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MILD NORMALITY OF FINITE PRODUCTS OF SUBSPACES OF ω_1

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Abstract. It is known that products of arbitrary many ordinals are mildly normal [4] and products of two subspaces of ordinals are also mildly normal [3]. It was asked if products of arbitrary many subspaces of ordinals are mildly normal. In this paper, we characterize the mild normality of products of finitely many subspaces of ω_1. Using this characterization, we show that there exist 3 subspaces of ω_1 whose product is not mildly normal.

1. Introduction

The closure of an open set in a topological space is called a *regular closed set*, and the interior of a closed set is called a *regular open set*. A space is called *mildly normal* (or *κ-normal*) if every pair of disjoint regular closed sets can be separated by disjoint open sets.

Obviously, every normal space is mildly normal. But mild normality does not imply normality. For instance, $\omega_1 \times (\omega_1 + 1)$ is mildly normal but not normal. Moreover, using elementary submodels, it is proved in [4] that products of arbitrary many ordinals are mildly normal. In [6], it is proved that for $A, B \subseteq \omega_1$, $A \times B$ is normal if and only if A or B is non-stationary or $A \cap B$ is stationary in ω_1. Since there are disjoint stationay sets A and B in ω_1, there is a non-normal product $A \times B$ of two subspaces of ω_1. On the other hand, $A \times B$ is mildly normal wherever A and B are arbitrary subsets of ordinals, see [3]. In [3], a subspace of ω_1^2 which is not mildly normal is given and they asked whether every finite product of subspaces of ordinals is mildly normal. On the other hand recently, it has been known that strong zero-dimensionality behaves like mild normality in the realm of products of ordinals. In particular, without using elementary submodels, a simultaneous proof of strong zero-dimensionality and mild normality of products of arbitrary many ordinals is given in [5]. Moreover in the same paper, it is proved that Σ-products and σ-products of arbitrary many ordinals are both strongly zero-dimensional and mildly normal. In [1], they proved that finite products of subspaces of ordinals are strongly zero-dimensional. Of course, they first proved it for two products and then extended for finite products.

In this paper, we characterize the mild normality of finite products of subspaces of ω_1 in terms of stationarity. Moreover, we show that there exist 3 subspaces of ω_1 whose product is not mildly normal.

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We call a sequence $\langle A_k : k < n\rangle$ of subsets of ω_1 with $2 \leq n < \omega$ a *stationary chain* if $A_{k-1} \cap A_k$ is stationary in ω_1 for every $0 < k < n$. A family $\{A_k : k \in N\}$ of subsets of ω_1 is *well-partitioned* if for every 1-1 function $w : n \longrightarrow N$ with $2 \leq n < \omega$, if $\langle A_{w(k)} : k < n\rangle$ is a stationary chain, then $\bigcap_{k<n} A_{w(k)}$ is stationary in ω_1. Note that if $|N| \leq 2$, then such a family $\{A_k : k \in N\}$ is well-partitioned moreover that if $\{A_k : k \in N\}$ is well-partitioned then $\{A_k : k \in N'\}$ is well-partitioned for every $N' \subseteq N$ and $\{A_k \cap [0, p(k)] : k \in N\}$ is also well-partitioned for every $p \in (\omega_1 + 1)^N$.

We prove the theorem below.

Theorem 1.1. *The finite product space $\Pi_{k\in N} A_k$ of non-empty subspaces of ω_1 is mildly normal if and only if the family $\{A_k : k \in N\}$ is well-partitioned.*

2. Preliminaries

We identify an ordinal α with the set of all ordinals less than α. We do not distinguish natural numbers from finite ordinals. Hence a natural number n is the set $\{0, 1, \ldots, n-1\}$. A sequence s of finite length n is a function of domain n, so $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$. In particular, A^n denotes the set of all functions from $\{0, 1, \ldots, n-1\}$ into A. For each sequence s, $\mathrm{lh}(s)$ denotes the length of s, and $\mathrm{ran}(s)$ denotes the set $\{s(i) : i < \mathrm{lh}(s)\}$.

Throughout the paper, each ordinal α is considered to be a space with the order-topology. For $X \subseteq \omega_1$, $\mathrm{Lim}(X)$ denotes the set $\{\alpha < \omega_1 : \alpha = \sup(X \cap \alpha)\}$ (i.e., the set of all cluster points of X in ω_1), where $\sup \emptyset = -\infty$ and $-\infty$ is considered as the immediate predecessor of the ordinal 0 for notational conveniences. Moreover $\mathrm{Succ}(X)$, Lim and Succ denote the sets $X \setminus \mathrm{Lim}(X)$, $\mathrm{Lim}(\omega_1)$ and $\mathrm{Succ}(\omega_1)$ respectively. Observe that $\mathrm{Lim}(X)$ is club (closed and unbounded) whenever X is unbounded in ω_1. Note that if X is not stationary, then X is covered by a pairwise disjoint family of bounded (in ω_1) clopen sets of X.

$\alpha^{\leq n}$ denotes the set $\bigcup_{k\leq n} \alpha^k$. Let A be a set of sequences of ordinals less than ω_1. We use the following notations.

- $A|_< = \{s \in A : \forall k_0, k_1 < \mathrm{lh}(s)(k_0 < k_1 \rightarrow s(k_0) < s(k_1))\}$,
- $A|_\leq = \{s \in A : \forall k_0, k_1 < \mathrm{lh}(s)(k_0 < k_1 \rightarrow s(k_0) \leq s(k_1))\}$,

A function $c : \omega_1 \longrightarrow \omega_1$ is said to be *normal* if it is strictly increasing, cofinal, and continuous. Note that if c is a normal function on ω_1, then $\mathrm{ran}(c)$ is a club set in ω_1 and conversely that the increasing enumeration c of a club set C as $C = \{c(\alpha) : \alpha < \omega_1\}$ is normal. Let c be a normal function, N a finite set and $\tau : N \longrightarrow m$ a function, where $m < \omega$. We define a function $\mathrm{pr}^c_\tau : \omega_1^m \longrightarrow \omega_1^N$, functions $\partial^c_k : \omega_1^N \longrightarrow \omega_1$ for each $k \in N$, moreover we define subsets $A|^c_{\tau,<}$ and $A|^c_{\tau,\leq}$ of A, where $A \subseteq \omega_1^N$, as follows.

- $\mathrm{pr}^c_\tau(y)(k) = c(y(\tau(k)))$ for each $y \in \omega_1^m$ and $k \in N$,
- $\partial^c_k(x) = \min\{\xi < \omega_1 : x(k) \leq c(\xi)\}$ for each $k \in N$ and $x \in \omega_1^N$,
- $A|^c_{\tau,<} = \{s \in A : \forall k_0, k_1 \in N(\tau(k_0) < \tau(k_1) \rightarrow \partial^c_{k_0}(s) < \partial^c_{k_1}(s))\}$,
- $A|^c_{\tau,\leq} = \{s \in A : \forall k_0, k_1 \in N(\tau(k_0) < \tau(k_1) \rightarrow \partial^c_{k_0}(s) \leq \partial^c_{k_1}(s))\}$.

If c is identity on ω_1, then we omit c and write $\mathrm{pr}_\tau(y)$, ∂_k, ...etc. Observe that pr^c_τ, and ∂^c_k are continuous and that for each $k_1, k_2 \in N$, $\{x \in \omega_1^N : \partial^c_{k_1}(x) < \partial^c_{k_2}(x)\}$ is open, and $\{x \in \omega_1^N : \partial^c_{k_1}(x) \le \partial^c_{k_2}(x)\}$ is closed in ω_1^N. So $A|^c_{\tau,<}$ is an open set in A and $A|^c_{\tau,\le}$ is a closed set in A. For convenience, let $c(\alpha-1) = -\infty$ if $\alpha = 0$.

3. Stationary open sets

Definition 3.1. Let $m < \omega$. We say that $X \subseteq \omega_1^m$ is *stationary* (in ω_1^m) if $X \cap C^m \neq \emptyset$ for every club set $C \subseteq \omega_1$. A function $f : X \longrightarrow (\omega_1 \cup \{-\infty\})^m$ is called a *regressive function* if $f(x)(j) < x(j)$ for every $x \in X$ and $j < m$. $T \subseteq \omega_1^{\le m}$ is called a *tree in* ω_1^m if $t \restriction j \in T$ for each $t \in T$ and $j < \mathrm{lh}(t)$. Let $\mathrm{Lv}_j(T)$ ($\mathrm{Lv}_{<j}(T), \mathrm{Lv}_{\le j}(T)$) denote the set of all elements $t \in T$ of height j ($< j, \le j$ respectively), that is $\mathrm{lh}(t) = j$ ($< j, \le j$ respectively). A tree in ω_1^m is called an *m-stationary tree* (*m-cofinal tree*) if $\emptyset \in T$ and $\{\alpha : t\hat{}\langle\alpha\rangle \in T\}$ is stationary (cofinal) in ω_1 for every $t \in \mathrm{Lv}_{<m}(T)$.

Lemma 3.2. *Let $m < \omega$. If $X \subseteq (\omega_1^m)|_<$ is stationary in ω_1^m and $f : X \longrightarrow (\omega_1 \cup \{-\infty\})^m$ is a regressive function, then there exist an m-stationary tree T in ω_1^m and a function $g : \mathrm{Lv}_{<m}(T) \longrightarrow \omega_1 \cup \{-\infty\}$ such that $\mathrm{Lv}_m(T) \subseteq X$ and $f(t)(j) = g(t \restriction j)$ for every $t \in \mathrm{Lv}_m(T)$ and $j < m$.*

Proof. Proofs of this lemma are seen in [1] and [2]. For reader's convenience, we give here a sketch of the proof. Put $X_m = X$ and $f_m = f$. Assume that $j < m$, $X_{j+1} \subseteq (\omega_1^{j+1})|_<$ is stationary in ω_1^{j+1} and f_{j+1} is a regressive function of domain X_{j+1}. Put $X_j = \{s \in (\omega_1^j)|_< : \{\alpha < \omega_1 : s\hat{}\langle\alpha\rangle \in X_{j+1}\}$ is stationary in $\omega_1\}$. Then X_j is stationary in ω_1^j by the normality of the club filter. By the Pressing Down Lemma for ω_1 and completeness of the club filter, we can pick $f_j(s) \in \Pi_{j'<j}(s(j') \cup \{-\infty\})$ and $g(s) < \omega_1$, for each $s \in X_j$, such that $A(s) = \{\alpha < \omega_1 : s\hat{}\langle\alpha\rangle \in X_{j+1}$ and $f_{j+1}(s\hat{}\langle\alpha\rangle) = f_j(s)\hat{}\langle g(s)\rangle\}$ is stationary in ω_1. $T = \{s \in \omega_1^{\le m} : s(j) \in A(s \restriction j)$ for each $j < \mathrm{lh}(s)\}$ satisfies the required condition. $\square$

Lemma 3.3. *Let X be a subspace of ω_1^n, where $n < \omega$, $\mathcal{P} = \{P_i : i \in \mathcal{I}\}$ a countable family of open sets of X such that $P_i|_\le$ is stationary in ω_1^n for each $i \in \mathcal{I}$. If T is an n-cofinal tree on ω_1 such that $\mathrm{Lv}_n(T) \subseteq X$, then $\bigcap_{i\in\mathcal{I}} P_i \cap \mathrm{Lv}_n(T) \neq \emptyset$.*

Proof. Fix $i \in \mathcal{I}$. Since

$$\omega_1^n|_\le = \bigcup\{\mathrm{pr}_\tau(\omega_1^m|_<) : m \le n, \tau : n \longrightarrow m \text{ is a non-decreasing, onto map }\},$$

we can pick an $m_i \le n$ and a non-decreasing, onto map $\tau_i : n \longrightarrow m_i$ such that $P_i \cap \mathrm{pr}_{\tau_i}(\omega_1^{m_i}|_<)$ is stationary. Put $Y_i = (\mathrm{pr}_{\tau_i}^{-1}(P_i))|_<$, then Y_i is a stationary subset of $\omega_1^{m_i}$. Since P_i is open, there is a regressive function $f_i : Y_i \longrightarrow (\omega_1 \cup \{-\infty\})^{m_i}$ such that $X \cap \Pi_{k<n}(f_i(y)(\tau_i(k)), y(\tau_i(k))] \subseteq P_i$ for every $y \in Y_i$. By Lemma 3.2, there are an m_i-stationary tree U_i in $\omega_1^{m_i}$ and a function $g_i : \mathrm{Lv}_{<m_i}(U_i) \longrightarrow \omega_1 \cup \{-\infty\}$ such that $\mathrm{Lv}_{m_i}(U_i) \subseteq Y_i$ and $f_i(u)(j) = g_i(u \restriction j)$ for each $u \in \mathrm{Lv}_{m_i}(U_i)$ and $j < m_i$.

We define inductively $t \in \mathrm{Lv}_n(T)$ and $u_i \in \mathrm{Lv}_{m_i} U_i$ for each $i \in \mathcal{I}$ as follows. Set $t \restriction 0 = u_i \restriction 0 = \emptyset$. Assume that $k < n$ and that $t \restriction k \in T$ and $u_i \restriction \tau_i(k) \in U_i$ for every $i \in \mathcal{I}$ are determined. Pick $t(k)$ such that $(t \restriction k)\hat{}\langle t(k)\rangle \in T$ and $g_i(u_i \restriction \tau_i(k)) < t(k)$ for all $i \in \mathcal{I}$. For each $i \in \mathcal{I}$ satisfying $k = \max \tau_i^{-1}(\{\tau_i(k)\})$, pick $u_i(\tau_i(k))$ such that $(u_i \restriction \tau_i(k))\hat{}\langle u_i(\tau_i(k))\rangle \in U_i$ and $t(k') \le u_i(\tau_i(k))$ for all $k' \in \tau_i^{-1}(\{\tau_i(k)\})$.

It follows from $t \in \mathrm{Lv}_n(T)$ that $t \in X$. For each $i \in \mathcal{I}$ and $k < n$, $f_i(u_i)(\tau_i(k)) = g_i(u_i \restriction \tau_i(k)) < t(k) \leq u_i(\tau_i(k))$. $t \in X \cap \Pi_{k<n}(f_i(u_i)(\tau_i(k)), u_i(\tau_i(k))] \subseteq P_i$ because of $u_i \in \mathrm{Lv}_{m_i}(U_i)$. Hence $t \in \bigcap_{i \in \mathcal{I}} P_i \cap \mathrm{Lv}_n(T)$. □

Lemma 3.4. *Let $\langle A_k : k < n\rangle$ be a stationary chain with $2 \leq n < \omega$ such that $\bigcap_{k<n} A_k$ is non-stationary in ω_1. Then $A = \Pi_{k<n} A_k$ is not mildly normal.*

Proof. Let C be a club set disjoint from $\bigcap_{k<n} A_k$ and $c : \omega_1 \longrightarrow \omega_1$ the increasing enumeration of C. Define non-decreasing, onto mappings $\tau_0^- : n \longrightarrow m_0$ and $\tau_1^- : n \longrightarrow m_1$ with $m_0, m_1 \leq n$ by, for each $0 < k < n$,

$$\tau_0^-(k-1) < \tau_0^-(k) \text{ iff } k \text{ is odd},$$

$$\tau_1^-(k-1) < \tau_1^-(k) \text{ iff } k \text{ is even}.$$

For each $i \in 2 = \{0, 1\}$ and $j < m_i$, put $\tau_i(k) = k$ if $(\tau_i^-)^{-1}(\{j\}) = \{k\}$ and put $\tau_i(k-1) = k$, $\tau_i(k) = k-1$ if $(\tau_i^-)^{-1}(\{j\}) = \{k-1, k\}$. Then $\tau_i : n \longrightarrow n$ is 1-1 onto. Put $U_i = A|^c_{\tau_i,<}$ and $F_i = A|^c_{\tau_i,\leq}$. Since U_i is open and F_i is closed, $\mathrm{cl}\, U_i$ is regular closed in A and $\mathrm{cl}\, U_i \subseteq F_i$.

Claim 1. $\mathrm{cl}\, U_0 \cap \mathrm{cl}\, U_1 = \emptyset$

Proof. Assume that $x \in \mathrm{cl}\, U_0 \cap \mathrm{cl}\, U_1$. If $0 < k < n$ is odd, then $\tau_0^-(k-1) < \tau_0^-(k)$ and $\tau_1^-(k-1) = \tau_1^-(k)$, so $\tau_0(k-1) < \tau_0(k)$ and $\tau_1(k-1) > \tau_1(k)$. Thus by $x \in F_0 \cap F_1$, we have $\partial^c_{k-1}(x) \leq \partial^c_k(x)$ and $\partial^c_{k-1}(x) \geq \partial^c_k(x)$, hence $\partial^c_{k-1}(x) = \partial^c_k(x)$. If $0 < k < n$ is even, then similarly we have $\partial^c_{k-1}(x) = \partial^c_k(x)$. Therefore there exists a $\xi < \omega_1$ such that $\partial^c_k(x) = \xi$ for all $k < n$. Since $x \in A = \Pi_{k<n} A_k$, $x(k) \leq c(\partial^c_k(x)) = c(\xi)$ for each $k < n$, and $c(\xi) \notin \bigcap_{k<n} A_k$, we have $x(k) < c(\xi)$ for some $k < n$. It follows from the minimality of $\partial^c_k(x)$ and the normality of c that $\xi \in \mathrm{Succ}$. Then $A \cap \Pi_{k<n}(c(\xi - 1), c(\xi)]$, where $\xi - 1$ denotes the immediate predecessor of ξ, is a neighborhood of x disjoint from U_0. Thus $x \notin \mathrm{cl}\, U_0$, a contradiction.

Now we go back to the proof of the lemma. Since $\langle A_k : k < n\rangle$ is a stationary chain, $A_{k-1} \cap A_k$ is stationary in ω_1 for every $0 < k < n$. For $i \in 2$ and $j < m_i$, put $Y'_{i,j} = c^{-1}(\bigcap\{A_k : k \in (\tau_i^-)^{-1}(\{j\})\})$, $Y_{i,j} = Y'_{i,j} \cap \mathrm{Lim}(Y'_{i,j})$, and $Y_i = (\Pi_{j<m_i} Y_{i,j})|_<$. Since $Y'_{i,j}$ is stationary in ω_1, $Y_{i,j}$ is also stationary, hence Y_i is stationary in $\omega_1^{m_i}$. It follows from $\mathrm{pr}^c_{\tau_i}(Y_i) \subseteq (\mathrm{cl}\, U_i)|_\leq$ that $(\mathrm{cl}\, U_i)|_\leq$ is stationary in ω_1^n. Obviously, there is an n-cofinal tree T in ω_1^n such that $\mathrm{Lv}_n(T) \subseteq A$. By Lemma 3.3, $\mathrm{cl}\, U_0$ and $\mathrm{cl}\, U_1$ cannot be separated by disjoint open sets. □

Since if $X \times Y \neq \emptyset$ is mildly normal then so is X, now by Lemma 3.4, we have one half of the proof of the main theorem.

Lemma 3.5. *If the finite product space $\Pi_{k \in N} A_k$ of non-empty subspaces of ω_1 is mildly normal, then $\{A_k : k \in N\}$ is well-partitioned.*

4. Mildly normal products

In this section, we prove another half of the proof of the main theorem.

Lemma 4.1. *If a finite family $\mathcal{A} = \{A_k : k \in N\}$ of non-empty subspaces of ω_1 is well-partitioned, then the product space $\Pi_{k \in N} A_k$ is mildly normal.*

Proof. For notational reasons, we assume $N \cap \omega_1 = \emptyset$. We prove this lemma by induction on $n = |N|$. Assume that this lemma holds for $< n$ and let $\mathcal{A} = \{A_k : k \in N\}$ be a well-partitioned family of non-empty subspaces of ω_1 with $|N| = n$.

Let $\prec$ be the well-founded relation on $(\omega_1+1)^N$ defined by:

$$p' \prec p \text{ iff } \forall k \in N(p'(k) \le p(k)) \text{ and } \exists k \in N(p'(k) < p(k)).$$

Note that $(\omega_1+1)^N$ has the $\prec$-largest element p_0 defined by $p_0(k) = \omega_1$ for each $k \in N$. Using $\prec$-induction on $p \in (\omega_1+1)^N$, we will prove that $A(p) = \Pi_{k\in N}(A_k \cap [0, p(k)])$ is mildly normal. Then the $\prec$-largest element witnesses the lemma.

Let $p \in (\omega_1+1)^N$ and assume that $A(p')$ is mildly normal for every $p' \prec p$. We call a clopen subset B of $A(p)$ *bounded* if $B \subseteq A(p')$ for some $p' \prec p$, moreover a subset B of $A(p)$ *small* if B is represented as the union of a locally finite family of bounded clopen sets. Note that by the $\prec$-inductive assumption, every small clopen subset of $A(p)$ is mildly normal.

Set $N_0 = \{k \in N : p(k) = \omega_1\}$ and $N_1 = N \setminus N_0$. We may assume $A(p) \neq \emptyset$ and $N_0 \neq \emptyset$, otherwise $A(p)$ is metrizable (since regular, Lindelöf, and 2nd-countable). In some special cases, we can immediately show the mild normality of $A(p)$.

Case 1. *There is $k_0 \in N_1$ such that $p(k_0) = 0$.*

Since $A(p)$ is homeomorphic to $\Pi_{k\in N\setminus\{k_0\}}(A_k \cap [0, p(k)])$, use the inductive assumption for $n-1$.

Case 2. *There is $k_0 \in N_1$ such that $p(k_0) = \lambda + 1$ for some λ.*

Since A_{k_0} is represented as the free union $A_{k_0} = (A_{k_0} \cap [0, \lambda]) \bigoplus (A_{k_0} \cap \{\lambda + 1\})$, $A(p)$ is mildly normal (more precisely, the mild normality of $\Pi_{k\in N\setminus\{k_0\}}(A_k \cap [0, p(k)]) \times (A_{k_0} \cap \{\lambda+1\})$ follows from the inductive assumption for $n-1$).

Case 3. *There is $k_0 \in N_1$ such that $p(k_0) \in \mathrm{Lim} \setminus A_{k_0}$.*

Fix a strictly increasing cofinal sequence $\langle \lambda_n : n \in \omega\rangle$ in $p(k_0)$. Then A_{k_0} can be represented as the free union $A_{k_0} = \bigoplus_{n\in\omega}(A_{k_0} \cap (\lambda_{n-1}, \lambda_n])$, where $\lambda_{0-1} = -\infty$. Therefore $A(p)$ is small.

Case 4. *There is $k_0 \in N_0$ such that A_{k_0} is non-stationary.*

Since A_{k_0} is covered by bounded clopen sets in A_{k_0}, $A(p)$ is small.

With the observation above, we may assume that

- $N_0 = \{k \in N : p(k) = \omega_1\}$ is non-empty,
- A_k is stationary for every $k \in N_0$,
- $p(k)$ is limit and $p(k) \in A_k$ for every $k \in N_1 = N \setminus N_0$.

Set $\mathrm{Stat} = \{N' \subseteq N_0 : \bigcap_{k\in N'} A_k$ is stationary in $\omega_1\}$. Let C be a club set disjoint from $\bigcup\{\bigcap_{k\in N'} A_k : N' \subseteq N_0, N' \notin \mathrm{Stat}\}$ and $c : \omega_1 \longrightarrow \omega_1$ the increasing enumeration of C. For each $r = \{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat}$ and $\xi \in r \cup \mathrm{Succ}$, let

$$F_{r\xi} = \begin{cases} \{s \in A(p) : \partial^c_{k_0}(s) < \partial^c_{k_1}(s)\} & \text{if } \xi = k_0, \\ \{s \in A(p) : \partial^c_{k_0}(s) > \partial^c_{k_1}(s)\} & \text{if } \xi = k_1, \\ \{s \in A(p) : \partial^c_{k_0}(s) = \partial^c_{k_1}(s) = \xi\} & \text{if } \xi \in \mathrm{Succ}. \end{cases}$$

Claim 1. *For each $r \in [N_0]^2 \setminus \mathrm{Stat}$, $\mathcal{F}_r = \{F_{r\xi} : \xi \in r \cup \mathrm{Succ}\}$ is a discrete clopen cover of $A(p)$.*

Proof. Let $r = \{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat}$. It is evident that $\mathcal{F}_r$ is disjoint and $F_{r\xi}$ is open in $A(p)$ for each $\xi \in r$. Let $\xi \in \mathrm{Succ}$ and $s_0 \in F_{r\xi}$. Then $\{s \in A(p) : \forall i \in 2(c(\xi - 1) < s(k_i) \leq c(\xi))\}$ is a neighborhood of s_0 contained in $F_{r\xi}$, thus $F_{r\xi}$ is open. To show that $\mathcal{F}_r$ is cover, let $s \in A(p) \setminus (F_{rk_0} \cup F_{rk_1})$. Assume $\xi = \partial^c_{k_0}(s) = \partial^c_{k_1}(s) \in \mathrm{Lim}$. Since c is normal, we have $c(\xi) = s(k_0) = s(k_1) \in A_{k_0} \cap A_{k_1} \cap C$, a contradiction. Thus $\xi \in \mathrm{Succ}$ and $s \in F_{r\xi}$. This completes the proof of the claim.

Now set

$$H = \{h \in \Pi_{r\in[N_0]^2\setminus\mathrm{Stat}}(r \cup \mathrm{Succ}) : \forall r \in [N_0]^2 \setminus \mathrm{Stat}(F_{rh(r)} \neq \emptyset)\},$$

$$F_h = \bigcap_{r\in[N_0]^2\setminus\mathrm{Stat}} F_{rh(r)} \text{ for each } h \in H \text{ , and } \mathcal{F} = \{F_h : h \in H\}.$$

By Claim 1 above, $\mathcal{F}$ is a discrete clopen cover of $A(p)$. It suffices to show that F_h is mildly normal for each $h \in H$. Set

$$H_0 = \{h \in H : \exists r \in [N_0]^2 \setminus \mathrm{Stat}(h(r) \in \mathrm{Succ})\} \text{ and } H_1 = H \setminus H_0.$$

First let $h \in H_0$ and take $r \in [N_0]^2 \setminus \mathrm{Stat}$ with $h(r) \in \mathrm{Succ}$. Then it follows from

$$F_h \subseteq F_{rh(r)} \subseteq \Pi_{k\in N\setminus r}(A_k \cap [0, p(k)]) \times \Pi_{k\in r}(A_k \cap [0, c(h(r))])$$

that F_h is a small clopen set of $A(p)$, thus F_h is mildly normal.

Next fix $h \in H_1$. Note $h(r) \in r$ for each $r \in [N_0]^2 \setminus \mathrm{Stat}$.

Claim 2. *There is an onto function $\tau : N_0 \longrightarrow m$ for some $m < \omega$ such that*

(1) $\bigcap_{k\in\tau^{-1}(\{j\})} A_k$ *is stationary for each $j < m$,*
(2) $F_h = (\Pi_{k\in N_0} A_k)|^c_{\tau,<} \times \Pi_{k\in N_1}(A_k \cap [0, p(k)])$.

Proof. Define a binary relation $\triangleleft$ on N_0 by:

$$k_0 \triangleleft k_1 \text{ iff } \{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat} \text{ and } h(\{k_0, k_1\}) = k_0.$$

Then note that either $k_0 \triangleleft k_1$ or $k_1 \triangleleft k_0$ iff $\{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat}$. Fix $s \in F_h$ and let $k_0, k_1 \in N_0$ be satisfying $k_0 \triangleleft k_1$. By $h(\{k_0, k_1\}) = k_0$, we have $s \in F_h \subseteq F_{\{k_0,k_1\}k_0}$. Therefore $\partial^c_{k_0}(s) < \partial^c_{k_1}(s)$, this shows that $\triangleleft$ is a well-founded relation on N_0. Let $\tau : N_0 \longrightarrow m$ be the rank function of the well-founded set $\langle N_0, \triangleleft\rangle$. Evidently $m < \omega$ because N_0 is finite. We will show that τ is the desired one. Let $j < m$ and $w : n_0 \longrightarrow \tau^{-1}(\{j\})$ be 1-1 and onto, where $n_0 = |\tau^{-1}(\{j\})|$. For every k with $0 < k < n_0$, by $\tau(w(k-1)) = \tau(w(k)) = j$, we have that neither $w(k-1) \triangleleft w(k)$ nor $w(k) \triangleleft w(k-1)$ iff $\{w(k-1), w(k)\} \in \mathrm{Stat}$ iff $A_{w(k-1)} \cap A_{w(k)}$ is stationary. Therefore $\langle A_{w(k)} : k < n_0\rangle$ is stationary chain (if $2 \leq n_0$), thus $\bigcap_{k\in\tau^{-1}(\{j\})} A_k = \bigcap_{k<n_0} A_{w(k)}$ is stationary by the well-partitionedness of $\mathcal{A}$.

Fact 1. *Let $0 < j < m$. Then for every $k_0 \in \tau^{-1}(\{j-1\})$ and $k_1 \in \tau^{-1}(\{j\})$, $\{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat}$ and $k_0 \triangleleft k_1$ hold.*

Proof. If once we prove $\{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat}$, then we have $k_0 \triangleleft k_1$ or $k_1 \triangleleft k_0$. However since $\tau(k_0) = j - 1 < j = \tau(k_1)$, $k_1 \triangleleft k_0$ does not hold, so we have $k_0 \triangleleft k_1$.

Now assume $\{k_0, k_1\} \in \mathrm{Stat}$. Since τ is a rank function and $\tau(k_1) = j$, there is $k_2 \in \tau^{-1}(\{j-1\})$ with $k_2 \triangleleft k_1$. It follows from $\{k_0, k_1\} \in \mathrm{Stat}$ and $\{k_1, k_2\} \notin \mathrm{Stat}$ that k_0, k_1 and k_2 are distinct. Since by (1) $A_{k_0} \cap A_{k_2} \supseteq \bigcap_{k\in\tau^{-1}(\{j-1\})} A_k$ is stationary, we have $\{k_0, k_2\} \in \mathrm{Stat}$. Therefore $\langle A_{k_2}, A_{k_0}, A_{k_1}\rangle$ is a stationary chain, so $A_{k_2} \cap A_{k_0} \cap A_{k_1}$ is stationary by the well-partitionedness. On the other hand by

$\{k_1, k_2\} \notin \mathrm{Stat}$, we have $A_{k_2} \cap A_{k_0} \cap A_{k_1}$ is not stationary, a contradiction. This completes the proof of fact.

To prove (2), first let $s \in (\Pi_{k\in N_0} A_k)|^c_{\tau,<} \times \Pi_{k\in N_1}(A_k \cap [0, p(k)])$. Now fix $r = \{k_0, k_1\} \in [N_0]^2 \setminus \mathrm{Stat}$. We may assume $h(r) = k_0$. Then we have $k_0 \lhd k_1$ so $\tau(k_0) < \tau(k_1)$. By $s \in (\Pi_{k\in N_0} A_k)|^c_{\tau,<} \times \Pi_{k\in N_1}(A_k \cap [0, p(k)])$, we have $\partial^c_{k_0}(s) < \partial^c_{k_1}(s)$. Therefore $s \in F_{rk_0} = F_{rh(r)}$. Moving $r \in [N_0]^2 \setminus \mathrm{Stat}$, we have $s \in F_h$.

Next let $s \in F_h$ and $k_0, k_1 \in N_0$ with $\tau(k_0) < \tau(k_1)$, say $j_0 = \tau(k_0)$ and $j_1 = \tau(k_1)$. Since τ is onto, for each j with $j_0 \le j \le j_1$, we can pick $k'_j \in \tau^{-1}(\{j\})$ such that $k'_{j_0} = k_0$ and $k'_{j_1} = k_1$. Let $j_0 < j \le j_1$. It follows from the fact above that $\{k'_{j-1}, k'_j\} \in [N_0]^2 \setminus \mathrm{Stat}$ and $k'_{j-1} \lhd k'_j$, so $h(\{k'_{j-1}, k'_j\}) = k'_{j-1}$. Since $s \in F_h \subseteq F_{\{k'_{j-1},k'_j\}k'_{j-1}}$, we have $\partial^c_{k'_{j-1}}(s) < \partial^c_{k'_j}(s)$. Moving j with $j_0 < j \le j_1$, we have $\partial^c_{k_0}(s) < \partial^c_{k_1}(s)$. This shows $s \in (\Pi_{k\in N_0} A_k)|^c_{\tau,<} \times \Pi_{k\in N_1}(A_k \cap [0, p(k)])$. This completes the proof of Claim 2.

Set

$$X = (\Pi_{k\in N_0} A_k)|^c_{\tau,<} \text{ and } Z = \Pi_{k\in N_1}(A_k \cap [0, p(k)]).$$

If we show that $F_h = X \times Z$ is mildly normal, the proof of the lemma is complete.

Claim 3. *$X \times Z$ is mildly normal.*

Proof. Let $\mathcal{U} = \{U_0, U_1\}$ be a regular open cover of $X \times Z$ and $z_0 = p \restriction N_1 \in Z$. We will show that $X \times Z \setminus U_i$ is contained in a small clopen set for some $i \in 2$. Noting $\mathrm{pr}_\tau : \omega_1^m \longrightarrow \omega_1^{N_0}$, it follows from $\tau^{-1}(\{j\}) \in \mathrm{Stat}$ for each $j < m$ that $\mathrm{pr}_\tau^{-1}(X)$ is stationary in ω_1^m and $\mathrm{pr}_\tau^{-1}(X) \subseteq \omega_1^m|_<$.

For each $i \in 2$, set $V_i = \{x \in X : \langle x, z_0\rangle \in U_i\}$. Then $\{V_0, V_1\}$ is an open cover of X. So we may assume that $Y_0 = \mathrm{pr}_\tau^{-1}(V_0)$ is stationary in ω_1^m. For each $y \in Y_0$, by $\langle \mathrm{pr}_\tau(y), z_0\rangle \in U_0$, we can fix $f_0(y) \in \Pi_{j<m}(y(j) \cup \{-\infty\})$ and $f_1(y) \in \Pi_{k\in N_1}(z_0(k) \cup \{-\infty\})$ such that

$$(X \times Z) \cap (\Pi_{k\in N_0}(f_0(y)(\tau(k)), y(\tau(k))] \times \Pi_{k\in N_1}(f_1(y)(k), z_0(k)]) \subseteq U_0.$$

Since $\Pi_{k\in N_1}(z_0(k) \cup \{-\infty\})$ is countable, there are a stationary set $Y_1 \subseteq Y_0$ in ω_1^m and $z_1 \in \Pi_{k\in N_1}(z_0(k)\cup\{-\infty\})$ such that $f_1(y) = z_1$ for each $y \in Y_1$. Moreover since $f_0 \restriction Y_1$ is regeressive on a stationary set Y_1, by Lemma 3.2, there are a stationary tree T in ω_1^m and a function $g : \mathrm{Lv}_{<m}(T) \longrightarrow \omega_1 \cup \{-\infty\}$ such that $\mathrm{Lv}_m(T) \subseteq Y_1$ and $f_0(t)(j) = g(t \restriction j)$ for every $t \in \mathrm{Lv}_m(T)$ and $j < m$.

Set

$$E = \{\xi < \omega_1 : \forall t \in \mathrm{Lv}_{<m} T \cap \xi^{\le m}(g(t) < \xi \in \mathrm{Lim}(\{\alpha < \omega_1 : t^\frown\langle\alpha\rangle \in T\}))\}$$

Then it is straightforward to show that E is club. Let

$$D = C \cap E \cap \bigcap_{k\in N_0} \mathrm{Lim}(A_k)$$

and let $d : \omega_1 \longrightarrow \omega_1$ be the increasing enumeration of D. Set $X' = X \cap (d(0), \omega_1)^{N_0}$ and $Z' = Z \cap \Pi_{k\in N_1}(z_1(k), z_0(k)]$. Then since $X \times Z - X' \times Z'$ is a small clopen set, it suffices to show that $X' \times Z' \setminus U_0$ is contained in a small clopen set.

For each $j < m$ and $x \in X$, let

$$\nu_j^-(x) = \min\{\partial^d_k(x) : k \in \tau^{-1}(\{j\})\}, \nu_j^+(x) = \max\{\partial^d_k(x) : k \in \tau^{-1}(\{j\})\}.$$

Obviously, ν_j^- and ν_j^+ are continuous.

Let $0 < j < m$. For each $\xi \in \mathrm{Succ}$, let

$$L_{j\xi} = \{x \in X' : \xi = \nu^+_{j-1}(x) \geq \nu^-_j(x)\} \times Z'.$$

Note that this $L_{j\xi}$ is actually $\{x \in X' : \xi = \nu^+_{j-1}(x) = \nu^-_j(x)\} \times Z'$. Moreover, let

$$L_{j\infty} = \{x \in X' : \nu^+_{j-1}(x) < \nu^-_j(x)\} \times Z'.$$

Fact 1. *For each j with $0 < j < m$, $\mathcal{L}_j = \{L_{j\xi} : \xi \in \mathrm{Succ} \cup \{\infty\}\}$ is a discrete clopen cover of $X' \times Z'$.*

Proof. Evidently $\mathcal{L}_j$ is pairwise disjoint. To show that $\mathcal{L}_j$ covers $X' \times Z'$, let $\langle x, z\rangle \in X' \times Z' \setminus L_{j\infty}$. Then $\nu^+_{j-1}(x) \geq \nu^-_j(x)$. Assume $\xi = \nu^+_{j-1}(x) \in \mathrm{Lim}$, then $x(k_0) = d(\xi)$ for some $k_0 \in \tau^{-1}(\{j-1\})$. Moreover it follows from $\nu^-_j(x) \leq \xi$ that $x(k_1) \leq d(\xi)$ for some $k_1 \in \tau^{-1}(\{j\})$. Then $x(k_1) \leq x(k_0)$, therefore $\partial^c_{k_1}(x) \leq \partial^c_{k_0}(x)$. But it follows from $x \in X$ and $\tau(k_0) < \tau(k_1)$ that $\partial^c_{k_0}(x) < \partial^c_{k_1}(x)$, a contradiction. Therefore $\xi \in \mathrm{Succ}$ and $\mathcal{L}_j$ covers $X' \times Z'$.

$L_{j\infty}$ is open by continuity of ν^-_j and ν^+_j. Fix $\xi \in \mathrm{Succ}$. To show that $L_{j\xi}$ is open, let $\langle x, z\rangle \in L_{j\xi}$. Then there are $k_0 \in \tau^{-1}(\{j-1\})$ and $k_1 \in \tau^{-1}(\{j\})$ such that $\partial^d_{k_0}(x) = \xi$ and $\partial^d_{k_1}(x) \leq \xi$. Since $\partial^d_k(x) \leq \xi$ for each $k \in \tau^{-1}(\{j-1\})$,

$$U = \{y \in X' : d(\xi - 1) < y(k_0), y(k_1) \leq x(k_1), \forall k \in \tau^{-1}(\{j-1\})(y(k) \leq d(\xi))\} \times Z'$$

is a neighborhood of $\langle x, z\rangle$ contained in $L_{j\xi}$, thus $L_{j\xi}$ is open.

By the fact above, $\{L_\varphi : \varphi \in (\mathrm{Succ} \cup \{\infty\})^{m \setminus \{0\}}\}$ is a discrete clopen cover of $X' \times Z'$ where $L_\varphi = \bigcap_{0<j<m} L_{j\varphi(j)}$. Let φ_0 be the function on $m \setminus \{0\}$ defined by $\varphi_0(j) = \infty$ for each $j < m$. Since L_φ is small for each $\varphi \in (\mathrm{Succ} \cup \{\infty\})^{m \setminus \{0\}} \setminus \{\varphi_0\}$, the following fact completes the proof.

Fact 2. $L_{\varphi_0} \subseteq U_0$.

Proof. Since U_0 is regular open, it suffices to show $L_{\varphi_0} \subseteq \mathrm{cl}\, U_0$. Let $\langle x, z\rangle \in L_{\varphi_0}$ moreover let $x' \in (\omega_1 \cup \{-\infty\})^{N_0}$ and $z' \in (\omega_1 \cup \{-\infty\})^{N_1}$ satisfy $x'(k) < x(k)$ for each $k \in N_0$ and $z'(k) < z(k)$ for each $k \in N_1$. We will show that $U_0 \cap (\Pi_{k \in N_0}(x'(k), x(k)] \times \Pi_{k \in N_1}(z'(k), z(k)]) \neq \emptyset$. Set $e(j) = d(\nu^+_j(x))$ for each $j < m$. For convenience, let $e(j-1) = d(0)$ if $j = 0$.

Let $j < m$ and $k \in \tau^{-1}(\{j\})$. Then $e(j-1) < x(k) \leq e(j)$ since $\nu^+_{j-1}(x) < \nu^-_j(x)$ if $0 < j$ by $\langle x, z\rangle \in L_{\varphi_0}$. $e(j) \in D \subseteq \mathrm{Lim}(A_k)$, so we can pick $y(k) \in A_k$ such that $\max\{e(j-1), x'(k)\} < y(k) \leq x(k)$ and $y(k) < e(j)$.

Since $d(0) \leq e(j-1) < y(k)$ for each $k \in \tau^{-1}(\{j\})$ with $j < m$ and $y(k_0) < e(j-1) < y(k_1)$ for each $k_0 \in \tau^{-1}(\{j-1\})$ and $k_1 \in \tau^{-1}(\{j\})$ with $0 < j < m$, we have $y \in X \cap \Pi_{k \in N_0}(x'(k), x(k)]$. It suffices to show $\langle y, z\rangle \in U_0$.

Now we will define $t \in T$ by induction on $j \leq m$ as follows. Let $j < m$ and assume that $t \restriction j \in T \cap \Pi_{j' < j} e(j')$ is already defined. It follows from $e(j) \in D$ that $e(j) \in \mathrm{Lim}(\{\alpha < \omega_1 : (t \restriction j)^\frown\langle\alpha\rangle \in T\})$. Therefore by $\max\{y(k) : k \in \tau^{-1}(\{j\})\} < e(j)$, we can find $t(j)$ with $(t \restriction j)^\frown\langle t(j)\rangle \in T$ and $\max\{y(k) : k \in \tau^{-1}(\{j\})\} \leq t(j) < e(j)$. Then for each $j < m$, it follows from $t \restriction j \in \mathrm{Lv}_j(T) \cap e(j-1)^j$ and $e(j-1) \in D \subseteq E$ that for each $k \in \tau^{-1}(\{j\})$,

$$g(t \restriction j) < e(j-1) < y(k) \leq t(j).$$

Thus by $z \in Z'$, we have

$$\langle y, z\rangle \in (X \times Z) \cap (\Pi_{k \in N_0}(f_0(t)(\tau(k)), t(\tau(k))] \times \Pi_{k \in N_1}(f_1(t)(k), z_0(k)]) \subseteq U_0.$$

□

5. Applications

In this section, we apply the main theorem. Immediately we have:

Corollary 5.1. *If $\{A_k : k \in N\}$ is a finite family of subsets of ω_1 with the pairwise non-stationary intersection, then $\Pi_{k\in N} A_k$ is mildly normal.*

Now, we answer the question of [4] and [3].

Corollary 5.2. *Let m and N be natural numbers with $2 \leq m < N$. Then there is a family $\{A_k : k \in N\}$ of subsets of ω_1 satisfying:*

(1) *$\Pi_{k\in r} A_k$ is mildly normal for each $r \in [N]^m$,*
(2) *$\Pi_{k\in r} A_k$ is not mildly normal for each $r \in [N]^{m+1}$.*

Proof. Enumerate $[N]^m$ as $[N]^m = \{r_j : j < K\}$, where $K = \frac{N!}{m! \times (N-m)!}$. Let $\mathcal{B} = \{B_j : j < K\}$ be a pairwise disjoint family of stationary sets in ω_1. For each $k < N$, set $A_k = \bigcup\{B_j : k \in r_j\}$.

It follows from $B_j \subseteq \bigcap_{k\in r_j} A_k$ that $\{A_k : k \in r_j\}$ is well-partitioned, this shows (1).

Let $r \in [N]^{m+1}$. Note that $\{A_k : k \in r\}$ has the pairwise stationary intersection. Assume $\alpha \in \bigcap_{k\in r} A_k$. Then for each $k \in r$, we can find $j(k) < K$ such that $k \in r_{j(k)}$ and $\alpha \in B_{j(k)}$. Since $\mathcal{B}$ is pairwise disjoint, for some $j < K$, $j(k) = j$ for each $k \in r$. This means $r \subseteq r_j$ and contradicts $|r| = m + 1$ and $|r_j| = m$, therefore $\bigcap_{k\in r} A_k = \emptyset$. So $\{A_k : k \in r\}$ is not well-partitioned, this shows (2). □

In particular, applying the corollary above for $m = 2$ and $N = 3$, we have:

Corollary 5.3. *There are subspaces A, B and C of ω_1 whose product $A \times B \times C$ is not mildly normal.*

Problem 5.4. *Let $\{A_k : k \in \omega\}$ be a pairwise disjoint infinite family of stationary sets in ω_1. Then is the product $\Pi_{k\in\omega} A_k$ mildly normal?*

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