, we have the following:

Implications 1. Let S be a stationary subset in a regular uncountable cardinal κ.

$$\begin{array}{ccc}
S \times Y\text{:} & & Y\text{:} \\
\text{orthocompact} & \Longrightarrow & \text{orthocaliber } \kappa \\
 & & \Downarrow \\
\text{normal and rectangular} & \Longrightarrow & \kappa\text{-dop property} \\
 & & \Downarrow \\
\text{normal} & \Longrightarrow & S\text{-docs property}
\end{array}$$

More about three neighborhood properties. In this subsection, we compare three neighborhood properties with some well-known concepts about neighborhoods. And several examples dividing these properties are found. Moreover, the κ-dop property is characterized by closedness of the projection. Most of our main results in this paper do not need the consequences in this subsection, though they probably makes three neighborhood properties more familiar to one. The reader may skip this subsection and go to Section 4.

Lemma 3.8. *Let κ be a regular cardinal and Y a space with $q \in Y$.*

(1) *If the character of Y at q is less than κ, then Y has orthocaliber κ at q.*
(2) *If the tightness of Y at q is less than κ, then Y has the κ-dop property at q.*

Proof. (1) is obvious. To see (2), assume that Y does not have the κ-dop property at q. Then there is a descending sequence $\{V_\alpha : \alpha \in \kappa\}$ of neighborhoods of q in Y such that $q \in \overline{Y \setminus \bigcap_{\alpha\in\kappa} V_\alpha}$. On the other hand, it is easy to see that for each $A \subset Y \setminus \bigcap_{\alpha\in\kappa} V_\alpha$ with $|A| < \kappa$, there is a $\beta \in \kappa$ with $V_\beta \cap A = \emptyset$, so we have $q \notin \overline{A}$. Hence the tightness of Y at q is at least κ. □

For an infinite cardinal κ and for a space Y with $q \in Y$, as a well known concept, recall that q is a *P_κ-point* in Y if for any collection $\mathcal{V}$ of open neighborhoods of q in X with $|\mathcal{V}| < \kappa$, the intersection $\bigcap \mathcal{V}$ is a neighborhood of q.

Proposition 3.9. *For an infinite cardinal κ and a space Y with $q \in Y$, the following are equivalent.*

(a) *q is a P_κ-point in Y.*
(b) *Y has orthocaliber τ at q for each regular cardinal $\tau < \kappa$.*

(c) *Y has the τ-dop property at q for each regular cardinal $\tau < \kappa$.*

Proof. (a)⇒(b): This is obvious.

(b)⇒(c): This follows from Lemma 3.4(1).

(c)⇒(a): By induction on $\kappa' < \kappa$, we show that for each collection $\mathcal{V}$ of open neighborhoods of q in Y with $|\mathcal{V}| = \kappa'$, $\bigcap \mathcal{V}$ is a neighborhood of q. We may consider $\kappa' \geq \omega$. Let $\tau = \mathrm{cf}(\kappa')$. Then τ is a regular cardinal with $\tau \leq \kappa' < \kappa$. We can express $\mathcal{V}$ as $\mathcal{V} = \bigcup_{\alpha \in \tau} \mathcal{V}_\alpha$, $|\mathcal{V}_\alpha| < \kappa'$, and $\mathcal{V}_\alpha \subset \mathcal{V}_\beta$ for every $\alpha < \beta < \tau$. Let $W_\alpha = \mathrm{Int}(\bigcap \mathcal{V}_\alpha)$ for each $\alpha \in \tau$. By inductive hypothesis, $\{W_\alpha : \alpha \in \tau\}$ is a descending sequence of open neighborhoods of q in Y. By (c), $\bigcap_{\alpha \in \tau} W_\alpha \subset \bigcap \mathcal{V}$ is a neighborhood of q. Hence, q is a P_κ-point in Y. □

Let Y be a space with $q \in Y$ and let κ and τ be regular cardinals with $\kappa < \tau$. If q is a P_τ-point in Y, then q is a P_κ-point in Y. On the other hand, it is possible that Y has orthocaliber τ (hence has the τ-dop property) at q, but Y does not have the κ-dop property (hence does not have orthocaliber κ) at q.

Example 3.10. Let κ be a regular cardinal. Then an ordinal $\kappa + 1 = [0, \kappa]$ is a compact space which has orthocaliber τ at κ for each regular cardinal $\tau \neq \kappa$, but does not have the κ-dop property at κ. Obviously, $[0, \kappa]$ has the character (tightness) κ at κ.

Let κ be a regular cardinal. Let Y be a space with $q \in Y$. As seen in Lemma 3.8 and Proposition 3.9, if q is a P_τ-point for some cardinal $\tau > \kappa$ or the character (tightness) of Y at q is less than κ, then Y has orthocaliber κ (the κ-dop property). The converse does not hold at all.

Example 3.11. Let μ, κ and ν be regular cardinals with $\mu < \kappa < \nu$. Then $Y = [0, \mu] \times [0, \nu]$ is a compact space which has orthocaliber κ at $q = \langle \mu, \nu \rangle$. But q is not a P_τ point for any cardinal $\tau > \kappa$. And the character (tightness) of Y at q is ν, which is larger than κ.

Example 3.12. For regular cardinals θ and κ with $\theta \leq \kappa$, let $A_\theta(\kappa)$ denote the space with $|A_\theta(\kappa)| = \kappa$, having only one non-isolated point q, such that $V \subset A_\theta(\kappa)$ is a neighborhood of q in $A_\theta(\kappa)$ iff $q \in V$ and $|A_\theta(\kappa) \setminus V| < \theta$. Then,

(1) $A_\theta(\kappa)$ has the τ-dop property for each regular cardinal τ with $\tau \neq \theta$.
(2) $A_\theta(\kappa)$ does not have the θ-dop property at q.
(3) $A_\theta(\kappa)$ does not have orthocaliber τ at q for each cardinal τ with $\theta \leq \tau \leq \kappa$.

Proof. (1) Let $\{V_\alpha : \alpha \in \tau\}$ be a descending sequence of open neighborhoods of q in $A_\theta(\kappa)$, where τ is a regular cardinal with $\tau \neq \theta$. By an easy cardinal arithmetic, whenever $\tau < \theta$, $\bigcap_{\alpha \in \tau} V_\alpha$ is obviously an open neighborhood of q. Also whenever $\tau > \theta$, we can find $\alpha_0 \in \tau$ such that $V_\alpha = V_{\alpha_0}$ fore every $\alpha \in \tau$ with $\alpha_0 \leq \alpha$. Otherwise, we can inductively choose a sequence $\{\alpha(\xi) : \xi \in \theta\}$ of τ such that $V_{\alpha(\xi)} \setminus V_{\alpha(\xi+1)} \neq \emptyset$ for each $\xi \in \theta$. Let $\zeta = \sup\{\alpha(\xi) : \xi \in \theta\}$. By $\zeta < \tau$, V_ζ is not a neighborhood of q. Hence $\bigcap_{\alpha \in \tau} V_\alpha = V_{\alpha_0}$ is a neighborhood of q. On the other hand, (2) and (3) are obvious. □

Remark 3.13. In the example above, we may consider that

- $A_\omega(\kappa)$ is the one-point compactification of a discrete space of cardinality κ. It has the τ-dop property for every regular uncountable cardinal τ, but does not have orthocaliber κ at q.
- $A_\kappa(\kappa)$ is the subspace in an ordinal $\kappa + 1$ defined by $A_\kappa(\kappa) = \{\kappa\} \cup \{\alpha + 1 : \alpha \in \kappa\}$ and $q = \kappa$, where κ is a regular cardinal. It does not have the κ-dop property at q.

By the first statement of the remark above, we see that the converse of Lemma 3.4(1) does not hold. Moreover, the following example shows that the converse of 3.4(2) does not also hold.

Example 3.14. Let κ be an infinite cardinal. There is a space $Y[\kappa]$ having exactly one non-isolated point q such that $Y[\kappa]$ has the τ-docs property for each regular uncountable cardinal τ, but does not have the τ-dop property for any regular cardinal τ with $\tau \leq \kappa$.

Proof. Ohta [12] constructed the space $Y[\kappa]$ defined by

(i) $Y[\kappa] = [\kappa]^{<\omega} \cup \{q\}$ as a set,
(ii) each $y \in [\kappa]^{<\omega}$ is an isolated point in $Y[\kappa]$,
(iii) $\{B(r) : r \in [\kappa]^{<\omega}\}$ is a neighborhood base at q in $Y[\kappa]$, where $B(r) = \{y \in [\kappa]^{<\omega} : r \subset y\} \cup \{q\}$.

He proved that $\kappa \times Y[\kappa]$ is normal, but not rectangular for each regular uncountable cardinal κ. We show that $Y[\kappa]$ is also an example required here. Note that every neighborhood of q is clopen in $Y[\kappa]$ since only q is a non-isolated point.

Let τ be a regular cardinal with $\tau \le \kappa$. For each $\alpha \in \tau$, let $W_\alpha = \{q\} \cup \{y \in [\kappa]^{<\omega} : y \cap (\alpha, \tau) \ne \emptyset\}$. Then $\{W_\alpha : \alpha \in \tau\}$ is a descending sequence of open neighborhoods of q in Y. Since $\bigcap_{\alpha\in\tau} W_\alpha = \{q\}$ is not a neighborhood of q, $Y[\kappa]$ does not have the τ-dop property at q.

We will show that the space $Y[\kappa]$ has the τ-docs property for each regular uncountable cardinal τ. Let $\{V_\xi : \xi \in \tau\}$ be a descending sequence of open neighborhoods of q in $Y[\kappa]$. For each $\zeta \in \tau$, take and fix a $r_\zeta \in [\kappa]^{<\omega}$ with $B(r_\zeta) \subset V_\zeta$. For each $r \in [\kappa]^{<\omega}$, let $S(r) = \{\zeta \in \tau : r_\zeta = r\}$.

In case that there is an $r \in [\kappa]^{<\omega}$ such that $S(r)$ is unbounded in τ: Letting $F_\xi = B(r)$ for each $\xi \in \tau$, we obtain a continuous shrinking $\{F_\xi : \xi \in \kappa\}$ of $\{V_\xi : \xi \in \kappa\}$ by closed neighborhoods of q. Actually, by taking $\zeta \in S(r)$ with $\xi \le \zeta$, we have $r_\zeta = r$, and so $F_\xi = B(r) = B(r_\zeta) \subset V_\zeta \subset V_\xi$.

In case that $S(r)$ is bounded in τ for any $r \in [\kappa]^{<\omega}$: Take a club set C of τ such that $\sup S(r_\zeta) < \xi$ for every $\xi \in C$ and $\zeta < \xi$. And let $F_\xi = \bigcup_{\eta\in\tau\setminus\xi} B(r_\eta)$ for each $\xi \in C$. It suffices from Lemma 3.2 to show that $\{F_\xi : \xi \in C\}$ is a continuous shrinking of $\{V_\xi : \xi \in C\}$. It is easy to see that $\{F_\xi : \xi \in C\}$ is a descending sequence of closed neighborhoods of q with $F_\xi \subset V_\xi$ for every $\xi \in C$. To see that $\{F_\xi : \xi \in C\}$ is continuously descending, let $\xi \in \mathrm{Lim}(C)$. Then $\sup(S(r) \cap \xi) < \xi$ for any $r \in [\kappa]^{<\omega}$. Actually, if $\zeta \in S(r) \cap \xi$, then $r_\zeta = r$, so $S(r) = S(r_\zeta)$. It follows from $\xi \in C$ and $\zeta < \xi$ that $\sup(S(r) \cap \xi) \le \sup S(r_\zeta) < \xi$. Let $y \in \bigcap_{\zeta\in C\cap\xi} F_\zeta$. If $y \ne q$, then $y \in [\kappa]^{<\omega}$ is a finite set, so it has at most finitely many subsets. By $\xi \in \mathrm{Lim}(C)$, we can take a $\zeta \in C \cap \xi$ such that $S(r) \cap [\zeta, \xi) = \emptyset$ for every $r \subset y$. By $y \in F_\zeta$, there is $\eta \in \tau \setminus \zeta$ with $y \in B(r_\eta)$. Then $r_\eta \subset y$. So $S(r_\eta) \cap [\zeta, \xi) = \emptyset$. Therefore $\eta \notin \zeta \cup [\zeta, \xi) = \xi$ since $\eta \in S(r_\eta)$. We have $y \in F_\xi$. Hence, $F_\xi = \bigcap_{\zeta\in C\cap\xi} F_\zeta$. So $\{F_\xi : \xi \in C\}$ is continuously descending. □

We close the section by showing another characterization of the κ-dop property.

Proposition 3.15. *Let Y be a space and κ a regular cardinal. Then the following are equivalent.*

(a) *Y has the κ-dop property.*
(b) *The projection $\pi : \kappa \times Y \to Y$ is closed.*
(c) *The projection $\pi_A : A \times Y \to Y$ is closed for some unbounded subset A in κ.*

Proof. (a)⇒(b): Take any closed set F in $\kappa \times Y$. Pick any $y \in Y \setminus \pi(F)$. For each $\alpha \in \kappa$, let

$$V_\alpha = \bigcup \{V : V \text{ is open in } Y \text{ with } ([0, \alpha] \times V) \cap F = \emptyset\}.$$

Then $\{V_\alpha : \alpha \in \kappa\}$ is a descending sequence of open sets in Y. Let $W = \bigcap_{\alpha\in\kappa} V_\alpha$. Then W is an open set in Y missing $\pi(F)$. Since each $[0, \alpha]$ is compact, W contains y. Hence $\pi(F)$ is closed in Y.

(b)⇒(c): This is obvious.

(c)⇒(a): Let $\{V_\alpha : \alpha \in \kappa\}$ be a descending sequence of open sets in Y. Let $W = \bigcap_{\alpha\in\kappa} V_\alpha$ and pick any $y \in W$. Let $F = (A \times Y) \setminus \bigcup_{\alpha\in\kappa} \big((A \cap [0, \alpha]) \times V_\alpha\big)$. Then F is a closed set in $A \times Y$ with $(A \times \{y\}) \cap F = \emptyset$. Since π_A is closed and $y \notin \pi_A(F)$, there is an open neighborhood Q of y in Y with $(A \times Q) \cap F = \emptyset$. Let $z \in Q$. For each $\alpha \in A$, since $\langle \alpha, z\rangle \in A \times Q \subset (A \times Y) \setminus F$, there is an $\alpha_0 \in A$ with $\langle \alpha, z\rangle \in (A \cap [0, \alpha_0]) \times V_{\alpha_0}$. By $\alpha \le \alpha_0$, we have $z \in V_{\alpha_0} \subset V_\alpha$. Hence $Q \subset \bigcap_{\alpha\in A} V_\alpha = W$ holds, which means that W is open in Y. □

4. Products with compact factors

Only this section has no subsection. A theorem here is a characterization for normality of products of a monotonically normal space and a compact space, which makes clearer than that of [17, Theorem 3.1].

Theorem 4.1. *Let X be a monotonically normal space and K a compact space. Then the following are equivalent.*

(a) *$X \times K$ is normal.*
(b) *$E \times K$ is normal for each $E \in \mathcal{S}(X)$.*
(c) *K has the κ-dop property for each $\kappa \in \mathcal{S}^*(X)$.*
(d) *K has the S_E-docs property for each $E \in \mathcal{S}(X)$.*

Lemma 4.2. *Let Y be a locally compact space with $q \in Y$. Let S be a stationary subset in a regular uncountable cardinal κ. Then Y has the S-docs property at q if and only if it has the κ-dop property at q.*

Proof. It suffices from Lemma 3.4(2) to show the "only if" part. Let $\{V_\alpha : \alpha \in \kappa\}$ be a descending sequence of open neighborhoods of q in Y. Take a compact neighborhood K of q in Y. By the assumption, there is a continuously descending sequence $\{F_\alpha : \alpha \in S\}$ of closed neighborhoods of q in Y such that $F_\alpha \subset V_\alpha$ for each $\alpha \in \kappa$. Note that there is $\gamma(\alpha) < \alpha$ with $K \cap F_{\gamma(\alpha)} \subset V_\alpha$ for each $\alpha \in S \cap \mathrm{Lim}(S)$. In fact, this follows from the compactness of K and $K \cap (\bigcap_{\beta \in S \cap \alpha} F_\beta) = K \cap F_\alpha \subset V_\alpha$. By PDL, there are $T \subset S \cap \mathrm{Lim}(S)$ and $\gamma \in \kappa$ such that T is stationary in κ and $\gamma(\alpha) = \gamma$ for each $\alpha \in T$. Pick any $\alpha \in \kappa$. Take a $\delta \in T$ with $\alpha < \delta$. Then we have $K \cap F_\gamma = K \cap F_{\gamma(\delta)} \subset V_\delta \subset V_\alpha$. Hence we obtain $q \in \mathrm{Int}(K \cap F_\gamma) \subset \mathrm{Int}(\bigcap_{\alpha \in \kappa} V_\alpha)$. □

Lemma 4.3. *Let κ be a regular uncountable cardinal. Let X be a space with $E \in \mathcal{S}(X, \kappa)$. Let Y be a space having the κ-dop property at $q \in Y$. If an open set G in $X \times Y$ contains $E \times \{q\}$, then there are a $\gamma \in \kappa$ and an open neighborhood W of q in Y such that $e_E\big(S_E \cap (\gamma, \kappa)\big) \times W \subset G$.*

Proof. Let $S = S_E$ and $e = e_E$. For each $\alpha \in S$, there are a $\gamma(\alpha) \in \alpha$ and an open neighborhood V_α of q in Y such that $e\big(S \cap (\gamma(\alpha), \alpha]\big) \times V_\alpha \subset G$. By PDL, there are $T \subset S$ and $\gamma \in \kappa$ such that T is stationary in κ and $\gamma(\alpha) = \gamma$ for each $\alpha \in T$. Let $W_\alpha = \mathrm{Int}\{y \in Y : e\big(S \cap (\gamma, \alpha]\big) \times \{y\} \subset G\}$ for each $\alpha \in T$. Then, $e(S \cap (\gamma, \alpha]) \times W_\alpha \subset G$ holds. And W_α contains V_α since $e\big(S \cap (\gamma, \alpha]\big) \times V_\alpha = e\big(S \cap (\gamma(\alpha), \alpha]\big) \times V_\alpha \subset G$. Hence $\{W_\alpha : \alpha \in T\}$ is a descending sequence of open neighborhoods of q in X. By the assumption, $W := \mathrm{Int}(\bigcap_{\alpha \in T} W_\alpha)$ is an open neighborhood of q in Y. We also have that $e\big(S \cap (\gamma, \kappa)\big) \times W \subset \bigcup_{\alpha \in T}(e(S \cap (\gamma, \alpha]) \times W_\alpha) \subset G$. □

Proof of Theorem 4.1. (a)⇒(b): This is obvious.

(b)⇒(c) : This immediately follows from Lemmas 2.1 and 3.6.

(c)⇔(d) : Since K is compact, this is an immediate consequence of Lemma 4.2.

(c)⇒(a): Let $\mathcal{G} = \{G_0, G_1\}$ be a binary open cover of $X \times K$. We show that $\mathcal{G}$ is normal. Take a $\kappa \in \mathcal{S}^*(X)$ and an $E \in \mathcal{S}(X, \kappa)$, and let $S = S_E$ and $e = e_E$. Pick any $q \in K$. By Fact 2.3, there are $\delta \in \kappa$ and $i \in 2$ such that $e\big(S \cap (\delta, \kappa)\big) \times \{q\} \subset G_i$. By the assumption of K, it follows from Lemma 4.3 that there are a $\gamma \in (\delta, \kappa)$ and an open neighborhood W of q in K such that $e\big(S \cap (\gamma, \kappa)\big) \times W \subset G_i$. Take an open neighborhood Q of q in K with $\overline{Q} \subset W$. Let $P = \{x \in X : \{x\} \times \overline{Q} \subset G_i\}$. Since $\overline{Q}$ is compact, P is an open set in X. Moreover, we conclude that $e\big(S \cap (\gamma, \kappa)\big) \subset P$ and $P \times Q \subset G_i$. It follows from Lemma 2.6 that $\mathcal{G}$ is normal. Hence $X \times K$ is normal. □

Moreover, we can obtain an analogue to Theorem 4.1 for orthocompactness of the same products.

Theorem 4.4. *Let X be a monotonically normal space and K a non-empty compact space. Then the following are equivalent.*

(a) *$X \times K$ is orthocompact.*
(b) *X is orthocompact and $E \times K$ is orthocompact for each $E \in \mathcal{S}(X)$.*
(c) *X is orthocompact and K has orthocaliber κ for each $\kappa \in \mathcal{S}^*(X)$.*

This will be proved by a more generalized form in terms of $\mathbb{DC}$-likeness in the next section (see Theorem 5.7).

5. Products with $\mathbb{DC}$-like factors

Telgársky [15] introduced and studied the topological game $G(\mathbb{K}, Y)$. This section devotes to generalize the results in the previous section in terms of topological games in the sense of Telgársky as studied in [18].

Topological games of Telgársky. Details of Telgársky's topological game $G(\mathbb{K}, Y)$ are described in [15, 18]. For reader's convenience, we give here only a sketch of the definition.

Definition 5. Let Y be a space, and $\mathbb{K}$ a class of spaces, for instance $\mathbb{K} = \mathbb{DC}$, where $\mathbb{DC}$ denotes the class of all spaces which have a discrete cover by compact sets. In the game $G(\mathbb{K}, Y)$, two players take closed subsets E_n and F_n in Y for each $n \in \omega$ in turn $E_0, F_0, E_1, F_1, \cdots$. Player I chooses $E_n \in \mathbb{K}$ with $E_n \subset F_{n-1}$, where $F_{-1} = Y$. Player II chooses F_n with $F_n \subset F_{n-1} \setminus E_n$. Player I wins if $\bigcap_{n\in\omega} F_n = \emptyset$. A space Y is said to be $\mathbb{DC}$-*like* if Player I has a winning strategy in the game $G(\mathbb{DC}, Y)$.

It is known that a space with a σ-closure preserving cover by compact sets and a subparacompact $\mathbb{C}$-scattered space are $\mathbb{DC}$-like. So the class of $\mathbb{DC}$-like spaces is much broader than that of compact spaces. The class of $\mathbb{DC}$-like spaces plays important roles in the study of covering properties of rectangular products. In fact, the following was proved.

Theorem 5.1 ([4, 15, 16])**.** *If X is a paracompact (metacompact) space and Y is a paracompact (metacompact regular) $\mathbb{DC}$-like space, then $X \times Y$ is paracompact and rectangular (metacompact).*

Arguments in this section is worth considering only in the case that a monotonically normal space X does not have a weak covering property such as the weak metalindelöf property. Otherwise, it follows from [1, Corollary 2.1] that X is paracompact, and so all results would be trivial from Theorem 5.1.

Normality and rectangularity of the products.

Theorem 5.2. *Let X be a monotonically normal and orthocompact space and Y a paracompact $\mathbb{DC}$-like space. Then the following are equivalent.*
- (a) *$X \times Y$ is normal and rectangular.*
- (b) *$E \times Y$ is normal and rectangular for each $E \in \mathcal{S}(X)$.*
- (c) *Y has the κ-dop property for each $\kappa \in \mathcal{S}^*(X)$.*

Remark 5.3. It is natural to ask whether orthocompactness of X in Theorem 5.2 can be taken off. However, Example 6.18 below shows that it is negative. Hence the compact factor of Theorem 4.1 cannot be extended to the $\mathbb{DC}$-like factor as in Theorem 5.2.

Let us begin to prove Theorem 5.2.

Lemma 5.4 ([18, Lemmas 8.2 and 8.3])**.** *Let X be a monotonically normal space and Y a paracompact (metacompact regular) $\mathbb{DC}$-like space. Let $\mathcal{G}$ be an open cover of $X \times Y$. Assume that, for each $E \in \mathcal{S}(X)$ and for each $y \in Y$, there is an open rectangle $U \times V$ in $X \times Y$ such that E is almost contained in U, $y \in V$ and $U \times V$ is contained in some member of $\mathcal{G}$. Then $\mathcal{G}$ has a σ-locally finite rectangular cozero refinement (a point-finite rectangular open refinement).*

Lemma 5.5. *Let X be a monotonically normal space and Y a paracompact $\mathbb{DC}$-like space. Assume that for each $E \in \mathcal{S}(X)$, for each $y \in Y$ and for each open set G in $X \times Y$ containing $E \times \{y\}$, there is an open rectangle $U \times V$ in $X \times Y$ such that E is almost contained in U, $y \in V$ and $U \times V \subset G$. Then $X \times Y$ is normal and rectangular.*

Proof. Let $\mathcal{G} = \{G_0, G_1\}$ be a binary open cover of $X \times Y$. Take any $E \in \mathcal{S}(X, \kappa)$ with $\kappa \in \mathcal{S}^*(X)$ and pick any $y \in Y$. By Fact 2.3, there are an $i \in 2$ and an $E' = e_E(S_E \cap (\delta, \kappa))$ for some $\delta \in \kappa$ such that $E' \times \{y\} \subset G_i$. Applying the assumption for $E' \in \mathcal{S}(X, \kappa)$, we obtain an open rectangle $U \times V$ in $X \times Y$ such that E' is (and so E is) almost contained in U, $y \in V$ and $U \times V \subset G_i$. By Lemma 5.4, $\mathcal{G}$ has a σ-locally finite rectangular cozero refinement. Hence $X \times Y$ is normal and rectangular. $\square$

Lemma 5.6. *Let X be an orthocompact space with $E \in \mathcal{S}(X,\kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let $\mathcal{U}$ be a collection of open sets in X with $E \subset \bigcup\mathcal{U}$. Then there are a $\gamma \in \kappa$ and an increasing open expansion $\{P(\alpha) : \alpha \in (\gamma,\kappa)\}$ of $\{e_E(S_E \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ in X which partially refines $\mathcal{U}$.*

Proof. Since $\mathcal{U} \cup \{X \setminus E\}$ is an open cover of X, it has an interior-preserving open refinement $\mathcal{V}$. Let $S = S_E$ and $e = e_E$. Pick any $\alpha \in S$. Choose a $V_\alpha \in \mathcal{V}$ with $e(\alpha) \in V_\alpha$ and a $\gamma(\alpha) \in \alpha \cup \{-1\}$ with $e(S \cap (\gamma(\alpha),\alpha]) \subset V_\alpha$. By PDL, there are $T \subset S$ and $\gamma \in \kappa$ such that T is stationary in κ and $\gamma(\alpha) = \gamma$ for each $\alpha \in T$. Let $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma,\kappa)\}$, where $P(\alpha) = \bigcap\{V \in \mathcal{V} : e(S \cap (\gamma,\alpha]) \subset V\}$. Then $\mathcal{P}$ is an increasing open expansion of $\{e(S \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ since $\mathcal{V}$ is interior preserving and $\{e(S \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ is increasing. Let $\alpha \in (\gamma,\kappa)$. Take a $\beta \in T$ with $\alpha \leq \beta$. By $e(S \cap (\gamma,\alpha]) \subset e(S \cap (\gamma(\beta),\beta]) \subset V_\beta$ with $V_\beta \in \mathcal{V}$, we have $P(\alpha) \subset V_\beta$. Since $\mathcal{V}$ refines $\mathcal{U} \cup \{X \setminus E\}$ and $e(\beta) \in V_\beta \cap E$, there is a $U \in \mathcal{U}$ with $P(\alpha) \subset V_\beta \subset U$. Hence $\mathcal{P}$ partially refines $\mathcal{U}$. □

Proof of Theorem 5.2. (a)⇒(b): This is obvious.

(b)⇒(c): This immediately follows from Lemma 3.6.

(c)⇒(a): Take any $E \in \mathcal{S}(X,\kappa)$ with $\kappa \in \mathcal{S}^*(X)$, and pick any $y \in Y$. By (c), Y has the κ-dop property at y. Moreover, take any open set G in $X \times Y$ containing $E \times \{y\}$. Let $\mathcal{U}$ be the family of all open sets U in X such that $U \times V \subset G$ for some open neighborhood V of y in Y. Then we have $E \subset \bigcup\mathcal{U}$. Let $S = S_E$ and $e = e_E$. It follows from Lemma 5.6 that there are a $\gamma \in \kappa$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma,\kappa)\}$ of $\{e(S \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ in X which partially refines $\mathcal{U}$. By the definition, $\mathcal{P}$ is a subcollection of $\mathcal{U}$. For each $\alpha \in (\gamma,\kappa)$, let $W_\alpha = \mathrm{Int}_Y(\{z \in Y : P(\alpha) \times \{z\} \subset G\})$. By $P(\alpha) \in \mathcal{U}$, we see that W_α is an open neighborhood of y in Y. Since $\{W_\alpha : \alpha \in (\gamma,\kappa)\}$ is descending, the κ-dop property of Y witnesses that $W := \mathrm{Int}(\bigcap_{\alpha \in (\gamma,\kappa)} W_\alpha)$ is an open neighborhood of y in Y. Let $U = \bigcup\{P(\alpha) : \alpha \in (\gamma,\kappa)\}$. Then U is an open set with $e(S \cap (\gamma,\kappa)) \subset U$, that is, E is almost contained in U. Moreover, we obtain that $U \times W \subset \bigcup\{P(\alpha) \times W_\alpha : \alpha \in (\gamma,\kappa)\} \subset G$. It follows from Lemma 5.5 that $X \times Y$ is normal and rectangular. □

Orthocompactness of the products. Now, we show a generalization of Theorem 4.4 in terms of $\mathbb{DC}$-likeness.

Theorem 5.7. *Let X be a monotonically normal space and Y a non-empty metacompact $\mathbb{DC}$-like regular space. Then the following are equivalent.*

(a) *$X \times Y$ is orthocompact.*
(b) *X is orthocompact and $E \times Y$ is orthocompact for each $E \in \mathcal{S}(X)$.*
(c) *X is orthocompact and Y has orthocaliber κ for each $\kappa \in \mathcal{S}^*(X)$.*

The following which is a generalization of Theorem 1.1 is an immediate consequence of Theorems 5.2, 5.7 and Lemma 3.4(1).

Corollary 5.8 ([18])**.** *Let X be a monotonically normal space and Y a paracompact $\mathbb{DC}$-like space. If $X \times Y$ is orthocompact, then it is normal and rectangular.*

Let us prove Theorem 5.7.

Lemma 5.9. *Let X be an orthocompact space with $E \in \mathcal{S}(X,\kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let Y be a space with orthocaliber κ at $q \in Y$. Let $\mathcal{G}$ be an open cover of $X \times Y$. Then there are a $\gamma \in \kappa$, an open neighborhood V of q in Y and an increasing open expansion $\{P(\alpha) : \alpha \in (\gamma,\kappa)\}$ of $\{e_E(S_E \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ in X such that $\{P(\alpha) \times V : \alpha \in (\gamma,\kappa)\}$ partially refines $\mathcal{G}$.*

Proof. Let $\mathcal{U}$ be the family of all open sets U in X such that $U \times V$ is contained in some member of $\mathcal{G}$ for some open neighborhood V of q in Y. Then $\mathcal{U}$ is an open cover of X. Let $S = S_E$ and $e = e_E$. It follows from Lemma 5.6 that there are a $\gamma \in \kappa$ and an increasing open expansion $\{P(\alpha) : \alpha \in (\gamma,\kappa)\}$ of $\{e(S \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ in X which partially refines $\mathcal{U}$. For each $\alpha \in (\gamma,\kappa)$, we can take an open neighborhood $V(\alpha)$ of q in Y such that $P(\alpha) \times V(\alpha)$ is contained in some member $G(\alpha)$ of $\mathcal{G}$. Since Y has orthocaliber κ, we can take an open neighborhood V of q in Y such that $V \subset V(\beta)$ holds for unbounded many $\beta \in (\gamma,\kappa)$ in κ. For each $\alpha \in (\gamma,\kappa)$, by taking $\beta \in [\alpha,\kappa)$ with $V \subset V(\beta)$, we have $P(\alpha) \times V \subset P(\beta) \times V(\beta) \subset G(\beta)$. Hence $\{P(\alpha) \times V : \alpha \in (\gamma,\kappa)\}$ partially refines $\mathcal{G}$. □

Proof of Theorem 5.7. (a)⇒(b): This is obvious.
(b)⇒(c): This immediately follows from Lemma 3.5.
(c)⇒(a): Let $\mathcal{G}$ be an open cover of $X \times Y$. Let

$$\mathcal{O} = \{O \subset X \times Y : O \text{ is open in } X \times Y,\ \mathcal{G} \restriction O \text{ has an interior-preserving open refinement in } O\}.$$

Then $\mathcal{O}$ is an open cover of $X \times Y$. It suffices to show that $\mathcal{O}$ has a point-finite open refinement. Take an $E \in \mathcal{S}(X,\kappa)$ and a $y \in Y$. Let $S = S_E$ and $e = e_E$. By the assumption, Y has orthocaliber κ at y. By Lemma 5.9, there are a $\gamma \in \kappa$, an open neighborhood V of y in Y and an increasing open expansion $\{P(\alpha) : \alpha \in (\gamma,\kappa)\}$ of $\{e(S \cap (\gamma,\alpha]) : \alpha \in (\gamma,\kappa)\}$ in X such that $\{P(\alpha) \times V : \alpha \in (\gamma,\kappa)\}$ partially refines $\mathcal{G}$. Let $U = \bigcup\{P(\alpha) : \alpha \in (\gamma,\kappa)\}$. Then U is an open set in X with $e(S \cap (\gamma,\kappa)) \subset U$. Since $\{P(\alpha) \times V : \alpha \in (\gamma,\kappa)\}$ is an increasing open cover of $U \times V$, it is an interior-preserving open refinement of $\mathcal{G} \restriction (U \times V)$. This means $U \times V \in \mathcal{O}$. It follows from the parenthetic part of Lemma 5.4 that $\mathcal{O}$ has a point-finite open refinement. □

6. Products with almost discrete factors

A space Y is said to be *almost discrete* if it has exactly one non-isolated point. Note that an almost discrete space is monotonically normal, paracompact and $\mathbb{DC}$-like. Moreover, observe that the spaces $A_\theta(\kappa)$ and $Y[\kappa]$ in Examples 3.12 and 3.14, respectively, are almost discrete.

Normality of the products. The first main result here is to characterize the normal products of a monotonically normal space and an almost discrete space in terms of the S-docs property as follows.

Theorem 6.1. *Let X be a monotonically normal space and Y an almost discrete space with a non-isolated point q. Then the following are equivalent.*

(a) *$X \times Y$ is normal.*
(b) *$E \times Y$ is normal for each $E \in \mathcal{S}(X)$.*
(c) *Y has the S_E-docs property at q for each $E \in \mathcal{S}(X)$.*

Proof. (a)⇒(b): This is obvious.
(b)⇒(c): This immediately follows from Lemma 3.7.
(c)⇒(a): This immediately follows from Lemmas 6.3 and 6.5 stated below. □

Recall that two disjoint sets E and F in a space X are *separated* if there are disjoint open sets U and V in X such that $E \subset U$ and $F \subset V$.

Lemma 6.2. *Let X and Y be monotonically normal spaces. Let E and F be closed sets in X and Y, respectively, and let O be an open set in $X \times Y$ with $E \times F \subset O$. If there are two open sets O_E and O_F in $X \times Y$ such that $E \times F \subset O_E \cap O_F$, $(E \times Y) \cap \overline{O_E} \subset O$ and $(X \times F) \cap \overline{O_F} \subset O$, then $E \times F$ and $(X \times Y) \setminus O$ are separated.*

Proof. Let us take two monotone normality assignments H_X and H_Y for X and Y, respectively (see Lemma 2.4). For each $\langle x,y\rangle \in E \times F$, take an open rectangle $U_{x,y} \times V_{x,y}$ in $X \times Y$ with $\langle x,y\rangle \in U_{x,y} \times V_{x,y} \subset O$, and let $O_{x,y} = H_X(x,U_{x,y}) \times H_Y(y,V_{x,y})$. Let $G = O_E \cap O_F \cap (\bigcup_{\langle x,y\rangle \in E\times F} O_{x,y})$. Obviously, G is an open set in $X \times Y$ with $E \times F \subset G$. So it suffices to show that $\overline{G} \subset O$. Pick any $\langle x,y\rangle \in \overline{G}$. In case $x \in E$, we have $\langle x,y\rangle \in (E \times Y) \cap \overline{O_E} \subset O$. In case $y \in F$, we have $\langle x,y\rangle \in (X \times F) \cap \overline{O_F} \subset O$. In case $x \notin E$ and $y \notin F$, since the neighborhood $H_X(x, X \setminus E) \times H_Y(y, Y \setminus F)$ of $\langle x,y\rangle$ meets G which is contained in $\bigcup_{\langle x',y'\rangle \in E\times F} O_{x',y'}$, there is $\langle p,q\rangle \in E \times F$ such that $H_X(p,U_{p,q})$ meets $H_X(x, X \setminus E)$ and $H_Y(q,V_{p,q})$ meets $H_Y(y, Y \setminus F)$. Then we have $\langle x,y\rangle \in U_{p,q} \times V_{p,q} \subset O$. □

Lemma 6.3. *Let X and Y be monotonically normal spaces with $E \in \mathcal{S}(X,\kappa)$, where $\kappa \in \mathcal{S}^*(X)$, and $q \in Y$. If Y has the S_E-docs property at q, then for each open set O in $X \times Y$ containing $E \times \{q\}$, there is $\gamma \in \kappa$ such that $e_E(S_E \cap (\gamma,\kappa)) \times \{q\}$ and $(X \times Y) \setminus O$ are separated.*

Proof. Let $S = S_E$ and $e = e_E$. Let O be an open set in $X \times Y$ containing $E \times \{q\}$. For each $\alpha \in S$, take a $\gamma(\alpha) < \alpha$ and an open neighborhood W_α of q in Y such that $e\big(S \cap (\gamma(\alpha),\alpha]\big) \times W_\alpha \subset O$. By PDL, there is $\gamma \in \kappa$ such that $e\big(S \cap (\gamma,\alpha]\big) \times W_\alpha \subset O$ holds for stationarily many $\alpha \in S \cap (\gamma,\kappa)$. Let $S_0 = S \cap (\gamma,\kappa)$ and $E_0 = e(S_0) = e(S \cap (\gamma,\kappa))$. Since $E_0 \times \{q\} \subset E \times \{q\} \subset O$, it suffice from Lemma 6.2 to show that there are two open sets O_E and O_q in $X \times Y$, satisfying

(i) $E_0 \times \{q\} \subset O_E$ and $(E_0 \times Y) \cap \overline{O_E} \subset O$ and
(ii) $E_0 \times \{q\} \subset O_q$ and $(X \times \{q\}) \cap \overline{O_q} \subset O$.

Let $P(O) = \{x \in X : \langle x,q\rangle \in O\}$. Since it is an open set in a normal space X and contains a closed set E, there is an open set U in X with $E \subset U \subset \overline{U} \subset P(O)$. Let $O_q = U \times Y$. Then O_q is an open set in $X \times Y$ satisfying (ii).

We will find an open set O_E in $X \times Y$ satisfying (i). Take a monotone normality operator M for X (see Definition 1). For each $\alpha \in \kappa$ with $\gamma < \alpha$, let $E_*(\alpha) = e(S \cap (\gamma,\alpha])$ and $E^*(\alpha) = e(S \cap (\alpha,\kappa))$. For each $\alpha \leq \gamma$, let $E_*(\alpha) = \emptyset$ and $E^*(\alpha) = e(S \cap (\gamma,\kappa))$. Since $E_*(\alpha)$ and $E^*(\alpha)$ are disjoint closed sets in X, we can take the open set $U_\alpha := M(E_*(\alpha), E^*(\alpha))$ in X for each $\alpha \in \kappa$. By the property of M, $\mathcal{U} :=\{U_\alpha : \alpha \in \kappa\}$ is an increasing sequence of open sets in X with $E_*(\alpha) \subset U_\alpha \subset \overline{U_\alpha} \subset X \setminus E^*(\alpha)$. For each $\alpha \in S$, we also let $V_\alpha = \mathrm{Int}\{y \in Y : E_*(\alpha) \times \{y\} \subset O\}$. It is easy to see that $\{V_\alpha : \alpha \in S\}$ is a descending sequence of open neighborhoods of q in Y with $E_*(\alpha) \times V_\alpha \subset O$. By the assumption, there is a continuously descending sequence $\{F_\alpha : \alpha \in S\}$ of closed neighborhoods of q in Y such that $F_\alpha \subset V_\alpha$ for each $\alpha \in S$. Here let $O_E = \bigcup_{\alpha \in S_0}(U_\alpha \times \mathrm{Int}_Y F_\alpha)$. Obviously, O_E is an open set of $X \times Y$ with $E_0 \times \{q\} \subset O_E$. For (i), it suffices to show that $(E_0 \times Y) \cap \overline{O_E} \subset O$.

Pick any $\langle x,y\rangle \in (E_0 \times Y) \cap \overline{O_E}$. There is $\xi \in S_0$ with $x = e(\xi)$. Let $\alpha^+ = \min\{\alpha' \in S_0 : \alpha < \alpha'\}$ for each $\alpha \in \kappa$.

Claim. $y \in F_{\alpha^+}$ for each $\alpha \in \xi$.

Proof. Pick an $\alpha \in \xi$. Since $x = e(\xi) \in E^*(\alpha)$ and $E^*(\alpha)$ misses $\overline{U_\alpha}$, $X \setminus \overline{U_\alpha}$ is an open neighborhood of x in X. Take any open neighborhood W of y in Y. Since $(X \setminus \overline{U_\alpha}) \times W$ meets O_E, there is $\delta \in S_0$ such that $(X \setminus \overline{U_\alpha}) \times W$ meets $U_\delta \times \mathrm{Int}\, F_\delta$. Then $X \setminus \overline{U_\alpha}$ meets U_δ and W meets $\mathrm{Int}\, F_\delta$. Since $\mathcal{U}$ is increasing, we have $\alpha < \delta$, so $\alpha^+ \leq \delta$. Hence we obtain $F_{\alpha^+} \supset F_\delta$. Thus we have $W \cap F_{\alpha^+} \supset W \cap F_\delta \supset W \cap \mathrm{Int}\, F_\delta \neq \emptyset$. This means that $y \in \overline{F_{\alpha^+}} = F_{\alpha^+}$. □

In case of $\xi \in \mathrm{Lim}(S)$, since $y \in F_{\alpha^+} \subset F_\alpha$ for each $\alpha \in S \cap \xi$, we have $y \in \bigcap_{\alpha \in S \cap \xi} F_\alpha = F_\xi$. In case $\xi \notin \mathrm{Lim}(S)$, by letting $\alpha = \sup(S \cap \xi) < \xi$, we have $\xi = \alpha^+$ and so $y \in F_{\alpha^+} = F_\xi$. In any case, we conclude that $\langle x,y\rangle \in E_*(\xi) \times F_\xi \subset E_*(\xi) \times V_\xi \subset O$. □

Lemma 6.4 (folklore)**.** *Let X be a normal space and Y an almost discrete space with a non-isolated point q. Then $X \times Y$ is normal if and only if for each closed set F in $X \times Y$ disjoint from $X \times \{q\}$, F and $X \times \{q\}$ are separated.*

Lemma 6.5. *Let X be a monotonically normal space and Y an almost discrete space with a non-isolated point q. Assume that for each $E \in \mathcal{S}(X,\kappa)$ with $\kappa \in \mathcal{S}^*(X)$ and for each open set G in $X \times Y$ with $E \times \{q\} \subset G$, there is $\gamma \in \kappa$ such that $e_E(S_E \cap (\gamma,\kappa)) \times \{q\}$ and $(X \times Y) \setminus G$ are separated. Then $X \times Y$ is normal.*

Proof. Let G be an open set in $X \times Y$ containing $X \times \{q\}$. It suffices to show from Lemma 6.4 that $X \times \{q\}$ and $(X \times Y) \setminus G$ are separated. Let

$$\mathcal{U} = \{U : U \text{ is open in } X \text{ such that } U \times \{q\} \text{ and } (X \times Y) \setminus G \text{ are separated}\}.$$

Since $X \times Y$ is regular, it is obvious that $\mathcal{U}$ is an open cover of X. By the assumption, each $E \in \mathcal{S}(X)$ is almost contained in some member of $\mathcal{U}$. It follows from Corollary 2.7 that $\mathcal{U}$ is normal. So there is a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$. By the choice of $\mathcal{U}$, for each $V \in \mathcal{V}$, one can find an open set W_V in $X \times Y$ such that $V \times \{q\} \subset W_V \subset \overline{W_V} \subset G$. Then $\mathcal{H} = \{(V \times Y) \cap W_V : V \in \mathcal{V}\}$ is a locally finite collection of open sets in $X \times Y$. Hence we obtain that $X \times \{q\} \subset \bigcup \mathcal{H} \subset \overline{\bigcup \mathcal{H}} \subset G$. □

Rectangularity implies normality for the products. For two subspaces A and B of an ordinal, if $A \times B$ is normal, it follows from Theorem 1.2 that it is rectangular. On the other hand, it follows from Lemma 2.1 that $\omega_1 \times (\omega_1 + 1)$ is rectangular, though this is not normal as well-known. So, normality of products seems to be stronger than its rectangularity. However, for the product of a monotonically normal space and an almost discrete space, such an implication suddenly becomes opposite as follows.

Theorem 6.6. *Let X be a monotonically normal space and Y an almost discrete space. If $X \times Y$ is rectangular, then it is normal.*

Remark 6.7. As stated in the proof of Example 3.14, Ohta [12] showed that the product $\kappa \times Y[\kappa]$ is normal but not rectangular. Since $Y[\kappa]$ is almost discrete, the converse of Theorem 6.6 is not true. Example 6.18 is also another example in this case.

In order to prove Theorem 6.6, we define a new neighborhood property which we call the S-codecop property. This property plays important roles also in Section 8 to characterize rectangularity and countable paracompactness of the product of a GO-space and a subspace of an ordinal.

Recall that a sequence $\{V_\alpha : \alpha \in S\}$ of subsets in a space Y, where S is an index set of ordinals, is continuously descending if $V_\alpha \supset V_{\alpha'}$ for each $\alpha, \alpha' \in S$ with $\alpha < \alpha'$ and $V_\alpha = \bigcap_{\beta \in S \cap \alpha} V_\beta$ for each $\alpha \in S \cap \mathrm{Lim}(S)$ (see Definition 4).

Definition 6. Let Y be a space and S a set of ordinals. We say that Y has the *S-continuously descending clopen preserving property* at $q \in Y$ (the *S-codecop property* at $q \in Y$ for short) if for each continuously descending sequence $\{V_\alpha : \alpha \in S\}$ of clopen neighborhoods of q in Y, $q \in \mathrm{Int}(\bigcap_{\alpha \in S} V_\alpha)$ holds. We also say that Y has the *S-codecop property* if Y has the S-codecop property at each point of Y.

Note that a space Y has the S-codecop property iff for each continuously descending sequence $\{V_\alpha : \alpha \in S\}$ of clopen sets in Y, $\bigcap_{\alpha \in S} V_\alpha$ is clopen in Y. The following is easy to see.

Lemma 6.8. *Let κ be a regular cardinal. If a space Y has the κ-dop property at $q \in Y$, then Y has the S-codecop property at q for any unbounded subset S in κ.*

Proposition 6.9. *Let κ be a regular cardinal. Let Y be an almost discrete space with a non-isolated point q. Then Y has the κ-dop property at q if and only if it has the S-codecop property at q for some (any) unbounded subset S in κ.*

Proof. We only show the "if" part. Let $\{V_\alpha : \alpha \in \kappa\}$ be a descending sequence of open neighborhoods of q in Y. Let $W_\alpha = V_\alpha$ for each $\alpha \in S \setminus \mathrm{Lim}(S)$ and let $W_\alpha = \bigcap_{\beta \in S \cap \alpha} V_\beta$ for each $\alpha \in S \cap \mathrm{Lim}(S)$. Note that any neighborhood of q is clopen in Y, so is each W_α. Since $\{W_\alpha : \alpha \in S\}$ is continuously descending, we have $q \in \mathrm{Int}(\bigcap_{\alpha \in S} W_\alpha) \subset \bigcap_{\alpha \in S} V_\alpha = \bigcap_{\alpha \in \kappa} V_\alpha$. □

Remark 6.10. We cannot remove the almost discreteness of Y in the proposition above. In fact, for a regular uncountable cardinal κ, let $Y = [0, \kappa]$ and $q = \kappa$. Then it follows from Lemma 8.19 below that Y has the S-codecop property at q for each stationary subset S in κ (notice that $c_{q,0}$ is the identity map of Y in there). On the other hand, it is easily seen that Y has not the κ-dop property at q. Moreover, for the case of $\kappa = \omega$, every connected and first countable space X with at least two points has the ω-codecop property, but does not have the ω-dop property at each point of X.

Lemma 6.11. *Let X be a monotonically normal space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. If $\{W_\alpha : \alpha \in \kappa\}$ is a continuously descending sequence of clopen sets in a space Y, then there is a cozero-set G in $X \times Y$ such that $G \cap (E \times Y) = \bigcup_{\alpha \in S_E} \big(\{e_E(\alpha)\} \times W_\alpha\big)$.*

Proof. Let $S = S_E$ and $e = e_E$. Let $\{W_\alpha : \alpha \in \kappa\}$ be as above. Let M be a monotone normality operator for X. For each $\alpha \in \kappa$, let $E_*(\alpha) = e(S \cap [0, \alpha])$, $E^*(\alpha) = e(S \cap (\alpha, \kappa))$ and $U_\alpha = M(E_*(\alpha), E^*(\alpha))$. Then $\{U_\alpha : \alpha \in \kappa\}$ is an increasing sequence of open sets in X with $E_*(\alpha) \subset U_\alpha \subset \overline{U_\alpha} \subset X \setminus E^*(\alpha)$. Here we let $U = \bigcup_{\alpha \in \kappa} U_\alpha$. Then U is an open set in X containing E. Let $\mathbb{I} = [0, 1]$ be the unit interval in the real line. Since X is normal, for each $\alpha \in \kappa$, there is a continuous function $f_\alpha : X \to \mathbb{I}$ such that $f_\alpha(x) = 1$ if $x \in \overline{U_\alpha}$ and $f_\alpha(x) = 0$ if $x \in E^*(\alpha)$. Moreover, there is a continuous function $g : X \to \mathbb{I}$ such that $g(x) = 1$ if $x \in E$ and $g(x) = 0$ if $x \in X \setminus U$. For convenience, let $f_{-1} \equiv 0$ and $f_\kappa \equiv 1$ be the constant functions on X, and let $W_{-1} = Y$, $W_\kappa = \bigcap_{\alpha \in \kappa} W_\alpha$ and $W_{\kappa+1} = \emptyset$. We let $L_\alpha = W_\alpha \setminus W_{\alpha+1}$ for each $\alpha \in [-1, \kappa]$. Since W_α is clopen in Y for each $\alpha \in \kappa$, note that $\{L_\alpha : \alpha \in [-1, \kappa]\}$ is a pairwise disjoint closed cover of Y. For each $y \in Y$, one can decide the unique $\alpha(y) \in [-1, \kappa]$ with $y \in L_{\alpha(y)}$. Now, we take the function $h : X \times Y \to \mathbb{I}$ defined by $h(x, y) = f_{\alpha(y)}(x)g(x)$ for each $\langle x, y\rangle \in X \times Y$.

Claim. h is continuous.

Proof. Note that $h \restriction (X \times L_\alpha) = (f_\alpha \cdot g) \circ \pi_X \restriction (X \times L_\alpha)$ for each $\alpha \in [-1, \kappa)$, where π_X denotes the projection from $X \times Y$ onto X. So h is continuous on $X \times L_\alpha$. Since L_α is clopen in Y for each $\alpha \in [-1, \kappa)$, h is continuous on $X \times \left(\bigcup_{\alpha \in [-1,\kappa)} L_\alpha\right)$. Since $0 \le h(x, y) \le g(x)$ for each $\langle x, y\rangle \in X \times Y$ and $g(x) = 0$ for each $x \in X \setminus U$, h is continuous at each $\langle x, y\rangle \in (X \setminus U) \times Y$. It remains to show that h is continuous at each point in $U \times L_\kappa \subset \bigcup_{\alpha \in \kappa}(U_\alpha \times W_\alpha)$. For that, it suffices to show that $h \restriction (U_\alpha \times W_\alpha) \equiv g \circ \pi_X \restriction (U_\alpha \times W_\alpha)$ for each $\alpha \in \kappa$. Pick any $\alpha \in \kappa$ and any $\langle x, y\rangle \in U_\alpha \times W_\alpha$. When $\alpha(y) = \kappa$, we have $f_{\alpha(y)}(x) = f_\kappa(x) = 1$. Assume that $\alpha(y) < \kappa$. Since $y \in L_{\alpha(y)}$ and $L_{\alpha(y)} \cap W_{\alpha'} = \emptyset$ if $\alpha(y) < \alpha'$, we have $y \notin \bigcup_{\alpha' > \alpha(y)} W_{\alpha'}$. By $y \in W_\alpha$, we obtain $\alpha \le \alpha(y)$. Since $x \in U_\alpha \subset U_{\alpha(y)}$ and $f_{\alpha(y)} \restriction \overline{U_{\alpha(y)}} \equiv 1$, we conclude that $f_{\alpha(y)}(x) = 1$. In any case, we have that $h(x, y) = f_{\alpha(y)}(x)g(x) = g(x)$. □

We put $G = \{\langle x, y\rangle \in X \times Y : h(x, y) > 0\}$. Then G is a cozero-set in $X \times Y$. Pick an $\alpha \in S$. Since $e(\alpha) \in E$ and $g(e(\alpha)) = 1$, it follows that

$$\langle e(\alpha), y\rangle \in G \Leftrightarrow h(e(\alpha), y) > 0 \Leftrightarrow f_{\alpha(y)}(e(\alpha)) > 0 \Leftrightarrow \alpha \le \alpha(y) \Leftrightarrow y \in W_\alpha$$

for each $y \in Y$. This means that $G \cap (E \times Y) = \bigcup_{\alpha \in S_E}\left(\{e_E(\alpha)\} \times W_\alpha\right)$. □

Proposition 6.12. *Let X be a monotonically normal space and Y a space. If $X \times Y$ is rectangular, then Y has the κ-codecop property for each $\kappa \in \mathcal{S}^*(X)$.*

Proof. Take a $\kappa \in \mathcal{S}^*(X)$. Take an $E \in \mathcal{S}(X, \kappa)$, and let $S = S_E$ and $e = e_E$. Let $\{V_\alpha : \alpha \in \kappa\}$ be a continuously descending sequence of clopen neighborhoods of $q \in Y$. By Lemma 6.11, there is a cozero-set G in $X \times Y$ such that $G \cap (E \times Y) = \bigcup_{\alpha \in S}\left(\{e(\alpha)\} \times V_\alpha\right)$. Then note that $e(S) \times \{q\} \subset G$. Since $X \times Y$ is rectangular, G is the union of a σ-locally finite collection by cozero rectangles in $X \times Y$. There are a cozero rectangle $U \times W$ in $X \times Y$ and a $\gamma \in \kappa$ such that $e\big(S \cap (\gamma, \kappa)\big) \subset U$, $q \in W$ and $U \times W \subset G$. Pick any $y \in W$ and any $\alpha \in \kappa$. Take a $\xi \in S$ with $\max\{\alpha, \gamma\} < \xi$. Then we have $\langle e(\xi), y\rangle \in e\big(S \cap (\gamma, \kappa)\big) \times W \subset U \times W \subset G$. By the choice of G, we have $y \in V_\xi \subset V_\alpha$. We obtain $q \in W \subset \bigcap_{\alpha \in \kappa} V_\alpha$. Hence Y has the κ-codecop property. □

Proof of Theorem 6.6. Let q be a non-isolated point of Y, and take an $E \in \mathcal{S}(X, \kappa)$ with $\kappa \in \mathcal{S}^*(X)$. By Proposition 6.12, rectangularity of $X \times Y$ implies that Y has the κ-codecop property at q. By Proposition 6.9, Y has the κ-dop property. It follows from Lemma 3.4(2) that Y has the S_E-docs property for every $E \in \mathcal{S}(X)$. Theorem 6.1 witnesses that $X \times Y$ is normal. □

Monotonically normal spaces which are not orthocompact. By Balogh-Rudin's Theorem (= Theorem 2.5), we see that monotonically normal spaces have a covering property which is near to paracompactness. On the other hand, GO-spaces are monotonically normal and orthocompact. It is natural to consider whether monotonically normal spaces are orthocompact. As a negative answer to this, we give a machine to produce, from a monotonically normal, non-paracompact space, a monotonically normal space but not orthocompact.

For a space Y with $A \subset Y$, we denote by Y_A (see [3, p.306]) the space of the set Y with the new topology

$$\{V \cup L : V \text{ is an open set in } Y \text{ and } L \subset Y \setminus A\}.$$

So $Y_{\{q\}}$ is an almost discrete space for each $q \in Y$.

Lemma 6.13. *If X is a monotonically normal space and Y is a space with $q \in Y$, then $(X \times Y)_{[X\times\{q\}]}$ is also monotonically normal.*

Proof. Let $Z = (X \times Y)_{[X\times\{q\}]}$. Let H be a monotone normality assignment for X. For each $A \subset Z$, let $U(A) = \{x \in X : \langle x, q\rangle \in A\}$. For each $\langle x, y\rangle \in Z$ and for each open set G in Z with $\langle x, y\rangle \in G$, let $H_Z(\langle x, y\rangle, G) = \big(H(x, U(G)) \times Y\big) \cap G$ if $y = q$ and let $H_Z(\langle x, y\rangle, G) = \{\langle x, y\rangle\}$ if $y \neq q$. Then it is easy to check that H_Z is a monotone normality assignment for Z. □

Let X be a space with a collection $\mathcal{W}$ of open subsets in X and $x \in X$. We denote by $\mathcal{W}_x$ or $(\mathcal{W})_x$ the collection consisting of all members of $\mathcal{W}$ containing x. Note that $\mathcal{W}$ is interior-preserving iff $x \in \mathrm{Int}(\bigcap \mathcal{W}_x)$ for each $x \in X$.

Proposition 6.14. *Let X be a space and Y an almost discrete space with a non-isolated point q. Then $X \times Y$ is orthocompact if and only if so is $(X \times Y)_{[X\times\{q\}]}$.*

Proof. Let $Y = D \cup \{q\}$ with $q \notin D$. Let $Z = (X \times Y)_{[X\times\{q\}]}$. Each open set in $X \times Y$ is also open in Z. For each $x \in X$, the neighborhood bases at $\langle x, q\rangle$ in $X \times Y$ and in Z coincide with each other.

Assume that $X \times Y$ is orthocompact. Let $\mathcal{G}$ be an open cover of Z. Since $\{\mathrm{Int}_{X\times Y}G : G \in \mathcal{G}\} \cup \{X \times D\}$ is an open cover of $X \times Y$, it has an interior preserving open refinement $\mathcal{H}_0$ in $X \times Y$. Let $\mathcal{H} = \{H \in \mathcal{H}_0 : H \text{ meets } X\times\{q\}\} \cup \{\{z\} : z \in X\times D\}$. Obviously, $\mathcal{H}$ is an open refinement of $\mathcal{G}$ in Z, and $\bigcap(\mathcal{H})_z = \{z\}$ is open in Z for each $z \in X \times D$. Pick any $x \in X$. Then $\bigcap(\mathcal{H})_{\langle x,q\rangle} = \bigcap(\mathcal{H}_0)_{\langle x,q\rangle}$ is open in $X \times Y$, so open also in Z. Hence $\mathcal{H}$ is an interior preserving open refinement of $\mathcal{G}$ in Z.

Conversely, assume that Z is orthocompact. Observe that X is orthocompact, since $X \times \{q\}$ is a closed subspace of Z. So $X \times D = \bigoplus_{y\in D}(X \times \{y\})$ is an orthocompact open subspace in $X \times Y$. Let $\mathcal{G}$ be an open cover of $X \times Y$. Then it is also an open cover of Z. We may assume without loss of generality that each member of $\mathcal{G}$ is an open rectangle in $X \times Y$. Since Z is orthocompact, there is an interior-preserving open refinement $\mathcal{H}^*$ of $\mathcal{G}$ in Z. For each $H \in \mathcal{H}^*$, take and fix a $P(H) \times Q(H) \in \mathcal{G}$ containing H, and let $U(H) = \{x \in X : \langle x, q\rangle \in H\}$. Then $U(H)$ is open in X and $U(H) \subset P(H)$. Let $\mathcal{H}_0 = \{U(H) \times Q(H) : H \in \mathcal{H}^*\}$. Then it is a partial open refinement of $\mathcal{G}$ in $X \times Y$ containing $X \times \{q\}$. Pick any $\langle x, y\rangle \in X \times Y$. Since $\mathcal{H}^*$ is interior-preserving at $\langle x, q\rangle$ in Z, there is an open rectangle $P \times Q$ in $X \times Y$ with $\langle x, q\rangle \in P \times Q \subset \bigcap(\mathcal{H}^*)_{\langle x,q\rangle}$. Assume $\langle x, y\rangle \in U(H) \times Q(H)$ with $H \in \mathcal{H}^*$. Then by $\langle x, q\rangle \in H$, we have $P \times Q \subset H \subset P(H) \times Q(H)$. It follows from $q \in Q$ that $P \times \{q\} \subset P \times Q \subset H$, thus $P \subset U(H)$ holds. By $Q \subset Q(H)$, we have $P \times Q \subset U(H) \times Q(H)$. Hence we see that $\langle x, q\rangle \in P \times Q \subset \bigcap(\mathcal{H}_0)_{\langle x,q\rangle}$. Now assume $y \neq q$. By $y \in Q(H)$, we have $P \times \{y\} \subset U(H) \times Q(H)$. This means that $\langle x, y\rangle \in P \times \{y\} \subset \bigcap(\mathcal{H}_0)_{\langle x,y\rangle}$. Therefore $\mathcal{H}_0$ is interior-preserving in $X\times Y$. Since $X\times D$ is orthocompact, $\mathcal{G}$ has an interior-preserving partial open refinement $\mathcal{H}_1$ in $X \times Y$ covering $X \times D$. Then $\mathcal{H}_0 \cup \mathcal{H}_1$ is an interior-preserving open refinement of $\mathcal{G}$ in $X \times Y$. □

Corollary 6.15. *If X is a monotonically normal space with $\kappa \in \mathcal{S}^*(X)$, and Y is an almost discrete space which does not have orthocaliber κ, then $\big(X \times Y\big)_{[X\times\{q\}]}$ is monotonically normal but not orthocompact, where q is the non-isolated point in Y.*

Proof. Let $Z = \big(X \times Y\big)_{[X\times\{q\}]}$. By Lemma 6.13, Z is monotonically normal. Since Y is almost discrete, it is paracompact $\mathbb{DC}$-like. By the assumption of Y, it follows from Theorem 5.7 that $X \times Y$ is not orthocompact. Hence it follows from Proposition 6.14 that Z is not orthocompact. □

Moreover, by this, we immediately obtain

Corollary 6.16. *If X is a monotonically normal space which is not paracompact with $\kappa \in \mathcal{S}^*(X)$, then $\big(X \times A_\omega(\kappa)\big)_{[X\times\{\kappa\}]}$ and $\big(X \times A_\kappa(\kappa)\big)_{[X\times\{\kappa\}]}$ are monotonically normal but not orthocompact, where $A_\omega(\kappa)$ and $A_\kappa(\kappa)$ are defined in Example 3.12.*

Remark 6.17. There are many monotonically normal spaces that are not paracompact. A typical example is a stationary subset of a regular uncountable cardinal.

Normal products which are not rectangular. As stated in Example 3.14, Ohta [12] constructed the almost discrete space $Y[\kappa]$, where κ is a regular uncountable cardinal, and showed that the product $\kappa \times Y[\kappa]$ is normal but not rectangular (see Remark 6.7). We give here another such example $X \times Y$. The difference is that: Ohta's example $Y[\kappa]$ does not have the κ-dop property, and it is a key to refuting rectangularity. On the other hand, our example Y has the τ-dop property for every regular uncountable cardinal τ, and it witnesses that in Theorem 5.2, the assumption of X being orthocompact cannot be removed (see Remark 5.3).

Example 6.18. For a regular uncountable cardinal κ, there are a monotonically normal space X and an almost discrete space Y, satisfying that

(1) $|X| = |Y| = \kappa$, and κ is embedded into X as a closed subset,
(2) Y has the τ-dop property for each regular uncountable cardinal τ,
(3) $X \times Y$ is normal,
(4) $X \times Y$ is not rectangular.

Proof. Let κ be a regular uncountable cardinal. First, by Theorem 5.2, note that the required space X must be a monotonically normal space which is not orthocompact. Now, we let $X = (\kappa\times A_\kappa(\kappa))_{[\kappa\times\{\kappa\}]}$, where $A_\kappa(\kappa) = \{\alpha+1 : \alpha \in \kappa\} \cup \{\kappa\}$. By Corollary 6.16, we see that X is such a space. Obviously, $|X| = \kappa$ holds, and κ is homeomorphic to a closed subset $\kappa \times \{\kappa\}$ in X.

Let $S^\kappa_\omega = \{\beta \in \kappa : \mathrm{cf}(\beta) = \omega\}$. Then S^κ_ω is stationary in κ (see [11, Lemma II.6.10]). Let $Y = \{q\} \cup \big(\bigcup_{\beta\in S^\kappa_\omega} (\{\beta\} \times \beta)\big)$ as a set, and the topology of Y is defined by

(i) each point of $Y \setminus \{q\}$ is an isolated point in Y,
(ii) a neighborhood base of q in Y is defined by

$$\Big\{\{q\} \cup \Big(\bigcup_{\beta\in S^\kappa_\omega} (\{\beta\} \times [\varphi(\beta), \beta))\Big) : \varphi \in \prod_{\beta\in S^\kappa_\omega} \beta\Big\}.$$

Then Y is clearly an almost discrete space with a non-isolated point q. It is obvious that $|Y| = \kappa$, so (1) holds. Normality of $X \times Y$ is assured by Lemma 3.4(2) and Theorem 6.1 if (2) is satisfied. It remains to show that (2) and (4) are satisfied.

(2): Let τ be a regular uncountable cardinal. Let $\{V_\xi : \xi \in \tau\}$ be a descending sequence of open neighborhoods of q in Y. For each $\xi \in \tau$, there is $\varphi_\xi \in \prod_{\beta\in S^\kappa_\omega} \beta$ such that V_ξ contains $\{q\} \cup \big(\bigcup_{\beta\in S^\kappa_\omega} (\{\beta\} \times [\varphi_\xi(\beta), \beta))\big)$. We may assume that $\varphi_\xi(\beta) = \min\{\delta \in \beta : \{\beta\} \times [\delta, \beta) \subset V_\xi\}$ for each $\beta \in S^\kappa_\omega$. Then notice that $\varphi_\xi(\beta) \le \varphi_\eta(\beta)$ for each $\beta \in S^\kappa_\omega$ and for each $\xi, \eta \in \tau$ with $\xi < \eta$. Since $\mathrm{cf}(\beta) = \omega < \tau$, there is $\varphi(\beta) \in \beta$ such that $\{\xi \in \tau : \varphi_\xi(\beta) \le \varphi(\beta)\}$ is unbounded in τ. Since the sequence $\{\varphi_\xi(\beta) : \xi \in \tau\}$ is increasing, we have $\varphi_\xi(\beta) \le \varphi(\beta)$ for each $\xi \in \tau$. Hence we obtain that

$$\{q\} \cup \Big(\bigcup_{\beta\in S^\kappa_\omega} (\{\beta\} \times [\varphi(\beta), \beta))\Big) \subset \bigcap_{\xi\in\tau} \Big(\{q\} \cup \Big(\bigcup_{\beta\in S^\kappa_\omega} (\{\beta\} \times [\varphi_\xi(\beta), \beta))\Big)\Big) \subset \bigcap_{\xi\in\tau} V_\xi.$$

This means that $q \in \mathrm{Int}(\bigcap_{\xi\in\tau} V_\xi)$.

(4): For each $\alpha \in \kappa$, let $P_\alpha = [0, \alpha] \times (A_\kappa(\kappa) \cap (\alpha, \kappa])$ and let $Q_\alpha = \{q\} \cup \big(\bigcup_{\beta\in S^\kappa_\omega} (\{\beta\} \times W_{\alpha,\beta})\big)$, where let $W_{\alpha,\beta} = [\alpha, \beta)$ for each $\beta \in S^\kappa_\omega \cap (\alpha, \kappa)$ and $W_{\alpha,\beta} = [0, \beta)$ for each $\beta \in S^\kappa_\omega \cap [0, \alpha]$. Let $G = \bigcup_{\alpha\in\kappa} (P_\alpha \times Q_\alpha)$. Then G is open in $X \times Y$ and $(\kappa \times \{\kappa\}) \times Y \subset G$ holds. Now, pick any $\langle x, y\rangle \notin G$. Note that $\{x\}$ is open in X. When $y = q$, $\{x\} \times Y$ misses G since $x \notin \bigcup_{\alpha\in\kappa} P_\alpha$. When $y \ne q$, $\{\langle x, y\rangle\}$ is open in $X \times Y$ and misses G. So G is a clopen set in $X \times Y$ containing $(\kappa \times \{\kappa\}) \times Y$.

Assume that $X \times Y$ is rectangular. Then there is a σ-locally finite collection $\mathcal{H}$ by open (cozero) rectangles in $X \times Y$ with $G = \bigcup \mathcal{H}$. Let $\mathcal{U} = \{U \cap (\kappa \times \{\kappa\}) : U \times V \in \mathcal{H} \text{ with } q \in V\}$. Then $\mathcal{U}$ is a σ-locally finite open cover of $\kappa \times \{\kappa\}$. By Fact 2.3, there are an open rectangle $U_0 \times V_0$ in $X \times Y$ and a $\gamma \in \kappa$ such that $(\gamma, \kappa) \times \{\kappa\} \subset U_0$, $q \in V_0$ and $U_0 \times V_0 \subset G$. There is $\psi \in \prod_{\beta \in S^\kappa_\omega} \beta$ with $\{q\} \cup \big(\bigcup_{\beta \in S^\kappa_\omega} (\{\beta\} \times [\psi(\beta), \beta))\big) \subset V_0$. Take a function $g : (\gamma, \kappa) \to \kappa$ defined by $\langle \alpha, g(\alpha) \rangle \in U_0$ and $g(\alpha) > \alpha$ for each $\alpha \in (\gamma, \kappa)$. Let $C = \{\beta \in (\gamma, \kappa) : \alpha \in (\gamma, \beta) \text{ implies } g(\alpha) < \beta\}$. By [11, Lemma II.6.13], C is a club set in κ. Since S^κ_ω is stationary in κ, take a $\beta_0 \in S^\kappa_\omega \cap C$. Pick an $\alpha_0 \in (\psi(\beta_0), \beta_0)$ with $\gamma < \alpha_0$. By $\beta_0 \in C$, note that $g(\alpha_0) < \beta_0$. Since $(\langle \alpha_0, g(\alpha_0) \rangle, \langle \beta_0, \psi(\beta_0) \rangle) \in U_0 \times V_0 \subset G$, it follows from the choice of G that $(\langle \alpha_0, g(\alpha_0) \rangle, \langle \beta_0, \psi(\beta_0) \rangle) \in P_\delta \times Q_\delta$ for some $\delta \in \kappa$. By $\langle \alpha_0, g(\alpha_0) \rangle \in P_\delta$, we have $\alpha_0 \le \delta < g(\alpha_0)$. Hence we obtain that $\psi(\beta_0) < \alpha_0 \le \delta < g(\alpha_0) < \beta_0$. By $\langle \beta_0, \psi(\beta_0) \rangle \in Q_\delta$ and $\delta < \beta_0$, we have $\psi(\beta_0) \in W_{\delta, \beta_0} = [\delta, \beta_0)$. This contradicts $\psi(\beta_0) < \delta$. □

Normal products and non-normal products. As an immediate consequence of Theorems 5.2 and 5.7, we obtain

Proposition 6.19. *Let X and X' be monotonically normal and orthocompact spaces with $\mathcal{S}^*(X) = \mathcal{S}^*(X')$. Let Y be an almost discrete space. Then the following are true.*

(1) *$X \times Y$ is normal and rectangular if and only if so is $X' \times Y$.*
(2) *$X \times Y$ is orthocompact if and only if so is $X' \times Y$.*

It is natural to ask whether $X \times Y$ is normal if and only if so is $X' \times Y$, when $\mathcal{S}^*(X) = \mathcal{S}^*(X')$. However, it is negative as shown in Corollary 6.21 below. Note that each stationary subsets S and T in ω_1 are monotonically normal and orthocompact spaces with $\mathcal{S}^*(S) = \mathcal{S}^*(T) = \{\omega_1\}$.

Example 6.20. Let κ be a regular uncountable cardinal. For each stationary subset T in κ, there is an almost discrete space Y_T with a non-isolated point q and $|Y_T| = \kappa$, satisfying that for each $S \subset \kappa$, Y_T has the S-docs property at q iff $S \cap T$ is non-stationary in κ.

Proof. Let us define an almost discrete space Y_T with a non-isolated point q, satisfying that

(i) $Y_T = T \cup \{q\}$ with $q \notin T$ as a set,
(ii) $\{\{q\} \cup (T \cap C) : C \text{ is a club set in } \kappa\}$ is a neighborhood base at q.

Take an $S \subset \kappa$ such that $S \cap T$ is non-stationary in κ. We show that Y_T has the S-docs property at q. Let $\{V_\alpha : \alpha \in S\}$ be a descending sequence of open neighborhoods of q in Y_T. For each $\alpha \in S$, there is a club set C_α in κ with $\{q\} \cup (T \cap C_\alpha) \subset V_\alpha$. We can take (by [11, Lemma II.6.14]) a club set $C \subset \{\xi \in \kappa : \xi \in \bigcap_{\alpha \in S \cap \xi} C_\alpha\}$ in κ missing $S \cap T$. For each $\alpha \in S$, let $F_\alpha = \{q\} \cup \big(T \cap C \cap [\alpha, \kappa)\big)$. Then note that each F_α is a closed neighborhood of q in Y_T such that $F_\alpha = \bigcap_{\beta \in S \cap \alpha} F_\beta$ if $\alpha \in S \cap \mathrm{Lim}(S)$. Pick an $\alpha \in S$ and any $\xi \in F_\alpha \setminus \{q\}$. Then we obtain $\alpha < \xi$, otherwise we have the contradiction that $\xi = \alpha \in S \cap (T \cap C) = \emptyset$. By $\alpha \in S \cap \xi$ and $\xi \in C$, we have that $\xi \in T \cap C_\alpha \subset V_\alpha$. So we obtain $F_\alpha \subset V_\alpha$. Hence Y_T has the S-docs property at q.

Next, take an $S \subset \kappa$ such that $S \cap T$ is stationary in κ. Assume that Y_T has the S-docs property at q. For each $\alpha \in S$, let $V_\alpha = \{q\} \cup \big(T \cap (\alpha, \kappa)\big)$. Then $\{V_\alpha : \alpha \in S\}$ is a descending sequence of open neighborhoods of q in Y_T. There is a continuously descending sequence $\{F_\alpha : \alpha \in S\}$ of closed neighborhoods of q in Y_T such that $F_\alpha \subset V_\alpha$ for each $\alpha \in S$. For each $\alpha \in S$, there is a club set C_α in κ with $\{q\} \cup (T \cap C_\alpha) \subset F_\alpha$. Let $C = \{\xi \in \kappa : \xi \in \bigcap_{\alpha \in S \cap \xi} C_\alpha\}$. Since C is a club set in κ, there is $\xi_0 \in (S \cap T) \cap \mathrm{Lim}(S) \cap C$. Then it follows that $\xi_0 \in T \cap (\bigcap_{\alpha \in S \cap \xi_0} C_\alpha) \subset \bigcap_{\alpha \in S \cap \xi_0} F_\alpha = F_{\xi_0} \subset V_{\xi_0}$. This contradicts $\xi_0 \notin V_{\xi_0}$. Hence Y_T does not have the S-docs property at q. □

Corollary 6.21. *There are two stationary subsets S and T in ω_1 and an almost discrete space Y_T such that $S \times Y_T$ is normal but $T \times Y_T$ is not normal.*

Proof. Let S and T be disjoint stationary subsets in ω_1 (the existence is assured by [11, Lemma II.6.12]). Take the almost discrete space Y_T described in Example 6.20. Take any $E \in \mathcal{S}(S)$. Since E is closed in S and uncountable, there is a club set C in ω_1 with $E = S \cap C$. So E is a stationary subset of ω_1 . Since the subsets E and S_E in ω_1 is homeomorphic, by Fact 2.2, there is a club set $C_0 \subset C$ in ω_1 such that $E \cap C_0 = S_E \cap C_0$. Since $(S_E \cap T) \cap C_0 = (E \cap C_0) \cap T \subset S \cap T = \emptyset$, $S_E \cap T$ is non-stationary in ω_1. It follows from Example 6.20 that Y_T has the S_E-docs property at q. By Theorem 6.1, $S \times Y_T$ is normal. On the other hand, since $T \cap T = T$ is stationary in ω_1, it follows from Example 6.20 that Y_T has not the T-docs property at q. By Lemma 3.7, $T \times Y_T$ is not normal. □

7. Products with ordinal factors

In this section, we characterize the normal and rectangular products of monotonically normal spaces with ordinal factors. Similarly, we also characterize orthocompactness of such products. As a consequence, we can extend one ordinal factor of the products in Theorem 1.2 to a monotonically normal factor.

Dop products and orthocaliber products. For a set T of ordinals, we denote by $\mathrm{otp}(T)$ the order type of T. Note that if S is a stationary subset of an ordinal λ with $\mathrm{cf}\,\lambda = \kappa > \omega$, then there is $T \subset S$ with $\mathrm{otp}(T) = \kappa$ such that T is stationary in λ.

For a limit ordinal λ, a function $c : \mathrm{cf}(\lambda) \to \lambda$ is called a *normal function* for λ if it is strictly increasing, continuous and the range $\{c(\xi) : \xi \in \mathrm{cf}(\lambda)\}$ is unbounded in λ. In particular, if κ is a regular cardinal, then we can fix the identity map on κ as the normal function.

Definition 7. A product space $X \times Y$ is called a *diagonal stationary product* if for each $\kappa \in \mathcal{S}^*(X) \cap \mathcal{S}^*(Y)$, whenever $E \in \mathcal{S}(X, \kappa)$ and $F \in \mathcal{S}(Y, \kappa)$, $S_E \cap S_F$ is stationary in κ.

By Fact 2.2, the definition above does not depend on the choices of S_E and S_F. We will sometimes use the following. The parenthetic part follows from Theorem 8.1 below.

Lemma 7.1 ([7, Theorem B]). *Let S and T be two stationary subsets in a regular uncountable cardinal κ. If $S \times T$ is countably paracompact (or equivalently, rectangular), then $S \cap T$ is stationary in κ.*

Lemma 7.2. *A product space $X \times Y$ is a diagonal stationary product if one of the following conditions holds:*

(1) *$X \times Y$ is orthocompact,*
(2) *$X \times Y$ is normal,*
(3) *$X \times Y$ is countably paracompact.*

Proof. Pick any $\kappa \in \mathcal{S}^*(X) \cap \mathcal{S}^*(Y)$. Take an $E \in \mathcal{S}(X, \kappa)$ and an $F \in \mathcal{S}(Y, \kappa)$. Let $X \times Y$ be orthocompact (normal, countably paracompact). Since $E \times F$ is closed in $X \times Y$, $S_E \times S_F$ has the same property. By Theorem 1.2(e), $S_E \times S_F$ is at least countably paracompact. It follows from Lemma 7.1 that $S_E \cap S_F$ is stationary in κ. □

Definition 8. A product space $X \times Y$ is called a *dop product* (an *orthocaliber product*) if $X \times Y$ is a diagonal stationary product, satisfying that

(i) Y has the κ-dop property (orthocaliber κ) for each $\kappa \in \mathcal{S}^*(X)$,
(ii) X has the κ-dop property (orthocaliber κ) for each $\kappa \in \mathcal{S}^*(Y)$.

It follows from Lemma 3.4(1) that every orthocaliber product is a dop product.

Proposition 7.3. *Let X and Y be spaces.*

(1) *If $X \times Y$ is normal and rectangular, then it is a dop product.*
(2) *If $X \times Y$ is orthocompact, then it is an orthocaliber product.*

Proof. By Lemma 7.2(1) and (2), $X \times Y$ is a diagonal stationary product. Take any $\kappa \in \mathcal{S}^*(X)$ and an $E \in \mathcal{S}(X, \kappa)$. Since $X \times Y$ is normal and rectangular (orthocompact), so is $S_E \times Y$. By Lemma 3.6 (Lemma 3.5), Y has the κ-dop property (orthocaliber κ). Similarly, (ii) in Definition 8 is satisfied. So (1) and (2) are true. □

Normality and rectangularity of the products revisited.

Theorem 7.4. *Let X be a monotonically normal space and B a subspace of an ordinal. Then $X \times B$ is normal and rectangular if and only if it is a dop product.*

In order to prove this theorem, we need several lemmas below.

For a subspace B of an ordinal, we use the notation $\Gamma(B)$ defined by

$$\Gamma(B) = \{\mu : \mu \text{ is an ordinal such that } B \cap \mu \text{ is stationary in } \mu,\ \mu \notin B \text{ and } \operatorname{cf}(\mu) > \omega\}.$$

Notice that $\mathcal{S}(B, \kappa) \neq \emptyset$ holds iff $\mu \in \Gamma(B)$ with $\operatorname{cf}\mu = \kappa$ exists (use Proposition 8.3 if necessary).

Lemma 7.5. *Let X be a monotonically normal space and B a subspace of an ordinal. Let $\mathcal{G}$ be an open cover of $X \times B$, satisfying the following two conditions;*

(1) *for each $x \in X$ and for each $\mu \in \Gamma(B)$, there are an open neighborhood U of x in X and a $\delta_0 \in \mu$ such that $U \times \big(B \cap (\delta_0, \mu)\big)$ is contained in some member of $\mathcal{G}$,*

(2) *for each $E \in \mathcal{S}(X)$ and for each $\mu \in B \cup \Gamma(B)$, there are an open set W in X and a $\delta_1 \in \mu \cup \{-1\}$ such that E is almost contained in W and $W \times \big(B \cap (\delta_1, \mu]\big)$ is contained in some member of $\mathcal{G}$.*

Then $\mathcal{G}$ has a locally finite rectangular cozero refinement.

Proof. Let λ be an ordinal with $B \subset [0, \lambda]$. Let $B_\mu = B \cap [0, \mu]$ for each $\mu \leq \lambda$. Using induction by $\mu \leq \lambda$, we shall construct a locally finite rectangular cozero refinement $\mathcal{H}_\mu$ of $\mathcal{G} \restriction (X \times B_\mu)$. Take any $\mu \leq \lambda$. Assume that such a $\mathcal{H}_\xi$ has been already constructed for each $\xi < \mu$.

Case 1. In case $\mu \notin B \cup \Gamma(B)$: By $\mu \notin B$, we have $B \cap \mu = B \cap (\mu + 1) = B_\mu$. By $\mu \notin \Gamma(B)$, $\operatorname{cf}(\mu) \leq \omega$ or B_μ is non-stationary in μ, where $\operatorname{cf}(\mu) > \omega$. We may express $B_\mu = \bigoplus_{\xi \in \operatorname{cf}(\mu)} B(\xi)$ such that $B(\xi) = B \cap (\gamma_\xi, \beta_\xi]$, where $-1 \leq \gamma_\xi < \beta_\xi < \mu$, for each $\xi \in \operatorname{cf}(\mu)$. Since each $\mathcal{H}_{\beta_\xi}$ has been already constructed, we may let

$$\mathcal{H}_\mu = \{H \cap (X \times B(\xi)) : H \in \mathcal{H}_{\beta_\xi} \text{ and } \xi \in \operatorname{cf}(\mu)\}.$$

Then $\mathcal{H}_\mu$ is a locally finite rectangular cozero refinement of $\mathcal{G} \restriction (X \times B_\mu)$.

Case 2. In case $\mu \in B \cup \Gamma(B)$: Let

$$\mathcal{U} = \{U : U \text{ is open in } X \text{ such that there are } \delta_U \in \mu \cup \{-1\} \text{ and } G_U \in \mathcal{G} \text{ with } U \times (B \cap (\delta_U, \mu]) \subset G_U\}.$$

By (1), note that $\mathcal{U}$ is an open cover of X. Since each $E \in \mathcal{S}(X)$ is almost contained in a member of $\mathcal{U}$ by (2), it follows from Corollary 2.7 that $\mathcal{U}$ is a normal cover of X. There is a locally finite cozero refinement $\mathcal{U}^*$ of $\mathcal{U}$. By the choice of $\mathcal{U}$, for each $U \in \mathcal{U}^*$, there are $\delta_U \in \mu \cup \{-1\}$ and $G_U \in \mathcal{G}$ such that $U \times (B \cap (\delta_U, \mu]) \subset G_U$. By the inductive assumption, there is a locally finite rectangular cozero partial refinement $\mathcal{H}_{\delta_U}$ of $\mathcal{G}$ such that $X \times B_{\delta_U} = \bigcup \mathcal{H}_{\delta_U}$. Now, we let

$$\mathcal{H}_\mu = \{U \times (B \cap (\delta_U, \mu]) : U \in \mathcal{U}^*\} \cup \{H \cap (U \times B_{\delta_U}) : H \in \mathcal{H}_{\delta_U} \text{ and } U \in \mathcal{U}^*\}.$$

Then $\mathcal{H}_\mu$ is a locally finite rectangular cozero partial refinement of $\mathcal{G}$. Moreover, it is easily checked that $\mathcal{H}_\mu$ covers $X \times B_\mu$. Thus we have constructed the desired $\{\mathcal{H}_\mu : \mu \leq \lambda\}$. Then $\mathcal{H}_\lambda$ is a locally finite rectangular cozero refinement of $\mathcal{G}$. □

Lemma 7.6. *Let B be a stationary subset in an ordinal μ with $\operatorname{cf}(\mu) = \tau > \omega$. And let X be a space with the τ-dop property at $x \in X$. If $\{G_0, G_1\}$ is a binary open cover of $X \times B$, then there are an open neighborhood U of x in X, a $\delta \in \mu$ and an $i \in 2$ such that $U \times (B \cap (\delta, \mu)) \subset G_i$.*

Proof. Pick an $x \in X$. For each $\beta \in B$, there are an open neighborhood W_β of x in X, a $\delta(\beta) < \beta$ and an $i(\beta) \in 2$ such that $W_\beta \times (B \cap (\delta(\beta), \beta]) \subset G_{i(\beta)}$. It follows from PDL that there are $T \subset B$,

$\delta \in \mu$ and $i \in 2$ such that T is stationary in μ with $\mathrm{otp}(T) = \tau$ and that $\delta(\beta) < \delta < \beta$ and $i(\beta) = i$ for each $\beta \in T$. Moreover, for each $\beta \in T$, we let

$$U_\beta = \bigcup\{U : U \text{ is open in } X \text{ such that } U \times (B \cap (\delta, \beta]) \subset G_i\}.$$

Then $\{U_\beta : \beta \in T\}$ is a descending sequence of open sets in X with $x \in W_\beta \subset U_\beta$ for each $\beta \in T$. Let $U = \mathrm{Int}_X(\bigcap_{\beta\in T} U_\beta)$. By $\mathrm{otp}(T) = \tau$, the τ-dop property implies that U is an open neighborhood of x in X. And we have $U \times \big(B \cap (\delta, \mu)\big) \subset \bigcup_{\beta\in T} \big(U_\beta \times (B \cap (\delta, \beta])\big) \subset G_i$. □

The following is well known and convenient to use.

Fact 7.7 (folklore)**.** *Let μ and ν be ordinals with $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$, where $\mathrm{cf}(\mu) \geq \omega$ $(\mathrm{cf}(\mu) > \omega)$. Let A be an unbounded (stationary) subset of μ, and let $f : A \to \nu$ be a function. Then there is a $\delta \in \nu$ such that $\{\alpha \in A : f(\alpha) \leq \delta\}$ is unbounded (stationary) in μ.*

Lemma 7.8. *Let X be a space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let B be a stationary subset in an ordinal μ with $\tau = \mathrm{cf}(\mu) > \omega$. Assume that $\kappa \neq \tau$ and X has the τ-dop property. If $\{G_0, G_1\}$ is a binary open cover of $X \times B$, then there are an open set U in X, a $\delta \in \mu$ and an $i \in 2$ such that E is almost contained in U and $U \times \big(B \cap (\delta, \mu)\big) \subset G_i$.*

Proof. Let $S = S_E$ and $e = e_E$. By Lemma 7.6, for each $\alpha \in S$, there are an open neighborhood U_α of $e(\alpha)$ in X, a $\delta(\alpha) \in \mu$ and an $i(\alpha) \in 2$ such that $U_\alpha \times \big(B \cap (\delta(\alpha), \mu)\big) \subset G_{i(\alpha)}$. Take a $\gamma(\alpha) < \alpha$ such that $e(S \cap (\gamma(\alpha), \alpha]) \subset U_\alpha$. By PDL and Fact 7.7 with $\kappa \neq \tau$, there are a $T \subset S$, a $\gamma \in \kappa$, a $\delta \in \mu$ and an $i \in 2$ such that T is stationary in κ and that $\gamma(\alpha) = \gamma$, $\delta(\alpha) \leq \delta$ and $i(\alpha) = i$ for each $\alpha \in T$. Let $U = \bigcup_{\alpha\in T} U_\alpha$. Then U is open in X. We have $e(S \cap (\gamma, \kappa)) = \bigcup_{\alpha\in T} e(S \cap (\gamma(\alpha), \alpha]) \subset \bigcup_{\alpha\in T} U_\alpha \subset U$ and $U \times (B \cap (\delta, \mu)) \subset \bigcup_{\alpha\in T}(U_\alpha \times (B \cap (\delta(\alpha), \mu))) \subset G_i$. □

Lemma 7.9. *Let X be a space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let B be a stationary subset in an ordinal μ with $\kappa = \mathrm{cf}(\mu)$. Assume that $S_E \cap c^{-1}(B)$ is stationary in κ, where $c : \kappa \to \mu$ is a normal function, and assume that X has the κ-dop property. If $\{G_0, G_1\}$ is a binary open cover of $X \times B$, then there are an open set U in X, a $\delta \in \mu$ and an $i \in 2$ such that E is almost contained in U and $U \times \big(B \cap (\delta, \mu)\big) \subset G_i$.*

Proof. Let $S = S_E$ and $e = e_E$. For each limit $\alpha \in S \cap c^{-1}(B)$, take $i(\alpha) \in 2$ with $\langle e(\alpha), c(\alpha)\rangle \in G_{i(\alpha)}$. We can take an open neighborhood W_α of $e(\alpha)$ in X and a $\gamma(\alpha) < \alpha$ such that $W_\alpha \times \big(B \cap (c(\gamma(\alpha)), c(\alpha)]\big) \subset G_{i(\alpha)}$ and $e(S \cap (\gamma(\alpha), \alpha]) \subset W_\alpha$. By PDL, there are a $T \subset S \cap c^{-1}(B)$, a $\gamma \in \kappa$ and an $i \in 2$ such that T is stationary in κ and that $\gamma(\alpha) = \gamma$ and $i(\alpha) = i$ for each $\alpha \in T$. Let $\delta = c(\gamma) \in \mu$. For each $\alpha \in T$, let

$$U_\alpha = \bigcup\{W : W \text{ is an open set in } X \text{ with } W \times \big(B \cap (\delta, c(\alpha)]\big) \subset G_i\}.$$

Then $W_\alpha \subset U_\alpha$ and $U_\alpha \times (B \cap (\delta, c(\alpha)]) \subset G_i$ hold. Let $U = \bigcap_{\alpha\in T} U_\alpha$. Since $\{U_\alpha : \alpha \in T\}$ is a descending sequence of open sets in X with $\mathrm{otp}(T) = \kappa$, the κ-dop property implies that U is an open set in X. We have $U \times \big(B \cap (\delta, \mu)\big) \subset \bigcup_{\alpha\in T} \big(U_\alpha \times (B \cap (\delta, c(\alpha)])\big) \subset G_i$. Pick any $\xi \in S \cap (\gamma, \kappa)$ and any $\alpha \in T$. Choose $\eta \in T$ with $\eta > \max\{\xi, \alpha\}$. By $\gamma(\eta) = \gamma < \xi < \eta$, we have $e(\xi) \in e(S \cap (\gamma(\eta), \eta]) \subset W_\eta \subset U_\eta \subset U_\alpha$. Hence U contains $e(S \cap (\gamma, \kappa))$. □

Lemma 7.10. *Let X be a space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let μ be an ordinal with $\tau = \mathrm{cf}(\mu)$ and $\mu \in B \subset \mu + 1$. Assume $\kappa \neq \tau$. If $\{G_0, G_1\}$ is a binary open cover of $X \times B$, then there are an open set U in X, a $\delta \in \mu$ and an $i \in 2$ such that E is almost contained in U and $U \times (B \cap (\delta, \mu]) \subset G_i$.*

Proof. Let $S = S_E$ and $e = e_E$. Pick an $\alpha \in S$. By $\langle e(\alpha), \mu\rangle \in X \times B$, there are an open neighborhood U_α of $e(\alpha)$ in X, a $\delta(\alpha) \in \mu$ and an $i(\alpha) \in 2$ such that $U_\alpha \times (B \cap (\delta(\alpha), \mu]) \subset G_{i(\alpha)}$. Take a $\gamma(\alpha) < \alpha$ with $e(S \cap (\gamma(\alpha), \alpha]) \subset U_\alpha$. By PDL and Fact 7.7 with $\kappa \neq \tau$ (containing the case of $\tau \leq \omega$), there are $T \subset S$, $\gamma \in \kappa$, $\delta \in \mu$ and $i \in 2$ such that T is stationary in κ and that $\gamma(\alpha) = \gamma$, $\delta(\alpha) \leq \delta$ and

$i(\alpha) = i$ for each $\alpha \in T$. Let $U = \bigcup_{\alpha \in T} U_\alpha$. Then U is an open set in X. As in the proof of Lemma 7.8, we can easily verify that $e(S \cap (\gamma, \kappa)) \subset U$ and $U \times \big(B \cap (\delta, \mu]\big) \subset G_i$. □

Proof of Theorem 7.4. Let X be a monotonically normal space and B a subspace of an ordinal. The "only if" part immediately follows from Proposition 7.3(1).

Assume that $X \times B$ is a dop product. Let $\mathcal{G} = \{G_0, G_1\}$ be a binary open cover of $X \times B$. We show that $\mathcal{G}$ has a σ-locally finite rectangular cozero refinement. For that, it suffices to show that (1) and (2) in Lemma 7.5 are satisfied. In the proof, (1) and (2) mean that conditions.

First we consider the case of $\mu \in \Gamma(B)$. Take a normal function $c : \tau \to \mu$, where $\tau = \mathrm{cf}(\mu)$. Then $F = B \cap \{c(\xi) : \xi \in \tau\}$ is a closed subset in B which is homeomorphic to a stationary subset $c^{-1}(B)$ in τ, so we have $F \in \mathcal{S}(B, \tau)$ and $\tau \in \mathcal{S}^*(B)$. By the assumption, X has the τ-dop property. Applying Lemma 7.6, for each $x \in X$, we see that (1) is satisfied. And Lemma 7.8 assures (2) for each $E \in \mathcal{S}(X, \kappa)$ with $\kappa \neq \tau$. In case $E \in \mathcal{S}(X, \kappa)$ with $\kappa = \tau$, since $X \times B$ is a diagonal stationary product, $S_E \cap S_F$ is stationary in $\kappa = \tau$. Since S_F and $c^{-1}(B)$ are homeomorphic, by Fact 2.2, we see that $S_E \cap c^{-1}(B)$ is stationary in $\kappa = \tau$. Then Lemma 7.9 assures (2) for each $E \in \mathcal{S}(X, \kappa)$ with $\kappa = \tau$.

Next we consider the case of $\mu \in B$. Let $\tau = \mathrm{cf}(\mu)$ and let $E \in \mathcal{S}(X, \kappa)$. If $\kappa \neq \tau$, then (2) is assured by Lemma 7.10. If $\kappa = \tau$, then B has the κ-dop property, so it is easy to see that $B \cap \mu$ is bounded in μ. Hence μ is an isolated point of B. Then Fact 2.3 assures (2) in this case. □

Collectionwise normality and the shrinking property.

Theorem 7.11. *Let X be a monotonically normal space and B a subspace of an ordinal. If $X \times B$ is normal and rectangular, then it is collectionwise normal and has the shrinking property.*

In order to prove this theorem, we also need several lemmas.

Lemma 7.12 ([18, Lemma 10.3])**.** *Let X and Y be two spaces such that $X \times Y$ is normal and rectangular. Let $F \in \mathcal{S}(Y)$. If $\mathcal{D}$ is a discrete collection of closed sets in $X \times Y$, then for each $x \in X$, there is an open rectangle $U_x \times V_x$ in $X \times Y$ such that $x \in U_x$, F is almost contained in V_x and $U_x \times V_x$ meets at most one member of $\mathcal{D}$.*

Lemma 7.13 ([10, Lemma 6.4])**.** *Let κ and τ be regular uncountable cardinals. Let S and T be stationary subsets in κ and τ, respectively. If $\mathcal{G}$ is a σ-locally finite rectangular open cover of $S \times T$, then there are $\gamma \in \kappa$ and $\delta \in \tau$ such that $\big(S \cap (\gamma, \kappa)\big) \times \big(T \cap (\delta, \tau)\big)$ is contained in some member of $\mathcal{G}$.*

Lemma 7.14. *Let X and Y be spaces with $E \in \mathcal{S}(X, \kappa)$ and $F \in \mathcal{S}(Y, \tau)$, where $\kappa \in \mathcal{S}^*(X)$ and $\tau \in \mathcal{S}^*(Y)$ such that $X \times Y$ is normal and rectangular. If G is an open set in $X \times Y$ containing $E \times F$, then there is an open rectangle $U \times V$ in $X \times Y$ such that E and F are almost contained in U and V, respectively, and that $U \times V \subset G$.*

Proof. Since $X \times Y$ is normal and rectangular and $E \times F$ is closed in $X \times Y$, it follows that there is a σ-locally finite collection $\mathcal{H}$ of cozero rectangles in $X \times Y$ with $E \times F \subset \bigcup \mathcal{H} \subset G$. Since $\{e_E^{-1}(U) \times e_F^{-1}(V) : U \times V \in \mathcal{H}\}$ is a σ-locally finite rectangular cozero cover of $S_E \times S_F$, it follows from Lemma 7.13 that there are $U \times V \in \mathcal{H}$, $\gamma \in \kappa$ and $\delta \in \tau$ such that $(S_E \cap (\gamma, \kappa)) \times (S_F \cap (\delta, \tau)) \subset e_E^{-1}(U) \times e_F^{-1}(V)$. Then we have $e_E(S_E \cap (\gamma, \kappa)) \times e_F(S_F \cap (\delta, \tau)) \subset U \times V \subset G$. □

Lemma 7.15. *Let X and Y be spaces with $E \in \mathcal{S}(X, \kappa)$ and $F \in \mathcal{S}(Y, \tau)$, where $\kappa \in \mathcal{S}^*(X)$ and $\tau \in \mathcal{S}^*(Y)$ such that $X \times Y$ is normal and rectangular. If $\mathcal{D}$ is a discrete collection of closed sets in $X \times Y$, then there is an open rectangle $U \times V$ in $X \times Y$ such that E and F are almost contained in U and V, respectively, and $U \times V$ meets at most one member of $\mathcal{D}$.*

Proof. Let $S = S_E$, $e = e_E$, $T = S_F$ and $f = e_F$. By Lemma 7.14, it suffices to find $E' = e(S \cap (\gamma, \kappa))$ and $F' = f(T \cap (\delta, \tau))$ for some $\gamma \in \kappa$ and $\delta \in \tau$ such that $E' \times F'$ meets at most one member of $\mathcal{D}$.

Case 1. Assume that $\kappa < \tau$: It follows from Lemma 7.12 that for each $\alpha \in S$, there is an open rectangle $U_\alpha \times V_\alpha$ in $X \times Y$ such that $\alpha \in U_\alpha$, F is almost contained in V_α and $U_\alpha \times V_\alpha$ meets at most one member of $\mathcal{D}$. Take $\gamma(\alpha) < \alpha$ with $e(S \cap (\gamma(\alpha), \alpha])) \subset U_\alpha$. By PDL, there are $S_0 \subset S$ and $\gamma \in \kappa$ such that S_0 is stationary in κ and $\gamma(\alpha) = \gamma$ for each $\alpha \in S_0$. By $|S_0| = \kappa < \tau$, we find some $\delta \in \tau$ with $f(T \cap (\delta, \tau)) \subset \bigcap_{\alpha \in S_0} V_\alpha$. Let $E' = e(S \cap (\gamma, \kappa))$ and $F' = f(T \cap (\delta, \tau))$. It is easy to see that for each $z_0, z_1 \in E' \times F'$, there is an $\alpha_0 \in S_0$ with $z_0, z_1 \in U_{\alpha_0} \times V_{\alpha_0}$. Hence $E' \times F'$ meets at most one member of $\mathcal{D}$. The case of $\kappa > \tau$ is the same from the symmetry.

Case 2. Assume that $\kappa = \tau$: Since $X \times Y$ is normal, so is $S \times T$. Since $S \times T$ has the shrinking property by Theorem 1.2, it is countably paracompact. It follows from Lemma 7.1 that $S \cap T$ is stationary in κ. For each $\xi \in S \cap T$, there is an open rectangle $U_\xi \times V_\xi$ in $X \times Y$ with $\langle e(\xi), f(\xi)\rangle \in U_\xi \times V_\xi$ such that $U_\xi \times V_\xi$ meets at most one member of $\mathcal{D}$. Take $\gamma(\xi) < \xi$ such that $e\big(S \cap (\gamma(\xi), \xi]\big) \subset U_\xi$ and $f\big(T \cap (\gamma(\xi), \xi]\big) \subset V_\xi$. By PDL, there are $R \subset S \cap T$ and $\gamma \in \kappa$ such that R is stationary in κ and $\gamma(\xi) = \gamma$ for each $\xi \in R$. Let $E' = e\big(S \cap (\gamma, \kappa)\big)$ and $F' = f\big(T \cap (\gamma, \kappa)\big)$. In a similar way to the above case, it is verified that $E' \times F'$ meets at most one member of $\mathcal{D}$. □

Lemma 7.16 ([18, Lemma 11.1])**.** *Let $X \times Y$ be normal and rectangular. If $\mathcal{G}$ is an open cover of $X \times Y$, then for each $x \in X$ and for each $F \in \mathcal{S}(Y)$, there is an open rectangle $U \times V$ in $X \times Y$ such that $x \in U, F$ is almost contained in V and $\mathcal{G} \restriction (U \times V)$ has a shrinking.*

Lemma 7.17. *Let X and Y be spaces with $E \in \mathcal{S}(X, \kappa)$ and $F \in \mathcal{S}(Y, \tau)$, where $\kappa \in \mathcal{S}^*(X)$ and $\tau \in \mathcal{S}^*(Y)$ such that $X \times Y$ is normal and rectangular. Let $\mathcal{G}$ be an open cover of $X \times Y$. Then there is an open rectangle $U \times V$ in $X \times Y$ such that E and F are almost contained in U and V, respectively, and $\mathcal{G} \restriction (U \times V)$ has a shrinking.*

Proof. Since $E \times F$ is normal, it follows from Theorem 1.2 that $E \times F$ has the shrinking property. There is a shrinking $\{L(G) : G \in \mathcal{G}\}$ of $\mathcal{G} \restriction (E \times F)$. Since $X \times Y$ is normal, for each $G \in \mathcal{G}$, there is an open set $H(G)$ in $X \times Y$ such that $L(G) \subset H(G) \subset \overline{H(G)} \subset G$. Let $H = \bigcup\{H(G) : G \in \mathcal{G}\}$. Then H is an open set in $X \times Y$ containing $E \times F$. Since $X \times Y$ is normal and rectangular, it follows from Lemma 7.14 that there is an open rectangle $U \times V$ in $X \times Y$ such that E and F are almost contained in U and V, respectively, and that $U \times V \subset H$. Then $\mathcal{G} \restriction (U \times V)$ has a shrinking $\{\overline{H(G)} \cap (U \times V) : G \in \mathcal{G}\}$. □

Proof of Theorem 7.11. Let $\mathcal{D}$ be a discrete collection of closed sets in $X \times B$. Then

$$\mathcal{G} = \{G \subset X \times B : G \text{ is an open set meeting at most one member of } \mathcal{D}\}$$

is an open cover of $X \times B$. In case $\mu \in \Gamma(B)$: Let $\tau = \mathrm{cf}(\mu)$. Then $F = B \cap D \in \mathcal{S}(B, \tau)$ by taking a club set D in μ with $\mathrm{otp}(D) = \tau$. For each $x \in X$, it follows from Lemma 7.12 that there is an open rectangle $U_x \times V_x$ in $X \times B$ such that $x \in U_x$, F is almost contained in V_x and that $U_x \times V_x$ is contained in some member of $\mathcal{G}$. Then Lemma 7.5(1) is satisfied. Take an $E \in \mathcal{S}(X, \kappa)$. It follows from Lemma 7.15 that there is an open rectangle $U \times V$ in $X \times Y$ such that E and F are almost contained in U and V, respectively, and that $U \times V$ is contained in some member of $\mathcal{G}$. Then Lemma 7.5(2) is also satisfied. In case $\mu \in B$: It follows from Lemma 7.12 that Lemma 7.5(2) is similarly satisfied. Hence $\mathcal{G}$ is normal. This implies that $X \times B$ is collectionwise normal.

Next, let $\mathcal{G}$ be an arbitrary open cover of $X \times B$. Obviously,

$$\mathcal{O} = \{O \subset X \times B : O \text{ is an open set}, \mathcal{G} \restriction O \text{ has a shrinking}\}$$

is an open cover of $X \times B$. We show that $\mathcal{O}$ is normal. For each $x \in X$ and each $\mu \in \Gamma(B)$, it follows from Lemma 7.16 that Lemma 7.5(1) is satisfied. For each $E \in \mathcal{S}(X, \kappa)$ and $\mu \in B \cup \Gamma(B)$, it follows from Lemmas 7.16 and 7.17 that Lemma 7.5(2) is satisfied. Hence $\mathcal{O}$ is normal. So there is a locally finite shrinking $\{F(O) : O \in \mathcal{O}\}$ of $\mathcal{O}$. Hence it is easy to show that $\mathcal{G}$ has a shrinking. □

Orthocompactness of the products revisited.

Theorem 7.18. *Let X be a monotonically normal and orthocompact space and B a subspace of an ordinal. Then $X \times B$ is orthocompact if and only if it is an orthocaliber product.*

Since each orthocaliber product is a dop product, the following analogue to Theorem 1.1 is an immediate consequence of Theorems 7.4 and 7.18.

Corollary 7.19. *Let X be a monotonically normal space and B a subspace of an ordinal. If $X \times B$ is orthocompact, then it is normal and rectangular.*

Let us begin the proof of Theorem 7.18.

Lemma 7.20. *Let B be a stationary subset in an ordinal μ with $\mathrm{cf}(\mu) = \tau > \omega$. Let X be a space with orthocaliber τ at $x \in X$. Then for each open cover $\mathcal{G}$ of $X \times B$, there are an open neighborhood U of x in X and a $\delta \in \mu$ such that $\{U \times (B \cap (\delta, \beta]) : \beta \in (\delta, \mu)\}$ partially refines $\mathcal{G}$.*

Proof. For each $\beta \in B$, there are a $\delta(\beta) < \beta$, an open neighborhood $U(\beta)$ of x in X and a $G(\beta) \in \mathcal{G}$ such that $U(\beta) \times \big(B \cap (\delta(\beta), \beta]\big) \subset G(\beta)$. By PDL, there are a $\delta \in \mu$ and a $T \subset B$ such that T is stationary in μ with $\mathrm{otp}(T) = \tau$ and $\delta(\beta) \leq \delta < \beta$ for each $\beta \in T$. By the assumption, there is a $T_0 \subset T$ such that T_0 is unbounded in μ and $x \in U := \mathrm{Int}\big(\bigcap_{\beta \in T_0} U(\beta)\big)$. Obviously, U and δ satisfy the required condition. □

Lemma 7.21. *Let X be an orthocompact space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let B be a stationary subset in an ordinal μ with $\mathrm{cf}(\mu) = \tau > \omega$. Assume that $\kappa \neq \tau$ and X has orthocaliber τ. Then for each open cover $\mathcal{G}$ of $X \times B$, there are a $\gamma \in \kappa$, a $\delta \in \mu$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e_E(S_E \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ in X such that*

$$\mathcal{H} = \{P(\alpha) \times \big(B \cap (\delta, \beta]\big) : \alpha \in (\gamma, \kappa) \text{ and } \beta \in (\delta, \mu)\}$$

is a partial open refinement of $\mathcal{G}$.

Proof. Applying Lemma 7.20, for each $\alpha \in S_E$, there are an open neighborhood U_α of $e(\alpha)$ in X and a $\delta_\alpha \in \mu$ such that $\{U_\alpha \times (B \cap (\delta_\alpha, \beta]) : \beta \in (\delta_\alpha, \mu)\}$ partially refines $\mathcal{G}$. Next applying Lemma 5.6 to E and $\mathcal{U} := \{U_\alpha : \alpha \in S_E\}$, there are a $\gamma \in \kappa$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e_E(S_E \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ such that $\mathcal{P}$ partially refines $\mathcal{U}$. Then we can take $\delta(\alpha) \in \mu$ such that $\mathcal{H}_\alpha := \{P(\alpha) \times (B \cap (\delta(\alpha), \beta]) : \beta \in (\delta(\alpha), \mu)\}$ partially refines $\mathcal{G}$ for each $\alpha \in (\gamma, \kappa)$. By $\kappa \neq \tau$, using Fact 7.7, we find a $\delta \in \mu$ such that $\{\alpha \in (\gamma, \kappa) : \delta(\alpha) \leq \delta\}$ is unbounded in κ. Then, $\mathcal{H} := \{P(\alpha) \times \big(B \cap (\delta, \beta]\big) : \alpha \in (\gamma, \kappa) \text{ and } \beta \in (\delta, \mu)\}$ partially refines $\mathcal{G}$. Actually, pick any $\langle \alpha, \beta \rangle \in (\gamma, \kappa) \times (\delta, \mu)$. Choose an $\alpha' \in \kappa$ with $\alpha \leq \alpha'$ and $\delta(\alpha') \leq \delta$. Since $\alpha' \in (\gamma, \kappa)$, $\beta \in (\delta(\alpha'), \mu)$ and $\mathcal{H}_{\alpha'}$ partially refines $\mathcal{G}$, we obtain a $G \in \mathcal{G}$ with $P(\alpha) \times (B \cap (\delta, \beta]) \subset P(\alpha') \times (B \cap (\delta(\alpha'), \beta]) \subset G$. □

Lemma 7.22. *Let X be an orthocompact space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let B be a stationary subset in an ordinal μ with $\mathrm{cf}(\mu) = \kappa$. Assume that $S_E \cap c^{-1}(B)$ is stationary in κ, where $c\colon \kappa \to \mu$ is a normal function. Then for each open cover $\mathcal{G}$ of $X \times B$, there are a $\gamma \in \kappa$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e_E(S_E \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ in X such that*

$$\mathcal{H} = \{P(\alpha) \times \big(B \cap (c(\gamma), c(\alpha)]\big) : \alpha \in (\gamma, \kappa)\}$$

is a partial open refinement of $\mathcal{G}$.

Proof. Let $S = S_E$ and $e = e_E$. Pick any limit $\alpha \in S \cap c^{-1}(B)$. By $\langle e(\alpha), c(\alpha) \rangle \in E \times B$, choose a $G_\alpha \in \mathcal{G}$ with $\langle e(\alpha), c(\alpha) \rangle \in G_\alpha$. Take an open neighborhood U_α of $e(\alpha)$ with $E \cap U_\alpha \subset e(S \cap [0, \alpha])$, and $\gamma(\alpha) < \alpha$ such that $U_\alpha \times (B \cap (c(\gamma(\alpha)), c(\alpha)]) \subset G_\alpha$ and $e(S \cap (\gamma(\alpha), \alpha]) \subset U_\alpha$. Letting α run over $S \cap c^{-1}(B)$, it follows from PDL that there are $\gamma_0 \in \kappa$ and $T \subset S \cap c^{-1}(B)$ such that T is stationary in κ and $\gamma(\alpha) = \gamma_0$ for each $\alpha \in T$. Then, $\mathcal{U} := \{U_\alpha : \alpha \in T\}$ covers $e(S \cap (\gamma_0, \kappa))$. It follows from Lemma 5.6 that there are a $\gamma \in \kappa$ with $\gamma_0 \leq \gamma$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e(S \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ in X such that $\mathcal{P}$ partially refines $\mathcal{U}$. Pick any $\alpha \in (\gamma, \kappa)$. Take $\alpha' \in S$ with $\alpha \leq \alpha'$ and choose $\eta \in T$ with $P(\alpha') \subset U_\eta$. Since $e(\alpha') \in e(S \cap (\gamma, \alpha']) \subset E \cap P(\alpha') \subset E \cap U_\eta \subset e(S \cap [0, \eta])$, we obtain $\alpha' \leq \eta$. By $P(\alpha) \subset P(\alpha') \subset U_\eta$ and $\gamma(\eta) = \gamma_0 \leq \gamma$, it follows that

$$P(\alpha) \times \big(B \cap (c(\gamma), c(\alpha)]\big) \subset U_\eta \times \big(B \cap (c(\gamma(\eta)), c(\eta)]\big) \subset G_\eta \in \mathcal{G}.$$

Hence $\mathcal{H}$ partially refines $\mathcal{G}$. □

Proof of Theorem 7.18. Since the "only if" part follows from Lemma 7.3(2), assume that $X \times B$ is an orthocaliber product. Let $\mathcal{G}$ be an open cover of $X \times B$. Let

$$\mathcal{O} = \{\bigcup \mathcal{W} : \mathcal{W} \text{ is an interior-preserving partial open refinement of } \mathcal{G}\}.$$

Note that $\mathcal{O}$ is an open cover of $X \times B$. It suffices to show that $\mathcal{O}$ has a point-finite open refinement. Using Lemma 7.5, we shall show that $\mathcal{O}$ is normal.

Claim 1. For each $x \in X$ and each $\mu \in \Gamma(B)$, there are an open neighborhood U of x in X and a $\delta \in \mu$ such that $U \times \big(B \cap (\delta, \mu)\big) \in \mathcal{O}$.

Proof. Let $\tau = \mathrm{cf}(\mu)$. Since $\mu \in \Gamma(B)$ implies $\tau \in \mathcal{S}^*(B)$, X has orthocaliber τ. Pick an $x \in X$. It follows from Lemma 7.20 that there are $\delta \in \mu$ and an open neighborhood U of x in X such that $\mathcal{W} := \{U \times (B \cap (\delta, \beta]) : \beta \in (\delta, \mu)\}$ partially refines $\mathcal{G}$. Since $\mathcal{W}$ is an increasing open cover of $U \times \big(B \cap (\delta, \mu)\big)$, we obtain $U \times \big(B \cap (\delta, \mu)\big) = \bigcup \mathcal{W} \in \mathcal{O}$. □

Claim 2. For each $E \in \mathcal{S}(X, \kappa)$ and each $\beta \in B$, there are an open set U in X and a $\delta \in \beta \cup \{-1\}$ such that E is almost contained in U and $U \times (B \cap (\delta, \beta]) \in \mathcal{O}$.

Proof. Take an $E \in \mathcal{S}(X, \kappa)$ and pick any $\beta \in B$. Then B has orthocaliber κ. It follows from Lemma 5.9 that there are a $\gamma \in \kappa$, an open neighborhood V of β in B and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e_E(S_E \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ such that $\{P(\alpha) \times V : \alpha \in (\gamma, \kappa)\}$ partially refines $\mathcal{G}$. Let $U = \bigcup_{\alpha \in (\gamma, \kappa)} P(\alpha)$. Then U is an open set in X which contains $e_E(S_E \cap (\gamma, \kappa))$. Take $\delta < \beta$ with $B \cap (\delta, \beta] \subset V$. Let $\mathcal{H}_0 = \{P(\alpha) \times (B \cap (\delta, \beta]) : \alpha \in (\gamma, \kappa)\}$. Since $\mathcal{P}$ is increasing, $\mathcal{H}_0$ is an increasing open cover of $U \times (B \cap (\delta, \beta])$ which partially refines $\mathcal{G}$. Hence we obtain $U \times (B \cap (\delta, \beta]) = \bigcup \mathcal{H}_0 \in \mathcal{O}$. □

Claim 3. For each $E \in \mathcal{S}(X, \kappa)$ and each $\mu \in \Gamma(B)$, there are an open set U in X and a $\delta \in \mu$ such that E is almost contained in U and $U \times \big(B \cap (\delta, \mu)\big) \in \mathcal{O}$.

Proof. We divide into the two cases of $\kappa \neq \mathrm{cf}(\mu)$ and $\kappa = \mathrm{cf}(\mu)$.

Case 1. In case $\kappa \neq \mathrm{cf}(\mu)$: Let $\tau = \mathrm{cf}(\mu)$. Since $\tau \in \mathcal{S}^*(B)$, X has orthocaliber τ. It follows from Lemma 7.21 that there are a $\gamma \in \kappa$, a $\delta \in \mu$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e_E(S_E \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ in X such that

$$\mathcal{H}_1 = \{P(\alpha) \times (B \cap (\delta, \beta]) : \alpha \in (\gamma, \kappa) \text{ and } \beta \in (\delta, \mu)\}$$

partially refines $\mathcal{G}$. Let $U = \bigcup_{\alpha \in (\gamma, \kappa)} P(\alpha)$. Then U is an open set in X which contains $e_E(S_E \cap (\gamma, \kappa))$. Since $\mathcal{P}$ and $\{B \cap (\delta, \beta] : \beta \in (\delta, \mu)\}$ are both increasing, it is easily seen that $\mathcal{H}_1$ is an interior-preserving open cover of $U \times \big(B \cap (\delta, \mu)\big)$ which partially refines $\mathcal{G}$. Hence we obtain $U \times \big(B \cap (\delta, \mu)\big) = \bigcup \mathcal{H}_1 \in \mathcal{O}$.

Case 2. In case $\kappa = \mathrm{cf}(\mu)$: Let $c : \kappa \to \mu$ be a normal function and let $F = B \cap \{c(\xi) : \xi \in \kappa\}$. Then $F \in \mathcal{S}(B, \kappa)$. Since $c^{-1}(B)$ is homeomorphic to S_F and $X \times B$ is a diagonal stationary product, $S_E \cap c^{-1}(B)$ is stationary in κ. It follows from Lemma 7.22 that there are a $\gamma \in \kappa$ and an increasing open expansion $\mathcal{P} = \{P(\alpha) : \alpha \in (\gamma, \kappa)\}$ of $\{e_E(S_E \cap (\gamma, \alpha]) : \alpha \in (\gamma, \kappa)\}$ in X such that

$$\mathcal{H}_2 = \{P(\alpha) \times (B \cap (c(\gamma), c(\alpha)]) : \alpha \in (\gamma, \kappa)\}$$

partially refines $\mathcal{G}$. Let $U = \bigcup_{\alpha \in (\gamma, \kappa)} P(\alpha)$. Then U is an open set in X which contains $e_E(S_E \cap (\gamma, \kappa))$. Similarly, $\mathcal{H}_2$ is interior-preserving. Hence we obtain $U \times \big(B \cap (\delta, \mu)\big) = \bigcup \mathcal{H}_2 \in \mathcal{O}$. □

By the Claims 1, 2 and 3, $\mathcal{O}$ satisfies (1) and (2) in Lemma 7.5. Hence $\mathcal{O}$ is normal. Thus we complete the proof of Theorem 7.18. □

Remark 7.23. A space X is *weakly suborthocompact* if every open cover $\mathcal{U}$ of X has an open refinement $\bigcup_{n \in \omega} \mathcal{V}_n$, satisfying that for each $x \in X$, there is $n_x \in \omega$ such that $x \in \bigcup \mathcal{V}_{n_x}$ and $\bigcap\{V \in \mathcal{V}_{n_x} : x \in V\}$ is an open neighborhood of x in X. Obviously, every orthocompact space is weakly suborthocompact. Using [10, Lemma 3.1] in stead of Lemma 3.5, we see that every weakly suborthocompact product is an orthocaliber product. As a consequence, we obtain the following analogue to [18, Theorems 1.3

and 4.3]: *Let X be a monotonically normal space and B a subspace of an ordinal. Then $X \times B$ is orthocompact if and only if it is weakly suborthocompact.*

8. Products of GO-spaces with ordinal factors

As a similar result to Theorem 1.2, we can recall the following result.

Theorem 8.1 ([7, 10]). *Let A and B be two subspaces of an ordinal. Then the following are equivalent.*

(a) *$A \times B$ is expandable.*
(b) *$A \times B$ is countably paracomapct.*
(c) *$A \times B$ is rectangular.*

It is natural to consider whether this result can be extended by the same way as in the previous section. That is, it is the problem whether A can be replaced by a monotonically normal space X in there. In this section, we introduce the concept called codecop products, and discuss the rectangular products of GO-spaces. Moreover, using this concept, we give a characterization of the rectangular (equivalently, countably paracompact) products of a GO-space and a subspace of an ordinal. This gives a partial answer to the above problem by the case of X being a GO-space.

We also show that normality of such products is equivalent to their orthocompactness and implies their rectangularity. This gives an extension of (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) in Theorem 1.2 in the introduction.

GO-spaces and continuous maps. Recall that a space X is called a *GO-space* (= *generalized ordered space*) if there is a linear order $<$ on X such that the topology τ of X is generated by some family of convex subsets by $<$. In particular, a linearly ordered set $L = (L, <)$ having the interval topolgy λ generated by the base $\{(a, b) : a, b \in L \cup \{\leftarrow, \rightarrow\}, a < b\}$ is called a LOTS (= *linearly ordered topological space*), where $(a, \rightarrow)$ and $(\leftarrow, b)$ denote $\{x \in L : a < x\}$ and $\{x \in L : x < b\}$ respectively. Note that the topology of a GO-space $(X, <)$ is stronger than the interval topology by $<$ as above because X is Hausdorff.

It is well known that for each GO-space $X = (X, <_X, \tau)$, there is a compact LOTS $L_X = (L_X, <, \lambda)$ with $X \subset L_X$ such that (X, τ) is a dense subspace of (L_X, λ), and the order $<$ coincides with $<_X$ on X. We will fix such L_X with no mention. For $A \subset X$, we denote by $\mathrm{Cl}_{L_X} A$ the closure of A in L_X. It is well-known that in any compact LOTS L, (in particular $L = L_X$ for a GO-space X), each subset A has the least upper bound and the greatest lower bound in L, and they are denoted by $\sup A$ and $\inf A$, respectively.

Let X be a GO-space. For $A \subset B \subset L_X$, we say that A is 0-*cofinal* (1-*cofinal*) in B if for each $b \in B$, there is an $a \in A$ with $b \le a$ ($b \ge a$). For each $p \in L_X$, let

$$0\text{-}\mathrm{cf}(p) = \min\{|A| : A \text{ is a 0-cofinal subset in } (\leftarrow, p)_{L_X}\},$$

$$1\text{-}\mathrm{cf}(p) = \min\{|A| : A \text{ is a 1-cofinal subset in } (p, \rightarrow)_{L_X}\}.$$

Then $0\text{-}\mathrm{cf}(p) = 0$ in case $p = \min L_X$, $0\text{-}\mathrm{cf}(p) = 1$ in case $(\leftarrow, p)$ has a maximum, and $0\text{-}\mathrm{cf}(p)$ is a regular infinite cardinal in the other case. Since X is dense in L_X, the following is easy to see.

Fact 8.2. *Let X be a GO-space with $p \in L_X$. If $0\text{-}\mathrm{cf}(p) = \kappa \ge \omega$, then there is a strictly increasing continuous function $c : \kappa \to L_X$ such that*

(i) *$p = \sup_{L_X}\{c(\xi) : \xi \in \kappa\}$,*
(ii) *$c(\xi) \in X$ for each $\xi \in \kappa \setminus \mathrm{Lim}(\kappa)$.*

For each $p \in L_X$, we will fix such a function $c_{p,0} = c$. Similarly, we define $c_{p,1}$. Such $c_{p,0}$ and $c_{p,1}$ are called *normal functions at p*. The details for $i\text{-}\mathrm{cf}(p), i = 0, 1$, are seen in [6].

The following seems to be known. However, as we cannot find any citation for it, we only give an outline of the proof.

Proposition 8.3 (folklore). *Let $f : S \to L$ be a continuous map from a stationary subset S of a regular uncountable cardinal κ into a LOTS L. Then there is a club set C of κ such that $f \restriction (S \cap C)$ is either constant, strictly increasing or strictly decreasing.*

Outline of Proof. For each $\alpha \in S$, let $T_0(\alpha) = \{\beta \in S : f(\alpha) = f(\beta)\}, T_1(\alpha) = \{\beta \in S : f(\alpha) < f(\beta)\}$ and $T_2(\alpha) = \{\beta \in S : f(\alpha) > f(\beta)\}$. Let $S_i = \{\alpha \in S : T_i(\alpha)$ is stationary in $\kappa\}$ $(i \in 3)$. Then S_i is stationary in κ for some $i \in 3$.

In case that S_0 is stationary in κ: Let $C = \{\alpha \in \kappa : \alpha \in \bigcap_{\beta \in S_0 \cap \alpha} \mathrm{Lim}(T_0(\beta))\} \cap \mathrm{Lim}(S_0)$. By [11, Lemma II.6.14], C is a club set in κ. We can show that $f \restriction (S \cap C)$ is constant. In case that S_0 is non-stationary and that S_1 is stationary in κ: Take a club set C_0 of κ disjoint from S_0. For each $\alpha \in S \setminus S_0$, take a club set $C_1(\alpha)$ of κ disjoint from $T_0(\alpha)$. Let

$$C = C_0 \cap \mathrm{Lim}(S_1) \cap \{\alpha \in \kappa : \alpha \in \bigcap\{C_1(\beta) : \beta \in (S \setminus S_0) \cap \alpha\}\}$$
$$\cap \{\alpha \in \kappa : \alpha \in \bigcap\{\mathrm{Lim}(T_1(\beta)) : \beta \in S_1 \cap \alpha\}\}.$$

Then C is as well as a club set of κ. We can show that $f \restriction (S \cap C)$ is strictly increasing. The other cases are similar.

Considering e_E and S_E as f and S, respectively, in Proposition 8.3, we immediately have

Corollary 8.4. *Let X be a GO-space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Then there is a club set C in κ such that $e_E \restriction (S_E \cap C)$ is strictly increasing for X or strictly decreasing for X.*

Observe that each $E \in \mathcal{S}(X)$ may have another order $<_E$ introduced by S_E. Corollary 8.4 states an implication between these orders $<_X$ and $<_E$ on E.

Codecop products. Let Y be a space and S a set of ordinals. Recall that Y has the S-codecop property at $q \in Y$ if for each continuously descending sequence $\{V_\alpha : \alpha \in S\}$ of clopen neighborhoods of q in Y, $q \in \mathrm{Int}(\bigcap_{\alpha \in S} V_\alpha)$ holds (see Definition 6). The following are easy to see.

Fact 8.5. *Let S and T be unbounded subsets in a limit ordinal λ with $T \subset S$. If a space Y has the T-codecop property at $q \in Y$, then it has the S-codecop property at q.*

Lemma 8.6. *Let S be a stationary subset in a regular uncountable cardinal κ. Then a space Y has the S-codecop property at $q \in Y$ if and only if Y has the $(S \cap C)$-codecop property at q for some (any) club set C in κ.*

By Fact 2.2, we see that the S_E-codecop property does not depend on the choice of S_E for each $E \in \mathcal{S}(X, \kappa)$. Moreover, we can get an analogue to Lemmas 3.5, 3.6 and 3.7.

Lemma 8.7. *Let S be a stationary subset in a regular uncountable cardinal κ and Y a space. If $S \times Y$ is rectangular, then Y has the S-codecop property.*

Proof. Pick a $q \in Y$. Let $\{V_\alpha : \alpha \in S\}$ be a continuously descending sequence of clopen neighborhoods of q in Y. For each $\alpha \in S$, let $G_\alpha = (S \cap [0, \alpha]) \times V_\alpha$, and let $G = \bigcup_{\alpha \in S} G_\alpha$. To show that G is closed in $S \times Y$, pick any $\langle \beta, z \rangle \in (S \times Y) \setminus G$. Then we have $z \notin V_\beta$. In case $\beta \in S \cap \mathrm{Lim}(S)$, there is $\beta_0 \in S \cap \beta$ with $z \notin V_{\beta_0}$. Then let $O = (S \cap (\beta_0, \beta]) \times (Y \setminus V_{\beta_0})$. In case $\beta \in S \setminus \mathrm{Lim}(S)$, there is $\beta_1 \in \beta \cup \{-1\}$ with $S \cap (\beta_1, \beta] = \{\beta\}$. Then let $O = (S \cap (\beta_1, \beta]) \times (Y \setminus V_\beta)$. In any case, O is an open neighborhood of $\langle \beta, z \rangle$ in $S \times Y$ disjoint from G. So G is a clopen set in $S \times Y$ containing $S \times \{q\}$. Since G is the union of a σ-locally finite collection of open (cozero) rectangles in $S \times Y$, there are an open rectangle $U \times W$ in $S \times Y$ and a $\gamma \in \kappa$ such that $U \times W \subset G$, $q \in W$ and $S \cap (\gamma, \kappa) \subset U$. To show $W \subset \bigcap_{\alpha \in S} V_\alpha$, pick any $y \in W$ and any $\alpha \in S$. Take a $\xi \in S$ with $\xi > \max\{\alpha, \gamma\}$. Since $\langle \xi, y \rangle \in U \times W \subset G$, there is $\eta \in S$ with $\langle \xi, y \rangle \in G_\eta = (S \cap [0, \eta]) \times V_\eta$. By $\alpha < \xi \leq \eta$, we have $y \in V_\eta \subset V_\alpha$. Hence we conclude that $q \in W \subset \bigcap_{\alpha \in S} V_\alpha$. □

Definition 9. A product space $X \times Y$ is called a *codecop product* (a *docs product*) if $X \times Y$ is a diagonal stationary product, satisfying

(i) Y has the S_E-codecop property (S_E-docs property) for each $E \in \mathcal{S}(X)$,
(ii) X has the S_F-codecop property (S_F-docs property) for each $F \in \mathcal{S}(Y)$.

Recall that a diagonal stationary product is defined in Definition 7.

Definition 10. A product space $X \times Y$ is called a *weak codecop product* if

(i) Y has the κ-codecop property for each $\kappa \in \mathcal{S}^*(X)$,
(ii) X has the τ-codecop property for each $\tau \in \mathcal{S}^*(Y)$.

The following is an immediate consequence of Lemma 6.8 and Fact 8.5.

Proposition 8.8. *The following are true.*

(1) *Every dop product is a codecop product.*
(2) *Every codecop product is a weak codecop product.*

Proposition 8.9. *The following are true.*

(1) *If the product of two spaces is normal, then it is a docs product.*
(2) *If the product of two monotonically normal spaces is rectangular, then it is a weak codecop product.*

These immediately follow from Lemmas 3.7 and 7.2 for (1) and from Proposition 6.12 for (2).

Rectangular products of GO-spaces. In Proposition 8.9, if we strengthen monotone normality of X and Y with GO-spaces, then we can take off the "weak" condition of $X \times Y$ (see Theorem 8.14). To prove this, we need the concept of retract.

Recall that a closed set E in a space X is called a *retract* of X if there is a continuous map $f : X \to E$ such that $f(x) = x$ for each $x \in E$. Such a map f is called a *retraction* from X to E.

Lemma 8.10. *Let X and Y be spaces with $E \subset X$ and $F \subset Y$. Suppose that E and F are retracts of X and Y, respectively. If $X \times Y$ is rectangular, then so is $E \times F$.*

Proof. Let G be a cozero-set in $E \times F$. Since $E \times F$ is also a retract of $X \times Y$, there is a cozero-set G^* in $X \times Y$ such that $G^* \cap (E \times F) = G$. Then there is a σ-locally finite collection $\mathcal{G}$ of cozero rectangles in $X \times Y$ such that $\bigcup \mathcal{G} = G^*$. Then

$$\mathcal{G} \restriction (E \times F) = \{(U \times V) \cap (E \times F) : U \times V \in \mathcal{G}\}$$

is a σ-locally finite collection of cozero rectangle in $E \times F$ such that $\bigcup\big(\mathcal{G} \restriction (E \times F)\big) = G$. Hence $E \times F$ is rectangular. □

Lemma 8.11. *Let A be a subspace of an ordinal. Then each non-empty closed subspace is a retract of A.*

Proof. Let $A \subset \lambda + 1$. Take any non-empty closed subspace F in A. Let $\mu = \sup_{\lambda+1} F$. When $\mu \in A$, we have $\mu \in F$. Pick a fixed $\alpha_0 \in F$. Take the function $g : A \to F$ defined by $g(\alpha) = \alpha$ for each $\alpha \in F$, $g(\alpha) = \min\{\alpha' \in F : \alpha < \alpha'\}$ for each $\alpha \in (A \cap [0, \mu]) \setminus F$ and $g(\alpha) = \alpha_0$ for each $\alpha \in A \cap (\mu, \lambda]$. It suffices to show that g is continuous at each point of A. Pick an $\alpha \in A$. In case $\alpha \in F$: Pick a $\beta < \alpha$. Then it is easily checked that $g(A \cap (\beta, \alpha]) \subset F \cap (\beta, \alpha]$. In case $\alpha \in (A \cap [0, \mu]) \setminus F$: Take $\gamma < \alpha$ with $F \cap (\gamma, \alpha] = \emptyset$. Then $(\gamma, g(\alpha))$ misses F. So we have $g(A \cap (\gamma, \alpha]) = \{g(\alpha)\}$. In case $\alpha \in A \cap (\mu, \lambda]$: Then $A \cap (\mu, \lambda]$ is clopen in A and $g(A \cap (\mu, \lambda]) = \{\alpha_0\} = \{g(\alpha)\}$. Hence g is continuous on A. This means that g is a retraction of A onto F. □

Lemma 8.12. *Let X be a GO-space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Then there is a club set C in κ such that either $C \subset S_E$ or $S_E \cap C$ is homeomorphic to a retract of X.*

Proof. Let $S = S_E$ and $e = e_E$. If $\kappa \setminus S$ is non-stationary in κ, then there is a club set C in κ with $C \subset S$. So we may assume that $\kappa \setminus S$ is stationary in κ. By Corollary 8.4, there is a club set D in κ such that $e \restriction (S \cap D)$ is strictly increasing or strictly decreasing. We may assume that $e \restriction (S \cap D)$ is strictly increasing because the other case is similar. Let $T = \mathrm{Lim}(S \cap D) \cap (\kappa \setminus S)$, $C = \mathrm{Lim}(T)$ and $E^* = e(S \cap C)$. Then C is a club set in κ since T is stationary in κ. And $C \subset D$ holds. Obviously, E^* is homeomorphic to $S \cap C$ and closed in X. It suffices to show that E^* is a retract of X.

Claim. For each $y, z \in E^*$ with $y < z$, there is a $d(y, z) \in L_X \setminus X$ with $y < d(y, z) < z$.

Proof. Take $\alpha, \beta \in S \cap C$ with $y = e(\alpha)$ and $z = e(\beta)$. Since $e \restriction (S \cap D)$ is strictly increasing, $C \subset D$ and $\beta \in C = \mathrm{Lim}(T)$ with $\alpha < \beta$, there is $\xi \in T$ with $\alpha < \xi < \beta$. Let $d(y, z) = \sup_{L_X} e(S \cap D \cap \xi) := d$. Then it is easily checked that $y < d < z$. By $\xi \notin S$, $S \cap D \cap \xi$ is closed in S. Assume that $d \in X$. Then we have $d \in \overline{e(S \cap D \cap \xi)} = e(S \cap D \cap \xi)$. Take $\eta \in S \cap D \cap \xi$ with $e(\eta) = d$. By $\xi \in T \subset \mathrm{Lim}(S \cap D)$, there is $\zeta \in S \cap D$ with $\eta < \zeta < \xi$. Then we obtain a contradiction that $d = e(\eta) < e(\zeta) \le d$. Hence $d \in L_X \setminus X$. □

Set

$$E(-,+) = \{\langle y, z\rangle : y \in \mathrm{Cl}_{L_X}(E^*) \cup \{\leftarrow\}, z \in E^* \cup \{\rightarrow\}, y < z \text{ and } E^* \cap (y, z) = \emptyset\}.$$

Pick an $x \in X$. Put $x^+ = \min(E^* \cap (x, \rightarrow))$ if $E^* \cap (x, \rightarrow) \neq \emptyset$, otherwise put $x^+ =\rightarrow$. Put $x^- = \sup_{L_X}(E^* \cap (\leftarrow, x))$ if $E^* \cap (\leftarrow, x) \neq \emptyset$, otherwise put $x^- =\leftarrow$. Then the following are easily checked.

(1) If $a \in E^* \cup \{\leftarrow\}$, $b \in E^* \cup \{\rightarrow\}$ and $x \in (a, b)$, then $a \le x^- \le x < x^+ \le b$.
(2) If $x^- \in X$, then $x^- \in \mathrm{Cl}_X(E^*) = E^*$ holds.
(3) If $x \in X \setminus E^*$, then $\langle x^-, x^+\rangle$ is a unique member of $E(-,+)$ satisfying $x \in (x^-, x^+)$. Hence $\{X \cap (y, z) : \langle y, z\rangle \in E(-,+)\}$ is a pairwise disjoint open cover of $X \setminus E^*$.
(4) If $x \in E^*$, then $x^+ \in E^*$ and $\langle x, x^+\rangle \in E(-,+)$ always hold, and $\langle x^-, x\rangle \in E(-,+)$ holds except the case that $x = \sup_{L_X}(E^* \cap (\leftarrow, x))$ with $E^* \cap (\leftarrow, x) \neq \emptyset$.

We will define a retraction $g : X \to E^*$. Let $g(x) = x$ for each $x \in E^*$. Pick any $\langle y, z\rangle \in E(-,+)$. We will define a continuous function $g \restriction (X \cap (y, z))$. In case $z =\rightarrow$: Fix some $x^* \in E^*$ and let $g(x) = x^*$ for each $x \in X \cap (y, z)$. In case $y \notin E^*$ and $z \in E^*$: Let $g(x) = z$ for each $x \in X \cap (y, z)$. In case $y, z \in E^*$, we define $g(x) = y$ for each $x \in X \cap (y, d(y, z))$ and define $g(x) = z$ for each $x \in X \cap (d(y, z), z)$. Thus $g : X \to E^*$ has been defined. It follows from (3) that $g \restriction (X \setminus E^*)$ is continuous. So it suffices to show that g is continuous at each point in E^*.

Pick an $x \in E^*$. By (4), we have $\langle x, x^+\rangle \in E(-,+)$ and $x, x^+ \in E^*$. By the above claim, $x < d(x, x^+) < x^+$ holds, and we have $g(X \cap (x, d(x, x^+))) = \{x\} = \{g(x)\}$. In case $\langle x^-, x\rangle \in E(-,+)$ for some x^-: When $x^- \notin E^*$, we have $g(X \cap (x^-, x)) = \{x\} = \{g(x)\}$. When $x^- \in E^*$, by the above claim, $x^- < d(x^-, x) < x$ holds. So we have $g\big(X \cap (d(x^-, x), x)\big) = \{x\} = \{g(x)\}$.

If there is not such x^-, then $x = \sup_{L_X}(E^* \cap (\leftarrow, x))$ with $E^* \cap (\leftarrow, x) \neq \emptyset$. Each neighborhood V of $g(x) = x$ in E^* contains $E^* \cap [a, x]$ for some $a \in E^* \cap (\leftarrow, x)$. Pick any $y \in X \cap (a, x)$. When $y \in E^*$, obviously, $g(y) = y \in E^* \cap (a, x)$. When $y \in X \setminus E^*$, by (1), $a \le y^- \le g(y) < y^+ \le x$ holds. Hence we obtain $g(X \cap (a, x)) \subset E^* \cap [a, x] \subset V$. Thus g is continuous at x in any case. □

Remark 8.13. We cannot remove "$C \subset S_E$" in Lemma 8.12, because there is a GO-space X with an $E \in \mathcal{S}(X, \kappa)$ such that $S_E \cap C$ is not homeomorphic to a retaract of X for any club set C in κ. In fact, consider the space $X = \kappa \times [0, 1)$ topologized by the lexicographic order, where $[0, 1)$ is an interval in the real line. Then X is a connected LOTS (hence a GO-space) with the closed subspace $E := \kappa \times \{0\} \in \mathcal{S}(X, \kappa)$. Since connectedness is preserved by continuous maps, any retract of X is connected. On the other hand, E is not connected, so it is not a retract of X. In the same reason, $S_E \cap C$ is not homeomorphic to a retract of X for any club set C in κ.

Theorem 8.14. *If the product of two GO-spaces is rectangular, then it is a codecop product.*

Proof. Let X and Y be GO-spaces such that $X \times Y$ is rectangular. First we show that $X \times Y$ is a diagonal stationary product. Pick a $\kappa \in \mathcal{S}^*(X) \cap \mathcal{S}^*(Y)$. Take $E \in \mathcal{S}(X, \kappa)$ and $F \in \mathcal{S}(Y, \kappa)$. By Lemma 8.12, there are two club sets C and D in κ such that $S_E \supset C$ or $S_E \cap C$ is homeomorphic to a retract in X and that $S_F \supset D$ or $S_F \cap D$ is homeomorphic to a retract in Y. When $S_E \supset C$ or $S_F \supset D$, it is obvious that $S_E \cap S_F$ is stationary in κ. Otherwise, let E' and F' be retracts in X and Y, respectively, such that $S_E \cap C$ is homeomorphic to E' and that $S_F \cap D$ is homeomorphic to F'. By Lemma 8.10, since $X \times Y$ is rectangular, so is $E' \times F'$. So $(S_E \cap C) \times (S_F \cap D)$ is rectangular. It follows from the parenthetic part of Lemma 7.1 that $(S_E \cap C) \cap (S_F \cap D)$ is stationary in κ. Hence so is $S_E \cap S_F$.

Since GO-spaces are monotonically normal, it follows from Proposition 8.9 that $X \times Y$ is a weak codecop product. So Y has the κ-codecop property for each $\kappa \in \mathcal{S}^*(X)$. Take an $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. By Lemma 8.12, there is a club set C in κ such that either $C \subset S_E$ or $S_E \cap C$ is homeomorphic to a retract E' of X. In case $C \subset S_E$: By $\kappa \cap C = C$, it follows from Lemma 8.6, Y has the C-codecop property. By Fact 8.5, Y has the S_E-codecop property. Otherwise, since E' is a retract of X and $X \times Y$ is rectangular, we see by applying Lemma 8.10 that $E' \times Y$ is rectangular. Hence $(S_E \cap C) \times Y$ is rectangular. Since $S_E \cap C$ is stationary in κ, it follows from Lemma 8.7 that Y has the $(S_E \cap C)$-codecop property. It follows from Fact 8.5 again that Y has the S_E-codecop property. Similarly, (ii) in Definition 9 is satisfied. □

Normality, orthocompactness and rectangularity. From this subsection, we deal with the products of a GO-space and a subspace of an ordinal. Here we show the equivalence of normality and orthocompactness for such products.

Proposition 8.15. *Let X be a GO-space with $x \in X$ and κ a regular uncountable cardinal. The following are equivalent.*

(a) *X has orthocaliber κ at x.*
(b) *X has the κ-dop property at x.*
(c) *X has the S-docs property at x for any stationary subset S in κ.*
(d) *X has the S-docs property at x for some stationary subset S in κ.*
(e) *$0\text{-}\mathrm{cf}(x) \neq \kappa$ and $1\text{-}\mathrm{cf}(x) \neq \kappa$ in L_X.*

Proof. (a)⇒(b)⇒(c): These are immediate consequences of Lemma 3.4.

(c)⇒(d): Obvious.

(d)⇒(e): Let $L = L_X$. Assume that $0\text{-}\mathrm{cf}(x) = \kappa$. Then there is a normal function $c_{x,0} : \kappa \to L$ at x described in Fact 8.2. Let $c = c_{x,0}$. Let $V_\alpha = X \cap (c(\alpha + 1), \to)_L$ for each $\alpha \in \kappa$, where note that $c(\alpha + 1) \in X$ by Fact 8.2(ii). Since $\{V_\alpha : \alpha \in S\}$ is a descending sequence of open neighborhoods of x in X, there is a continuously descending sequence $\{F_\alpha : \alpha \in S\}$ of closed neighborhoods of x in X such that $F_\alpha \subset V_\alpha$ for each $\alpha \in S$. For each $\alpha \in S$, choose $\xi_\alpha \in \kappa$ such that $X \cap (c(\xi_\alpha), x]_L \subset F_\alpha$ and $c(\alpha + 1) < c(\xi_\alpha) < x$. Let

$$C = \{\alpha \in \kappa : \beta \in S \cap \alpha \ \text{ implies } \ \xi_\beta < \alpha\}.$$

Since C is a club set in κ by [11, Lemma II.6.13], we can choose $\alpha_0 \in S \cap \mathrm{Lim}(S) \cap C$. Then we have that

$$c(\alpha_0 + 1) \in X \cap [c(\alpha_0 + 1), x]_L \subset X \cap \Big(\bigcap_{\beta \in S \cap \alpha_0} \big(c(\xi_\beta), x]\Big)_L \subset \bigcap_{\beta \in S \cap \alpha_0} F_\beta = F_{\alpha_0} \subset V_{\alpha_0}.$$

Hence we obtain $c(\alpha_0 + 1) \in V_{\alpha_0} = X \cap (c(\alpha_0 + 1), \to)_L$, which is a contradiction.

Assume that $1\text{-}\mathrm{cf}(x) = \kappa$. Then there is another normal function $c_{x,1} : \kappa \to L$ at x similarly described in Fact 8.2. Let $c' = c_{x,1}$. Let $W_\alpha = X \cap (\leftarrow, c'(\alpha + 1))_L$ for each $\alpha \in \kappa$. The remaining argument is similar to the above.

(e)⇒(a): Let $\{V_\alpha : \alpha \in \kappa\}$ be a sequence of open neighborhoods of x of X. It suffices to show that (i) and (ii) below hold.

(i) For each unbounded subset S of κ, there are $S_0 \subset S$ and $a \in [\leftarrow, x)_L$ such that S_0 is unbounded in κ and $X \cap (a, x]_L \subset \bigcap_{\alpha \in S_0} V_\alpha$.

(ii) For each unbounded subset S of κ, there are $S_1 \subset S$ and $b \in X \cap (x, \rightarrow]_L$ such that S_1 is unbounded in κ and $X \cap [x, b)_L \subset \bigcap_{\alpha \in S_1} V_\alpha$.

Actually, if (i) and (ii) are true, then by applying (i) for $S = \kappa$ and by applying (ii) for $S = S_0$, we obtain an unbounded subset S_1 of κ and an open neighborhood $X \cap (a, b)_L$ of x in X which is contained in $\bigcap_{\alpha \in S_1} V_\alpha$, that is, $x \in \mathrm{Int}(\bigcap_{\alpha \in S_1} V_\alpha)$ holds. Hence X has orthocaliber κ at x.

Because (i) and (ii) are similar, we prove only (i). For each $\alpha \in \kappa$, we can take $x(\alpha) \in [\leftarrow, x)_L$ with $X \cap (x(\alpha), x]_L \subset V_\alpha$. Let S be an unbounded subset of κ, and let $S(\beta) = \{\alpha \in S : x(\alpha) \leq_L x(\beta)\}$ for each $\beta \in S$. If $S(\beta)$ is unbounded in κ for some $\beta \in S$, then $a = x(\beta)$ and $S_0 = S(\beta)$ satisfy the required condition. If $S(\beta)$ is bounded in κ for every $\beta \in S$, then by induction, we can take a strictly increasing sequence $\{\alpha_\xi : \xi \in \kappa\}$ by members of S such that $\alpha_\xi \notin \bigcup_{\zeta \in \xi} S(\alpha_\zeta)$ for each $\xi \in \kappa$. Let $a = \sup_L \{x(\alpha_\xi) : \xi \in \kappa\} \in L$. Since $\{x(\alpha_\xi) : \xi \in \kappa\}$ is strictly increasing in L and κ is regular, we have $0\text{-}\mathrm{cf}(a) = \kappa \neq 0\text{-}\mathrm{cf}(x)$, thus $a <_L x$ holds. Hence a and $S_0 = \{\alpha_\xi : \xi \in \kappa\}$ satisfy the required condition. □

Corollary 8.16. *Let X be a GO-space and B a subspace of an ordinal. Then $X \times B$ is normal if and only if it is orthocompact.*

Proof. The "if" part is an immediate consequence of Corollary 7.19, because X is monotonically normal. Assume that $X \times B$ is normal. By Proposition 8.9(1), $X \times B$ is a docs product. In particular, it is a diagonal stationary product. Pick a $\kappa \in \mathcal{S}^*(X)$ and take an $E \in \mathcal{S}(X, \kappa)$. Then B has the S_E-docs property. By Proposition 8.15, B has orthocaliber κ. Similarly, X has orthocaliber κ for each $\kappa \in \mathcal{S}^*(B)$. Hence $X \times B$ is an orthocaliber product. It follows from Theorem 7.18 that $X \times B$ is orthocompact. □

The following is an immediate consequence of Corollaries 7.19 and 8.16.

Corollary 8.17. *Let X be a GO-space and B a subspace of an ordinal. If $X \times B$ is normal, then it is rectangular.*

Countable paracompactness and rectangularity. Recall that a space X is *expandable* if for every locally finite collection $\mathcal{F}$ of closed sets in X, there is a locally finite open expansion of $\mathcal{F}$. It is well known that a space X is countably paracompact iff every locally finite countable collection $\{F_n : n \in \omega\}$ of closed sets in X has a locally finite open expansion $\{U(F_n) : n \in \omega\}$.

Here we prove the equivalence of expandability, countable paracompactness and rectangularity of the products of a GO-space and a subspace of an ordinal as an extension of Theorem 8.1.

Theorem 8.18. *Let X be a GO-space and B a subspace of an ordinal. Then the following are equivalent.*

(a) *$X \times B$ is expandable.*
(b) *$X \times B$ is countably paracompact.*
(c) *$X \times B$ is rectangular.*
(d) *$X \times B$ is a codecop product.*

For the proof of this, we need some lemmas below.

Lemma 8.19. *Let X be a GO-space with $x \in X$. Let S be a stationary subset in a regular uncountable cardinal κ. Then X has the S-codecop property at x if and only if the following conditions are satisfied;*

(i) *if $0\text{-}\mathrm{cf}(x) = \kappa$, then $S \cap c_{x,0}^{-1}(X)$ is stationary in κ,*
(ii) *if $1\text{-}\mathrm{cf}(x) = \kappa$, then $S \cap c_{x,1}^{-1}(X)$ is stationary in κ.*

Proof. Let $c = c_{x,0}$ and $L = L_X$. The "only if" part: We show only (i), because (ii) is similar. Assuming the contrary of (i), there is a club set C in κ such that $S \cap c^{-1}(X) \cap C = \emptyset$. Pick an $\alpha \in S \cap C$. Let $V_\alpha = X \cap (c(\alpha), \rightarrow)_L$. By $c(\alpha) \in L \setminus X$, V_α is a clopen neighborhood of x in X.

When $\alpha \in \mathrm{Lim}(S \cap C)$, by the continuity of c, we have $V_\alpha = \bigcap_{\beta \in S\cap C\cap\alpha} V_\beta$. Hence $\{V_\alpha : \alpha \in S \cap C\}$ is a continuously descending sequence of clopen neighborhoods of x in X. By $x = \sup_L\{c(\alpha) : \alpha \in S \cap C\}$, we have $\bigcap_{\alpha\in S\cap C} V_\alpha \subset X \cap [x, \rightarrow)_L$. Since $x \in \overline{X \cap (\leftarrow, x)_L}$ by $0\text{-}\mathrm{cf}(x) = \kappa > 1$, we have $x \notin \mathrm{Int}(\bigcap_{\alpha\in S\cap C} V_\alpha)$. Hence X does not have the $(S\cap C)$-codecop property at x. By Lemma 8.6, this is a contradiction.

The "if" part: Let $\{V_\alpha : \alpha \in S\}$ be a continuously descending sequence of clopen neighborhoods of x in X. Assuming (i), we show that there is $a \in [\leftarrow, x)_L$ with $X \cap (a, x) \subset \bigcap_{\alpha\in S} V_\alpha$. This is easily seen for the case of $0\text{-}\mathrm{cf}(x) < \kappa$ or $0\text{-}\mathrm{cf}(x) > \kappa$. So we may assume that $0\text{-}\mathrm{cf}(x) = \kappa$. For each $\alpha \in S$, since V_α is a neighborhood of x, there is $f(\alpha) \in \kappa$ with $f(\alpha) > \alpha$ and $X \cap (c(f(\alpha)), x] \subset V_\alpha$. Let $C = \{\alpha \in \kappa : \beta \in S\cap\alpha \text{ implies } f(\beta) < \alpha\}$, then C is a club set in κ. Let $T = S\cap c^{-1}(X)\cap\mathrm{Lim}(S)\cap C$. By the assumption, T is stationary in κ. Pick an $\alpha \in T$. For each $\beta \in S \cap \alpha$, by the choice of C, we have $c(\alpha) \in X \cap \big(c(f(\beta)), x\big] \subset V_\beta$. Since $\{V_\alpha : \alpha \in S\}$ is continuously descending and α is a limit, we have that $c(\alpha) \in \bigcap_{\beta\in S\cap\alpha} V_\beta = V_\alpha$. Since c is continuous, we can take $g(\alpha) < \alpha$ with $X \cap \big(c(g(\alpha)), c(\alpha)\big] \subset V_\alpha$. By PDL, there is $T_0 \subset T$ and $\alpha_0 \in \kappa$ such that T_0 is stationary in κ and $g(\alpha) = \alpha_0$ for each $\alpha \in T_0$. Let $a = c(\alpha_0)$. Then $a < x$. Let $x' \in X \cap (a, x)$ and $\alpha \in S$. By taking $\alpha_1 \in T_0$ with $\alpha < \alpha_1$ and $x' < c(\alpha_1)$, we have $x' \in X \cap (c(g(\alpha_1)), c(\alpha_1)] \subset V_{\alpha_1} \subset V_\alpha$. Hence, $x' \in \bigcap_{\alpha\in S} V_\alpha$, and so $X \cap (a, x) \subset \bigcap_{\alpha\in S} V_\alpha$ holds.

Assuming (ii), by the similar argument, we can show that there is $b \in (x, \rightarrow]_L$ such that $X \cap (x, b) \subset \bigcap_{\alpha\in S} V_\alpha$. We obtain an open neighborhood $X \cap (a, b)$ of x which is contained in $\bigcap_{\alpha\in S} V_\alpha$, so $x \in \mathrm{Int}_X(\bigcap_{\alpha\in S} V_\alpha)$. Thus X has the S-codecop property at x. □

Lemma 8.20 ([7, Theorem B])**.** *Let κ be a regular uncountable cardinal. Let S be a stationary subset in κ and T an unbounded subset in κ. If $S \times (T \cup [\kappa\})$ is countably paracompact, then $S \cap T$ is stationary in κ.*

Lemma 8.21. *Let X and Y be GO-spaces. If $X \times Y$ is countably paracompact, then it is a codecop product.*

Proof. By Lemma 7.2, note that $X\times Y$ is a diagonal stationary product. Take an $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. We show that Y has the S_E-codecop property. Pick any $y \in Y$ with $0\text{-}\mathrm{cf}(y) = \kappa$. Let $c = c_{y,0}$. Since $c^{-1}(Y)$ is unbounded in κ and $c(c^{-1}(Y))$ is closed in $Y \cap (\leftarrow, y)_{L_Y}$, $c(c^{-1}(Y)) \cup \{y\}$ is closed in Y. So $c^{-1}(Y) \cup \{\kappa\}$ is homeomorphic to a closed subspace in Y. Since $S_E \times (c^{-1}(Y) \cup \{\kappa\})$ is homeomorphic to a closed subspace in $X \times Y$, it is countably paracompact. It follows from Lemma 8.20 that $S_E\cap c^{-1}(Y)$ is stationary in κ. Hence Lemma 8.19(i) is satisfied. Similarly, we can show that Lemma 8.19(ii) is satisfied. Thus Y has the S_E-codecop property. Similarly, X has the S_F-codecop property for each $F \in \mathcal{S}(Y)$. Hence $X \times Y$ is a codecop product. □

Fact 8.22. *Let X be a space with $E \in \mathcal{S}(X, \kappa)$, where $\kappa \in \mathcal{S}^*(X)$. Let E' be a closed subset of E with $|E'| = \kappa$. Then $E' \in \mathcal{S}(X, \kappa)$ holds, and for each open set U in X, E is almost contained in U iff E' is almost contained in U.*

The proof is a routine by PDL. So it is omitted here.

Fact 8.23. *Let X be a GO-space with $E \in \mathcal{S}(X, \kappa)$ and an open set U in X. If E is almost contained in U, then there is an open interval I in L_X such that E is almost contained in I and $X \cap I \subset U$.*

Proof. Let $e = e_E$ and $S = S_E$. By Proposition 8.3, there is a club set C in κ such that $e \restriction (S \cap C)$ is strictly increasing or strictly decreasing. We may assume that $e \restriction (S\cap C)$ is strictly increasing, because the other case is similar. And we may also assume that $e(S \cap C) \subset U$. Let $p = \sup_{L_X}(S \cap C)$ and $S_0 = S\cap C\cap\mathrm{Lim}(S\cap C)$. For each $\alpha \in S_0$, take a $\gamma(\alpha) \in S\cap C\cap\alpha$ such that $X\cap(e(\gamma(\alpha)), e(\alpha)]_{L_X} \subset U$. By PDL, there are an $S_1 \subset S_0$ and a $\gamma \in S \cap C$ such that S_1 is stationary in κ and $\gamma(\alpha) = \gamma$ for each $\alpha \in S_1$. Define an open interval I in L_X by putting $I = (e(\gamma), p)_{L_X}$ and let $U' = X \cap I$. Then U' is an open set of X such that $U' \subset \bigcup_{\alpha\in S_1} \big(X \cap (e(\gamma(\alpha)), e(\alpha)]_{L_X}\big) \subset U$. Obviously, $e(S \cap C)$ is a closed subset of E with $|e(S \cap C)| = \kappa$. And it is almost contained in an open set $U' \cap E$ of E since $e(S \cap C \cap (\gamma, \kappa)) \subset U' \cap E$. By Fact 8.22, E is almost contained in $U' \cap E$, and so in U'. □

Definition 11. We say that a family $\mathcal{G}$ of subsets of a space Z is *countably determined* if the following condition holds:

> for each subset Z' of Z, if every countable subset of Z' is contained in some member of $\mathcal{G}$, then Z' is contained in some member of $\mathcal{G}$.

Lemma 8.24. *A family $\mathcal{G}$ of subsets of a space Z is countably determined if one of the following conditions holds.*

(1) *$\mathcal{G}$ is point-countable.*
(2) *$\mathcal{G} = \{G \subset Z : G$ is an open set meeting at most finitely many members of $\mathcal{L}\}$ for some locally finite collection $\mathcal{L}$ of closed sets of Z.*

Proof. Assume that a subset Z' of Z is not contained in any member of $\mathcal{G}$. It suffices to find a countable subset M of Z' which is not contained in any member of $\mathcal{G}$.

(1): We may assume that Z' is uncountable. Take $z_0 \in Z'$ and let $\mathcal{G}_0 = \{G \in \mathcal{G} : z_0 \in G\}$. Then $\mathcal{G}_0$ is countable. For each $G \in \mathcal{G}_0$, we can take a $z_G \in Z' \setminus G$. Let $M = \{z_0\} \cup \{z_G : G \in \mathcal{G}_0\}$. It is easily seen that M is a desired countable subset of Z'.

(2): Let $\mathcal{L}_0 = \{L \in \mathcal{L} : Z' \cap L \neq \emptyset\}$ and $G_0 = Z \setminus \bigcup(\mathcal{L} \setminus \mathcal{L}_0)$. Then G_0 is an open set of Z with $Z' \subset G_0$. Let $\mathcal{L}_1 = \{L \in \mathcal{L}_0 : L \cap G_0 \neq \emptyset\}$. By $G_0 \notin \mathcal{G}$, we have $|\mathcal{L}_1| \geq \omega$. Take a $\mathcal{L}_2 \subset \mathcal{L}_1$ with $|\mathcal{L}_2| = \omega$. For each $L \in \mathcal{L}_2$, pick a $z_L \in L \cap Z'$. Let $M = \{z_L : L \in \mathcal{L}_2\}$. It is also easily seen that M is a desired countable subset of Z'. □

Lemma 8.25. *Let X be a space with $E \in \mathcal{S}(X, \kappa)$, and $\mathcal{G}$ a countably determined family of open sets of X with $E \subset \bigcup \mathcal{G}$. Then E is almost contained in some member of $\mathcal{G}$.*

Proof. Let $e = e_E$ and $S = S_E$. For each $\xi \in S$, take $G_\xi \in \mathcal{G}$ with $e(\xi) \in G_\xi$ and take $\gamma(\xi) < \xi$ with $e(S \cap (\gamma(\xi), \xi]) \subset G_\xi$. By PDL, there are an $S_0 \subset S$ and a $\gamma \in \kappa$ such that S_0 is stationary in κ and $\gamma(\xi) = \gamma$ for each $\xi \in S_0$. Put $E_0 = e(S \cap (\gamma, \kappa))$. Let M be a countable subset of E_0. Since there is $\xi_0 \in S_0$ with $e^{-1}(M) \subset \xi_0$, it follows that $M \subset e(S \cap (\gamma, \xi_0]) = e(S \cap (\gamma(\xi_0), \xi_0])$ is contained in $G_{\xi_0} \in \mathcal{G}$. This means that every countable subset of E_0 is contained in some member of $\mathcal{G}$. By the assumption of $\mathcal{G}$, E_0 is contained in some member G of $\mathcal{G}$. □

Note that if $\mathcal{G}$ is countably determined in Z and $A \subset Z$, then so is $\mathcal{G} \restriction A$ in A. Considering as $Z = X \times Y$ and $A = \{x\} \times Y$, Lemma 8.25 immediately yields

Lemma 8.26. *Let X and Y be spaces with $x \in X$ and $F \in \mathcal{S}(Y, \tau)$. Let $\mathcal{G}$ be a countably determined family of open sets of $X \times Y$ with $\{x\} \times F \subset \bigcup \mathcal{G}$. Then there is an open set V of Y, satisfying that*

(i) *F is almost contained in V,*
(ii) *$\{x\} \times V$ is contained in some member of $\mathcal{G}$.*

Lemma 8.27. *Let X be a GO-space with $x \in X$ and Y a space with $F \in \mathcal{S}(Y, \tau)$. And let G be an open set in $X \times Y$ such that $\{x\} \times F \subset G$.*

(1) *If $0\text{-cf}(x) \neq \tau$, then there are an $a \in [\leftarrow, x)_{L_X}$ and an open set V in Y which almost contains F such that $\big(X \cap (a, x]_{L_X}\big) \times V \subset G$.*
(2) *If $1\text{-cf}(x) \neq \tau$, then there are a $b \in (x, \rightarrow]_{L_X}$ and an open set W in Y which almost contains F such that $\big(X \cap [x, b)_{L_X}\big) \times W \subset G$.*

Proof. (1) and (2) are proved in a similar way, so we prove only (1). Let $f = e_F$ and $T = S_F$. Let $\kappa = 0\text{-cf}(x)$ and $c = c_{x,0}$. For each $\beta \in T$, take a $\gamma(\beta) \in \kappa \cup \{-1\}$, an open neighborhood V_β of $f(\beta)$ in Y and a $\delta(\beta) \in \beta \cup \{-1\}$ such that $\big(X \cap (c(\gamma(\beta)), x]_{L_X}\big) \times V_\beta \subset G$ and $f\big(T \cap (\delta(\beta), \beta]\big) \subset V_\beta$. By PDL and $\kappa \neq \tau$, it follows from Fact 7.7 that there are a $T_0 \subset T$, a $\gamma \in \kappa \cup \{-1\}$, and a $\delta \in \tau \cup \{-1\}$

such that T_0 is stationary in τ, and for each $\beta \in T_0$, $\gamma(\beta) \leq \gamma$ and $\delta(\beta) = \delta$ hold. We obtain required a and V by letting $a = c(\gamma)$ and $V = \bigcup_{\beta \in T_0} V_\beta$. □

Lemma 8.28. *Let X and Y be GO-spaces with $E \in \mathcal{S}(X,\kappa)$, $F \in \mathcal{S}(Y,\tau)$ and $\kappa \neq \tau$. Let $\mathcal{G}$ be a countably determined family of open sets of $X \times Y$ with $E \times F \subset \bigcup \mathcal{G}$. Then there is an open rectangle $U \times V$ in $X \times Y$, satisfying that*

(i) *E and F are almost contained in U and V, respectively,*
(ii) *$U \times V$ is contained in some member of $\mathcal{G}$.*

Proof. Let $e = e_E$, $S = S_E$, $f = e_F$ and $T = S_F$. By Proposition 8.3, take club sets C and D of κ and τ, respectively, such that each of $e \restriction (S \cap C)$ and $f \restriction (T \cap D)$ is strictly increasing or strictly decreasing. By $\kappa \neq \tau$, either $\kappa < \tau$ or $\kappa > \tau$ holds. We may assume that $e \restriction (S \cap C)$ and $f \restriction (T \cap D)$ are strictly increasing and $\kappa < \tau$, because the other cases are similar. Let $p = \sup_{L_X} e(S \cap C)$ and $q = \sup_{L_Y} f(T \cap D)$. By Lemmas 8.26, 8.27, Fact 8.23 and PDL, we can take a $\gamma \in S \cap C$, $\delta \in T \cap D$ and $R \subset S \cap C \cap \operatorname{Lim}(S \cap C) \cap (\gamma, \kappa)$ such that R is stationary in κ and $Z_\alpha = \big(X \cap (e(\gamma), e(\alpha)]_{L_X}\big) \times \big(Y \cap (f(\delta), q)_{L_Y}\big)$ is contained in some member of $\mathcal{G}$ for each $\alpha \in R$. Let $U = X \cap (e(\gamma), p)_{L_X}$ and $V = Y \cap (f(\delta), q)_{L_Y}$. By Fact 8.22, E and F are almost contained in I and J, respectively. Each countable subset of $U \times V$ is contained in Z_α for some $\alpha \in R$, so it is contained in some member of $\mathcal{G}$. Since $\mathcal{G}$ is countably determined, $U \times V$ is also contained in some member of $\mathcal{G}$. □

Lemma 8.29. *Let X and Y be GO-spaces with $x \in X$ and $F \in \mathcal{S}(Y,\kappa)$. Assume that $\kappa = 0\text{-}\operatorname{cf}(x)$ ($\kappa = 1\text{-}\operatorname{cf}(x)$) and $S_F \cap c^{-1}(X)$ is stationary in κ, where $c = c_{x,0}$($c = c_{x,1}$). Then for each countably determined family $\mathcal{G}$ of open sets of $X \times Y$ with $(c(\kappa) \cap X) \times F \subset \bigcup \mathcal{G}$, there are $\gamma \in \kappa$ and an open set V in Y such that*

(i) *F is almost contained in V,*
(ii) *$\big(X \cap (c(\gamma), x)_{L_X}\big) \times V$ (respectively, $\big(X \cap (x, c(\gamma))_{L_X}\big) \times V$) is contained in some member of $\mathcal{G}$.*

Proof. We consider the "$\kappa = 0\text{-}\operatorname{cf}(x)$" case. Let $S = c^{-1}(X)$, $T = S_F$ and $f = e_F$. By Proposition 8.3, there is a club set C in κ such that $f \restriction (T \cap C)$ is strictly increasing or strictly decreasing. We may assume that it is strictly increasing, because the other cases are similar. Let $q = \sup_{L_Y} f(T \cap C)$. Since $T \cap S$ is stationary, by PDL, we can take a $\gamma \in S \cap T \cap C$ and an $R_0 \subset S \cap T \cap C \cap \operatorname{Lim}(S \cap T \cap C) \cap (\gamma, \kappa)$ such that R_0 is stationary in κ and $Z_\alpha = \big(X \cap (c(\gamma), c(\alpha)]_{L_X}\big) \times \big(Y \cap (f(\gamma), f(\alpha)]_{L_Y}\big)$ is contained in some member of $\mathcal{G}$ for every $\alpha \in R_0$. Let $V = Y \cap (f(\gamma), q)_{L_Y}$. By Fact 8.22, note that F is almost contained in F. Since each countable subset of $\big(X \cap (c(\gamma), x)_{L_X}\big) \times V$ is contained in Z_α for some $\alpha \in R_0$, it is contained in some member of $\mathcal{G}$. Since $\mathcal{G}$ is countably determined, $\big(X \cap (c(\gamma), x)_{L_X}\big) \times V$ is also contained in some member of $\mathcal{G}$. Thus γ and V witness the lemma. □

Lemma 8.30. *Let X and Y be GO-spaces with $E \in \mathcal{S}(X,k)$ and $F \in \mathcal{S}(Y,\kappa)$. Assume that $S_E \cap S_F$ is stationary in κ. Then for each countably determined family $\mathcal{G}$ of open sets of $X \times Y$ with $E \times F \subset \bigcup \mathcal{G}$, there is an open rectangle $U \times V$ in $X \times Y$, satisfying that*

(i) *E and F are almost contained in U and V, respectively,*
(ii) *$U \times V$ is contained in some member of $\mathcal{G}$.*

Proof. Let $S = S_E, e = e_E, T = S_F$ and $f = e_F$. By Proposition 8.3, we may assume that $e \restriction (S \cap C)$ and $f \restriction (S \cap C)$ are strictly increasing for some club set C in κ. Let $p = \sup_{L_X} e(S \cap C)$ and $q = \sup_{L_Y} f(S \cap C)$. Then, as in the proof of the lemma above, we can find $\gamma \in S \cap T \cap C$ such that $U \times V$ is contained in some member of $\mathcal{G}$, where $U = X \cap (e(\gamma), p)$ and $V = Y \cap (f(\gamma), q)$. Then $U \times V$ is the desired open rectangle. □

Lemma 8.31. *Let X and Y be GO-spaces with $x \in X$ and $F \in \mathcal{S}(Y,\tau)$. Assume that X has the S_F-codecop property at x and that $\mathcal{L}$ is a locally finite collection of closed sets in $X \times Y$. Let*

$$\mathcal{G} = \{G \subset X \times Y : G \text{ is an open set meeting at most finitely many members of } \mathcal{L}\}.$$

Then there is an open rectangle $U \times V$ in $X \times Y$, satisfying that

(i) $x \in U$ *and F is almost contained in V,*
(ii) $U \times V$ *is contained in some member of $\mathcal{G}$.*

Proof. By Lemmas 8.24(2) and 8.26, there are an open set V_0 of Y which almost contains F, and $G_0 \in \mathcal{G}$ such that $\{x\} \times V_0 \subset G_0$. It suffices to show that (1) and (2) below hold.

(1) There are an $a \in [\leftarrow, x)_{L_X}$, an open set V_1 in Y which almost contains F, and a $G_1 \in \mathcal{G}$ such that $\big(X \cap (a,x)_{L_X}\big) \times V_1 \subset G_1$.
(2) There are a $b \in (x, \rightarrow]_{L_X}$, an open set V_2 in Y which almost contains F, and a $G_2 \in \mathcal{G}$ such that $\big(X \cap (x,b)_{L_X}\big) \times V_2 \subset G_2$.

In fact, when (1) and (2) above are true, by Fact 8.23, we can take an open set V in Y such that F is almost contained in V and $V \subset V_0 \cap V_1 \cap V_2$ holds. Let $U = X \cap (a,b)_{L_X}$. Then U and V witness the lemma because of $G_0 \cup G_1 \cup G_2 \in \mathcal{G}$. We prove only (1), because (2) is similar. In case $0\text{-}\mathrm{cf}(x) \neq \tau$, we obtain (1) by letting $G_1 = G_0$ and applying Lemma 8.27. So assume that $0\text{-}\mathrm{cf}(x) = \tau$. Let $c = c_{x,0}$. Since X has the S_F-codecop property at x, it follows from Lemma 8.19 that $S_F \cap c^{-1}(X)$ is stationary in τ. Since $\mathcal{G}$ covers $X \times Y$, Lemma 8.29 shows (1). □

Lemma 8.32. *Let X and Y be GO-spaces with $x \in X$ and $F \in \mathcal{S}(Y, \tau)$. Assume that X has the S_F-codecop property at x. If $\mathcal{G}$ is a binary cozero cover of $X \times Y$, then there is an open rectangle $U \times V$ in $X \times Y$, satisfying that*

(i) $x \in I$ *and F is almost contained in J,*
(ii) $U \times V$ *is contained in some member of $\mathcal{G}$.*

Proof. By Lemma 8.26, there are an open set V_0 of Y which almost contains F, and $G \in \mathcal{G}$ such that $\{x\} \times V_0 \subset G$. It suffices to show that (1) and (2) below hold.

(1) There are an $a \in [\leftarrow, x)_{L_X}$ and an open set V_1 of Y which almost contains F such that $\big(X \cap (a,x)_{L_X}\big) \times V_1 \subset G$.
(2) There are a $b \in (x, \rightarrow]_{L_X}$ and an open set V_2 of Y which almost contains F such that $\big(X \cap (x,b)_{L_X}\big) \times V_2 \subset G$.

In fact, when (1) and (2) above are true, by Fact 8.23, we can take an open set V in Y such that F is almost contained in V and $V \subset V_0 \cap V_1 \cap V_2$ holds. Let $U = X \cap (a,b)_{L_X}$. Then U and V witness the lemma. Since (1) and (2) are similar, we prove only (1). In case $0\text{-}\mathrm{cf}(x) \neq \tau$: Lemma 8.27 shows (1). So we may assume $0\text{-}\mathrm{cf}(x) = \tau$. Let $c = c_{x,0}$, $S = c^{-1}(X)$, and $T = S_F$. Since X has the T-codecop property at x, it follows from Lemma 8.19 that $S \cap T$ is stationary in τ. Since a cozero set G is an F_σ-set, there is a sequence $\{H_n : n \in \omega\}$ of open sets in $X \times Y$ such that $(X \times Y) \setminus G = \bigcap_{n\in\omega} H_n$. For each $n \in \omega$, $\mathcal{H}_n := \{G, H_n\}$ is a binary open cover of $X \times Y$. By Lemma 8.29, there are $\gamma_n \in \tau$ and an open set W_n in Y such that

(i) F is almost contained in W_n,
(ii) $U_n \times W_n$ is contained in some member G_n of $\mathcal{H}_n$, where $U_n = X \cap (c(\gamma_n), x)_{L_X}$.

Since F is almost contained in V_0 and W_n's, one can take $y_0 \in F \cap V_0 \cap \bigcap_{n\in\omega} W_n$. It follows from $\langle x, y_0\rangle \in \{x\} \times V_0 \subset G$ that there is $\delta \in \tau$ with $(X \cap (c(\delta), x)) \times \{y_0\} \subset G$. Pick $\gamma \in \tau$ with $\delta < \gamma$ and $\sup\{\gamma_n : n \in \omega\} < \gamma$, moreover pick $x_0 \in X \cap (c(\gamma), x)$. Then we have $\langle x_0, y_0\rangle \in G \cap \bigcap_{n\in\omega} (U_n \times W_n) \subset \bigcap_{n\in\omega} G_n$. Hence $\bigcap_{n\in\omega} G_n \neq (X \times Y) \setminus G = \bigcap_{n\in\omega} H_n$ holds. Therefore, there is $m \in \omega$ with $G_m \neq H_m$, that is, $G_m = G$ holds. Since $U_m \times W_m \subset G_m = G$, $a := c(\gamma_m)$ and $V_1 := W_m$ witness (1). □

Proof of Theorem 8.18. (a)⇒(b): This is obvious because every expandable space is countably paracompact.

(b)⇒(d): This immediately follows from Lemma 8.21.

(c)⇒(d): This immediately follows from Theorem 8.14.

(d)⇒(a): Let $\mathcal{L}$ be a locally finite collection of closed sets in $X \times B$. And let

$$\mathcal{G} = \{G \subset X \times Y : G \text{ is an open set meeting at most finitely many members of } \mathcal{L}\}.$$

Then $\mathcal{G}$ is an open cover of $X \times B$. To show that $\mathcal{L}$ has a locally finite open expansion, it suffices to show that $\mathcal{G}$ is a normal cover. Note that $\mathcal{G}$ is countably determined by Lemma 8.24. Let $\mu \in \Gamma(B)$. Take a club set D of μ having order type $\tau = \mathrm{cf}(\mu)$, and let $F = B \cap D$. Then $F \in \mathcal{S}(B, \tau)$. Since $X \times B$ is a codecop product, X has the S_F-codecop property. By Lemma 8.31, (1) in Lemma 7.5 holds. Let $E \in \mathcal{S}(X, \kappa)$. In case $\kappa \neq \tau$, by Lemma 8.28, (2) in Lemma 7.5 holds for E and μ. In case $\kappa = \tau$, since $X \times B$ is a diagonal stationary product, $S_E \cap S_F$ is stationary in $\kappa = \tau$. By Lemma 8.30, (2) in Lemma 7.5 holds for E and μ. Let $E \in \mathcal{S}(X, \kappa)$ and $\mu \in B$. Since $X \times B$ is a codecop product, B has the S_E-codecop property at μ. By Lemma 8.31, (2) in Lemma 7.5 holds for E and μ. By Lemma 7.5, $\mathcal{G}$ is a normal cover.

(d)⇒(c): Let $\mathcal{G} = \{G_0, G_1\}$ be a binary cozero cover of $X \times B$. In the similar way to the case of (d)⇒(a), applying Lemmas 8.28, 8.30 and 8.32, we can verify that $\mathcal{G}$ satisfies (1) and (2) in Lemma 7.5. Hence $\mathcal{G}$ has a locally finite rectangular cozero refinement. □

9. Problems

Refuting rectangularity of $X \times Y$, Example 6.18 shows that orthocompactness of X in Theorem 5.2 cannot be taken off from the assumption. On the other hand, the product space $X \times Y$ in that example is normal. So it is natural to ask

Problem 9.1. *Let X be a monotonically normal space and Y a paracompact $\mathbb{DC}$-like space. Assume that Y has the κ-dop property for each $\kappa \in \mathcal{S}^*(X)$. Is $X \times Y$ normal?*

By Theorem 6.1, Y cannot be almost discrete if a counterexample exists for this.

It is also natural to ask whether the arguments for ordinal factors can be extended to GO-space factors. That is, we raise

Problem 9.2. *Let X and Y be GO-spaces (or monotonically normal and orthocompact spaces).*

(1) *If $X \times Y$ is orthocompact, is it normal?*
(2) *If $X \times Y$ is normal, is it collectionwise normal?*
(3) *If $X \times Y$ is normal, does it have the shrinking property?*
(4) *If $X \times Y$ is rectangular, is it countably paracompact?*
(5) *If $X \times Y$ is countably paracompact, is it rectangular?*
(6) *If $X \times Y$ is a codecop product, is it rectangular?*
(7) *If $X \times Y$ is a codecop product, is it countably paracompact?*

Our arguments are separated in several sections.

Problem 9.3. *Is it possible to have a unified approach for some of our sections?*

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References

[1] Z. Balogh and M .E. Rudin, *Monotone Normality*, Topology and Appl. **47** (1992), 115–127.
[2] T. Chiba and K. Chiba, *Q-paracompactness and closed mappings*, Sci. Rep. Tokyo Kyouiku Daigaku Sect. A **11** (1972), 159-162.
[3] R. Engelking, *General Topology.* Herdermann Verlag, Berlin (1989).
[4] G. Gruenhage and Y. Yajima, *A filter property of submetacompactness and its application to products*, Topology and Appl. **36** (1990), 43–55.
[5] R. W. Heath, D. Lutzer and P. Zenor, *Monotonically normal spaces*, Trans. Amer. Math. Soc. **178** (1973), 481–493.
[6] N. Kemoto, *Normality of products of GO-space and cardinals*, Topology Proc. **18** (1993), 133–142.
[7] N. Kemoto, H. Ohta and K. Tamano, *Products of spaces of ordinal numbers*, Topology and Appl. **45** (1992), 245–260.
[8] N. Kemoto and Y. Yajima, *Orthocompactness in products*, Tsukuba J. Math. **16** (1992), 407–422.
[9] N. Kemoto and Y. Yajima, *Orthocompactness and normality of products with a cardinal factor*, Topology and Appl. **49** (1993), 141–148.

[10] N. Kemoto and Y. Yajima, *Rectangular products with ordinal factors*, Topology Appl. **154** (2007), 758-770.
[11] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, North-Holland, Amsterdam (1980).
[12] H. Ohta, *On normal non–rectangular products*, Quart. J. Math. **32** (1981), 339–344.
[13] B. A. Pasynkov, *On the dimension of rectangular products*, Soviet Math. Dokl. **16** (1975), 344-347.
[14] B. M. Scott, *Toward a product theory of orthocompactness*, Studies in Topology (N.M.Stavrakas and K.R.Allen eds) Academic Press, New York 517–537 (1975).
[15] R. Telgársky, *Spaces defined by tological games*, Fund. Math. **88** (1975), 193–223.
[16] Y. Yajima, *Topological games and products I*, Fund. Math. **113** (1981), 141–153.
[17] Y. Yajima, *Normality of products of monotonically normal spaces with compact spaces*, Topology and Appl. **158** (2011), 2085–2089.
[18] Y. Yajima, *Products of monotonically normal spaces with factors defined by topological games*, Topology and Appl. **159** (2012), 1223–1235.

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