Strong Zero-dimensionality of Products of Ordinals

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Abstract

We show that the product of finitely many subspaces of ordinals is strongly zero-dimensional. In contrast, for each natural number n, there is a subspace of $(\omega+1)\times\mathfrak{c}$ of dimension n.

1 Introduction

All spaces are assumed to be completely regular and T_1.

A space X is said to be *zero-dimensional* if it has a base of clopen sets. A space X is said to be *strongly zero-dimensional* if for every disjoint pair of zero-sets Z_0 and Z_1, there is a clopen set W with $Z_0 \subset W \subset X \setminus Z_1$. In this situation, we say that Z_0 and Z_1 are separated by a clopen set. It is well-known that a space X is strongly zero-dimensional if and only if βX is zero-dimensional, see [3, 7.1.17]. It is straightforward to verify that a space X is normal and strongly zero-dimensional iff every pair of disjoint *closed* sets of X are separated by a clopen set.

In [7], it was proved that for every subspace of the product space of two ordinals, normality, collectionwise normality, and the shrinking property are equivalent. While extending this equivalence to subspaces X of the product of finitely many ordinals, the first author [5] found it convenient to first prove that if X is normal, then X is strongly zero-dimensional. Moreover, it was shown earlier (see [8]) that $X \times Y$ is not normal when X and Y are disjoint

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stationary sets in ω_1. So it is natural to ask if this $X \times Y$ is strongly zero-dimensional. More generally, since all subspaces of product spaces of ordinals are zero-dimensional, it is also natural to ask if such subspaces are strongly zero-dimensional.

We answer all these questions in the present paper.

First we generalize the notion of stationary sets in Section 2, and show a Generalized Pressing Down Lemma (Theorem 3.2) in Section 3. One corollary is that if κ_i, $i < n$, is an n-tuple of distinct, regular, uncountable cardinals, then every continuous function $\varphi : \prod_{i<n} \kappa_i \to \mathbb{R}$ is constant on a final segment. Example 3.9 shows that this result is not true when the κ_i's are not distinct. In Theorem 4.2, we show that after a small clopen set is deleted from the domain, φ has finite range. ("Small" is defined precisely in Definition 4.1).

Using Theorem 4.2, we prove that the product of finitely many subspaces of ordinals is strongly zero-dimensional (Theorem 5.1), thus answering the first question above in the affirmative.

In Section 6, however, we present a negative solution to the second question. Namely, subspaces of the product of finitely many subspaces of ordinals are not necessarily strongly zero-dimensional. More precisely, we prove that for every natural number n, there is a subspace K of $(\omega+1) \times \mathfrak{c}$ such that $\dim K = n$ (Theorem 6.9). An important step in proving that theorem is to establish that for every maximal almost disjoint family $\mathcal{R}$ of subsets of ω, $\beta\Psi(\mathcal{R})$ is embedded in the remainder of such a subspace K (Theorem 6.1). Here $\Psi(\mathcal{R})$ is a so-called Ψ-space generated by $\mathcal{R}$ (see [3, 3.6.I] or [6, 5.I]). Section 6 can be read independently of other sections.

2 Generalized Stationary Sets

We will use set theoretical notation described in [10, Chapter I]. For example, 0 denotes the empty set, an ordinal is the set of smaller ordinals, thus $n = \{0, 1, ..., n-1\}$ for each natural number n.

For an n-tuple $t = \langle t_0, \ldots, t_{n-1} \rangle$ and an n'-tuple $t' = \langle t'_0, \ldots, t'_{n'-1} \rangle$, $t^\frown t'$ denotes the $(n+n')$-tuple $s = \langle s_0, \ldots, s_{n+n'-1} \rangle$, where $s_i = t_i$ for $i < n$ and $s_{n+i} = t'_i$ for $i < n'$. The 0-tuple is considered as the empty sequence $0 = \emptyset$ as usual. For an n-tuple $t = \langle t_0, \ldots, t_{n-1} \rangle$ of subsets $t_0, \ldots, t_{n-1}$ of ordinals, $\prod t$ denotes the usual product $t_0 \times \cdots \times t_{n-1}$ and $\nabla t = \{x \in \prod t : x_0 < \cdots < x_{n-1}\}$ its subspace.

For $s \subset n$, $t \restriction s$ denotes the sub-tuple $\langle t_i : i \in s \rangle$ of t. For $A \subset \prod t$, $A \restriction s$ denotes the set $\{x \restriction s : x \in A\}$. Note $A \restriction 0 = \{0\}$ if $A \neq \emptyset$. For $m \leq n$ and $x \in \prod_{i<m} t_i$, $A[x]$ denotes the set $\{y \in \prod_{m \leq i<n} t_i : x^\frown y \in A\}$.

Observe that $A[x] = A$ if $m = 0$ and $A[x] = \{0\}$ if $m = n$ and $x \in A$. When $m = 1$ and $\alpha \in t_0$, we write $A[\alpha]$ instead of $A[\langle\alpha\rangle]$.

For $s \subset n$, $x \in \prod_{i \in s} t_i$, $A \subset \prod t$ and $j \notin s$, we let

$$\pi_j^x[A] = \{a_j : a \in A \text{ and } a \restriction s = x\}.$$

When $s = 0$, this is the usual projection $\pi_j[A]$ of A to the t_j-axis.

Let $x = \langle x_0, \ldots, x_{n-1}\rangle$, $y = \langle y_0, \ldots, y_{n-1}\rangle$ be n-tuples of ordinals. If $x_i < y_i$ for each $i < n$, then we write $x < y$. We let $x \leq y$ have the analogous meaning. The generalized intervals $(x, y) = \prod_{i<n}(x_i, y_i)$ and $(x, y] = \prod_{i<n}(x_i, y_i]$ should be understood in terms of these orders. In Sections 4 and 5 we will write $x \prec y$ when $x \leq y$ and $x \neq y$. All these relations are well-founded on the class of all n-tuples of ordinals in the sense of [10, III Definition 5.1].

For a subset S of an ordinal μ, let $\operatorname{Lim}_\mu(S) = \{\gamma < \mu : \sup(S \cap \gamma) = \gamma\}$, in other words, $\operatorname{Lim}_\mu(S)$ is the closed set of all cluster points of S in the space μ. We will also use the symbol $\operatorname{Succ}_\mu(S) = S \setminus \operatorname{Lim}_\mu(S)$. When the situation is clear in its context, we simply write $\operatorname{Lim} S$ or $\operatorname{Succ} S$ instead of $\operatorname{Lim}_\mu(S)$ or $\operatorname{Succ}_\mu(S)$, respectively. Observe that if $\operatorname{cf} \mu \geq \omega_1$ and S is unbounded in μ, then $\operatorname{Lim} S$ is cub (i.e., closed and unbounded) in μ.

Let C_α, $\alpha \in A \subset \kappa$, be cub sets of an uncountable regular cardinal κ. Its *diagonal intersection* is defined by

$$\triangle_{\alpha \in A} C_\alpha = \big\{\beta \in \kappa : (\forall \alpha \in A \cap \beta)(\beta \in C_\alpha)\big\}.$$

Then $\triangle_{\alpha \in A} C_\alpha$ is a cub set in κ (see [10, II Lemma 6.14]).

As usual (see [10, II Definition 6.9]), a subset Y of an uncountable regular cardinal κ is called *stationary* (or *κ-stationary*) iff it meets every cub subset C of κ. The question arises how we should define *κ-stationary set* when $\kappa = \langle \kappa_0, \ldots, \kappa_{n-1}\rangle$ is not just a cardinal but a finite-tuple of non-decreasing uncountable regular cardinals. There are two ways to do this, namely,

- ($\prod$-type stationary) Y meets every $\prod C$,
- (∇-type stationary) Y meets every ∇C,

where C is an n-tuple of cub sets C_i of κ_i. When the κ is strictly increasing, the two notions are equivalent (different filter bases generate the same filter) and have a satisfactory theory. When $\kappa_i = \kappa_{i+1}$ for some i, however, the notions are not equivalent. The prototypic result, "an open stationary set contains a final segment", has a useful generalization (Theorem 3.5) for the notion ∇-type stationary. In contrast, there can be disjoint open $\prod$-type stationary sets – this is the essential idea of Example 3.9.

In the present paper we will develop the theory of ∇-type stationary sets.

For expository reasons, we prefer to start with concepts equivalent to ∇-type stationarity. However, we soon prove (Proposition 2.4) that Y is κ-stationary iff $Y \cap \nabla C_i \neq \emptyset$ for every n-tuple C with C_i a cub subset of κ_i. Here is our official definition.

Definition 2.1. Let $\kappa = \langle \kappa_0, \kappa_1, \ldots, \kappa_{n-1} \rangle$ be an n-tuple of non-decreasing uncountable regular cardinals.

$Y \subset \prod \kappa$ is called κ*-stationary* if there is $Z \subset Y$ such that, for all $z \in Z$ and $i < n$, the set $\pi_i^{z \restriction i}[Z] = \pi_i^{\langle z_0, \ldots, z_{i-1} \rangle}[Z]$ is κ_i-stationary.

We call the set Z in the above *pruned.* Z is obviously itself κ-stationary. Note that, if κ is an uncountable regular cardinal, "$\langle \kappa \rangle$-stationary" and "κ-stationary" are synonymous.

In the discussion of κ-stationary sets, it is often useful to use induction on the length of the tuple κ.

Proposition 2.2. *For an* n*-tuple* $\kappa = \langle \kappa_0, \kappa_1, \ldots, \kappa_{n-1} \rangle$ *and* $Y \subset \prod \kappa$*, the following are equivalent.*

(1) Y *is* κ*-stationary,*

(2) there are a κ_0*-stationary set* K *and, for each* $\gamma \in K$*,* $\langle \kappa_1, \ldots, \kappa_{n-1} \rangle$*-stationary set* L_γ *such that* $\{\gamma\} \times L_\gamma \subset Y$ *for each* $\gamma \in K$*,*

(3) there are a $\langle \kappa_0, \ldots, \kappa_{n-2} \rangle$*-stationary set* S *and, for each* $s \in S$*, a* κ_{n-1}*-stationary set* T_s *such that* $\{s\} \times T_s \subset Y$ *for each* $s \in S$*.*

Proof. We show the equivalence of (1) and (2). The equivalence of (1) and (3) is seen quite similarly.

We proceed by induction and suppose that (1) and (2) are shown to be equivalent for κ of length $\leq (n-1)$.

Let κ be of length n, and suppose Y is κ-stationary. Let $Z \subset Y$ be pruned and $K = \pi_0[Z]$. Then K is κ_0-stationary. For each $\gamma \in K$, let $L_\gamma = Z[\gamma]$. Then, for each $\zeta \in L_\gamma$ and $0 < i < n$, $\pi_i^{\langle \zeta_1, \ldots, \zeta_{i-1} \rangle}[L_\gamma] = \pi_i^{\langle \gamma, \zeta_1, \ldots, \zeta_{i-1} \rangle}[Z] = \pi_i^{(\{\gamma\}^\frown \zeta) \restriction i}[Z]$ is κ_i-stationary. Hence, by the definition, each L_γ is $\langle \kappa_1, \ldots, \kappa_{n-1} \rangle$-stationary.

Suppose that (2) holds. Then, by induction hypothesis, there is a set $Z_\gamma \subset L_\gamma$ so that, for each $z \in Z_\gamma$ and $0 < i < n$, $\pi_i^{\langle z_1, \ldots, z_{i-1} \rangle}[Z_\gamma]$ is κ_i-stationary. This set is identical to $\pi_i^{\langle \gamma, z_1, \ldots, z_{i-1} \rangle}[Z]$ where $Z = \bigcup_{\gamma \in K} \{\gamma\} \times Z_\gamma$. Obviously $Z \subset Y$ holds and hence, the induction is complete. $\square$

For convenience we will call singletons 0-stationary for the 0-tuple.

Proposition 2.3. *Let $\kappa = \langle \kappa_0, \ldots, \kappa_{n-1} \rangle$ be an n-tuple and Y κ-stationary.*

(1) If $Y \subset X$, then X is also κ-stationary.

(2) If $Y = \bigcup_{\alpha<\lambda} Z_\alpha$ and $\lambda < \kappa_0$, then some Z_α is κ-stationary.

(3) If C_i is cub in κ_i for each $i < n$, then $Y \cap \prod_{i<n} C_i$ is also κ-stationary. In particular, if $s < \kappa$ is an n-tuple, then $Y \cap (s, \kappa)$ is also κ-stationary.

(4) $Y \cap \nabla\kappa$ is κ-stationary.

Proof. We prove this by induction on n. Suppose this is true for i-tuples for all $i < n$. Let $\kappa' = \langle \kappa_1, \ldots, \kappa_{n-1} \rangle$, and take K and L_γ as in Proposition 2.2 (2). Then K is κ_0-stationary and each L_γ is κ'-stationary.

(1) Obvious from the definition.

(2) For each $\gamma \in K$, let $Z_{\alpha,\gamma} = Z_\alpha[\gamma]$. Then $L_\gamma \subset \bigcup_\alpha Z_{\alpha,\gamma}$ and hence, by induction hypothesis, $Z_{\alpha,\gamma}$ is κ'-stationary for some $\alpha = \alpha(\gamma)$. Since $K = \bigcup_{\alpha<\lambda} \{\gamma : \alpha(\gamma) = \alpha\}$ is κ_0-stationary and $\lambda < \kappa_0$, there is a δ so that $\{\gamma : \alpha(\gamma) = \delta\} = H$ is κ_0-stationary ([10, II Lemma 6.8]). Then, by definition, $Z_\delta \supset \bigcup_{\gamma\in H} \{\gamma\} \times Z_{\delta,\gamma}$ is κ-stationary.

(3) For each $\gamma \in K$, $L_\gamma \cap \prod_{0<i<n} C_i$ is κ'-stationary, by induction hypothesis, and $\{\gamma\} \times (L_\gamma \cap \prod_{i<n} C_i) \subset Y \cap \prod_{i<n} C_i$ holds.

(4) By induction hypothesis, for each $\gamma \in K$, there is a κ'-stationary set $Y_\gamma \subset L_\gamma$ such that $t_1 < \cdots < t_{n-1}$ for each $t = \langle t_1, \ldots, t_{n-1} \rangle \in Y_\gamma$. By (3), $Z_\gamma = Y_\gamma \cap (\gamma, \kappa_1) \times \kappa_2 \times \cdots \times \kappa_{n-1}$ is κ'-stationary. Then $Z = \bigcup_{\gamma\in K} \{\gamma\} \times Z_\gamma$ is κ-stationary and contained in Y. □

We have developed enough machinery to prove that official definition of κ-stationary is equivalent to the motivating notion, ∇-type stationary.

Proposition 2.4. *Y is κ-stationary iff $Y \cap \nabla C \neq \emptyset$ for every n-tuple C with C_i a cub subset of κ_i. Therefore the collection of all non-κ-stationary subsets of $\prod \kappa$ forms a σ-complete ideal.*

Proof. It suffices to show only the sufficiency part, the necessity part being included in Proposition 2.3.

Assume the sufficiency part for i-tuples for all $i < n$ and let $\kappa' = \langle \kappa_1, \ldots, \kappa_{n-1} \rangle$.

Suppose that Y is not κ-stationary. For each $\alpha \in \pi_0[Y]$, let us consider the subset $L_\alpha = Y[\alpha]$, and let $K = \{\alpha : L_\alpha \text{ is } \kappa'\text{-stationary}\}$. Since K is not κ_0-stationary by Proposition 2.2 (2), there is a cub set C_0 disjoint from K. For each $\alpha \in C_0$, L_α is not κ'-stationary. Then, by induction hypothesis, there is a cub set $C_{\alpha,i} \subset \kappa_i$ for each $0 < i < n$ such that $L_\alpha \cap \nabla_{0<i<n} C_{\alpha,i} = \emptyset$. Let $\kappa_0 = \cdots = \kappa_{m-1} < \kappa_m$. Then define $C_i = \triangle_{\alpha \in C_0} C_{\alpha,i}$ for $0 < i < m$, and $C_i = \bigcap_{\alpha \in C_0} C_{\alpha,i}$ for $m \le i < n$. Obviously each C_i is cub in κ_i.

To show $Y \cap \nabla_{i<n} C_i = \emptyset$, suppose $t = \langle t_0, \ldots, t_{n-1} \rangle \in Y \cap \nabla_{i<n} C_i$. Then $\langle t_1, \ldots, t_{n-1} \rangle \in L_{t_0} \cap \nabla_{0<i<n} C_i$. Thus $t_0 < t_i \in C_i = \triangle_{\alpha \in C_0} C_{\alpha,i}$ for $0 < i < m$, and $t_0 < t_i \in C_i = \bigcap_{\alpha \in C_0} C_{\alpha,i}$ for $m \le i < n$. This implies that $t_i \in C_{t_0,i}$ for $0 < i < n$, and hence $\langle t_1, \ldots, t_{n-1} \rangle \in L_{t_0} \cap \nabla_{0<i<n} C_{t_0,i}$, a contradiction. Thus $Y \cap \nabla_{i<n} C_i = \emptyset$. □

Corollary 2.5. *Let $\kappa = \langle \kappa_0, \ldots, \kappa_{n-1} \rangle$ be an n-tuple and $A_i \subset \kappa_i$ for each $i < n$. Then $Y = \prod_{i<n} A_i$ is κ-stationary iff each A_i is κ_i-stationary.*

3 Generalized Pressing Down Lemma

The usual Pressing Down Lemma [10, II Lemma 6.15] says that a function $f : S \to \kappa$ defined on a stationary subset S of an uncountable regular cardinal κ is constant on a stationary subset of S if $f(\alpha) < \alpha$ for each α. We now generalize this.

Definition 3.1. Let $\alpha = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ be an n-tuple of ordinals, and suppose that a function f sends $x \in \prod \alpha$ to $f(x) \in \prod \alpha$.

We call f *regressive* if $f(x) < x$ for all $x \in \operatorname{dom} f$, and a *stem function* if $f(x)_j = f(x')_j$ whenever $x \upharpoonright j = x' \upharpoonright j$.

Observe that if f is a stem function then $f(x)_0$ is constant, and that a stem function defined on a set of 1-tuples is constant. Hence the $n = 1$ case of the following theorem is the Pressing Down Lemma.

Theorem 3.2 (Generalized Pressing Down Lemma). *Let κ be an n-tuple, R a κ-stationary subset of $\prod \kappa$, and $f : R \to \prod \kappa$ regressive. Then there is a κ-stationary subset Y of R so that f restricted to Y is a stem function.*

Proof. Assume this theorem for i-tuples for all $i < n$, and let us consider $\kappa = \langle \kappa_0, \ldots, \kappa_{n-1} \rangle$. Let $\kappa' = \kappa \upharpoonright n - 1$.

By Proposition 2.2 (3), there are a κ'-stationary subset S, and κ_{n-1}-stationary sets T_s, $s \in S$, so that $\{s\} \times T_s \subset R$.

For each $s \in S$ and each $\tau \in T_s$, note that the point $f(s^\frown\tau)$ consists of the first $(n-1)$ coordinates $f_1(s^\frown\tau)$ and the last coordinate $f_2(s^\frown\tau)$. We have $f_1(s^\frown\tau) < s$ and $f_2(s^\frown\tau) < \tau$. Since $|S| \le \kappa_{n-2}$, the set of all $f_1(s^\frown\tau)$ has cardinality $< \kappa_{n-1}$. By Proposition 2.3(2) and the Pressing Down Lemma applied to T_s, there are a stationary subset Y_s of T_s, $g(s) \in$

$\prod \kappa'$ and $\gamma_s \in \kappa_{n-1}$ such that $f_1(s^\frown\tau) = g(s)$ and $f_2(s^\frown\tau) = \gamma_s$, that is, $f(s^\frown\tau) = g(s)^\frown\gamma_s$ for all $s \in S$ and $\tau \in Y_s$.

Apply the induction hypothesis to the regressive function g to get a stationary subset Y' of S so that g restricted to Y' is a stem function. Let $Y = \bigcup_{s\in Y'}\{s\} \times Y_s$. To verify that f restricted to Y is a stem function, let z, $z' \in Y$. If $z \restriction j = z' \restriction j$ and $j < n-1$, then $f(z)_j = g(z \restriction n-1)_j = g(z' \restriction n-1)_j = f(z')_j$. If $z \restriction n-1 = z' \restriction n-1 = s$, then $f(z)_{n-1} = \gamma_s = f(z')_{n-1}$ holds. □

A consequence of the Pressing Down Lemma is that a real-valued continuous function on a stationary subset of a regular uncountable cardinal is constant on its tail (= its intersection with a final segment). We can generalize this result for a non-decreasing n-tuple of regular uncountable cardinals (Theorem 3.7).

We begin with definitions.

Definition 3.3. Let $\kappa = \langle\kappa_0, \dots, \kappa_{n-1}\rangle$ be an n-tuple. Let us say that an n-tuple $C = \langle C_0, \dots, C_{n-1}\rangle$ of cub sets C_j of κ_j is *attuned to* κ, or simply, κ*-attuned*, if the following holds:

(1) $C_j \subset \operatorname{Lim} \kappa_j$ for all $j < n$,

(2) if $\kappa_j < \kappa_{j+1}$, then $C_{j+1} \subset (\kappa_j, \kappa_{j+1})$,

(3) if $\kappa_j = \kappa_{j+1}$, then $C_j = C_{j+1}$.

Note that every n-tuple $\langle D_0, \dots, D_{n-1}\rangle$ of cub sets can be attuned to κ, that is, there is a κ-attuned tuple $\langle C_0, \dots, C_{n-1}\rangle$ such that $C_j \subset D_j$ for each j. In fact, when $\kappa_{\ell-1} < \kappa_\ell = \cdots = \kappa_m < \kappa_{m+1}$, let $C_\ell = \cdots = C_m = \bigcap_{\ell\le j\le m} D_j \cap (\kappa_{\ell-1}, \kappa_\ell) \cap \operatorname{Lim} \kappa_\ell$.

Definition 3.4. We say that an n-tuple x is *entwined with* another n-tuple c if

$$c_0 < x_0 < c_1 < \cdots < c_j < x_j < c_{j+1} < \cdots < c_{n-1} < x_{n-1}.$$

Let κ be a non-decreasing n-tuple of uncountable regular cardinals, and C be an attuned n-tuple of cubs. Then we let $E(C)$ denote the collection of all $x \in \nabla\kappa$ which are entwined with some $c \in \prod C$.

Observe that the set of x which are entwined with a specific c is an open set. Hence $E(C)$ is an open set. Further observe that, if $\kappa_0 < \cdots < \kappa_{n-1}$, then $x \in E(C)$ iff $\min C_j < x_j$ for all j, that is, $E(C) = (s, \kappa)$ is a final segment where $s = \min(\prod C)$ denotes the minimum of the set $\prod C$ in the sense of the order $\le$.

It is easily seen that the set $E(C)$ is never empty. More precisely, by taking $D_j = \operatorname{Lim} C_j$ for each j, we have $E(C) \supset \nabla D = \prod D \cap \nabla\kappa$. Thus by Proposition 2.3, for a κ-stationary set Y, $Y \cap E(C)$ is always κ-stationary.

Theorem 3.5. *Let κ be an n-tuple and U an open κ-stationary subset of $\prod\kappa$. Then there is an attuned n-tuple C of cub sets, so that $E(C)$ is contained in U.*

Proof. Let $f : U \to \prod\kappa$ be regressive so that for each $u \in U$, the half-open interval $(f(u), u] \subset U$. By our definition and Theorem 3.2, there is a pruned stationary subset Y of U so that $f \restriction Y$ is a stem function. For each $j < n$, let D_j be the set of $\gamma < \kappa_j$ satisfying: if $y \in Y$ and $y_0, \ldots, y_{j-1} < \gamma$, then

(1) $f(y)_j < \gamma$,

(2) $\operatorname{Lim}\big(\pi_j^{y\restriction j}[Y]\big) \ni \gamma$.

Note that, because $f \restriction Y$ is a stem function, to know $f(y)_j$ it suffices to know $y \restriction j$; in particular, we know the constant value $f(y)_0$ at the start. Also note that each $\operatorname{Lim}\big(\pi_j^{y\restriction j}[Y]\big)$ is cub. Then D_j is a cub set of κ_j (see, e.g., the proof of [10, II Lemma 6.13]). Let C be attuned to κ with $D_j \supset C_j$ for each $j < n$.

To verify the conclusion, let $x \in \prod\kappa$ be entwined with $c \in \prod C$. By induction on $j < n$, we shall define y_j and verify that

$$f(y)_j < c_j < x_j < y_j < c_{j+1}.$$

Let y_j be the least element of $\pi_j^{\langle y_0,\ldots,y_{j-1}\rangle}[Y]$ greater than x_j. This is possible because $\pi_j^{\langle y_0,\ldots,y_{j-1}\rangle}[Y] = \pi_j^{z\restriction j}[Y]$ for any $z \in Y$ with $z \restriction j = \langle y_0, \ldots, y_{j-1}\rangle$, and is κ_j-stationary.

We verify the inequalities left to right. First, $f(y)_j < c_j$ because of (1). Second, $c_j < x_j$ because x is entwined with c. Third, $x_j < y_j$ by our choice of y_j. Finally, $y_j < c_{j+1}$ is seen in the following way. It is obvious if $\kappa_j < \kappa_{j+1}$. If $\kappa_j = \kappa_{j+1}$, then (2) implies this because $c_{j+1} \in C_{j+1} = C_j \subset D_j$. Thus, we have verified that $x \in (f(y), y] \subset U$, as required. □

Corollary 3.6. *Let κ be a strictly increasing n-tuple and U an open κ-stationary subset of $\prod\kappa$. Then there is an $s \in \nabla\kappa$ so that the final segment (s, κ) is contained in U.*

Theorem 3.7. *If κ is an n-tuple, and $\varphi : Y \to \mathbb{R}$ is a continuous function defined on a κ-stationary set Y, then there is a κ-attuned n-tuple C so that φ is constant on $E(C) \cap Y$.*

Proof. For each $i \in \omega$, $\mathbb{R}$ is covered by countably many open sets $B(i,k)$, $k \in \omega$, of diameter $\leq 1/(i+1)$. By Proposition 2.3, for each i, there is a k_i such that $\varphi^{\leftarrow}[B(i,k_i)]$ is κ-stationary. Let $U(i)$ be an open set of $\prod \kappa$ in which $\varphi^{\leftarrow}[B(i,k_i)] = U(i) \cap Y$. Obviously, $U(i)$ is κ-stationary, and by Theorem 3.5, there is a κ-attuned n-tuple $C(i)$ of cub sets such that $E(C(i)) \subset U(i)$.

Define $C_j = \bigcap_i C(i)_j$. Then $C = \langle C_0, \ldots, C_{n-1} \rangle$ is attuned to κ and $E(C) \subset \bigcap_i E(C(i)) \subset \bigcap_i U(i)$. Thus we have $E(C) \cap Y \subset \bigcap_i \varphi^{\leftarrow}[B(i,k_i)] = \varphi^{\leftarrow}\left[\bigcap_i B(i,k_i)\right]$. Since $E(C) \cap Y$ is κ-stationary and hence non-empty as we have noted above, $\bigcap_i B(i,k_i)$ is a singleton. This means that $\varphi \restriction E(C) \cap Y$ is constant. □

In case the tuple κ is strictly increasing, we have

Corollary 3.8. *If κ is a strictly increasing n-tuple and $\varphi : Y \to \mathbb{R}$ is a continuous function defined on a κ-stationary set Y, then there is an $s \in \nabla\kappa$ so that φ is constant on the final segment $(s, \kappa) \cap Y$ of Y.*

As Proposition 2.3 (4) shows, the essential part of κ-stationary set lies in its intersection with $\nabla\kappa$. The above set $E(C)$ also lies in $\nabla\kappa$. In particular, if κ is strictly increasing, $E(C)$ is a final segment itself and its complement is seen to be small (i.e., related to smaller cardinals). If $\kappa_i = \kappa_{i+1}$ for some $i < n$, however, the complement is not small enough and we must partition $\prod \kappa$. The partition is suggested by the following two examples. We will develop the idea of partitioning in the next section. (The idea of partitioning ω_1^n appears in [9], which also contains the equivalence of "inductively" stationary and ∇-type stationary for $\kappa = \langle \omega_1, \ldots, \omega_1 \rangle$).

Example 3.9. Let $X = A_0 \times A_1$, where each A_i is stationary in ω_1 and $A_0 \cap A_1 = \{\xi + 1 : \xi \in \omega_1\}$, call it N. Let $\hat{\varphi} : N \to \mathbb{R}$ have uncountable range. Define $\varphi : X \to \mathbb{R}$ by cases: $\varphi(x_0, x_1) = 0$ if $x_0 < x_1$; $\varphi(x_0, x_1) = 1$ if $x_0 > x_1$; $\varphi(x_0, x_1) = \hat{\varphi}(\xi + 1)$ if $x_0 = x_1 = \xi + 1$. Now φ is continuous, but is not constant on any final segment. That is, the conclusion of Corollary 3.8 fails for φ. Theorem 4.2 will give more information on this; we must be able to discard the diagonal from a final segment and be satisfied with a finite range.

Here is a space on which every real-valued continuous function is constant on a final segment. The technique of applying the Pressing Down Lemma on a subset of our space to obtain a a final segment of the whole space will reappear in Lemma 4.4.

Example 3.10. Let $\kappa = \langle \omega_1, \omega_1 \rangle$. Let $X = \prod \kappa = \omega_1 \times \omega_1$. Let $\varphi : X \to \mathbb{R}$ be continuous. Define $\delta : \omega_1 \to X$ by $\delta(\xi) = \langle \xi, \xi \rangle$. To prove that φ is constant on a final segment of X, it suffices (by the proof of Theorem 3.7) to assume that U is open in X and $\delta^{\leftarrow}[U]$ is stationary, and then show that U contains a final segment (s, κ) of X.

For each ξ such that $\delta(\xi) \in U$, define $f(\xi) < \xi$ so that $((f(\xi), \xi] \times (f(\xi), \xi]) \subset U$. By the Pressing Down Lemma, there is ζ so that $f(\xi) = \zeta$ for a stationary set of ξ's. Now U contains the final segment $(\delta(\zeta), \kappa)$.

4 Finite Range

Throughout this section, we fix α, an n-tuple of ordinals of uncountable cofinality. For each $i < n$, let A_i be a stationary subset of α_i, and define the n-tuple κ via $\kappa_i = \operatorname{cf} \alpha_i$. We fix the space $X = \prod_{i<n} A_i$.

The next notion "small" includes not only sets bounded in (at least) one coordinate, but also sets like the diagonal in Example 3.9.

Definition 4.1. Let $X \subset \prod \alpha$ $(=\prod_{i<n} \alpha_i)$. We say that a clopen subset V of X is *bounded* if $V \subset \prod \beta$ for some n-tuple $\beta \prec \alpha$ (i.e., $\beta \leq \alpha$ but $\beta \neq \alpha$). Moreover V is *small* if V is represented as the union of a locally finite family of bounded clopen subsets of X.

Note that when $n = 1$, the complement of a small set contains a final segment. So the next theorem is the promised generalization. We devote this section to its proof.

Theorem 4.2. *Let $X = \prod_{i<n} A_i$, where each A_i is stationary. Let $\varphi : X \to \mathbb{R}$ be continuous. Then there is a small clopen subset V of X such that $\varphi \restriction (X \setminus V)$ has finite range.*

The strategy of the proof is as follows. After more notation, we partition the space X into a small clopen subset V^* and finitely many subspaces X_θ, $\theta \in \Theta$, and classify these subspaces. A first approximation to the desired small set V is V^* together with the subspaces of Type 1. We prove that φ is constant on "almost all" of each subspace of Type 2. Finally, we define V and verify the conclusion of our theorem.

Let us establish more notation, also fixed throughout this section. For each i, let $M_i : \operatorname{cf} \alpha_i = \kappa_i \to \alpha_i$ be a strictly increasing continuous function whose range is cofinal in α_i. We call M_i normal functions. For each $i < n$, let $\mu_i : \alpha_i \to \kappa_i$ be the function defined by $\mu_i(\gamma) = \min\{\beta < \kappa_i : \gamma \leq M_i(\beta)\}$. Observe that μ_i almost is an inverse to M_i. In particular, $\mu_i(M_i(\xi)) = \xi$ and $\gamma \leq M_i(\mu_i(\gamma))$ always hold, and $\gamma = M_i(\mu_i(\gamma))$ holds whenever

$\mu_i(\gamma) \in \operatorname{Lim} \kappa_i$. Note that each μ_i is continuous. Therefore the product map $\mu : \prod \alpha \to \prod \kappa$ defined by $\mu(x)_i = \mu_i(x_i)$ is continuous.

For each $i < n$, set $\kappa_i^- = \sup\{\kappa_{i'} : \kappa_{i'} < \kappa_i\}$ (by convention, $\sup \emptyset = 0$). Then $V^* = \{x \in X : (\exists i < n)(\mu(x)_i \leq \kappa_i^-)\}$ is a small clopen set. We consider the case where $A_i = (\kappa_i^-, \kappa_i)$ for each $i < n$ to be very important, and in this case we write Σ in place of X. In symbols,

$$\Sigma = \{x \in \prod \alpha : (\forall i < n)(\kappa_i^- < \mu(x)_i < \kappa_i)\}.$$

The simplest case for general n is where $\alpha_i = \kappa_i$, the κ_i's are strictly increasing, and μ is an identity function. In this case, $\Sigma \subset \nabla\kappa$. In the general case, it helps to discard V^*, or, equivalently, to work within Σ.

Let Θ be the family of functions θ from n onto some m_θ, (necessarily $m_\theta \leq n$), which additionally satisfy

$$\text{if } \kappa_i < \kappa_{i'}, \text{ then } \theta(i) < \theta(i').$$

We say that θ is *coarser* than θ', or θ' is *finer* than θ, if $\theta(i) < \theta(i')$ implies that $\theta'(i) < \theta'(i')$.

For example, when all the κ's are equal, then the constant 0 function is the coarsest θ, the permutations are the finest θ's. At the other extreme, if the κ_i's are distinct, then Θ has only one element: the permutation of n which arranges the κ_i's in increasing order.

Now we can define the partition. For $\theta \in \Theta$, let

$$X_\theta = \{x \in X \setminus V^* : \theta(i) < \theta(i') \iff \mu(x)_i < \mu(x)_{i'}\}.$$

Observe that $X \setminus V^* = \bigcup\{X_\theta : \theta \in \Theta\}$.

Next, we define the m_θ-tuple κ^θ by $\kappa^\theta_{\theta(i)} = \kappa_i$. (So κ^θ is formed from κ by possibly identifying some equal coordinates.) And we define, for $x \in X$ and $j < m_\theta$,

$$\mu_\theta^-(x)_j = \min\{\mu(x)_i : \theta(i) = j\} \quad \text{and} \quad \mu_\theta^+(x)_j = \max\{\mu(x)_i : \theta(i) = j\}.$$

Then the maps μ_θ^-, $\mu_\theta^+ : X \to \prod \kappa^\theta$ are continuous. By the definition, these maps coincide on X_θ and give us a map $\mu_\theta : X_\theta \to \nabla\kappa^\theta$.

The next lemma basically repeats Theorems 3.5 with more notation and a stronger conclusion. The prototype is Example 3.10 above. Note that it is true for all $X \subset \prod \alpha$, not just those of the form $\prod A$. We need the following notation to express this stronger conclusion in a general setting.

Definition 4.3. For C, a m_θ-tuple of cub sets attuned to κ^θ, let $E_\theta(C)$ be the set of $x \in X$ such that there is $c \in \prod C$ satisfying

$$c_0 < \mu_\theta^-(x)_0 \leq \mu_\theta^+(x)_0 < c_1 < \cdots < c_{m_\theta - 1} < \mu_\theta^-(x)_{m_\theta - 1}.$$

In this case we say that x is *θ-entwined with c*. Notice that $E_\theta(C) \subset \Sigma \cap X$ because C is attuned. Observe that $E_\theta(D) \subset E_\theta(C)$ if $D_j \subset C_j$ for all $j < m_\theta$. Note that the set of $x \in X$ which are θ-entwined with a specific $c \in \prod C$ is an open subset of X; hence $E_\theta(C)$ is open in X. If $X = \Sigma$, then $E_\theta(C)$ is open in $\prod \alpha$.

Lemma 4.4. *Let U be an open subset of X such that $\mu_\theta[U \cap X_\theta]$ is a κ^θ-stationary subset of $\nabla\kappa^\theta$. Then there is an attuned m_θ-tuple C of cub sets so that $E_\theta(C)$ is contained in U.*

Proof. Let Y be the set of elements y of $\mu_\theta[U \cap X_\theta]$ such that every coordinate y_j is a limit ordinal. By Proposition 2.3, Y is κ^θ-stationary. Because each y_j is limit, there is a unique $\tilde{y} \in X_\theta$ such that $\mu_\theta(\tilde{y}) = y$. Choose $b(\tilde{y}) < \tilde{y}$ so that $(b(\tilde{y}), \tilde{y}] \cap X \subset U$. Define $f(y) \in \nabla\kappa^\theta$ via $f(y)_j = \mu_\theta^+\big(b(\tilde{y})\big)_j$. Because each y_j is a limit, $f(y)_j < y_j$. In other words, $f(y) < y$ and f is regressive.

Now we follow the proof of Theorem 3.5 closely. We point out only differences. There is a pruned stationary subset Y' of Y so that f restricted to Y' is a stem function. Find an attuned C to satisfy (1) and (2). Let x be an arbitrary element of $E_\theta(C)$. Define y_j to be the least element of $\pi_j^{\langle y_0,\ldots,y_{j-1}\rangle}[Y']$ greater than $\mu_\theta^+(x)_j$. Verify that $f(y)_j < \mu_\theta^-(x)_j \le \mu_\theta^+(x)_j < y_j$ for each $j < m_\theta$, which yields $b(\tilde{y}) < x_i < \tilde{y}_i$ for all $i < n$. We have verified that $x \in (b(\tilde{y}), \tilde{y}] \subset U$, as required. □

And this implies, as before (see Theorem 3.7),

Lemma 4.5. *Let $\theta \in \Theta$ satisfy $\mu_\theta[X_\theta]$ is κ^θ-stationary, and let $\psi : X \to \mathbb{R}$ be continuous. Then ψ is constant on $E_\theta(C)$ for some attuned m_θ-tuple C of cub sets.*

Now we return to the proof of Theorem 4.2.

For a carefully chosen C, φ will be constant on $E_\theta(C)$. However, we cannot ensure that $X_\theta \setminus E_\theta(C)$ is small. So we introduce a slightly larger set.

Definition 4.6. Let $\overline{E_\theta}(C)$ be the set of $x \in X$ such that there is $c \in \prod C$ satisfying

$$c_0 < \mu_\theta^-(x)_0 \le \mu_\theta^+(x)_0 \le c_1 < \cdots \le c_{m_\theta-1} < \mu_\theta^-(x)_{m_\theta-1}.$$

We say that x is *weakly θ-entwined with c*.

Notice that $\overline{E_\theta}(C) \subset \Sigma \cap X$ because C is attuned. Observe that $\overline{E_\theta}(D) \subset \overline{E_\theta}(C)$ if $D_j \subset C_j$ for all $j < m_\theta$. Note that the set of $x \in X$ which are

θ-entwined with a specific $c \in \prod C$ is an open subset of X; hence $\overline{E_\theta}(C)$ is open in X. If $X = \Sigma$, then $\overline{E_\theta}(C)$ is open in $\prod \alpha$.

Let ζ be the coarsest element of Θ; in other words, κ^ζ lists the coordinates of κ in strictly increasing order. For example, when all the κ_i's are equal, then ζ is constant 0 function and κ^ζ is a 1-tuple. At the other extreme, if the κ_i's are distinct, then ζ is the unique element of Θ.

For $\ell < m_\zeta$, let S_ℓ be the collection of $s \subset \zeta^{\leftarrow}[\{\ell\}]$ such that $\bigcap_{i \in s} \mu_i[A_i]$ is not stationary in κ^ζ_ℓ. Let $S = \bigcup_\ell S_\ell$. We now classify the elements of the partition.

Definition 4.7. We say that θ is *Type 1* if $\theta^{\leftarrow}[\{j\}] \in S$ for some $j < m_\theta$. We say that θ is *Type 2* otherwise.

Since θ corresponds to subspace X_θ in a unique way, we can say X_θ is Type 1 or 2 when θ is Type 1 or 2, respectively.

Note that by Corollary 2.5, $\mu_\theta[X_\theta]$ is κ^θ-stationary iff θ is Type 2. If θ' is coarser than θ and θ is Type 1, then θ' is Type 1.

The next lemma is where we use that X has the form $\prod_{i<n} A_i$.

Lemma 4.8. *Let $D = \langle D_0, \dots, D_{m_\theta - 1}\rangle$ be a κ^θ-attuned tuple of cub sets which additionally satisfies: for all $i < n$, $D_{\theta(i)} \subset M_i^{\leftarrow}[\operatorname{Lim} A_i]$. Then*

$$\overline{E_\theta}(D) \subset \operatorname{Cl}_X\bigl(E_\theta(D)\bigr).$$

Proof. Take $y \in \overline{E_\theta}(D)$ arbitrarily and suppose that y is weakly θ-entwined with c. Let

$$H = \{i < n : \theta(i) < m_\theta - 1 \text{ and } \mu_i(y_i) = c_{\theta(i)+1}\}$$

We claim that if $i \in H$, then $\kappa^\theta_{\theta(i)} = \kappa^\theta_{\theta(i)+1}$. Indeed, if $i \in H$ and $\kappa^\theta_{\theta(i)} < \kappa^\theta_{\theta(i)+1}$, then $c_{\theta(i)+1} \in D_{\theta(i)+1} \subset (\kappa^\theta_{\theta(i)}, \kappa^\theta_{\theta(i)+1})$ and $c_{\theta(i)+1} = \mu_i(y_i) < \kappa^\theta_{\theta(i)}$, which is a contradiction. Thus we have $\kappa^\theta_{\theta(i)} = \kappa^\theta_{\theta(i)+1}$. Since $\mu_i(y_i) = c_{\theta(i)+1} \in D_{\theta(i)+1} = D_{\theta(i)} \subset M_i^{\leftarrow}[\operatorname{Lim} A_i]$, we have $y_i = M_i(\mu_i(y_i)) \in \operatorname{Lim} A_i$. Let $(z, y]$ be an arbitrary neighborhood of y. We seek $x \in (z, y] \cap E_\theta(D)$. If $i \notin H$, let $x_i = y_i$. If $i \in H$, choose $x_i \in A_i$ so that $\max\{z_i, M_i(c_{\theta(i)})\} < x_i < y_i$. It is possible because $M_i(c_{\theta(i)}) < M_i(\mu_i(y_i)) = y_i$. Now it is clear that $x \in (z, y] \cap X$, and routine to verify that $x \in E_\theta(D)$. $\square$

By Lemmas 4.5 and 4.8, we have

Lemma 4.9. *Let θ be Type 2, and $\varphi : X \to \mathbb{R}$ be continuous. Then φ is constant on $\overline{E_\theta}(C)$ for some C.*

For each θ of Type 2, let us fix C^θ so that φ is constant on $\overline{E_\theta}(C^\theta)$. Let $E = \bigcup\{\overline{E_\theta}(C^\theta) : \theta \text{ is Type 2}\}$. Then $\varphi \restriction E$ has finite range. We must show that $X \setminus E$ is contained in a small clopen set.

Fix a κ^ζ-attuned tuple $\langle G_0, \dots, G_{m_\zeta - 1}\rangle$ of cub sets satisfying

(1) if θ is Type 2 and $\kappa_j^\theta = \kappa_\ell^\zeta$, then $G_\ell \subset C_j^\theta$,

(2) if $s \in S_\ell$, then $G_\ell \cap \bigcap_{i\in s} \mu_i[A_i] = \emptyset$.

Let $V^\dagger = \{x \in X : (\exists i < n)(\mu(x)_i \leq \min G_{\zeta(i)})\}$. Then $V^\dagger$ is a small clopen set and $V^* \subset V^\dagger$.

For each $s \in S$, we will define a small clopen set V_s. Let $s \in S_\ell$. For $\gamma \in G_\ell$, let γ^+ be the least element of G_ℓ greater than γ. If $\xi \notin G_\ell$, then either $\xi < \min G_\ell$, or there is $\gamma \in G_\ell$ such that $\gamma < \xi < \gamma^+$. Let

$$\begin{aligned} V_\gamma =& \{x \in X : \gamma < \mu(x)_i \leq \gamma^+ \text{ for all } i \in s\}, \\ V_s =& \bigcup\{V_\gamma : \gamma \in G_\ell\}. \end{aligned}$$

Lemma 4.10. *For each $s \in S$, V_s is a small clopen set.*

Proof. Fix $s \in S_\ell$. Observe that each V_γ is clopen. We must show that $\{V_\gamma : \gamma \in G_\ell\}$ is discrete. Towards that end, let $x \in X$ be arbitrary. First consider the case that $\mu(x)_i \notin G_\ell$ for some $i \in s$. If $\mu(x)_i < \min G_\ell$, then the clopen set $V^\dagger \ni x$ misses V_s. Otherwise, for some $\gamma \in G_\ell$, the clopen set $\{y \in X : \gamma < \mu(y)_i \leq \gamma^+\} \ni x$ meets only V_γ.

Next consider the case that $\mu(x)_i \in G_\ell$ for all $i \in s$. If $\mu(x)_i = \mu(x)_{i'}$ for all $i, i' \in s$, then $\mu(x)_i \in G_\ell \cap \bigcap_{i\in s} \mu_i[A_i] = \emptyset$. So let $\mu(x)_i < \mu(x)_{i'}$ for some $i, i' \in s$. Then the clopen set $\{y \in X : \mu(x)_i < \mu(y)_{i'} \text{ and } \mu(y)_i \leq \mu(x)_i\}$ contains x and misses V_s. □

Set $V = V^\dagger \cup \bigcup_{s\in S} V_s$. We claim that V satisfies the conclusion of Thorem 4.2. So we fix an arbitrary $x \in X$ and prove that $x \in V \cup E$. We assume that $x \notin V^*$. Define $\eta \in \Theta$ so that (informally) $\eta(i) < \eta(i')$ iff G separates $\mu(x)_i$ and $\mu(x)_{i'}$. Formally, $\eta(i) < \eta(i')$ iff $\mu(x)_i \leq \gamma < \mu(x)_{i'}$ for some $\gamma \in G_{\zeta(i)}$. If $x \in X_\theta$, then η is coarser than (possibly, but not necessarily, equal to) θ.

Lemma 4.11. *If η is Type 1, then $x \in V$. If η is Type 2 and $x \notin V^\dagger$, then $x \in E$.*

Proof. Assume that η is Type 1. By Definition 4.7, there are j, s, and ℓ so that $\eta^{\leftarrow}[\{j\}] = s \in S_\ell$. From the definition of η, there is $\gamma' \in G_\ell$ so that $\min\{\gamma \in G_\ell : \mu(x)_i \leq \gamma\}$ are equal to γ' for all $i \in s$. The assumption that

$\gamma' \in \operatorname{Lim} G_\ell$ together with (2) of the definition of G_ℓ leads to a contradiction, so $\gamma' = \gamma^+$ for some $\gamma \in G_\ell$. Then $x \in V_\gamma \subset V_s \subset V$.

Assume that η is Type 2 and $x \notin V^\dagger$. We will show that $x \in \overline{E_\eta}(C^\eta)$. We define $c \in \prod C^\eta$ by cases. If $j = 0$ or if $\kappa^\eta_{j-1} < \kappa^\eta_j$, then set $c_j = \min C^\eta_j$. In this case, $c_j < \mu^-_\eta(x)_j$ because $x \notin V^\dagger$. If $\kappa^\eta_{j-1} = \kappa^\eta_j$, let c_j be the least element of C^η_j greater than or equal to $\mu^+_\eta(x)_{j-1}$. In this case, $c_j < \mu^-_\eta(x)_j$, because by definition of η, there is $\gamma \in G_\ell$ such that $\mu^+_\eta(x)_{j-1} \leq \gamma < \mu^-_\eta(x)_j$, and $G_\ell \subset C^\eta_j$. In both cases, $\mu^+_\eta(x)_{j-1} \leq c_j$ is obvious. So x is weakly η-entwined with c, and $x \in \overline{E_\eta}(C^\eta) \subset E$. □

Thus ends our proof of Theorem 4.2.

To end this section, we calculate the upper bound of $\left|\varphi \restriction (X \setminus V)\right|$ in Theorem 4.2. For that we need to find a standard form of sets $\overline{E_\eta}(C^\eta)$ on which $\varphi \restriction (X \setminus V)$ is constant.

Take any $\theta \in \Theta$. Since ζ is coarser than θ, there is, for each $j < m_\theta$, a unique $\ell < m_\zeta$ such that $\kappa^\theta_j = \kappa^\zeta_\ell$ and hence, we can define a κ^θ-attuned tuple D^θ by $D^\theta_j = G_\ell$.

Lemma 4.12. *If η is coarser than θ, then $\overline{E_\theta}(D^\theta) \subset \overline{E_\eta}(C^\eta)$.*

Proof. For each $k < m_\eta$, let $j(k) = \min \theta\big[\eta^{\leftarrow}[\{k\}]\big]$. Note that $\kappa^\theta_{j(k)} = \kappa^\eta_k$.

Let x be weakly θ-entwined with $d \in \prod D^\theta$ and define $c_k = d_{j(k)}$ for $k < m_\eta$. Since ζ is coarser than η, there is a unique $\ell < m_\zeta$ so that $\kappa^\zeta_\ell = \kappa^\eta_k$. This implies $c_k = d_{j(k)} \in D^\theta_{j(k)} = G_\ell \subset C^\eta_k$, and $c = \langle c_0, \ldots, c_{m_\eta - 1}\rangle \in \prod C^\eta$.

Let us see that x is weakly η-entwined with c. Let $k < m_\eta$, $\eta(i) = k$ and $\theta(i) = j$. Then $j \in \theta\big[\eta^{\leftarrow}[\{k\}]\big]$ implies $j \geq j(k)$, which further implies $c_k = d_{j(k)} \leq d_j < \mu^-_\theta(x)_j \leq \mu(x)_i$, and hence $c_k < \mu^-_\eta(x)_k$. When $k < m_\eta - 1$, observe that $j + 1 \leq j(k+1)$ because η is coarser than θ. Then $\mu(x)_i \leq \mu^+_\theta(x)_j \leq d_{j+1} \leq d_{j(k+1)} = c_{k+1}$, and hence $\mu^+_\eta(x)_k \leq c_{k+1}$. This shows $x \in \overline{E_\eta}(C^\eta)$. □

Corollary 4.13. *Under the assumptions of Theorem 4.2, there is a small clopen set V of X such that*

$$\left|\varphi \restriction (X \setminus V)\right| \leq \prod_{\ell < m_\zeta} \left(\left|\zeta^{\leftarrow}[\{\ell\}]\right|!\right) \leq n!\,.$$

Proof. By the proof of Lemma 4.11, the values of $\varphi \restriction (X \setminus V)$ are given by constant values $\varphi\big[\overline{E_\eta}(C^\eta)\big]$, where η is determined by $x \notin V^\dagger$. Let $\theta(\eta) \in \Theta$ be a permutation finer than such η. Then, by Lemma 4.12, $\varphi\big[\overline{E_\eta}\,(C^\eta)\big] = \varphi\big[\overline{E_{\theta(\eta)}}\,\big(D^{\theta(\eta)}\big)\big]$. Since there are at most $\prod_{\ell < m_\zeta}\big(\left|\zeta^{\leftarrow}[\{\ell\}]\right|!\big)$-many permutations in Θ, we have $\left|\varphi \restriction (X \setminus V)\right| \leq \prod_{\ell < m_\zeta}\big(\left|\zeta^{\leftarrow}[\{\ell\}]\right|!\big) \leq n!$. □

Let $\{A_i : i < n\}$ be a pairwise disjoint collection of stationary sets in ω_1. Then $X = \prod_{i<n} A_i$ is the free union of X_θ's, where θ is a permutation on n. So we can define a continuous map φ on X such that $\big|\varphi \restriction (X \setminus V)\big| = n!$ for each small clopen set V.

5 Main Theorem

In this section, we state and prove

Theorem 5.1 (Main). *The product of finitely many subspaces of ordinals is strongly zero-dimensional. In other words, if $A_i \subset \alpha_i$ for all $i < n$, then $X = \prod_{i<n} A_i$ is strongly zero-dimensional.*

Proof. Here is our induction hypothesis. For a tuple $\alpha = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ of ordinals, let $SZD(\alpha)$ abbreviate "if $A_i \subset \alpha_i$ for all $i < n$, then $X = \prod_{i<n} A_i$ is strongly zero-dimensional".

We will prove $SZD(\alpha)$ for all finite-tuples of ordinals by induction on the order $\prec$.

Assuming $SZD(\beta)$ for all $\beta \prec \alpha$, we will show $SZD(\alpha)$. Let Z_0 and Z_1 be disjoint zero-sets of X. By [6, 1.15], we may assume that $Z_0 = h^{\leftarrow}\big[\{0\}\big]$ and $Z_1 = h^{\leftarrow}\big[\{1\}\big]$ for some continuous function $h : X \to [0,1]$.

Case 1. For some $i < n$, α_i has the form $\beta + 2$, or $\operatorname{cf} \alpha_i = \omega$, or $\operatorname{cf} \alpha_i > \omega$ and A_i is not stationary in α_i.

We shall show that X is the free sum of spaces known to be strongly zero-dimensional by induction hypothesis, and hence is itself strongly zero-dimensional.

Indeed, for notational convenience, we may assume $i = 0$. Let $Y = \prod_{1 \le i < n} A_i$.

The first case ($\alpha_0 = \beta + 2$): We have $X = (A_0 \cap (\beta+1)) \times Y \bigoplus (A_0 \cap \{\beta + 1\}) \times Y$ and $(A_0 \cap \{\beta+1\}) \times Y$ is homeomorphic to $\{0\} \times Y$ if $\beta + 1 \in A_0$.

The second case ($\operatorname{cf} \alpha_0 = \omega$): Fix a normal function $M : \omega \to \alpha_0$. Then we have $X = \bigoplus_{n \in \omega} (A_0 \cap (M(n-1), M(n)]) \times Y$, where $M(-1)$ is considered as -1.

The third case ($\operatorname{cf} \alpha_0 > \omega$ and A is not stationary in α_0): Since A_0 is not stationary in α_0 and $\operatorname{cf} \alpha_0 > \omega$, one can fix a normal function $M : \operatorname{cf} \alpha_0 \to \alpha_0$ such that $\operatorname{ran} M \cap A_0 = \emptyset$. Then $X = \bigoplus_{\gamma < \operatorname{cf} \alpha_0} (A_0 \cap (M(\gamma - 1), M(\gamma)]) \times Y$.

Case 2. For some $i < n$, α_i has the form $\lambda + 1$, where λ is a limit ordinal.

For notational convenience, we may assume that $i = 0$. Moreover by induction hypothesis, we may assume that $\lambda \in A_0$. Set $Y = \prod_{1 \le i < n} A_i$ and $X_1 = \{\lambda\} \times Y$. Set $h_1 = h \restriction X_1$.

By the induction hypothesis, there is a clopen set W of Y so that $h_1^{\leftarrow}\big[[0,1/3]\big] \subseteq \{\lambda\} \times W$ and $h_1^{\leftarrow}\big[[2/3,1]\big] \subset \{\lambda\} \times (Y \setminus W)$.

Let $X_2 = (A_0 \setminus \{\lambda\}) \times W$ and $h_2 = h \restriction X_2$. By induction hypothesis, there is a clopen subset V_2 of X_2 such that $h_2^{\leftarrow}\big[[0,5/6]\big] \subset V_2$ and $h_2^{\leftarrow}[\{1\}] \subset X_2 \setminus V_2$. Analogously, by letting $X_3 = (A_0 \setminus \{\lambda\}) \times (Y \setminus W)$ and $h_3 = h \restriction X_3$, one can find a clopen set V_3 of X_3 such that $h_3^{\leftarrow}[\{0\}] \subset V_3$ and $h_3^{\leftarrow}[\{1\}] \subset X_3 \setminus V_3$. Then $V = (\{\lambda\} \times W) \cup V_2 \cup V_3$ obviously contains Z_0 and is disjoint from Z_1.

To show that V is open, let $x \in V$. Since V_2 and V_3 are open in X, it suffices to consider the case that $x = \langle \lambda, y \rangle \in \{\lambda\} \times W$. It follows from $h(x) < 2/3 < 5/6$ that there are $\alpha < \lambda$ and a neighborhood U of y such that $U \subset W$ and $((\alpha,\lambda] \cap A_0) \times U \subset h^{\leftarrow}\big[[0,2/3)\big]$. Then it is straightforward to show that $\big((\alpha,\lambda] \cap A_0\big) \times U \subset V$, thus V is open in X. Similarly we can show that $X \setminus V$ is open in X, and hence V is clopen.

Case 3. For all $i < n$, $\operatorname{cf} \alpha_i > \omega$ and A_i is stationary in α_i.

We apply Theorem 4.2 to the function h and obtain a small clopen set V so that $h \restriction (X \setminus V)$ has finite range. Note that $W^* = Z_0 \cap (X \setminus V)$ is clopen in X.

By the definition of small, $V = \bigcup \{V_\lambda : \lambda \in \Lambda\}$, where the induction hypothesis applies to each V_λ. That is, for each λ, there is W_λ^0, clopen in V_λ (hence clopen in X) such that $Z_0 \cap V_\lambda \subset W_\lambda^0 \subset V_\lambda \setminus Z_1$. Because $\{V_\lambda : \lambda \in \Lambda\}$ is locally finite in X, $W^0 = \bigcup \{W_\lambda^0 : \lambda \in \Lambda\}$ is clopen in X. Then $W^* \cup W^0$ is the desired clopen set separating Z_0 and Z_1. □

6 Subspaces of the Product Space $(\omega+1) \times \mathfrak{c}$ Which Are Not Strongly Zero-Dimensional

We begin by considering a MAD family $\mathcal{R}$ of subsets of ω. Here $\mathcal{R}$ is called MAD (=maximal almost disjoint) if it is almost-disjoint ($|s \cap s'| < \omega$ for distinct $s, s' \in \mathcal{R}$), and not contained properly in any other almost-disjoint family. For such $\mathcal{R}$, let $\Psi(\mathcal{R})$ denote the space which is defined on the set $\omega \cup \mathcal{R}$ and has the so-called Ψ-space topology, [6, 5.I], [3, 3.6.I]. That is, a subset U of $\Psi(\mathcal{R})$ is open iff

$$\forall s \in \mathcal{R} \quad (s \in U \Rightarrow |s \setminus U| < \omega).$$

Let $L = \{\lambda \in \mathfrak{c} : \lambda \text{ is a limit}\}$, and let $S = \mathfrak{c} \setminus L$. Note that $|L| = |S| = \mathfrak{c}$. Since $|\mathcal{R}| \times \mathfrak{c} \approx \mathfrak{c}$, the unindexed family can be indexed $\mathcal{R} = \{s_\alpha : \alpha \in S\}$ in such a way that, for each $s \in \mathcal{R}$, $|\{\alpha \in S : s = s_\alpha\}| = \mathfrak{c}$. With $\mathcal{R}$ thus indexed, we consider the subspace

$$K(\mathcal{R}) = \omega \times L \cup \bigcup_{\alpha \in S} \Big(\big(s_\alpha \times \{\alpha\} \big) \cup \big\{ \langle \omega,\, \alpha \rangle \big\} \Big)$$

of the product space $(\omega + 1) \times \mathfrak{c}$.

Theorem 6.1. *For every MAD family $\mathcal{R}$, $\beta\Psi(\mathcal{R})$ is embedded in $K(\mathcal{R})^* = \beta K(\mathcal{R}) \setminus K(\mathcal{R})$.*

As noted in [11, Concluding remarks], where the symbol $N \cup \mathcal{R}$ denotes our space $\Psi(\mathcal{R})$, every first-countable separable compact space as well as the space $\omega_1 + 1$ is homeomorphic to $\Psi(\mathcal{R})^*$ for some $\mathcal{R}$. (Let us take this opportunity to point out that extensions of this result, which were subsequently obtained by a few authors, remain mostly unpublished, and that some of their zero-dimensional versions are found in [1]. However, [1] is written in Boolean algebra terms, and, naturally, concerned with Banaschewski compactification (i.e., maximal zero-dimensional compactification) of $\Psi(\mathcal{R})$ instead of [11]'s Stone-Čech one. Hence results of [1] and [11] overlap only in the case that $\Psi(\mathcal{R})$ is strongly zero-dimensional, or, equivalently, $\Psi(\mathcal{R})^*$ is zero-dimensional, see [11, Lemma 1.1]. See also the interesting paper [2].)

Therefore

Corollary 6.2. *Every first-countable, separable, compact space is embedded in $K(\mathcal{R})^*$ for some $\mathcal{R}$.*

Corollary 6.3. *The space $\omega_1 + 1$ is embedded in $K(\mathcal{R})^*$ for some $\mathcal{R}$.*

Throughout the rest of this section we will fix $\mathcal{R}$ a MAD family indexed by S in the special way described above. We will often write K or Ψ in place of $K(\mathcal{R})$ or $\Psi(\mathcal{R})$, respectively, for simplicity's sake.

For the proof of Theorem 6.1, we define in the space $K = K(\mathcal{R})$

$$H_\alpha = (\omega + 1) \times (\alpha,\, \mathfrak{c}) \cap K, \quad \alpha < \mathfrak{c}$$

and in the space βK

$$Y = \bigcap_{\alpha < \mathfrak{c}} \mathrm{Cl}_\beta\, H_\alpha\,, \quad Y' = Y \cap \mathrm{Cl}_\beta\, [\{\omega\} \times S]\,.$$

(Here and below Cl_β denotes the closure in a Stone-Čech compactification.) Obviously each H_α is a clopen subset of K. Being the intersection of compact sets, Y and Y' are compact subspaces of K^*.

The following well-known lemma, which is a consequence of the Pressing Down Lemma, is central to our argument.

Lemma 6.4. *Let $L \subseteq A \subseteq \mathfrak{c}$. Then every continuous map $f \in C(A)$ is constant on $A \cap (\lambda, \mathfrak{c})$ for some $\lambda < \mathfrak{c}$. Hence A^* is a singleton.*

Now Theorem 6.1 follows from

Proposition 6.5. *Y and $\beta\Psi$ are homeomorphic.*

We present two proofs of this fact. The first one gives a direct construction of a homeomorphism $Y \to \beta\Psi$, while the alternate one shows that $C(Y)$ and $C(\beta\Psi)$ are isomorphic.

Proof. In the sequel, for a Tychonoff space X, we will identify points of βX with z-ultrafilters on X. Thus, for a zero-set Z of X and $p \in \beta X$, $Z \in p$ is equivalent to $p \in \mathrm{Cl}_\beta Z$, that is, $\{p\} = \bigcap_{Z \in p} \mathrm{Cl}_\beta Z$ (cf. [6, 6.5(c)]). Note that $H_\alpha \in u$ for every α and $u \in Y$.

First let $\pi : K \to \Psi$ be the natural continuous map defined by

$$\pi(\langle n, \varphi \rangle) = n$$
$$\pi(\langle \omega, \alpha \rangle) = s_\alpha .$$

Let $\tilde{\pi} : \beta K \to \beta\Psi$ be the unique continuous extension of π.

It suffices to see that $\tilde{\pi}[Y] = \beta\Psi$ and that $\tilde{\pi} \restriction Y$ is one-to-one.

To see the former, take any point $p \in \beta\Psi \setminus \omega$. For any $Z \in p$, $\pi^{\leftarrow}[Z]$ is a zero set of K and Z meets $\mathcal{R}$. By the careful indexing of points of $\mathcal{R}$, we have that $\pi^{\leftarrow}[Z] \cap H_\alpha$ is a non-empty zero-set of K for each $\alpha < \mathfrak{c}$. Let Z_α denote this set. Since $\{Z_\alpha : \alpha, Z\}$ has the finite intersection property, it is contained in a z-ultrafilter u. Obviously u belongs to Y. Since $u \in \mathrm{Cl}_\beta Z_\alpha \subseteq \mathrm{Cl}_\beta \pi^{\leftarrow}[Z]$, we have that $\tilde{\pi}(u) \in \tilde{\pi}[\mathrm{Cl}_\beta \pi^{\leftarrow}[Z]] = \mathrm{Cl}_\beta Z$ for any Z and hence that $\tilde{\pi}(u) = p$.

Let us show that $\tilde{\pi} \restriction Y$ is one-to-one. Suppose that there are points $u_0 \neq u_1$ in Y so that $\tilde{\pi}(u_0) = \tilde{\pi}(u_1) = p$. Since $u_0 \neq u_1$, there are zero-sets $Z_i \in u_i$ for $i = 0, 1$ such that $Z_0 \cap Z_1 = \emptyset$. Then there is a continuous map $f : K \to [0, 1]$ in which $f^{\leftarrow}[\{i\}] = Z_i$.

By Lemma 6.4, for each n, there is an ordinal $\lambda_n < \mathfrak{c}$ in which $f \restriction \{n\} \times (\lambda_n, \mathfrak{c}) \cap K$ is constant. Let $\lambda = \sup\{\lambda_n : n\}$. Then we have that $f \restriction \{n\} \times (\lambda, \mathfrak{c}) \cap K$ is constant for each $n < \omega$.

Let $Z'_i = Z_i \cap H_\lambda$.

We claim that $\pi[Z'_i]$ is a zero-set of Ψ. In fact, we can define a map $F : \Psi \to [0, 1]$ by $F(n) =$ the constant value of $f \restriction \{n\} \times (\lambda, \mathfrak{c}) \cap K$, and $F(s) = f(\langle \omega, \alpha \rangle)$ when $s = s_\alpha$ and $\alpha > \lambda$. It is clear that F is well-defined and continuous. And it is not too difficult either to see that $\pi[Z'_i] = F^{\leftarrow}[\{i\}]$.

Now we are almost done. Since $Z'_i \in u_i$, we have that $u_i \in \mathrm{Cl}_\beta Z'_i$, that $p = \tilde{\pi}(u_i) \in \tilde{\pi}[\mathrm{Cl}_\beta Z'_i] = \mathrm{Cl}_\beta \pi[Z'_i]$, and hence that $\pi[Z'_i] \in p$. Obviously p

does not belong to ω, and $\pi[Z_0'] \cap \pi[Z_1'] \neq \emptyset$. So take s from this intersection and find points $\langle \omega, a \rangle \in Z_0'$ and $\langle \omega, b \rangle \in Z_1'$ in which $s = s_a = s_b$. Let $s = \{k_n : n < \omega\}$. Then since $a > \lambda$ and $b > \lambda$, $f(\langle k_n, a \rangle) = f(\langle k_n, b \rangle)$ holds for each n. However, we know that the sequences $\{\langle k_n, a \rangle\}_n$ and $\{\langle k_n, b \rangle\}_n$ converge to $\langle \omega, a \rangle$ and $\langle \omega, b \rangle$, respectively. Here is a contradiction, because all this implies that $0 = f(\langle \omega, a \rangle) = f(\langle \omega, b \rangle) = 1$. □

Alternate Proof. For this approach, we need two results from the theory of rings of continuous functions. First, $C^*(X) \cong C(\beta X)$; in words, the ring of bounded, continuous real-valued functions on a space X is isomorphic to the ring of continuous real-valued functions on the Stone-Čech compactification of X [6, 6.6(b)]. Second, compact spaces X and Y are homeomorphic iff $C(X)$ and $C(Y)$ are isomorphic [6, 4.9].

Set $\Omega = (\omega \times \mathfrak{c}) \cap K$, and $\Omega_n = \{\xi \in \mathfrak{c} : \langle n, \xi \rangle \in \Omega\}$. Our first goal is to show that $C(\beta\Psi)$ is isomorphic to Q, the subring of $C^*(K)$ of functions f for which there is $f^\circ \in C^*(\omega)$ satisfying

(1) for all $n \in \omega$ and all $\xi \in \Omega_n \quad f(n, \xi) = f^\circ(n)$.

(2) for all $s \in \mathcal{R} \ \ \lim\{f^\circ(n) : n \in s\}$ exists.

We know that $f \mapsto f \restriction \Psi$ is an isomorphism of $C(\beta\Psi)$ onto $C^*(\Psi)$. When we further restrict to ω, the map is injective (because ω is dense in Ψ) but not surjective. It is routine to check that the image of $f \mapsto f \restriction \omega$ is

$$Q' = \{f \in C^*(\omega) : (\forall s \in \mathcal{R}) \ \lim\{f(n) : n \in s\} \text{ exists}\}.$$

Moreover, it is obvious that $f \mapsto f^\circ$ is an isomorphism of Q onto Q'.

Because of condition (2), each $f \in Q$ can by extended to $f' \in C^*(K)$ and then further to $f'' \in C(\beta K)$. Clearly, $\eta : Q \to C(Y)$ defined by $\eta(f) = f'' \restriction Y$ is a homomorphism; we must show that η is injective and surjective.

Notice that for each $n \in \omega$, Ω_n is clopen in K. Hence $\beta(\Omega_n) \cong \mathrm{Cl}_\beta\, \Omega_n$; so $\mathrm{Cl}_\beta\, \Omega_n \setminus \Omega_n$ is a singleton $\{y_n\}$. Because Ω_n meets H_α for all $\alpha < \mathfrak{c}$, this y_n belongs to Y. Clearly, $\eta(f)(y_n) = f''(y_n) = f^\circ(n)$ for all $n \in \omega$ and $f \in Q$. Hence η is injective.

Fix $g \in C(Y)$. Because Y is closed in βK, normal, we can apply Tietze's Extension Theorem to extend g to $g^\flat \in C(\beta K)$. By Lemma 6.4, for each $n \in \omega$, there are t_n and λ_n satisfying $g^\flat(\langle n, \xi \rangle) = t_n$ for all $\xi \in \Omega_n \cap (\lambda_n, \mathfrak{c})$. Set $\lambda = \sup\{\lambda_n : n \in \omega\}$. By the special indexing, for each $s \in \mathcal{R}$, there is $\delta > \lambda$ with $s_\delta = s$. By continuity, $\lim\{t_n : n \in s\} = g^\flat(\langle \omega, \delta \rangle)$. Therefore the function $n \mapsto t_n$ is f° for some $f \in Q$, and $g^\flat \restriction (\Omega \cap H_\lambda) = f \restriction (\Omega \cap H_\lambda)$. Because $\Omega \cap H_\lambda$ is dense in $\mathrm{Cl}_\beta\, H_\lambda$, we conclude that $f'' \restriction Y = g$; that is, η is surjective. □

From the first proof, we see

Corollary 6.6. *Y' is homeomorphic to Ψ^*.*

It appears that Y' is the main part of the space βK and, accordingly, of K. That is,

Proposition 6.7. *Any compact subspace of βK is strongly zero-dimensional if it is disjoint from Y'.*

Proof. Let us take any compact subspace $E \subseteq \beta K \setminus Y'$. Since E and Y' are disjoint closed sets in a compact space βK, there is a continuous map $f : \beta K \to [0, 1]$ so that $Y' = f^{\leftarrow}[\{0\}]$ and $E = f^{\leftarrow}[\{1\}]$. Similarly to the later part of the proof of Proposition 6.5, there is an ordinal λ in which $f \restriction \{n\} \times (\lambda, \mathfrak{c}) \cap K$ is constant $= t_n$ for each $n \in \omega$.

We claim that $\lim_{n\to\infty} t_n = 0$. Suppose to the contrary and choose its subsequence $\{t_{i_n}\}_n$ which converges to $t \neq 0$. By the maximality of the MAD family $\mathcal{R}$, we can find $s = \{k_n\}_n \in \mathcal{R}$ which contains infinitely many i_n. Let $A = \{\alpha : s = s_\alpha\}$. Then A is cofinal in $\mathfrak{c}$. For each $\alpha \in A$, we have that the sequence $\langle k, \alpha\rangle$, $k \in s_\alpha$, converges to $\langle \omega, \alpha\rangle$, and hence that $f(\langle \omega, \alpha\rangle) = t$ if $\alpha > \lambda$. Therefore f should send the subset $C = \{\langle \omega, \alpha\rangle : \alpha \in A \text{ and } \lambda < \alpha\}$ of K to $t \neq 0$, and this contradicts with the definition of Y', because C meets H_α for each $\alpha < \mathfrak{c}$.

Since $\lim_{n\to\infty} t_n = 0$, we have that $f(\langle \omega, \alpha\rangle) = 0$ for every $\alpha > \lambda$, and that there is an integer N such that $t_n < 1/3$ for $n > N$.

Since K is dense in βK, E is contained in $\mathrm{Cl}_\beta [f^{\leftarrow}(1/2, 1] \cap K]$, and the latter set $f^{\leftarrow}(1/2, 1] \cap K$ is contained in the union of $(\omega+1) \times [0, \lambda+1]$ and $\bigcup_{n<N} \{n\} \times [0, \mathfrak{c})$. Let $U = (\omega+1) \times [0, \lambda+1] \cap K$ and $G_n = \{n\} \times [0, \mathfrak{c}) \cap K$. These are clopen sets of K and C^*-embedded in it. Thus each of $\mathrm{Cl}_\beta U$ and $\mathrm{Cl}_\beta G_n$ is equivalent, as an extention, to the Stone-Čech compactification of U and G_n, respectively.

Since U consists of less-than-$\mathfrak{c}$ many points, it has no continuous map onto $[0, 1]$. This means that $0 = \dim U = \dim \beta U = \dim \mathrm{Cl}_\beta U$. And, on the other hand, it is well-known that each G_n is normal and strongly zero-dimensional, and hence that $\dim \mathrm{Cl}_\beta G_n = 0$. Therefore $E \subseteq \mathrm{Cl}_\beta U \cup \bigcup_{n<N} \mathrm{Cl}_\beta G_n$ is strongly zero-dimensional. □

We note the following Dowker-Morita's Generalized Sum Theorem (see, e.g., [3, Problem 7.4.11]): "Let X be a normal space and M its closed subspace such that $\dim M \leq n$. If every closed set $F \subseteq X$ disjoint from M satisfies $\dim F \leq n$, then $\dim X \leq n$." Then Proposition 6.7 assures us

Proposition 6.8. $\dim Y' = \dim \beta K$.

Now Theorem 6.1, and Propositions 6.5 and 6.8 imply the following, which establishes what we have intended in this section.

Theorem 6.9. *For every non-negative integer n, we can choose $\mathcal{R}$ so that* $\dim \beta K(\mathcal{R}) = n$.

As we have pointed out in the last part of the proof of Proposition 6.7, every space of cardinality $< \mathfrak{c}$ is strongly zero-dimensional. Hence

Theorem 6.10. $\mathfrak{c}$ *is the minimum cardinal such that* $(\omega + 1) \times \mathfrak{c}$ *is not hereditarily strongly zero-dimensional.*

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