“Higher separation axioms”

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All spaces are assumed to be a T_1-space, i.e. each singleton is closed.

Definitions and basic facts. Subsets F_0 and F_1 in a space X are said to be **completely separated in** X if there exists a continuous mapping $f : X \to I$ such that $F_i \subset f^{-1}(\{i\})$ for each $i \in 2 = \{0, 1\}$, where I denotes the unit interval $[0, 1]$ in the reals $\mathbb{R}$. A subset F in a space X is a **zero-set** (**cozero-set**) if $F = f^{-1}(\{0\})$ ($F = f^{-1}((0, 1])$, respectively) for some continuous mapping $f : X \to I$. Subsets F_0 and F_1 in a space X are said to be **separated by disjoint open sets in** X if there are disjoint open sets U_0 and U_1 such that $F_i \subset U_i$ for each $i \in 2$. Completely separated sets are separated by disjoint open sets, indeed $U_0 = f^{-1}([0, \frac{1}{2}))$ and $U_1 = f^{-1}((\frac{1}{2}, 1])$ are such open sets.

A space X is called **Tychonoff**, **completely regular** or $T_{3\frac{1}{2}}$ if for each pair of a point $x \in X$ and a closed set $F \subset X$ with $x \notin F$, $\{x\}$ and F are completely separated, equivalently, the collection of all cozero-sets forms a base for X. Tychonoff spaces are *regular* or T_3 (i.e., such $\{x\}$ and F are separated by disjoint open sets), but not vice versa. The class of Tychonoff (as well as T_i, $i \leq 3$) spaces is closed under taking arbitrary products and subspaces. Here a space is T_2 or *Hausdorff* if each pair of distinct points are separated by disjoint open sets.

A space is **normal** or T_4 if disjoint closed sets are separated by disjoint open sets. Note that disjoint closed sets in a normal space are completely separated (Urysohn’s Lemma), so normal spaces are Tychonoff. Also note that closed subspaces of a normal space is normal and C-embedded in X (Tieze-Urysohn’s Lemma), where a subspace A of a space X is C**-embedded in** X if every $f \in C(A)$ can be extended to a mapping in $C(X)$, and $C(X)$ denotes the collection of all continuous mappings of X to the reals $\mathbb{R}$. Likewise, a subspace A of a space X is C^***-embedded in** X if every $f \in C^*(A)$ can be extended to a mapping in $C^*(X)$, where $C^*(X)$ denotes the collection of all continuous mappings of X to the unit interval I. Note that C-embedded subspaces are C^*-embedded.

A **perfectly normal** or T_6 space is a *perfect* (i.e., closed sets are G_δ-sets) and normal space. A perfectly normal space is a **hereditarily normal** ($= T_5$) space, that is, all subspaces are normal.

A collection $\mathcal{F}$ of subsets of a space X is **discrete** if for every point $x \in X$, there is a neighborhood U of x such that $|\{F \in \mathcal{F} : U \cap F \neq \emptyset\}| \leq 1$. A space X is called κ**-collectionwise normal** (κ**-CWN** for short), where κ is a cardinal, if for every discrete collection of $\mathcal{F}$ with $|\mathcal{F}| \leq \kappa$, there is a pairwise disjoint collection $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ of open sets such that $F \subset U(F)$ for each $F \in \mathcal{F}$. It is not difficult to show “normal iff ω-CWN”, where ω is the smallest infinite cardinal. A space is **collectionwise normal** (**CWN** for short) if it is κ-CWN for any cardinal κ.

Of course, CWN spaces are normal. *Compact Hausdorff* (more generally *Lindelöf regular*) spaces are CWN and hence normal, where *compact* (*Lindelöf*) means "every open cover has a finite (countable) subcover". Moreover *GO-spaces* (i.e., subspaces of a linerarly ordered topological space) are CWN, therefore subspaces of ordinal numbers with the usual order topology are CWN. In particular, ω_1 is hereditarily CWN, but not perfect. Indeed, the subset of all limit ordinals in ω_1 is closed but not G_δ. Most of definitions and basic results of this article can be found in [E].

Tychonoff spaces. First observe that $X = \mathbb{R} \setminus \{0\}$ is not C^*-embedded in $\mathbb{R}$. Indeed, the continuous mapping $f : X \to I$, defined by $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x < 0$, cannot be extended over X as a continuous mapping. Moreover we mention that a subset A of X is C^*-embedded in X iff any two completely separated sets in A are completley separated in X (Urysohn's extension theorem) and that a C^*-embedded subspace A of X is C-embedded in X iff for every zero-set Z of X with $A \cap Z = \emptyset$, A and Z are completely separated in X. As will be mentioned below, there is a space with a C^*-embedded but not C-embedded subspace.

Next, consider $C(X)$ as a ring with pointwise operations. For every topological space X we have a Tychonoff space Y by identifying points x, x' in which $f(x) = f(x')$ for every real-valued continuous mapping f on X. Then there is a natural quotient map $\tau : X \to Y$ so that every real-valued continuous mapping on X is factored through τ.

The class of Tychonoff spaces has important roles when we discuss *compactifications* (a compactification of a space X is a compact space containing X as a dense subspace). Let X be a Tychonoff space. Consider a collection $\mathcal{F}$ of continuous mappings on X to the unit interval I such that $\{f^{-1}((0,1]) : f \in \mathcal{F}\}$ forms a basis for X. Indeed, such a collection $\mathcal{F}$ exists because X is Tychonoff, e.g. $\mathcal{F} = C^*(X)$ is such one. Furthermore one can take such an $\mathcal{F}$ with $|\mathcal{F}| = w(X)$, the *weight* of X. Moreover if $\mathcal{F}$ is a such a collection and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G}$ is also such one. Consider the mapping $F_\mathcal{F} : X \to \prod_{f\in\mathcal{F}} I_f$, where $I_f = I$ for each $f \in \mathcal{F}$, defined by $F_\mathcal{F}(x) = \langle f(x) : f \in \mathcal{F}\rangle$. Then obviously the mapping $F_\mathcal{F}$ is an embedding of X into $\prod_{f\in\mathcal{F}} I_f = I^{|\mathcal{F}|}$. By taking the closure $\mathrm{Cl}F_\mathcal{F}(X)$ in $I^{|\mathcal{F}|}$, one can obtain a (homeomorphic copy of) T_2-compactification of X. Therefore X is Tychonoff iff X has a T_2-compactification (of the weight $w(X)$) iff X can be embedded in a product space of ($w(X)$-many) copies of the the unit interval I. Moreover taking $\mathcal{F}$ as $\mathcal{F} = C^*(X)$, one obtains the *Stone-Čech compactification* $\beta X = \mathrm{Cl}F_{C^*(X)}(X)$ of X. By the construction, one can understand that βX is characterized as a T_2-compactification of X in which X is C^*-embedded, indeed for a given $f \in C^*(X)$, consider the mapping $\beta f \in C^*(\beta X)$ defined by $\beta f(z) = \pi_f(z)$ for each $z \in \beta X = \mathrm{Cl}F_{C^*(X)}(X)$, where $\pi_f : \prod_{f\in\mathcal{F}} I_f \to I_f$ is the f-th projection. Furthermore βX can be characterized in such a way that every continuous mapping $f : X \to K$ of X to a T_2-compact space K can be extended to a continuous mapping $\beta f : \beta X \to K$, details of this theory are given in [E]. Notice that X is normal iff every pair of disjoint closed sets have disjoint closures in βX.

There is an analogous theory on realcompactifications of Tychonoff spaces, where a space is called *realcompact* if it is homeomorphic to a closed subspace of a product of copies of $\mathbb{R}$. Replace, in the above argument, $C^*(X)$ and I

by $C(X)$ and $\mathbb{R}$ respectively, and consider $\mathcal{F}$ as a subcollection of $C(X)$ such that $\{f^{-1}((0,\infty)) : f \in \mathcal{F}\}$ forms a basis for X. Then one obtains a real compactification $\mathrm{Cl}F_{\mathcal{F}}(X) \subseteq \mathbb{R}^{|\mathcal{F}|}$. In particular, $\mathrm{Cl}F_{C(X)}(X)$, denoted by υX, is called the *Hewitt realcompactification* of the Tychonoff space X. Observe that υX is chracterized as a realcompactification of X in which X is C-embedded. In this sense, compactness and realcompactness are called *I-compactness* and *$\mathbb{R}$-compactness* respectively. There is also a more general theory of E-compactness for a T_2-space E, see [MN, Ch.12].

Now, we describe a normal space with a C^*-embedded but not C-embedded subspace. Let $\beta\omega$ be the Stone-Čech compactification of the discrete space ω, $p \in \beta\omega$ and $X = \omega \cup \{p\}$. Since $\omega \subseteq X \subseteq \beta\omega$, ω is C^*-embedded in X. Note that the mapping $f : X \to I$, defined by $f(n) = \frac{1}{n}$ and $f(p) = 0$, witnesses that the one point set $\{p\}$ is a zero-set in X. But since ω is dense in X, ω and $\{p\}$ are not completely separated, thus ω is not C-embedded in X.

Wallman also constructed a T_1-compactification wX of a T_1-space X such that every continuous mapping $f : X \to K$ of X to a T_2-compact space K can be extended to a continuous mapping $wf : wX \to K$. He showed that wX is T_2 iff X is normal, in this case $wX = \beta X$.

Let X be a set. A mapping $d : X \times X \to \mathbb{R}$ is called a *pseudometric* on X if $d(x, y) \geq 0$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$ for every points x, y, z. A pseudometric d is *metric* if $d(x, y) = 0$ implies $x = y$. A collection P of pseudometrics on a set X is called a *base for a uniformity* on X if (1) $\max\{d_0, d_1\} \in P$ whenever $d_0, d_1 \in P$ (where $\max\{d_0, d_1\} \in P$ is the pseudometric d defined by $d(x,y) = \max\{d_0(x,y), d_1(x,y)\}$), and (2) for every pair x, y of distinct points of X, there is $d \in P$ such that $d(x,y) > 0$. Moreover a base P for a uniformity is a *uniformity* if (3) a pseudometric e belongs to P whenever, for every $\varepsilon > 0$, there are $d \in P$ and $\delta > 0$ so that $d(x, y) \leq \delta$ implies $e(x, y) \leq \varepsilon$. Set $U_d(x,\varepsilon) = \{y : d(x, y) < \varepsilon\}$. Then a base for a uniformity P induces the Tychonoff topology $\tau_P = \{G \subseteq X : (\forall x \in G)(\exists d \in P)(\exists \varepsilon > 0)(U_d(x,\varepsilon) \subseteq G)\}$. Note that for a given base for a uniformity P, $P' = \{e :\ e$ is a pseudometric on X such that for every $\varepsilon > 0$, there are $d \in P$ and $\delta > 0$ so that $d(x, y) \leq \delta$ implies $e(x, y) \leq \varepsilon$ $\}$ is a uniformity and the topology $\tau_{P'}$ coincides with τ_P. Observe that in the space $\langle X, \tau_P\rangle$, $U_d(x,\varepsilon)$ is open for every $x \in X$, $\varepsilon > 0$ and $d \in P$, and that every $d \in P$ is continuous with respect to the topology τ_P. On the other hand, for a given Tychonoff topological space $\langle X, \tau\rangle$ and a finite subset $\mathcal{F}$ of $C(X)$, $d_{\mathcal{F}}(x,y) = \max\{|f(x) - f(y)| : f \in \mathcal{F}\}$ defines a continuous (with respect to τ) pseudometric on X. Then it is not difficult to show that $P = \{d_{\mathcal{F}} : \mathcal{F}$ is a finite subset of $C(X)$ $\}$ is a base for a uniformity on the set X and $\tau_P = \tau$. Thus the topology of X can be induced by a uniformity on X iff it is Tychonoff.

The *function space* Y^X of all continuous mappings of X to a Tychonoff space Y with the *compact-open topology* is Tychonoff. A compact-open topology is the topology on Y^X generated by the collection $\{M(A, U) : A$ is a finite subset of X and $U \in \tau\}$ as a subbase, where $M(A, U) = \{f \in Y^X : f(A) \subseteq U\}$ and τ is the topology on Y. But note that the function space Y^X with the compact-open topology need not be normal even if Y is normal and X is a two points set.

It is straightforward to show that the properties T_i, $i = 4, 5, 6$, are invariants of closed mappings, i.e., if X has a property and $f : X \to Y$ is a closed continuous and onto mapping, then Y also has the same property. But the properties T_i, $i = 2, 3, 3\frac{1}{2}$, are not invariants of closed mappings. Indeed, take a non-normal Tychonoff space X and disjoint closed sets F_0 and F_1 which cannot be separated by disjoint open sets. Identify F_i to a point x_i, $i = 0, 1$, then one can obtain a closed mapping of the Tychonoff space X to a non-T_2-space. On the other hand, the properties T_i, $i = 2, 3, 4, 5, 6$, are invariants of perfect mappings, where a perfect mapping is a closed mapping such that each point inverse is compact. But $T_{3\frac{1}{2}}$ is not an invariant of perfect mappings.

Normality vs $T_i (i \leq 3\frac{1}{2})$. First we present two well-studied non-normal Tychonoff spaces. One is the *Niemytzki plane*. This is a space constructed on the upper half of the Euclidean plane with the topology: every point with the second coordinate > 0 has the Euclidean neighborhood and every point p on the x-axis has a neighborhood of the form $\{p\} \cup D$, where D is an open disc in the upper half plane which is tangent to the x-axis at the point p. Then the Niemytzki plane is Tychonoff and *separable*. Since the x-axis is a closed discrete set of size continuum $\mathfrak{c}$, the Jone's Lemma (1937) "No separable normal space contain closed discrete subsets of size $\mathfrak{c}$" shows that the Niemytzki plane is not normal. Another one is the *Sorgenfrey square* S^2: the underlying set of the *Sorgenfrey line* S is the reals $\mathbb{R}$, S has a subbase of the form $\{[a, b) : a < b\}$, and the Sorgenfrey square is the square S^2 of the Sogenfrey line S. S is perfectly normal, hereditarily separable and hereditarily Lindelöf, and S^ω is perfect. But $\{\langle x, y\rangle \in S^2 : y = -x\} \subseteq S^2$ is closed discrete, so the Jone's Lemma shows that S^2 is not normal. On the other hand, Przymusinski [25] showed that assuming the *Martin's Axiom* (MA for short) and $\omega_1 < \mathfrak{c}$, if $X \subseteq S$ and $\omega < |X| < \mathfrak{c}$, then X^2 is normal but not CWN.

Hereafter, *we assume that spaces are regular.* There is a large literature on the normality. One reason may be its incompleteness. For example, normality does not behave like T_i's, $i \leq 3\frac{1}{2}$: (a) product spaces of normal spaces need not be normal, (b) subspaces of normal spaces need not be normal. This can be seen by considering the space $X = (\omega_1 + 1) \times \omega_1$:(a) disjoint closed sets $\{\omega_1\} \times \omega_1$ and $\{\langle\alpha, \alpha\rangle : \alpha < \omega_1\}$ of X cannot be separated by disjoint open sets, and (b) X is a subspace of a compact space $(\omega_1 + 1)^2$. But of course, every closed subspace of a normal or CWN space is normal or CWN, respectively.

The relationships between normality and *countable paracompactness* (i.e., every countable open cover has a locally finite open refinement) are interesting in the product theory. The detailed proofs of the below on normality of product spaces and the discussion of related matters are found in [KV, Ch. 17, 18].

$\omega + 1$ is homeomorphic to a convergent sequence with its limit point. So it is considered as the simplest non-discrete topological space, moreover note that it is compact and *metrizable*. In 1951, Dowker showed that $X \times (\omega + 1)$ (equivalently, $X \times I$) is normal iff X is normal and countably paracompact. And he posed a question whether every normal space is countably paracompact. This proved to be a very hard problem, and remained open some twenty years untill

Rudin produced a counterexample, thus settling it negatively. Now a normal but not countably paracomapct space is called a *Dowker space*. For more detailed discussion on Dowker spaces, we wish to refer the readers to other parts of the Encyclopedia.

Normality of subspaces of ordinal products. Both ω_1^2 and $(\omega_1+1)^2$ are countably paracompact and normal. But $(\omega_1+1)\times\omega_1$ is countably paracompact but not normal. Normality as well as other topological properies of subspaces of ordinal products are interesting subjects. For subspaces A and B of ω_1, $A\times B$ is normal iff it is countably paracompact iff A is not stationary, B is not stationary or $A\cap B$ is stationary [19]. In particular, $A\times B$ is neither normal nor countably paracompact whenever A and B are disjoint stationary sets in ω_1 (such A and B exist, see [Ku, II 6.12]). On the other hand, every subspace of ω_1^2 (more generally, ω_1^n for every $n\in\omega$) is *countably metacompact* (i.e., every countable open cover has a point-finite open refinement). Observe that countably paracompact spaces are countably metacompact and that in normal spaces, countable paracompactness and countable metacompactness are equivalent, thus normal subspaces of ω_1^2 are countably paracompact. It is unknown whether countably paracompact subspaces of ω_1^2 are normal in ZFC. Only known result is that under the additional set theoretic assumption $V=L$, the answer is "yes" [20].

Discussing infinite products of copies of ω_1 is also interesting. For example, ω_1^ω is normal but $\omega_1^{\omega_1}$ is not. Also note that ω^{ω_1} is not normal. On the other hand, ω_1^κ is *countably compact* (i.e., every countable open cover has a finite subcover) for every infinite cardinal κ. Moreover there is a subspace of ω_1^ω which is not countably metacompact. It is also unknown whether normal subspaces of ω_1^ω are countably paracompact.

Now we mention that *strong zero-dimensionality* of subspaces of ordinal products. A Tychonoff space X is strongly zero-dimensional if every completely sparated sets F_0 and F_1 are separated by a clopen set, that is, there exists a clopen set W such that $F_0\subseteq W\subseteq X\setminus F_1$. Moreover a T_1-space X is *zero-dimensional* if for each pair of a point $x\in X$ and a closed set $F\subseteq X$ with $x\notin F$, $\{x\}$ and F are separated by a clopen set, equivalently, the collection of all clopen sets forms a base for X. Observe that zero-dimensional spaces are Tychonoff and strongly zero-dimensional spaces are zero-dimensional. As is well-known, the class of zero-dimensional spaces is closed under taking arbitrary products and subspaces, but Wage [29] constructed a strongly zero-dimensional space X such that X^2 is normal but not strongly zero-dimensional. Since all subspaces of an ordinal are strongly zero-dimensional, all subspaces of the square of an ordinal are zero-dimensional. Recently Fleissner, Kemoto and Terasawa [14] have shown that for subspaces A and B of ω_1, $A\times B$ is strongly zero-dimensional, and constructed a subspace of the product space $(\omega+1)\times\mathfrak{c}$ of the ordinal spaces $\omega+1$ and $\mathfrak{c}$ which is not strongly zero-dimensional.

Normality in products. Let κ be a cardinal. A space is called *κ-paracompact* if every open cover of size $\leq\kappa$ has a locally finite open refinement, and a space is *paracompact* if it is κ-paracompact for each cardinal κ. Metrizable spaces are paracompact and paracompact spaces are CWN, but as is witnessed by $(\omega_1+1)\times\omega_1$, ω-paracompact (= countably paracompact) spaces need not be

ω-CWN (= nomal) in general. It is not difficult to show that the product space of a paracompact space with a compact space is also paracompact. In 1962, Tamano established: (1) X is paracompact iff $X \times \beta X$ is normal iff $X \times \alpha X$ is normal for every compactification αX of X, (2) X is separable metrizable iff $X \times \alpha X$ is perfectly normal for some compactification αX of X, (3) X is CWN iff $F \times \beta X$ is C^*-embedded in $X \times \beta X$ for every closed subspace F of X. Afterwards, as an analogous result of the Dowker Theorem, Morita and Kunen proved that for an infinite cardinal κ, X is normal and κ-paracompact iff $X \times (\kappa + 1)$ is normal iff $X \times I^\kappa$ is normal iff $X \times 2^\kappa$ is normal iff $X \times Y$ is normal for each compact space Y with $w(Y) = \kappa$, see [KV, Ch.18]. Since the open cover $\{[0, \alpha) : \alpha < \omega_1\}$ of ω_1 witnesses that ω_1 is not ω_1-paracompact, this result also shows the non-normality of $\omega_1 \times (\omega_1 + 1)$. Another parallel of the Morita and Kunen's result is the Alas [1] and Rudin's [26] result that X is countably paracompact κ-CWN iff $X \times A(\kappa)$ is normal iff $X \times Y$ is normal for some compact space Y with $w(Y) = \kappa$, where $A(\kappa)$ denotes the one point compactification of the discrete space of size κ. Since ω_1 is countably paracompact and ω_1-CWN, $\omega_1 \times A(\omega_1)$ is normal. Here note that both $\omega_1 + 1$ and $A(\omega_1)$ are compact.

The product space X of a normal space and a non-discrete metric space is normal iff X is countably paracompact [28]. Moreover, the product space of a perfectly normal space and a metric space is normal. Another simple result is that the product space of a countably compact normal space and a metric space is normal. In 1964, Morita charaterized the normal space X in which every product of X and an arbitrary metrizable space is normal. That is, a space is a *P-space* if for every cardinal $\kappa \geq 1$ and for every collection $\{F(s) : s \in \bigcup_{n\in\omega} \kappa^n\}$ of closed sets such that $F(s) \supseteq F(t)$ whenever $s \subseteq t$, there exists a collection $\{U(s) : s \in \bigcup_{n\in\omega} \kappa^n\}$ of open sets with $U(s) \supseteq F(s)$ for each s, and $\bigcap_{n\in\omega} F(f|n) = \emptyset$ implies $\bigcap_{n\in\omega} U(f|n) = \emptyset$ for each $f \in \kappa^\omega$. Observe that P-spaces are countably metacompact, therefore normal P-spaces are countably paracompact. Morita asked whether any one of the following is true: (1) if $X \times Y$ is normal for every metric space Y, then X is discrete, (2) if $X \times Y$ is normal for every normal P-space Y, then X is metrizable, and (3) if $X \times Y$ is normal for every countably paracompact normal space Y, then X is metrizable and σ-locally compact. Atsuji [5] showed that the answer of (1) is "yes" if there is a κ-Dowker space for each infinite cardinal κ, where a κ-Dowker space is a normal space having an open cover $\{U_\alpha : \alpha < \kappa\}$ which does not have a closed cover $\{F_\alpha : \alpha < \kappa\}$ such that $F_\alpha \subseteq U_\alpha$ for each $\alpha < \kappa$. Finally Rudin [27] solved (1) affirmatively by showing that κ-Dowker spaces do exist. Chiba, Przymusinski and Rudin [11] showed that assuming $V = L$, the answers to (2) and (3) are "yes". Finally Balogh [7] proved that the answer to (3) is "yes" in ZFC.

Normality of a product space of a normal space with the irrationals $\mathbb{P}$ has been also investigated. Historically, Michael constructed a normal space M, so called **Michael line** [E,5.1.32], whose product space with $\mathbb{P}$ is not normal. M is obtained from the reals $\mathbb{R}$ by isolating each irrational point. M is of weight $\mathfrak{c}$, hereditarily paracompact but not Lindelöf. Moreover M^n is paracompact for every $n \in \omega$ but M^ω is not normal. A **Michael space** is a Lindelöf space whose product with $\mathbb{P}$ is not normal and the **Michael problem** is: is there a Michael space in ZFC? The existence of a Michael space follows from $\omega_1 = \mathfrak{c}$ [23] or MA [2]. Lawrence [22] showed that Michael problem is equivalent to the existence of a Lindelöf space and a separable completely metrizable space with a non-normal product, and that it is not possible in ZFC to construct a Michael space of weight ω_1. Afterwards he proved that there is a ZFC example of a Lindelöf space and a completely metrizable (but not separable) space whose product is non-normal and weight ω_1.

Spaces considered here are assumed to have at least two points and κ is an infinite cardinal. It follows from the non-normality of ω^{ω_1} that if an infinite product space $X = \prod_{\alpha<\kappa} X_\alpha$ is normal, then all X_α's, except for at most countably many, are countably compact. Thus if an infinite product space $X = \prod_{\alpha<\kappa} X_\alpha$ of metric spaces is normal, then all X_α's, except for at most countably many, are compact. So in this sense, countable product spaces are essential for discussing infinite products. Zenor and Nagami established that if all finite subproduct of a product space $X = \prod_{n\in\omega} X_n$ are normal (i.e., $\prod_{n\in F} X_n$ is normal for each finite subset $F \subseteq \omega$), then X is normal iff it is countably paracompact. Aoki [3] proved that if a product space $X = \prod_{\alpha<\kappa} X_\alpha$ is κ-paracompact, then X is normal iff all finite subproducts of X are normal. Moreover Bešlagiċ [9] proved that if $X = \prod_{\alpha<\kappa} X_\alpha$ is normal, then it is κ-paracompact. Thus these results extend the Zenor and Nagami's one to uncountable products.

Collectionwise normality. Normal spaces are ω-CWN, but Bing constructed an example of a normal not ω_1-CWN space. One of good articles on collectionwise normality is Tall's one [KV, Ch.15]. Moreover good articles on covering property and metrization theory are [KV, Ch.9] and [KV, Ch.16]. Most of results below are found in these articles.

Collectionwise normality has been studied in connection with metrization theorems. Historically, Jones conjectured that normal **Moore** spaces are metrizable, where a space is Moore if it a has a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers such that for each point x, $\{\bigcup\{G \in \mathcal{G}_n : x \in G\} : n \in \omega\}$ forms a neighborhood base at x. This conjecture was called the *Normal Moore Space Conjecture* (NMSC for short). Observe that Moore spaces are first countable and *subparacompact* (i.e., every open cover has a σ-discrete closed refinement), and that metrizable spaces are paracompact, hence CWN, and Moore. Assuming $2^\omega < 2^{\omega_1}$, Jones showed in 1937 that separable normal spaces have no uncountable closed discrete subspaces, and as a corollary, that separable normal Moore spaces are metrizable. Afterwards, Bing established that a space is metrizable iff it is a CWN Moore space. Therefore the problem on the NMSC was focused on the difference between normality and collectionwise normality of Moore spaces. In 1964, Heath showed that there is a separable normal non-metrizable Moore space iff there exists an uncountable **Q-set** in the reals $\mathbb{R}$, where $E \subset \mathbb{R}$ is a Q-set if every subset $E' \subset E$ is G_δ in E. The "if" part of the Heath's result was shown by considering a subspace of the Niemytzki plane. Around 1967-1968, Silver and Tall proved that MA + $\omega_1 < \mathfrak{c}$ yields a non-metrizable separable normal Moore space. So the existence of a separable normal non-metrizable

Moore space is consitent with and independent of ZFC. For non-separable case, the situation was more complicated. Nyikos [24] proved that the Product Measure Extension Axiom (PMEA for short, i.e., for every cardinal κ, the product measure on 2^κ extends to a $\mathfrak{c}$-additive full measure) implies that normal first countable spaces are CWN, therefore normal Moore spaces are metrizable. It is known that the consistency of PMEA follows from the existence of a strongly compact cardinal and that the consistency of PMEA implies the existence of a measurable cardinal. A remarkable result of this line is, by Fleissner [13], that the statement "normal Moore spaces are metrizable" implies the consistency of the existence of a measurable cardinal. In a sense, this completes the NMSC problem. An interesting open problem is whether NMSC implies the statement "normal first countable spaces are CWN".

The problem whether normal locally compact spaces are collectionwise normal is also interesting. Arhangelskiĭ [4] showed that locally compact *metacompact* (i.e., every open cover has an point-finite open refinement) perfectly normal spaces are paracompact. Paracompact spaces are subparacompact and metacompact, moreover subparacompact spaces and also metacompact spaces are *submetacompact* (i.e., for every open cover $\mathcal{U}$, there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open refinements of $\mathcal{U}$ such that for every point x, there is an $n \in \omega$, such that $\{U \in \mathcal{U}_n : x \in U\}$ is finite). On the other hand, CWN submetacompact spaces are paracompact. Then Watson [30] proved that $V = L$ implies that locally compact normal sapces are collectionwise normal with respect to compact sets (i.e., every discrete collection of compact sets are separated by disjoint open sets), hence locally compact submetacompact normal spaces are paracompact. A best possible ZFC result might be that locally compact submetacompact normal spaces are subparacompact [18]. Tall asked if there exists a locally compact normal non-CWN space under various additional set theoretical or topological assumptions. Then Daniels and Gruenhage [12] constructed in the constructible universe L, a locally compact collectionwise Hausdorff perfectly normal non-CWN space. Afterwards, Balogh [6] gave an answer that in a model adding super compact many random or Cohen reals to a model of ZFC, locally compact normal spaces are CWN.

Hereditary normality ($= T_5$) and perfect normality ($= T_6$). Metrizable spaces are perfectly normal, perfectly normal spaces are hereditarily normal and there is a hereditarily normal but not perfectly normal space. Evidently subspaces of a hereditarily normal space are hereditarily normal, and it is not difficult to show that subspaces of a perfectly normal space are also perfectly normal. The Sorgenfrey square S^2 witnesses that these properties are not productive. Characterizations of these properties are well-known: (1) a space X is hereditarily normal iff subsets F_0 and F_1 with $\mathrm{Cl} F_0 \cap F_1 = F_0 \cap \mathrm{Cl} F_1 = \emptyset$ are separated by disjoint open sets, (2) a space X is perfectly normal iff every closed set is a zero-set.

If the product space $X \times Y$ is hereditarily normal, then the factor spaces X and Y have stronger properties, that is, either X is perfectly normal or countable subsets of Y are closed discrete. This is due to Katětov [17]. In case that both X and Y are compact, this means that both X and Y are perfectly normal. Another result of Katětov is: if X is compact and X^3 is hereditarily normal, then X is metrizable. Related to this, it is known that, if X is compact and the diagonal $\{\langle x, x\rangle \in X \times X : x \in X\}$ is G_δ in X^2, then X is metrizable. In particular, a compact space X is metrizable whenever X^2 is perfectly normal. Chaber proved that if X is countably compact and X^2 is perfectly normal, then X is compact hence metrizable. So one can ask whether, if X^2 is hereditarily normal and X is compact, X is metrizable, and whether, if X^2 is hereditarily normal and X is countably compact, X is compact. The former was first asked by Katětov. For the latter, Bešlagić [10] constructed, assuming $\diamondsuit$, a countably compact non-compact space such that X^2 is hereditarily normal. For the Katětov's problem, Nyikos and Gruenhage constructed a compact non-metrizable space such that X^2 is hereditarily normal assuming MA + $\omega_1 < \mathfrak{c}$, or $\omega_1 = \mathfrak{c}$ [15]. Recently Larson and Todorocevic [21] has shown a consistently affirmative answer to the the Katětov's problem. An interesrting open problem is: if the product space $X \times Y$ of compact spaces X and Y is perfectly normal, then is at least one of X and Y metrizable?

Non-normality of ω^{ω_1} yields that the product space of uncountably many spaces having at least two points is not hereditarily normal. For perfect normality or hereditary normality of countable product spaces, the following are known: (1) a countable product space is perfect iff its all finite subproducts are perfect, (2) a countable product space is hereditarily normal iff it is perfectly normal iff it is hereditarily countably compact.

Separation axioms of hyperspaces. In this paragraph, we assume no separation axiom. For a topological space X, 2^X denotes the collection of all non-empty closed subsets of X. The Vietoris topology τ_V on 2^X is generated by the collection $\{V(U_0, ..., U_{n-1}) : U_i\text{'s are open in } X\}$ as a subbase, where $V(U_0, ..., U_{n-1}) = \{F \in 2^X : F \subseteq \bigcup_{i<n} U_i \text{ and, } F \cap U_i \neq \emptyset \text{ for each } i < n\}$. The Fell topology τ_F on 2^X is generated by the collection $\{W(U, K) : U \text{ is open in } X \text{ and } K \subseteq X \text{ is compact}\}$ as a subbase, where $W(U, K) = \{F \in 2^X : F \cap U \neq \emptyset, F \cap K = \emptyset\}$. It follows from $W(U, K) = V(U, X \setminus K)$ that $\tau_F \subseteq \tau_V$. For the hyperspace 2^X with the Vietoris topology: (1) if X is T_1, then 2^X is T_1, (2) for T_1-space X, 2^X is T_2 iff X is regular, (3) for T_1-space X, 2^X is regular iff X is normal, (4) for T_1-space X, 2^X is normal iff X is compact T_2, see [E]. For the hyperspace 2^X with the Fell topology: (1) 2^X is regular iff 2^X is Tychonoff iff X is locally compact [8], (2) 2^X is normal iff 2^X is paracompact iff 2^X is Lindelöf iff X is locally compact Lindelöf [16],

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