Mild Normality in Products of Ordinals

Lutfi N. H. Kalantan and Nobuyuki Kemoto

February 20, 2002

Abstract

A space is said to be *mildly normal* (or κ*-normal*) if every disjoint pair of regular closed sets are separated by disjoint open sets. In this paper, we will show:

(1) There is a compact linearly ordered topological space Y such that $\omega_1 \times Y$ is not mildly normal.

(2) $A \times B$ is mildly normal whenever A and B are subspaces of ordinals.

(3) There is a subspace of ω_1^2 which is not mildly normal.

(4) There is a closed subspace of $\omega_1 \times (\omega_1 + 1)$ which is not mildly normal.

A space is said to be *mildly normal* (or κ*-normal*) if every disjoint pair of regular closed sets are separated by disjoint open sets. Normal spaces are mildly normal and subspaces of linearly ordered toplogical spaces are normal. Closed subspaces of normal spaces are also normal. It is well known that ω_1^2 is normal but $\omega_1 \times (\omega_1 + 1)$ is not normal. Moreover it is known from [2] that $A \times B$ is not normal whenever A and B are disjoint stationary sets in ω_1. But as is shown as a corollary of Theorem 5 below, strangely, $\omega_1 \times (\omega_1 + 1)$ is mildly normal (the authors do not know whether someone have already mentioned this result or not). A. V. Arkhangel'skiĭ [1] asked if all subspaces of ω_1^2 are mildly normal, and if closed subspaces of $\omega_1 \times (\omega_1 + 1)$ are mildly normal. In this paper, we will show:

(1) There is a compact linearly ordered topological space Y such that $\omega_1 \times Y$ is not mildly normal.

2000 *Mathematics subject classification.* 54B10, 54D15.
Keywords and phrases. mildly normal, normal, ordinal, product space

(2) $A \times B$ is mildly normal whenever A and B are subspaces of ordinals.

(3) There is a subspace of ω_1^2 which is not mildly normal.

(4) There is a closed subspace of $\omega_1 \times (\omega_1 + 1)$ which is not mildly normal.

Let $X \subset (\rho+1) \times (\sigma+1)$ for some suitably large ordinals ρ and σ. In general, the letters μ and ν will stand for limit ordinals with $\mu \leq \rho$ and $\nu \leq \sigma$. For each $A \subset \rho+1$ and $B \subset \sigma+1$ put

$$X_A = A \times (\sigma+1) \cap X,\ X^B = (\rho+1) \times B \cap X,$$

and

$$X_A^B = X_A \cap X^B.$$

$\mathrm{cf}\,\mu$ denotes the cofinality of the ordinal μ. When $\mathrm{cf}\,\mu \geq \omega_1$, a subset S of μ is called *stationary in* μ if it intersects all cub (i.e., closed and unbounded) sets in μ. Moreover for each $A \subset \mu$, $\mathrm{Lim}_\mu(A)$ is the set $\{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of A in μ. For convenience, we consider $\sup \emptyset = -1$ and -1 is the immediate predecessor of the ordinal 0. Therefore $\mathrm{Lim}_\mu(A)$ is cub in μ whenever A is unbounded in μ. We will simply denote $\mathrm{Lim}_\mu(A)$ by $\mathrm{Lim}(A)$ if the situation is clear in its context. In particular, assume that C is a cub set in μ with $\mathrm{cf}\,\mu \geq \omega$, then $\mathrm{Lim}(C) \subset C$. In this case, we define $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}(C)$, and $p_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Note that, for each $\alpha \in C$, $p_C(\alpha) \in C \cup \{-1\}$, and $p_C(\alpha) < \alpha$ iff $\alpha \in \mathrm{Succ}(C)$. So $p_C(\alpha)$ is the immediate predecessor of α in $C \cup \{-1\}$ whenever $\alpha \in \mathrm{Succ}(C)$. Moreover observe that $\mu \setminus C$ is the union of the pairwise disjoint collection $\{(p_C(\alpha), \alpha) : \alpha \in \mathrm{Succ}(C)\}$ of open intervals of μ and that $\mu \setminus \mathrm{Lim}(C)$ is the union of the pairwise disjoint collection $\{(p_C(\alpha), \alpha] : \alpha \in \mathrm{Succ}(C)\}$ of clopen intervals of μ. For simplicity, Lim and Succ stand for $\mathrm{Lim}(\omega_1)$ and $\mathrm{Succ}(\omega_1)$ respectively.

A strictly increasing function $M : \mathrm{cf}\,\mu + 1 \to \mu + 1$ is said to be a *normal function for* μ if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma \leq \mathrm{cf}\,\mu$ and $M(\mathrm{cf}\,\mu) = \mu$. Observe that, if $\mathrm{cf}\,\mu \geq \omega_1$, then two normal functions for μ coincide on a cub set of $\mathrm{cf}\,\mu$. Note that a normal function for μ always exists if $\mathrm{cf}\,\mu \geq \omega$. So we always fix a normal function M for each ordinal μ with $\mathrm{cf}\,\mu \geq \omega$. Then M carries $\mathrm{cf}\,\mu + 1$ homeomorphically to the range $\mathrm{ran}\,M$ of M and $\mathrm{ran}\,M$ is closed in $\mu + 1$. Note that for all $S \subset \mu$ with $\mathrm{cf}\,\mu \geq \omega_1$, S is stationary in μ if and only if $M^{-1}(S)$ is stationary in $\mathrm{cf}\,\mu$. For convenience, we define $M(-1) = -1$. Moreover for a cub set $C \subset \mathrm{cf}\,\mu$, define $m_C : \mu+1 \to C \cup \{\mathrm{cf}\,\mu\}$ by $m_C(\alpha) = \min\{\gamma \in C \cup \{\mathrm{cf}\,\mu\} : \alpha \leq M(\gamma)\}$ for each $\alpha \leq \mu$. Then they are straightforward to show that $\alpha = M(m_C(\alpha))$ whenever $m_C(\alpha) \in \mathrm{Lim}(C)$ and that $M(p_C(m_C(\alpha))) < \alpha \leq M(m_C(\alpha))$

whenever $m_C(\alpha) \in \mathrm{Succ}(C)$. Similarly for a normal function N on a limit ordinal ν and a cub set D in $\mathrm{cf}\,\nu$, define $n_D : \nu + 1 \to D \cup \{\mathrm{cf}\,\nu\}$ by $n_D(\beta) = \min\{\delta \in D \cup \{\mathrm{cf}\,\nu\} : \beta \leq N(\delta)\}$ for each $\beta \leq \nu$.

As is shown as a corollary of Theorem 5 below, $\omega_1 \times (\omega_1 + 1)$ is mildly normal. In this respect, first we note that the mild normality is not productive in general, even if these spaces are in the class of linearly ordered spaces.

Example 1. There is a compact linerarly ordered space Y such that $\omega_1 \times Y$ is not mildly normal.

Let $\{y_n : n \in \omega\}$ be a countably infinite set such that $\{y_n : n \in \omega\}$ is disjoint from $\omega_1 + 1$. Let $Y = (\omega_1 + 1) \cup \{y_n : n \in \omega\}$. Define a linear order $\prec$ on Y as follows. For each $n < m < \omega$, $\omega_1 \prec y_m \prec y_n$, and $\prec$ on $\omega_1 + 1$ is the same as the usual order on $\omega_1 + 1$. Consider Y as a linearly ordered topological space whose toplogy is induced by $\prec$. Then Y is compact, because Y is the union of the two compact subspaces $\omega_1 + 1$ and $\{\omega_1\} \cup \{y_n : n \in \omega\} \cong \omega + 1$. Let $Z = \omega_1 \times Y$, $F(0) = \{\langle \alpha, \alpha \rangle : \alpha < \omega_1\}$ and $F(1) = \omega_1 \times (\{\omega_1\} \cup \{y_n : n \in \omega\})$. Then $F(0) = \mathrm{Cl}_Z\{\langle \alpha, \alpha \rangle : \alpha \in \mathrm{Succ}\}$ and $F(1) = \mathrm{Cl}_Z(\omega_1 \times \{y_n : n \in \omega\})$, so they are disjoint regular closed sets in Z. Let U be an open set in Z with $F(0) \subset U$. For each $\alpha < \omega_1$, fix $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha]^2 \subset U$. Applying the Pressing Down Lemma(PDL), we find $\alpha_0 < \omega_1$ such that $(\alpha_0, \omega_1)^2 \subset U$. Take $\alpha < \omega_1$ with $\alpha_0 < \alpha$. Then it is straightforward to show $\langle \alpha, \omega_1 \rangle \in \mathrm{Cl}_Z\, U \cap F(1)$. So Z is not mildly normal.

Next we will show that $A \times B$ is mildly normal whenever A and B are subspaces of ordinals, that is, A and B are in the class of subspaces of well-ordered spaces. To do this, we need several preparations. In our discussion, for each limit ordinals μ and ν, we fix normal functions M and N for μ and ν respectively. For each cub sets C and D in $\mathrm{cf}\,\mu$ and $\mathrm{cf}\,\nu$, define m_C and n_D as above.

Lemma 2. *Let $\mu \in A \subset \mu + 1$, $\nu \in B \subset \nu + 1$, $\kappa = \mathrm{cf}\,\mu = \mathrm{cf}\,\nu \geq \omega_1$ and $X = A \times B \setminus \{\langle \mu, \nu \rangle\}$. Assume that there is a cub set C in κ such that $C \cap M^{-1}(A) \cap N^{-1}(B) = \emptyset$. Then $\mathcal{X} = \{X^{(N(p_C(\gamma)), N(\gamma)]}_{(M(p_C(\gamma)), M(\gamma)]} : \gamma \in \mathrm{Succ}(C)\}$ is a discrete collection of clopen subspaces of X and $\bigcup \mathcal{X} = \{\langle \alpha, \beta \rangle \in X : m_C(\alpha) = n_C(\beta)\}$. Moreover $Y = \{\langle \alpha, \beta \rangle \in X : m_C(\alpha) < n_C(\beta)\}$ and $Z = \{\langle \alpha, \beta \rangle \in X : m_C(\alpha) > n_C(\beta)\}$ are clopen in X.*

Proof. First we will show that $\mathcal{X}$ is discrete. Let $\langle \alpha, \beta \rangle \in X$. If $\alpha = \mu$, then since $\langle \mu, \nu \rangle \notin X$, we have $\beta < \mu$, hence $U = X^{[0, N(n_C(\beta))]}_{(M(n_C(\beta)), \mu]}$ is a neighborhood of $\langle\alpha,\beta\rangle$. Let $\gamma\in\mathrm{Succ}(C)$. If $\gamma\le n_C(\beta)$, then $M(\gamma)\le M(n_C(\beta))$, so $U\cap X^{(N(p_C(\gamma)),N(\gamma)]}_{(M(p_C(\gamma)),M(\gamma)]}=\emptyset$. If $n_C(\beta)<\gamma$, then by $n_C(\beta)\in C$, we have $n_C(\beta)\le p_C(\gamma)$ and therefore $N(n_C(\beta))\le N(p_C(\gamma))$. So $U\cap X^{(N(p_C(\gamma)),N(\gamma)]}_{(M(p_C(\gamma)),M(\gamma)]}=\emptyset$. This witnesses the discreteness of $\mathcal{X}$ at $\langle\alpha,\beta\rangle$ with $\alpha=\mu$. Similarly we can show the discreteness of $\mathcal{X}$ at $\langle\alpha,\beta\rangle$ with $\beta=\nu$. So assume $\alpha<\mu$ and $\beta<\nu$. Put $\gamma=m_C(\alpha)$. Assume that $\gamma\in\mathrm{Succ}(C)$, then $U=X_{(M(p_C(\gamma)),M(\gamma)]}$ is a neighborhood of $\langle\alpha,\beta\rangle$ which meets at most one member of $\mathcal{X}$. So let $\gamma=m_C(\alpha)\in\mathrm{Lim}(C)$. It follows from $\alpha=M(m_C(\alpha))\in A$ that $\gamma=m_C(\alpha)\in M^{-1}(A)\cap C$. Since $C\cap M^{-1}(A)\cap N^{-1}(B)=\emptyset$, we have $\gamma\notin N^{-1}(B)$. If $m_C(\alpha)=n_C(\beta)$, then $N(\gamma)=N(n_C(\beta))=\beta\in B$, a contradiction. So we have $m_C(\alpha)\ne n_C(\beta)$. If $m_C(\alpha)<n_C(\beta)$, then $U=X^{(N(m_C(\alpha)),N(n_C(\beta))]}_{[0,M(m_C(\alpha))]}$ is a neighborhood of $\langle\alpha,\beta\rangle$ which is disjoint from each member of $\mathcal{X}$. If $m_C(\alpha)>n_C(\beta)$, then $U=X^{[0,N(n_C(\beta))]}_{(M(n_C(\beta)),M(m_C(\alpha))]}$ is a neighborhood of $\langle\alpha,\beta\rangle$ which is also disjoint from each member of $\mathcal{X}$. Therefore $\mathcal{X}$ is discrete in X and $\bigcup\mathcal{X}$ is clopen in X.

Next we will show $\bigcup\mathcal{X}=\{\langle\alpha,\beta\rangle\in X: m_C(\alpha)=n_C(\beta)\}$. Let $\langle\alpha,\beta\rangle\in\bigcup\mathcal{X}$. Then $\langle\alpha,\beta\rangle\in X^{(N(p_C(\gamma)),N(\gamma)]}_{(M(p_C(\gamma)),M(\gamma)]}$ for some $\gamma\in\mathrm{Succ}(C)$, so $m_C(\alpha)=n_C(\beta)=\gamma$. Now let $\langle\alpha,\beta\rangle\in X$ and $\gamma=m_C(\alpha)=n_C(\beta)$. It follows from $\langle\mu,\nu\rangle\notin X$ that $\gamma\in C$. Assume $\gamma\in\mathrm{Lim}(C)$. Then $M(\gamma)=\alpha\in A$ and $N(\gamma)=\beta\in B$. Therefore $\gamma\in\mathrm{Lim}(C)\cap M^{-1}(A)\cap N^{-1}(B)\subset C\cap M^{-1}(A)\cap N^{-1}(B)=\emptyset$, a contradiction. Hence $\gamma\in\mathrm{Succ}(C)$ and $\langle\alpha,\beta\rangle\in X^{(N(p_C(\gamma)),N(\gamma)]}_{(M(p_C(\gamma)),M(\gamma)]}\in\mathcal{X}$.

Finally, we will show that Y and Z are clopen in X. Since the proofs are identical, we only show it for Y. Let $\langle\alpha,\beta\rangle\in Y$. Then $X^{(N(m_C(\alpha)),N(n_C(\beta))]}_{[0,M(m_C(\alpha))]}$ is a neighborhood of $\langle\alpha,\beta\rangle$ contained in Y, so Y is open. Let $\langle\alpha,\beta\rangle\notin Y$. If $m_C(\alpha)=n_C(\beta)$, then $\bigcup\mathcal{X}$ is a neighborhood of $\langle\alpha,\beta\rangle$ which is disjoint from Y. If $m_C(\alpha)>n_C(\beta)$, then $X^{[0,N(n_C(\beta))]}_{(M(n_C(\beta)),M(m_C(\alpha))]}$ is a neighborhood of $\langle\alpha,\beta\rangle$ which is also disjoint from Y. Therefore Y is closed. □

Lemma 3. *Let $A\subset\mu$, $B\subset\nu$, $\kappa=\mathrm{cf}\,\mu=\mathrm{cf}\,\nu\ge\omega_1$ and $X=A\times B$. Assume that $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are mildly normal for each $\mu'<\mu$ and $\nu'<\nu$. Then X is mildly normal.*

Proof. If A is not stationary in μ, then by taking a cub set $C\subset\kappa$ missing $M^{-1}(A)$, X can be represented as the free union

$$X=\bigoplus_{\gamma\in\mathrm{Succ}(C)}X_{(M(p_C(\gamma)),M(\gamma)]}$$

of mildly normal clopen subspaces of X. So we may assume that A and similarly B are stationary in μ and ν respectively. Let $F(0)$ and $F(1)$ be disjoint regular closed sets in X.

Case 1. $M^{-1}(A) \cap N^{-1}(B)$ is stationary in κ.

In this case, for each $\gamma \in M^{-1}(A) \cap N^{-1}(B) \cap \mathrm{Lim}(\kappa)$, fix $i(\gamma) \in 2$ and $f(\gamma) < \gamma$ such that $X^{(N(f(\gamma)),N(\gamma)]}_{(M(f(\gamma)),M(\gamma)]} \cap F(i(\gamma)) = \emptyset$. Then by the PDL, we can find $\gamma_0 < \kappa$ and $i_0 \in 2$ such that $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \cap F(i_0) = \emptyset$. Since by the assumption, $X_{[0,M(\gamma_0)]}$ and $X^{[0,N(\gamma_0)]}$ are mildly normal clopen subspaces, $F(0)$ and $F(1)$ can be separated by disjoint open sets.

Case 2. $M^{-1}(A) \cap N^{-1}(B)$ is not stationary in κ.

Take a cub set $C \subset \mathrm{Lim}(\kappa)$ in κ such that $C \cap M^{-1}(A) \cap N^{-1}(B) = \emptyset$. Since X is a subspace of $\big(A \cup \{\mu\}\big) \times \big(B \cup \{\nu\}\big) \setminus \{\langle \mu, \nu \rangle\}$, by Lemma 2, putting $\mathcal{X} = \{X^{(N(p_C(\gamma)),N(\gamma)]}_{(M(p_C(\gamma)),M(\gamma)]} : \gamma \in \mathrm{Succ}(C)\}$, $\bigcup \mathcal{X}$, $Y = \{\langle \alpha, \beta \rangle \in X : m_C(\alpha) < n_C(\beta)\}$ and $Z = \{\langle \alpha, \beta \rangle \in X : m_C(\alpha) > n_C(\beta)\}$ are clopen in X and $X = Y \bigoplus (\bigcup \mathcal{X}) \bigoplus Z$. Since by the assumption, $\bigcup \mathcal{X}$ is mildly normal, without loss of generality, it suffices to show that $F(0) \cap Y$ and $F(1) \cap Y$ can be separated by disjoint open sets in Y.

Fix $\gamma \in M^{-1}(A) \cap C$. For each $\delta \in N^{-1}(B) \cap C \cap (\gamma, \kappa)$, by $\langle M(\gamma), N(\delta) \rangle \in Y$, we can take $i(\gamma, \delta) \in 2$, $f(\gamma, \delta) < \gamma$ and $g(\gamma, \delta) < \delta$ with $\gamma \leq g(\gamma, \delta)$ such that $Y^{(N(g(\gamma,\delta)),N(\delta)]}_{(M(f(\gamma,\delta)),M(\gamma)]} \cap F(i(\gamma, \delta)) = \emptyset$. Since $N^{-1}(B) \cap C \cap (\gamma, \kappa)$ is stationary in κ and $|\gamma| \leq \gamma < \kappa$, applying the PDL, we find a stationary set $T(\gamma) \subset N^{-1}(B) \cap C \cap (\gamma, \kappa)$ and $i(\gamma) \in 2$, $f(\gamma) < \gamma$ and $g(\gamma) < \kappa$ such that $i(\gamma, \delta) = i(\gamma)$, $f(\gamma, \delta) = f(\gamma)$, $g(\gamma, \delta) = g(\gamma)$ for each $\delta \in T(\gamma)$. Set $g(\gamma) = 0$ for each $\gamma \in \kappa \setminus (M^{-1}(A) \cap C)$. Again applying the PDL to $M^{-1}(A) \cap C$, find a stationary set $S \subset M^{-1}(A) \cap C$, $i_0 \in 2$ and $\gamma_0 < \kappa$ such that $i(\gamma) = i_0$ and $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Then we have:

$$(*)\ Y^{(N(g(\gamma)),\nu)}_{(M(\gamma_0)),M(\gamma)]} \cap F(i_0) = \emptyset \text{ for each } \gamma \in S.$$

Set $C' = \{\gamma < \kappa : \forall \gamma' < \gamma (g(\gamma') < \gamma)\}$ and $D = \mathrm{Lim}(S \cap C')$. Notice that $D \subset C$ by $S \subset C$.

Claim 1. $Y_{\{M(\gamma)\}} \cap \mathrm{Int}\, F(i_0) = \emptyset$ for each $\gamma \in D$.

Proof. Assume $\langle M(\gamma), \beta \rangle \in Y_{\{M(\gamma)\}} \cap \mathrm{Int}\, F(i_0)$ for some $\gamma \in D$ and β. Take $\gamma' < \gamma$ and $\beta' < \beta$ such that $\gamma_0 \leq \gamma'$, $N(\gamma) \leq \beta'$ and $Y^{(\beta',\beta]}_{(M(\gamma'),M(\gamma)]} \subset \mathrm{Int}\, F(i_0)$. By $\gamma' < \gamma \in D = \mathrm{Lim}(S \cap C')$, we can find $\gamma'' \in S \cap C'$ with $\gamma' < \gamma'' < \gamma$. It follows from $\gamma'' < \gamma \in D \subset C'$ that $g(\gamma'') < \gamma$, so $N(g(\gamma'')) < N(\gamma) \leq \beta' < \beta$. Then $\langle M(\gamma''), \beta \rangle \in Y^{(N(g(\gamma'')),\nu)}_{(M(\gamma_0),M(\gamma'')]} \cap F(i_0)$. This contradicts (*).

Claim 2. $Y_{\{M(\gamma)\}} \cap F(i_0) = \emptyset$ for each $\gamma \in D$.

Proof. Assume $\langle M(\gamma),\beta\rangle \in Y_{\{M(\gamma)\}} \cap F(i_0)$ for some $\gamma \in D$. Since $\langle M(\gamma),\beta\rangle \in F(i_0) = \operatorname{Cl}\operatorname{Int} F(i_0)$ and $Y^{(N(\gamma),\beta]}_{(M(\gamma_0),M(\gamma)]}$ is a neighborhood of $\langle M(\gamma),\beta\rangle$, we can find $\langle \alpha',\beta'\rangle \in Y^{(N(\gamma),\beta]}_{(M(\gamma_0),M(\gamma)]} \cap \operatorname{Int} F(i_0)$. It follows from Claim 1 that $\alpha' \notin M''D$, where $M''D$ denotes the range of D under M, so $\alpha' < M(\gamma)$. By $\gamma \in D = \operatorname{Lim}(S \cap C')$, we can fix $\gamma' \in S \cap C'$ with $\alpha' < M(\gamma') < M(\gamma)$. It follows from $\gamma' < \gamma \in D \subset C'$ that $g(\gamma') < \gamma$, so $N(g(\gamma')) < N(\gamma) < \beta'$. Then $\langle \alpha',\beta'\rangle \in Y^{(N(g(\gamma')),\nu)}_{(M(\gamma_0),M(\gamma')]} \cap F(i_0)$. This contradicts (*).

Claim 3. $Y_{(M(\gamma_0),\mu)} \cap F(i_0) \subset \bigcup_{\gamma\in\operatorname{Succ}(D)} Y^{(N(p_D(\gamma)),N(\gamma)]}_{(M(p_D(\gamma)),M(\gamma)]}$.

Proof. Let $\langle \alpha,\beta\rangle \in Y_{(M(\gamma_0),\mu)} \cap F(i_0)$. It follows from Claim 2 that $\alpha \notin M''D$. So there is $\gamma \in \operatorname{Succ}(D)$ with $\alpha \in (M(p_D(\gamma)), M(\gamma))$. By $\gamma \in D = \operatorname{Lim}(S\cap C')$ and $\alpha < M(\gamma)$, we can fix $\gamma' \in S\cap C'$ with $\alpha < M(\gamma') < M(\gamma)$. So it follows from $\gamma' < \gamma \in C'$ that $g(\gamma') < \gamma$. By (*), $Y^{(N(g(\gamma')),\nu)}_{(M(\gamma_0),M(\gamma')]} \cap F(i_0) = \emptyset$. Moreover by $\langle \alpha,\beta\rangle \in F(i_0)$ and $\alpha \in (M(\gamma_0), M(\gamma')]$, we have $\beta \le N(g(\gamma')) < N(\gamma)$. It follows from $\langle \alpha,\beta\rangle \in Y$ that $p_D(\gamma) < m_C(\alpha) < n_C(\beta)$, so $N(p_D(\gamma)) < N(m_C(\alpha)) \le \beta$. Therefore $\langle \alpha,\beta\rangle \in Y^{(N(p_D(\gamma)),N(\gamma)]}_{(M(p_D(\gamma)),M(\gamma)]}$.

Since $D \subset C$ and $M^{-1}(A) \cap N^{-1}(B) \cap C = \emptyset$ and $Y^{(N(p_D(\gamma)),N(\gamma)]}_{(M(p_D(\gamma)),M(\gamma)]} \subset X_{[0,M(\gamma)]}$, by Lemma 2, $\mathcal{Y} = \{Y^{(N(p_D(\gamma)),N(\gamma)]}_{(M(p_D(\gamma)),M(\gamma)]} : \gamma \in \operatorname{Succ}(D)\}$ is a discrete collection of clopen mildly normal subspaces in Y, so $\bigcup \mathcal{Y}$ is also mildly normal. Then since $Y_{[0,M(\gamma_0)]}$ is mildly normal, using Claim 3, we can find disjoint open sets which separate $F(0) \cap Y$ and $F(1) \cap Y$. This completes the proof of Lemma 3. □

Lemma 4. *Let $A \subset \mu$, $\nu \in B \subset \nu+1$, $\kappa = \operatorname{cf}\mu = \operatorname{cf}\nu \ge \omega_1$ and $X = A\times B$. Assume that $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are mildly normal for each $\mu' < \mu$ and $\nu' < \nu$. Then X is mildly normal.*

Proof. As in the first paragraph of the proof of Lemma 3, we may that assume A is stationary in μ. If $B \cap \nu$ is bounded by $\beta_0 < \nu$, then X can be represented as $X = X^{[0,\beta_0]} \bigoplus X^{\{\nu\}}$. So we may also assume that $B \cap \nu$ is unbounded in ν.

Let $F(0)$ and $F(1)$ be disjoint regular closed sets in X.

Case 1. $M^{-1}(A) \cap N^{-1}(B)$ is stationary in κ.

As in Case 1 of Lemma 3, we can find $\gamma_0 < \kappa$ and $i_0 \in 2$ such that $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \cap F(i_0) = \emptyset$.

Claim 0. $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \cap F(i_0) = \emptyset$.

Proof. Since $B \cap \nu$ is unbounded in ν, we have $\mathrm{Cl}\, X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} = X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)}$. Therefore $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \cap \mathrm{Int}\, F(i_0) = \emptyset$. But since $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)}$ is clopen in X, we have $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \cap F(i_0) = X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \cap \mathrm{Cl}\,\mathrm{Int}\, F(i_0) = \emptyset$.

Then in a usual way, $F(0)$ and $F(1)$ can be separated by disjoint open sets.

Case 2. $M^{-1}(A) \cap N^{-1}(B)$ is not stationary in κ.

Take a cub set $C \subset \mathrm{Lim}(\kappa)$ in κ such that $C \cap M^{-1}(A) \cap N^{-1}(B) = \emptyset$. By Lemma 2, putting $\mathcal{X} = \{X^{(N(p_C(\gamma)),N(\gamma)]}_{(M(p_C(\gamma)),M(\gamma)]} : \gamma \in \mathrm{Succ}(C)\}$, $\bigcup \mathcal{X}$, $Y = \{\langle\alpha,\beta\rangle \in X : m_C(\alpha) < n_C(\beta)\}$ and $Z = \{\langle\alpha,\beta\rangle \in X : m_C(\alpha) > n_C(\beta)\}$ are clopen in X and $X = Y \bigoplus (\bigcup \mathcal{X}) \bigoplus Z$. Note that $A \times \{\nu\} \subset Y$ and $Z \subset A \times (B \cap \nu)$. Of course, $\bigcup \mathcal{X}$ is mildly normal. Moreover as in the proof of Case 2 of Lemma 3, we can similarly show that $F(0) \cap Z$ and $F(1) \cap Z$ can be separated by disjoint open sets in Z. Therefore it suffices to show that $F(0) \cap Y$ and $F(1) \cap Y$ can be separated by disjoint open sets in Y.

Fix $\gamma \in M^{-1}(A) \cap C$. By $\langle M(\gamma), \nu\rangle \in Y$, we can find $i(\gamma) \in 2$, $f(\gamma) < \gamma$ and $g(\gamma) < \kappa$ with $\gamma \leq g(\gamma)$ such that $Y^{(N(g(\gamma)),\nu]}_{(M(f(\gamma)),M(\gamma)]} \cap F(i(\gamma)) = \emptyset$. Set $g(\gamma) = 0$ for each $\gamma \in \kappa \setminus (M^{-1}(A) \cap C)$. Applying the PDL, find a stationary set $S \subset M^{-1}(A) \cap C$, $i_0 \in 2$ and $\gamma_0 < \kappa$ such that $i(\gamma) = i_0$ and $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Then we have:

$$(*)\ Y^{(N(g(\gamma)),\nu]}_{(M(\gamma_0)),M(\gamma)]} \cap F(i_0) = \emptyset \text{ for each } \gamma \in S.$$

Set $C' = \{\gamma < \kappa : \forall \gamma' < \gamma (g(\gamma') < \gamma)\}$ and $D = \mathrm{Lim}(S \cap C')$. Notice that $D \subset C$ by $S \subset C$. The following Claims 1, 2 and 3 can be proved in a similar way as in Claims 1, 2 and 3 of Lemma 3 respectively.

Claim 1. $Y_{\{M(\gamma)\}} \cap \mathrm{Int}\, F(i_0) = \emptyset$ for each $\gamma \in D$.

Claim 2. $Y_{\{M(\gamma)\}} \cap F(i_0) = \emptyset$ for each $\gamma \in D$.

Claim 3. $Y_{(M(\gamma_0),\mu)} \cap F(i_0) \subset \bigcup_{\gamma \in \mathrm{Succ}(D)} Y^{(N(p_D(\gamma)),N(\gamma)]}_{(M(p_D(\gamma)),M(\gamma)]}$.

Then as in the final paragraph of the proof of Lemma 3, we can find disjoint open sets which separate $F(0) \cap Y$ and $F(1) \cap Y$. This completes the proof of Lemma 4. □

Theorem 5. *Let A and B be subspaces of ordinals. Then $X = A \times B$ is mildly normal.*

Proof. Assme that $X = A \times B$ is not mildly normal. Let

$$\mu = \min\{\alpha : X_{[0,\alpha]} \text{ is not mildly normal }\},$$

$$\nu = \min\{\beta : X^{[0,\beta]}_{[0,\mu]} \text{ is not mildly normal }\}.$$

Then μ and ν are limit ordinals. Without loss of generality, we may assume that $A \subset \mu+1$, $B \subset \nu+1$, $X = A \times B$ is not mildly normal but $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are mildly normal for each $\mu' < \mu$ and $\nu' < \nu$. Let $F(0)$ and $F(1)$ be disjoint regular closed sets which cannot be separated by disjoint open sets. We will consider several cases. In each case, we will derive a contradiction.

Case 1. $\mu \in A$ and $\nu \in B$.

In this case, we may assume $\langle \mu, \nu \rangle \notin F(0)$. Then we can take $\mu' < \mu$ and $\nu' < \nu$ with $X^{(\nu',\nu]}_{(\mu',\mu]} \cap F(0) = \emptyset$. Since $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are mildly normal, in a usual way, we can find disjoint open sets which separate $F(0)$ and $F(1)$, a contradiction.

Case 2. $\mu \notin A$ and $\nu \notin B$.

We have $\mathrm{cf}\,\mu \geq \omega_1$, otherwise X can be represented as the free union $X = \bigoplus_{n\in\omega} X_{(M(n-1),M(n)]}$ of mildly normal clopen subspaces, a contradiction. Moreover as in the first paragraph of the proof of Lemma 3, A is stationary in μ. Similarly $\mathrm{cf}\,\nu \geq \omega_1$ and B is stationary in ν. It follows from Lemma 3 that $\mathrm{cf}\,\mu \neq \mathrm{cf}\,\nu$, so we may assume $\mathrm{cf}\,\mu < \mathrm{cf}\,\nu$. Fix $\gamma \in M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}\,\mu)$. For each $\delta \in N^{-1}(B) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$, by $\langle M(\gamma), N(\delta) \rangle \in X$, we can take $i(\gamma,\delta) \in 2$, $f(\gamma,\delta) < \gamma$ and $g(\gamma,\delta) < \delta$ such that $X^{(N(g(\gamma,\delta)),N(\delta)]}_{(M(f(\gamma,\delta)),M(\gamma)]} \cap F(i(\gamma,\delta)) = \emptyset$. Applying the PDL to $N^{-1}(B) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$, by $|\gamma| < \mathrm{cf}\,\mu < \mathrm{cf}\,\nu$, we can find a stationary set $T(\gamma) \subset N^{-1}(B) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$ and $i(\gamma) \in 2$, $f(\gamma) < \gamma$ and $g(\gamma) < \mathrm{cf}\,\nu$ such that $i(\gamma,\delta) = i(\gamma)$, $f(\gamma,\delta) = f(\gamma)$, $g(\gamma,\delta) = g(\gamma)$ for each $\delta \in T(\gamma)$. Again applying the PDL to $M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}\,\mu)$, find a stationary set $S \subset M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}\,\mu)$, $i_0 \in 2$ and $\gamma_0 < \mathrm{cf}\,\mu$ such that $i(\gamma) = i_0$ and $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Let $\delta_0 = \sup\{g(\gamma) : \gamma \in S\}$. It follows from $\mathrm{cf}\,\mu < \mathrm{cf}\,\nu$ that $\delta_0 < \mathrm{cf}\,\nu$. Then $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)}$ is a clopen set in X which does not meet $F(i_0)$. Moreover since $X_{[0,M(\gamma_0)]}$ and $X^{[0,N(\delta_0)]}$ are mildly normal, $F(0)$ and $F(1)$ can be separated by disjoint open sets, a contradiction.

Case 3. $\mu \in A$ and $\nu \notin B$.

Since the proofs are identical, we only show the following case.

Case 4. $\mu \notin A$ and $\nu \in B$.

As in Case 2, we may assume that $\mathrm{cf}\,\mu \geq \omega_1$ and A is stationary in $\mathrm{cf}\,\mu$. Moreover as in the first paragraph of the proof of Lemma 4, we may assume that $B \cap \nu$ is unbounded in ν. By Lemma 4, we have $\mathrm{cf}\,\mu \neq \mathrm{cf}\,\nu$. For each $\gamma \in M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}\,\mu)$, by $\langle M(\gamma), \nu \rangle \in X = A \times B$, fix $i(\gamma) \in 2$, $f(\gamma) < \gamma$ and $g(\gamma) < \mathrm{cf}\,\nu$ such that $X^{(N(g(\gamma)),\nu]}_{(M(f(\gamma)),M(\gamma)]} \cap F(i(\gamma)) = \emptyset$.

First assume $\mathrm{cf}\,\nu < \mathrm{cf}\,\mu$. Find a stationary set $S \subset M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}\,\mu)$, $i_0 \in 2$, $\gamma_0 < \mathrm{cf}\,\mu$ and $\delta_0 < \mathrm{cf}\,\nu$ such that $i(\gamma) = i_0$, $f(\gamma) = \gamma_0$ and $g(\gamma) = \delta_0$ for each $\gamma \in S$.

Next assume $\mathrm{cf}\,\nu > \mathrm{cf}\,\mu$. Find a stationary set $S \subset M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}\,\mu)$, $i_0 \in 2$ and $\gamma_0 < \mathrm{cf}\,\mu$ such that $i(\gamma) = i_0$ and $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Moreover let $\delta_0 = \sup\{g(\gamma) : \gamma \in S\}$.

Then in either cases, we have $X^{(N(\delta_0),\nu]}_{(M(\gamma_0),\mu)} \cap F(i_0) = \emptyset$. Since $X_{[0,M(\gamma_0)]}$ and $X^{[0,N(\delta_0)]}$ are mildly normal, $F(0)$ and $F(1)$ can be separated by disjoint open sets, a contradiction.

This completes the proof of Theorem 5. □

Next we answer the Arkhangel'skiĭ's question whether all subspaces of ω_1^2 are mildly normal or not.

Example 6. There is a subspace of ω_1^2 which is not mildly normal.

Our space is $X = \omega \times \omega_1 \cup \{\omega\} \times \mathrm{Succ}$ which is a subspace of $(\omega+1) \times \omega_1$. This space is also well known to be not normal, by showing that $X_{\{\omega\}}$ and X^{Lim} cannot be separated by disjoint open sets. But both $X_{\{\omega\}}$ and X^{Lim} are not regular closed in X.

Decompose ω into two infinite subsets $A(0)$ and $A(1)$, moreover decompose Succ into two uncountable subsets $B(0)$ and $B(1)$. Since $A(i) \times B(i)$ consists of isolated points, it is open in X. So $F(i) = \mathrm{Cl}_X(A(i) \times B(i))$ is a regular closed set. It is easy to verify that $F(0) \cap F(1) = \emptyset$ and $\{\omega\} \times B(i) \subset F(i)$. Assume that $F(0)$ and $F(1)$ are separated by disjoint open sets $U(0)$ and $U(1)$, respectively. Let $n \in A(i)$ and $\beta \in \mathrm{Lim}(B(i))$. Then $\langle n, \beta\rangle \in \{n\} \times \mathrm{Lim}(B(i)) \subset \mathrm{Cl}_X(A(i) \times B(i)) = F(i) \subset U(i)$. So fix $g(n,\beta) < \beta$ such that $X^{(g(n,\beta),\beta]}_{\{n\}} \subset U(i)$. Since $\mathrm{Lim}(B(i))$ is stationary, by the PDL, there is $g(n) < \omega_1$ such that $X^{(g(n),\omega_1)}_{\{n\}} \subset U(i)$. Pick $\beta_0 > \sup(\bigcup_{i\in 2}\{g(n) : n \in A(i)\})$ with $\beta_0 < \omega_1$. Then $X^{(\beta_0,\omega_1)}_{\{n\}} \subset U(i)$ if $n \in A(i)$.

On the other hand, pick $\beta \in B(0)$ with $\beta > \beta_0$. Since $\langle \omega, \beta\rangle \in \{\omega\} \times B(0) \subset F(0) \subset U(0)$, there is $n_0 \in \omega$ such that $X^{\{\beta\}}_{(n_0,\omega]} \subset U(0)$. Moreover pick $n \in A(1)$ with $n_0 < n$. Then $\langle n, \beta\rangle \in X^{\{\beta\}}_{(n_0,\omega]} \cap X^{(\beta_0,\omega_1)}_{\{n\}} \subset U(0) \cap U(1)$, a contradiction.

Now we give here two examples of a closed subspace of a mildly normal product of ordinals which is not mildly normal. By using 1, we have the following:

Example 7. There is a closed subspace Z of the mildly normal space $X = \omega_1 \times (\omega_1 + 1) \times (\omega + 1)$ which is not mildly normal.

The “edge” $Y = \{\omega_1\}\times(\omega+1)\cup(\omega_1+1)\times\{\omega\}$ in the space $(\omega_1+1)\times(\omega+1)$ clearly closed in $(\omega_1+1)\times(\omega+1)$ and homeomorphic to the space Y described in Example 1. Therefore $Z = \omega_1 \times Y$ is a closed subspace of X which is not mildly normal. We will give a direct proof that X is mildly normal. Let $F(0)$ and $F(1)$ be disjoint regular closed subsets of X. For each $\alpha < \omega_1$, by $\langle\alpha,\alpha,\omega\rangle \in X$, we can find $f(\alpha) < \alpha$, $n(\alpha) < \omega$ and $i(\alpha) \in 2$ such that $(f(\alpha),\alpha] \times (f(\alpha),\alpha] \times (n(\alpha),\omega] \cap F(i(\alpha)) = \emptyset$. Applying the PDL, we find $\alpha_0 < \omega_1$, $n_0 < \omega$ and $i_0 \in 2$ such that $(\alpha_0,\omega_1) \times (\alpha_0,\omega_1) \times (n_0,\omega] \cap F(i_0) = \emptyset$. Since $F(i_0)$ is regular closed, we can show as in Claim 0 of Lemma 4, $(\alpha_0,\omega_1) \times (\alpha_0,\omega_1] \times (n_0,\omega] \cap F(i_0) = \emptyset$. Since $\omega_1 \times (\omega_1+1)$ is mildly normal, $\omega_1 \times (\omega_1+1) \times [0,n_0] = \bigoplus_{n\le n_0} \omega_1 \times (\omega_1+1) \times \{n\}$ is also mildly normal. $[0,\alpha_0] \times (\omega_1+1) \times (\omega+1)$ is normal because it is compact. Since ω_1 is normal countably compact and $[0,\alpha_0] \times (\omega+1)$ is metrizable, by [3], $\omega_1\times[0,\alpha_0]\times(\omega+1)$ is normal. Since X is coverd by the four clopen subspaces $(\alpha_0,\omega_1)\times(\alpha_0,\omega_1]\times(n_0,\omega]$, $\omega_1\times(\omega_1+1)\times[0,n_0]$, $[0,\alpha_0]\times(\omega_1+1)\times(\omega+1)$ and $\omega_1 \times [0,\alpha_0] \times (\omega+1)$, X is mildly normal.

Finally we answer the Arkhangel’skiĭ’s question whether closed subspaces of $\omega_1 \times (\omega_1+1)$ are mildly normal or not.

Example 8. There is a closed subspace X of $\omega_1 \times (\omega_1+1)$ which is not mildly normal.

Let $W = \bigcup_{\alpha\in\mathrm{Succ}}\{\alpha\} \times (\alpha,\omega_1)$. Since W is open in $\omega_1 \times (\omega_1+1)$, $X = \omega_1\times(\omega_1+1)\setminus W$ is closed in $\omega_1\times(\omega_1+1)$. Let $F(0) = \{\langle\alpha,\alpha\rangle : \alpha < \omega_1\}$ and $F(1) = \{\langle\alpha,\omega_1\rangle : \alpha < \omega_1\}$. Since $F(0) = \mathrm{Cl}_X\{\langle\alpha,\alpha\rangle : \alpha \in \mathrm{Succ}\}$ and $F(1) = \mathrm{Cl}_X\{\langle\alpha,\omega_1\rangle : \alpha \in \mathrm{Succ}\}$, they are regular closed in X. Let U be an open set with $F(0) \subset U$. By the PDL, there is $\alpha_0 < \omega_1$ such that $X^{(\alpha_0,\omega_1)}_{(\alpha_0,\omega_1)} \subset U$. Take $\alpha \in \mathrm{Lim}$ with $\alpha_0 < \alpha$. It follows from $\{\alpha\}\times(\alpha,\omega_1) \subset U$ that $\langle\alpha,\omega_1\rangle \in \mathrm{Cl}_X\, U \cap F(1)$. Therefore X is not mildly normal.

Problem 9. Let A_i’s be subspaces of ordinals, $i < n$. Is the finite product $\prod_{i<n} A_i$ mildly normal?

References

[1] A. V. Arkhangel’skiĭ, private communications with the first author.

[2] N. Kemoto, H. Ohta and K. Tamano, Products of spaces of ordinal numbers, Top. Appl. 45(1992), pp.245–260.

[3] A.H. Stone, Paracompactness and product spaces, Bull. AMS 54(1948),pp.997–982.

Department of Mathematics, Ohio University, Athens, Ohio 45701-2979 USA (Current Address: King Abdulaziz University, Department of Mathematics, P.O.Box 114641 Jeddah, 21381 Saudi Arabia)
Department of Mathematics, Faculty of Education, Oita University, Dannoharu, Oita, 870-1192, Japan

E-mail addresses:
lkalantan@hotmail.com
nkemoto@cc.oita-u.ac.jp