COUNTABLE METACOMPACTNESS OF PRODUCTS OF LOTS'

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Abstract. The second author and Smith proved that the product of two ordinals is hereditarily countably metacompact [5]. It is natural to ask whether $X \times Y$ is countably metacompact for every LOTS' X and Y. We answer the problem negatively, in fact, for every regular uncountable cardinal κ, we construct a hereditarily paracompact LOTS L_κ such that $L_\kappa \times S$ is not countably metacompact for any stationary set S in κ. Moreover we will find a condition on a GO-space X in order that $X \times \kappa$ is countably metacompact. As a corollary, we see that a subspace X of an ordinal is paracompact iff $X \times Y$ is countably metacompact for every GO-space Y.

A topological space X is said to be *countably metacompact* if each countable open cover has a point finite open refinement. It is well-known that every LOTS is hereditarily countably metacompact. The second author and Smith proved that the product of two ordinals is hereditarily countably metacompact [5]. It is natural to ask whether $X \times Y$ is countably metacompact for every LOTS' X and Y. We answer the problem negatively, in fact, for every regular uncountable cardinal κ, we construct a hereditarily paracompact LOTS L_κ such that $L_\kappa \times S$ is not countably metacompact for any stationary set S in κ. Moreover we will find a condition on a GO-space X in order that $X \times \kappa$ is countably metacompact. As a corollary, we see that a subspace X of an ordinal is paracompact iff $X \times Y$ is countably metacompact for every GO-space Y.

Spaces mean regular topological spaces having at least two points. Let $<$ be a linear order on a set X. $\lambda(<)$ denotes the usual order topology, that is, the topology generated by

$$\{(a, \rightarrow) : a \in X\} \cup \{(\leftarrow, b) : b \in X\}$$

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as a subbase, where $(a, \rightarrow) = \{x \in X : a < x\}$, $(a, b) = \{x \in X : a < x < b\}$,..., etc. If necessary, we write $<_X$ and $(a, b)_X$ instead of $<$ and (a, b) respectively. A *LOTS* X means the triple $\langle X, <, \lambda(<)\rangle$. LOTS is an abbreviation of "Linearly Ordered Topological Space". As usual, we consider an ordinal α as the set of smaller ordinals and as a LOTS with the order $\in$ (we identify it with $<$). Similarly a *Generalized Ordered space* (*GO-space*) means the triple $\langle X, <, \tau\rangle$, where τ is a topology on X with $\lambda(<) \subset \tau$ which has a base consisting convex sets. Here recall that a subset A is *convex* if $(a, b) \subset A$ whenever $a, b \in A$ with $a < b$.

It is well-known that:

- If $X = \langle X, <_X, \tau\rangle$ is a GO-space, then there is a compact LOTS $L = \langle L, <_L, \lambda(<_L)\rangle$ with $X \subset L$ and $<_X = <_L\restriction X$ such that the compact space $\langle L, \lambda(<_L)\rangle$ contains $\langle X, \tau\rangle$ as a dense subspace, where $<_L\restriction X$ is the restricted order of $<_L$ to X. We say this situation as "a GO space $\langle X, <_X, \tau\rangle$ has a linearly ordered compactification $\langle L, <_L, \lambda(<_L)\rangle$" or more simply "a GO-space X has a linearly ordered compactification L".

Let us say that a linearly ordered set X has the *Sorgenfrey topology* if at each $a \in X$, $\{(d, a]_X : d \in [\leftarrow, a)_X\}$ is a neighborhood base. Obviously, a linearly ordered set with the Sorgenfrey topology is a GO-space.

Let C be a subset of a regular uncountable cardinal κ. Define $p_C(\alpha) = \sup(C \cap \alpha)$ for $\alpha < \kappa$, $\mathrm{Lim}(C) = \{\alpha < \kappa : \alpha = p_C(\alpha)\}$ and $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}(C)$, where for convenience we consider that -1 is the immediate predecessor of the ordinal 0 and $\sup \emptyset = -1$. Note that $\mathrm{Lim}(C)$ is the set of all cluster points of C in κ therefore it is closed unbounded (club) in κ whenever C is unbounded in κ, also note that $\mathrm{Succ}(C)$ is the set of isolated points in the subspace C.

First, we will prove the theorem below.

Theorem 1. *Let κ be a regular uncountable cardinal. Then there is a hereditarily paracompact GO-space X_κ with $|X_\kappa| = 2^{<\kappa}$ such that $X_\kappa \times S$ is not countably metacompact for any stationary subset S of κ.*

Proof. Recall that a subset of κ is *stationary* if it intersects with all club subsets of κ. Let

$$\bar{L} = \{u : \ u \text{ is a function on } \kappa \text{ into } [0, \kappa]\}$$

be a linearly ordered set with the lexicographic order, that is,

$$u_0 <_{\bar{L}} u_1 \Leftrightarrow u_0 \restriction \mu = u_1 \restriction \mu \text{ and } u_0(\mu) < u_1(\mu) \text{ for some } \mu < \kappa,$$

where $u_0, u_1 \in \bar{L}$. For each $\mu < \kappa$, put

$$L(\mu) = \{s : \ s \text{ is a function on } \mu \text{ into } \kappa\}.$$

For each $s \in L(\mu)$, define $\bar{s} \in \bar{L}$ by

$$\bar{s}(\xi) = \begin{cases} s(\xi) & \text{for each } \xi \in \mu, \\ \kappa & \text{for each } \xi \in \kappa \setminus \mu. \end{cases}$$

Let $X = \bigcup_{\mu<\kappa} L(\mu)$ and consider the linear order $<_X$ on X such that

$$s_0 <_X s_1 \Leftrightarrow \overline{s_0} <_{\bar{L}} \overline{s_1}$$

for each $s_0, s_1 \in X$, and equip X with the Sorgenfrey topology. We will see that $X_\kappa = X$ is required.

For each $s \in X$, $\mathrm{lh}(s)$ denotes the length of s, i.e. $\mathrm{lh}(s) = \mu$ with $s \in L(\mu)$. For each $s \in X$ and ν with $\mathrm{lh}(s) < \nu < \kappa$, let

$$M(s, \nu) = \{t \in L(\nu) : \ t \restriction \mathrm{lh}(s) = s\},$$

and $M(s) = \bigcup_{\nu \in (\mathrm{lh}(s), \kappa)} M(s, \nu)$.

It is routine to check that $|X| = 2^{<\kappa}$ and three claims below hold.

Claim 1. For each $s_0, s_1 \in X$, $s_0 <_X s_1$ holds iff one of the following holds:

- $s_0 \restriction \xi = s_1 \restriction \xi$ and $s_0(\xi) < s_1(\xi)$ for some $\xi < \min\{\mathrm{lh}(s_0), \mathrm{lh}(s_1)\}$.
- $s_0 \in M(s_1)$.

Claim 2. Let $s \in X$ and $t \in M(s, \mathrm{lh}(s) + 1)$. Then, $t <_X s$ and

$$(t, s]_X = \{s\} \cup \{u \in M(s) : \ t(\mathrm{lh}(s)) < u(\mathrm{lh}(s))\}.$$

Claim 3. Let $s \in X$. Then

$$\mathcal{B}(s) = \{(t, s]_X : \ t \in M(s, \mathrm{lh}(s) + 1)\}$$

is a neighborhood base of s in X.

Claim 4. X is hereditarily paracompact.

Proof. Assume that a subspace Y of X is not paracompact. Then there is a closed subspace of Y which is homeomorphic to a stationary set of a regular uncountable cardinal θ, see [2]. There is an order preserving or reverse order preserving homeomorphism $\varphi : R \to E$ from some stationary subset R of θ onto some closed subset E of Y. We will derive a contradiction.

First assume that φ is reverse order preserving. By stationarity of R, there is an $\alpha \in R \cap \mathrm{Lim}(R)$. Since X has the Sorgenfrey topology,

$Y \cap (\leftarrow, \varphi(\alpha)]$ is an open neighborhood of $\varphi(\alpha)$ which is disjoint from $\{\varphi(\beta) : \beta \in R \cap \alpha\} \subset (\varphi(\alpha), \rightarrow)$. This contradicts continuity.

Next assume that φ is order preserving. For each $\alpha \in R$, put

$$\xi_\alpha = \min\{\xi \in \kappa : \overline{\varphi(\alpha)}(\xi) \neq \overline{\varphi(\beta)}(\xi) \text{ for some } \beta \in R \cap (\alpha, \theta)\},$$

$$\beta_\alpha = \min\{\beta \in R \cap (\alpha, \theta) : \overline{\varphi(\alpha)}(\xi_\alpha) \neq \overline{\varphi(\beta)}(\xi_\alpha)\}.$$

Because of $\overline{\varphi(\alpha)} < \overline{\varphi(\beta)}$ for every $\beta \in R \cap (\alpha, \theta)$, such ξ_α and β_α are well-defined. Moreover by the minimality of ξ_α, we see:

- $\overline{\varphi(\alpha)} \restriction \xi_\alpha = \overline{\varphi(\beta)} \restriction \xi_\alpha$ for every $\beta \in R \cap (\alpha, \theta)$,
- $\overline{\varphi(\alpha)}(\xi_\alpha) < \overline{\varphi(\beta_\alpha)}(\xi_\alpha) \leq \kappa$, therefore
- $\xi_\alpha < \mathrm{lh}(\varphi(\alpha))$.

Claim 4-1. For each $\alpha, \gamma \in R$, $\alpha < \gamma$ implies $\xi_\alpha \leq \xi_\gamma$.

Proof. Let $\alpha, \gamma \in R$ with $\alpha < \gamma$. Since $\gamma, \beta_\gamma \in R \cap (\alpha, \theta)$ and $\overline{\varphi(\gamma)}(\xi_\gamma) < \overline{\varphi(\beta_\gamma)}(\xi_\gamma)$, $\overline{\varphi(\alpha)}(\xi_\gamma)$ is different from either $\overline{\varphi(\gamma)}(\xi_\gamma)$ or $\overline{\varphi(\beta_\gamma)}(\xi_\gamma)$. Therefore by the minimality of ξ_α, we have $\xi_\alpha \leq \xi_\gamma$. □

Consider the stationary set $R' = \{\gamma \in R : \forall \alpha \in R \cap \gamma (\beta_\alpha < \gamma)\}$. Take $\gamma \in R' \cap \mathrm{Lim}(R')$ and $t \in M(\varphi(\gamma))$. Then $t <_X \varphi(\gamma)$ and $U = (t, \varphi(\gamma)] \cap Y$ is a neighborhood of $\varphi(\gamma)$ in Y. Using the continuity of φ with $\gamma \in \mathrm{Lim}(R')$, pick $\alpha \in R' \cap \gamma$ with $\varphi(\alpha) \in U$. Noting $\xi_\alpha \leq \xi_\gamma < \mathrm{lh}(\varphi(\gamma))$ and $\alpha < \beta_\alpha < \gamma$, we have:

- $\overline{\varphi(\alpha)} \restriction \xi_\alpha = \overline{\varphi(\beta_\alpha)} \restriction \xi_\alpha = \overline{\varphi(\gamma)} \restriction \xi_\alpha = \varphi(\gamma) \restriction \xi_\alpha = t \restriction \xi_\alpha$,
- $\overline{\varphi(\alpha)}(\xi_\alpha) < \overline{\varphi(\beta_\alpha)}(\xi_\alpha) \leq \overline{\varphi(\gamma)}(\xi_\alpha) = \varphi(\gamma)(\xi_\alpha) = t(\xi_\alpha)$.

Therefore $\varphi(\alpha) <_X t$, a contradiction. This completes the proof of Claim 4. □

Let S be a stationary subset of κ, and $Z = X \times S$. Put $e(s) = \sup\{s(\xi) : \xi < \mathrm{lh}(s)\}$ for each $s \in X$, and set

$$F = \{\langle s, \alpha \rangle \in Z : e(s) = \alpha\rangle\}.$$

Claim 5. F is a discrete closed set in Z.

Proof. Let $z = \langle s, \alpha \rangle \in Z$. Take $t \in M(s, \mathrm{lh}(s) + 1)$ with $\alpha \leq t(\mathrm{lh}(s))$. Then, $V = (t, s]_X \times (S \cap [0, \alpha])$ is a neighborhood of z in Z with $V \cap F \subseteq \{\langle s, e(s) \rangle\}$ because of $(t, s]_X = \{s\} \cup \{u \in M(s) : t(\mathrm{lh}(s)) < u(\mathrm{lh}(s))\} \subseteq \{s\} \cup \{u \in X : e(u) > \alpha\}$. □

Take a pairwise disjoint collection $\{S_n : n \in \omega\}$ of stationary subsets of S, and put $F_n = F \cap (X \times S_n)$ for each $n \in \omega$. Then $\mathcal{F} = \{F_n : n \in \omega\}$ is a discrete collection of closed sets in Z.

Claim 6. Z is not countably metacompact.

Proof. We would like to prove that Z is not countably metacompact. Remark that in a countably metacompact space, every countable discrete collection of closed sets has a point finite open expansion. Let $\mathcal{U} = \{U_n : n \in \omega\}$ be an open expansion of $\mathcal{F}$ in Z, i.e. U_n is open in Z and $F_n \subset U_n$ for every $n \in \omega$. To see that Z is not countably metacompact, it suffices to show that $\mathcal{U}$ is not point finite.

We will define a strictly increasing sequence $s = \{s(\xi) : \xi < \kappa\}$ of ordinals in κ by induction on $\xi \in \kappa$. Assume that $s_\xi = \{s(\zeta) : \zeta < \xi\} \in L(\xi)$ with $\xi < \kappa$ is already defined. Pick $s(\xi) \in \kappa$ with $e(s_\xi) = \sup\{s(\zeta) : \zeta < \xi\} < s(\xi)$ such that if $0 < \xi = e(s_\xi) \in S_n$, then $M(s_{\xi+1}) \times (S \cap (\gamma_n(\xi), \xi]) \subseteq U_n$ is satisfied for some $\gamma_n(\xi) < \xi$, where $s_{\xi+1} = \{s(\zeta) : \zeta \leq \xi\}$. We can take such $s(\xi)$ and $\gamma_n(\xi)$. Actually, if $0 < \xi = e(s_\xi) \in S_n$, then $\langle s_\xi, \xi\rangle \in F_n \subseteq U_n$, so there are $\gamma_n(\xi) < \xi$ and $t_\xi \in M(s_\xi, \xi + 1)$ with $(t_\xi, s_\xi]_X \times (S \cap (\gamma_n(\xi), \xi]) \subseteq U_n$ since $\mathcal{B}(s_\xi)$ is a neighborhood base at s_ξ. We obtain a required $s(\xi)$ by taking as $t_\xi(\xi) < s(\xi)$.

After finishing induction, we obtain a club set $C = \{\xi < \kappa : 0 < \xi = e(s_\xi)\}$ of κ. Let $n \in \omega$. For each $\xi \in S_n \cap C$, $\gamma_n(\xi) < \xi$ is defined. By the Pressing Down Lemma (PDL), there are $\gamma_n < \kappa$ and a stationary subset T_n of $S_n \cap C$ such that $\gamma_n(\xi) = \gamma_n$ for every $\xi \in T_n$. Take $\alpha \in S$ such that $\gamma_n < \alpha$ for all $n \in \omega$. For each $n \in \omega$, take $\xi_n \in T_n$ such that $\alpha \leq \xi_n$. And take $\xi \in \kappa$ such that $\xi_n + 1 < \xi$ for all $n \in \omega$.

Let $z = \langle s_\xi, \alpha\rangle$. Then, $z \in Z$. Let $n \in \omega$. Then, $\gamma_n(\xi_n) = \gamma_n < \alpha \leq \xi_n$, so $\alpha \in S \cap (\gamma_n(\xi_n), \xi_n]$. Since $s_\xi \in M(s_{\xi_n+1})$, we have

$$z \in M(s_{\xi_n+1}) \times (S \cap (\gamma_n(\xi_n), \xi_n]) \subseteq U_n.$$

It has been seen $z \in \bigcap_{n\in\omega} U_n$, which says that $\mathcal{U}$ is not point finite. Thus Z is not countably metacompact. □

The proof of the theorem is complete. □

As is shown in the above theorem with $S = \kappa$, Remark 4.2 of [4] is misstated. It is known that each GO-space X is contained in some LOTS L as a closed subspace. The construction of the LOTS L for a GO-space X discussed in [6, Definition 2.5] ensures that if X is hereditarily paracompact, then L is also hereditarily paracompact with $|L| \leq |X|$. So we have:

Corollary 2. *Let κ be a regular uncountable cardinal. Then there is a hereditarily paracompact LOTS L_κ with $|L_\kappa| = 2^{<\kappa}$ such that $L_\kappa \times S$ is not countably metacompact for any stationary subset S of κ.*

We now pose three questions which are raised by Theorem 1 and Corollary 2. For the product space $L_\kappa \times S$ in Corollary 2, L_κ is hereditarily paracompact, but a stationary subset S is not paracompact.

Question 3. Is there a product $X \times Y$ of (hereditarily) paracompact GO-spaces X and Y such that $X \times Y$ is not countably metacompact?

If either X or Y is a subspace of an ordinal, then the answer of the question above is negative, see Corollary 11.

The product space $L_\kappa \times S$ in Corollary 2 is not normal. In fact, it is known that if the product $X \times B$ of a GO-space X and a subspace B of an ordinal is normal, then $X \times B$ has the shrinking property [3, Corollary 8.16, 7.19, Theorem 7.11], in particular, $X \times B$ is countably metacompact. So it is natural to ask:

Question 4. Is there a normal product $X \times Y$ of GO-spaces X and Y such that $X \times Y$ is not countably metacompact?

It is well-known that a space is countably metacompact if and only if each countable increasing open cover has a countable closed refinement. And so the union of countably many countably metacompact closed subspaces is also countably metacompact. In particular, the product $X \times Y$ of countably metacompact spaces X and Y is countably metacompact if either $|X| \leq \omega$ or $|Y| \leq \omega$. On the other hand, by applying Corollary 2 for $\kappa = S = \omega_1$, there is a product $L \times S$ of LOTS' with $|L| = 2^{<\omega_1} = 2^\omega$ and $|S| = \omega_1$ which is not countably metacompact. So by assuming the Continuum Hypothesis, we obtain a product $L \times S$ of LOTS' with $|L| = |S| = \omega_1$ which is not countably metacompact.

Question 5. Is it derived only from ZFC that there are GO-spaces X and Y with $|X| = |Y| = \omega_1$ such that $X \times Y$ is not countably metacompact?

In Theorem 1, we found a product of two GO-spaces which is not countably metacompact. On the other hand, we know that the product of two subspaces of ordinals is countably metacompact, and we would like to generalize this result. We suggest the intermediate concept, countable 0-compactness (countable 1-compactness).

Definition 6. Let $X = \langle X, <, \tau \rangle$ be a GO-space. We say that X is *countably* 0*-compact* (*countably* 1*-compact*) if each strictly increasing (decreasing) sequence, of length ω, by points in X has a cluster point in X.

Observe that a subspace of an ordinal is vacuously countably 1-compact because there is no infinite strictly decreasing sequence. Also observe that countable compactness of a GO-space is equivalent to countable 0-compactness + countable 1-compactness.

We will show:

Theorem 7. *If X is a countably 1-compact GO-space and Y is a GO-space satisfying the both clauses (1) and (2), then $Z = X \times Y$ is countably metacompact.*

(1) *either (1A) X is paracompact, or (1B) Y is countably 1-compact,*
(2) *either (2A) X is well-ordered, or (2B) Y is countably compact.*

Remark 8. Even if L_0 and L_1 are compact LOTS', $L_0 \times L_1$ need not be *hereditarily* countably metacompact. Because let $X = X_\kappa$ and κ be defined in Theorem 1. Then $L_0 = lX$ and $L_1 = \kappa + 1$ are compact LOTS' but $X \times \kappa$ is not a countably metacompact subspace of $L_0 \times L_1$.

The following three corollaries are easy consequences of the theorem above.

Corollary 9. *If X is a subspace of an ordinal and Y is a countably 1-compact GO-space, then $X \times Y$ is countably metacompact.*

Corollary 10. *If X is a countably 1-compact GO-space and Y is a countably compact GO-space, then $X \times Y$ is countably metacompact. In particular, $X \times \kappa$ is countably metacompact whenever X is a countably 1-compact GO-space and κ is a regular uncountable cardinal.*

Corollary 11. *Let X be a subspace of an ordinal. If X is paracompact, then $X \times Y$ is countably metacompact for every GO-space Y.*

Remark 12. Let κ be a regular uncountable cardinal and X_κ the space defined in Theorem 1. Considering $X = \kappa$ and $Y = X_\kappa$, we see that paracompactness in Corollary 11 cannot be removed. Also considering $X = X_\kappa$ and $Y = \kappa$, we see that the condition "Let X be a subspace of an ordinal" cannot be weakened to "Let X be a GO-space" in Corollary 11.

In fact, applying Theorem 1, we see that the converse implication of Corollary 11 is also true, because non-paracompact GO-space contains a closed set which is homeomorphic to a stationary set in a regular uncountable cardinal.

Corollary 13. *Let X be a subspace of an ordinal. Then X is paracompact if and only if $X \times Y$ is countably metacompact for every (hereditarily paracompact) GO-space Y.*

Remark 14. The space $X = X_\kappa$ in Theorem 1 is neither countably 0-compact nor countably 1-compact. To see this, fix a strictly increasing sequence $\{\alpha_n : n \in \omega\}$ in κ. Let define $x_n \in L(1)$ by $x_n(0) = \alpha_n$, moreover define $y_n \in L(n)$ by $y_n(m) = 0$ for every $m < n$. Then it is not hard to see that in X, $\{x_n : n \in \omega\}$ is a strictly increasing sequence without cluster points and $\{y_n : n \in \omega\}$ is a strictly decreasing sequence without cluster points.

All subspaces of an ordinal are countably 1-compact GO-spaces. Therefore, the corollary below is immediately obtained from Corollary 9.

Corollary 15. [5] *If X and Y are subspaces of ordinals, then $X \times Y$ is countably metacompact.*

Since the union of countably many countably metacompact closed subspaces is also countably metacompact, Corollary 10 yields:

Corollary 16. *If X is represented as the union of countably many countably 1-compact closed GO-subspaces and κ is a regular uncountable cardinal, then $X \times \kappa$ is countably metacompact.*

Remark 17. $\mathbb{R}$ denotes the real line with the usual order topology and $\mathbb{S}$ denotes the Sorgenfrey line declaring that sets of type $[x, \rightarrow)$ are open in $\mathbb{S}$. Let X be either $\mathbb{R}$ or $\mathbb{S}$. Moreover let $X_n = [-n, \rightarrow)$ for every $n \in \omega$ and κ a regular uncountable cardinal. Obviously X_n is countably 1-compact and closed in both cases. By the corollary above, $X \times \kappa$ is countably metacompact (there are other approaches to see this). However X is not countably 1-compact, so the converse of Theorem 7 is false.

Question 18. Find a condition on a GO-space X that is equivalent to countable metacompactness of $X \times \kappa$, where κ is a regular uncountable cardinal. Remark that normality of $X \times \kappa$ is characterized in terms of lX, see [4, Theorem 4.3].

Question 19. Find conditions on GO-spaces X and Y that imply hereditary countable metacompactness of $X \times Y$.

To prove Theorem 7, we need some tools handling GO-spaces which are appeared in [4]. For reader's convenience, we give their abstracts here.

At first, note that every subset A of a compact LOTS L has the least upper bound $\sup_L A$ and the greatest lower bound $\inf_L A$ in L, where $\sup_L \emptyset = \min L$ (= the smallest element of L) and $\inf_L \emptyset = \max L$ (= the largest element of L), see [1, Problem 3.12.3(a)].

Definition 20. Let L be a compact LOTS and $x \in L$. A subset $A \subset (\leftarrow, x)_L$ is said to be 0-*unbounded* for x in L if for every $y < x$, there is $a \in A$ with $y \leq a$. Similarly for a subset $A \subset (x, \rightarrow)_L$, "1-unbounded for x" is defined. Now 0-cofinality $0\text{-}\mathrm{cf}_L\, x$ of x in L is defined by:

$$0\text{-}\mathrm{cf}_L\, x = \min\{|A| : A \text{ is 0-unbounded for } x \text{ in } L.\}.$$

Also $1\text{-}\mathrm{cf}_L\, x$ is defined. If there are no confusion, we write simply $0\text{-}\mathrm{cf}\, x$ and $1\text{-}\mathrm{cf}\, x$. Observe that

- if x is the smallest element of L, then $0\text{-}\mathrm{cf}\, x = 0$,
- if x has the immediate predecessor in L, then $0\text{-}\mathrm{cf}\, x = 1$,
- otherwise, then $0\text{-}\mathrm{cf}\, x$ is a regular infinite cardinal.

Moreover, remark:

- When $\min L < x$, $\omega \leq 0\text{-}\mathrm{cf}\, x$ iff $\sup_L(\leftarrow, x)_L = x$ iff $x \in \mathrm{Cl}_L(\leftarrow, x)_L$.

Let $x \in L$ and $\kappa = 0\text{-}\mathrm{cf}\, x$. Then we can take a sequence $c : \kappa \to L$ which is strictly increasing and continuous as a function, and the range $\{c(\alpha) : \alpha < \kappa\}$ is a subset of $(\leftarrow, x)_L$ which is 0-unbounded for x in L. We call such c a 0-*normal sequence* for x in L. Similarly, a 1-*normal sequence* for x in L is defined. Obviously, if L is a compact LOTS such that $1\text{-}\mathrm{cf}\, x$ is 0 or 1 for every $x \in L$, then the linearly ordered set $\langle L, < \rangle$ is well-ordered, that is, every non-empty subset A of L has the $<$-smallest element.

In our discussion, we fix a linearly ordered compactification lX for each GO-space X, apply these methods for $L = lX$, and consider $0\text{-}\mathrm{cf}_{lX}\, a$ or $1\text{-}\mathrm{cf}_{lX}\, a$ for every $a \in lX$. We always fix a 0-normal sequence (similarly 1-normal sequence) $\{a(\alpha) : \alpha < \kappa\}$ for a, where $\kappa = 0\text{-}\mathrm{cf}\, a$. Observe that stationarity of $\{\alpha < \kappa : a(\alpha) \in X\}$ does not depend on the choices of 0-normal sequences whenever $\kappa \geq \omega_1$, see [4, Lemma 3.3]. Therefore the following notion is well-defined.

Definition 21. Let $a \in lX$, where X is a GO-space. We say that X is 0-*stationary at* a if $\kappa = 0\text{-}\mathrm{cf}_{lX}\, a$ is uncountable and $\{\alpha < \kappa : a(\alpha) \in X\}$ is stationary in κ, where $\{a(\alpha) : \alpha < \kappa\}$ is a 0-normal sequence for a. Note that if a GO-space X is paracompact, then X is not 0-stationary at a for every $a \in lX \setminus X$.

In particular, if X is a subspace of an ordinal, say $X \subset [0, \gamma]$, with the usual order, then we may consider that X is a GO-space and $lX = \mathrm{Cl}_{[0,\gamma]}\, X$. Moreover in this case, for every $a \in lX$, obviously $1\text{-}\mathrm{cf}\, a$ is 0 or 1, furthermore we can easily check that $0\text{-}\mathrm{cf}\, a$ is equal to $\mathrm{cf}\, a$ in the usual sense whenever a is a cluster point of X.

Let $a \in lX$ with $\kappa = 0\text{-}\operatorname{cf} a \geq \omega_1$, where X is a GO-space. If $\{\alpha < \kappa : a(\alpha) \in X\}$ is non-stationary in κ, then we can take a club set C in κ such that $\{a(\alpha) : \alpha \in C\} \subset lX \setminus X$. Then note that $(\leftarrow, a) \cap X$ can be represented as the disjoint sum $\bigcup_{\alpha \in \mathrm{Succ}(C)}((a(p_C(\alpha)), a(\alpha)) \cap X)$ of open subspaces, where $a(-1) = \leftarrow$.

Now let us prove Theorem 7. In the proof of this theorem, we frequently use the fact that if $\mathcal{V}_\lambda$ is a point finite collection of subsets of a space Z for every $\lambda \in \Lambda$ and the collection $\{\bigcup \mathcal{V}_\lambda : \lambda \in \Lambda\}$ is point finite, then $\bigcup_{\lambda \in \Lambda} \mathcal{V}_\lambda$ is also point finite.

A proof of Theorem 7.

Proof. Let $\mathcal{U}$ be a countable open cover of Z. We will find a point finite open refinement of $\mathcal{U}$. There are 5 Claims (Claim 1–Claim 5) in the proof of this theorem. The proof is analogous to the Heine-Borel proof which uses the completeness of $\mathbb{R}$ to show that every open cover of the unit interval $[0,1]$ has a finite subcover. Let us explain our plan. Define the subset A of lX below. Claim 1 shows that A is not empty. Claim 2 is a technical result used in the hardest subcase of Claim 3. Claim 3 asserts $\sup A \in A$. Claim 4 asserts that if $a \in A$ and $a \neq \max lX$ hold, then there is $a^* \in A$ with $a < a^*$. If X is well-ordered, then the simplest case of Claim 4 suffices. The other cases of Claim 4 require the assumption that Y is countably compact. Claim 5 puts the previous claims together and concludes that $\max lX \in A$, hence $\mathcal{U}$ has a point finite open refinement, and $X \times Y$ is countably metacompact.

Set

$$A = \{a' \in lX : \quad \text{there is a point finite partial open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ with } ((\leftarrow, a'] \cap X) \times Y \subset \bigcup \mathcal{V}\}.$$

Note that A is an initial segment of lX, that is, if $a'' < a' \in A$ then $a'' \in A$.

Claim 1. The following hold:

(1) for every $x \in X$, there is a point finite partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $\{x\} \times Y \subset \bigcup \mathcal{V}$,
(2) for every $y \in Y$, there is a point finite partial open refinement $\mathcal{V}$ of $\mathcal{U}$ with $X \times \{y\} \subset \bigcup \mathcal{V}$.

Proof. (1): Since Y is countably metacompact, there is a point finite open cover $\{V(U) : U \in \mathcal{U}\}$ of Y such that $\{x\} \times V(U) \subset U$ for every $U \in \mathcal{U}$. Then $\mathcal{V} = \{(X \times V(U)) \cap U : U \in \mathcal{U}\}$ is as desired. (2) is similar. □

Claim 1 (1) implies $\min lX \in A$, so A is not empty. Now we show:

Claim 2. Let $a \in lX$ and $b \in lY$. Assume that $(\leftarrow, a) \subset A$, X is 0-stationary at a and Y is 0-stationary at b. Then $\mathcal{U}$ has a point finite partial open refinement which covers $((a^*, a) \cap X) \times ((b^*, b) \cap Y)$ for some $a^* \in (\leftarrow, a)_{lX}$ and $b^* \in (\leftarrow, b)_{lY}$.

Proof. Let $\lambda = 0\text{-}\operatorname{cf} a$, $\mu = 0\text{-}\operatorname{cf} b$, $S = \{\alpha \in \lambda : a(\alpha) \in X\}$ and $T = \{\beta \in \mu : b(\beta) \in Y\}$, where $\{a(\alpha) : \alpha < \lambda\}$ and $\{b(\beta) : \beta < \mu\}$ are 0-normal sequences for a and b respectively. By the assumption, λ and μ are uncountable, moreover S and T are stationary in λ and μ respectively. For each $\alpha \in S \cap \operatorname{Lim}(\lambda)$ and $\beta \in T \cap \operatorname{Lim}(\mu)$, take $U(\alpha, \beta) \in \mathcal{U}$, $f(\alpha, \beta) < \alpha$, and $g(\alpha, \beta) < \beta$ with

$$((a(f(\alpha, \beta)), a(\alpha)] \cap X) \times ((b(g(\alpha, \beta)), b(\beta)] \cap Y) \subset U(\alpha, \beta).$$

There are 3 small Claims (Claim 2-1 — Claim 2-3) in the proof of Claim 2.

Claim 2-1. If $\lambda \leq \mu$, then there are $U_0^* \in \mathcal{U}$, $a^* \in (\leftarrow, a)_{lX}$, $S^* \subset S \cap \{\alpha \in \lambda : a^* < a(\alpha)\}$ which is stationary in λ, and a function $g^* : S^* \to \mu$ such that $Z_0^* \subset U_0^*$ holds, where

$$Z_0^* = \bigcup_{\alpha \in S^*} (((a^*, a(\alpha)] \cap X) \times ((b(g^*(\alpha)), b) \cap Y)).$$

Proof. Let $\alpha \in S \cap \operatorname{Lim}(\lambda)$. By $|\mathcal{U}| \leq \omega < \mu$, $|\alpha| \leq \alpha < \lambda \leq \mu$ and PDL, we can take $U_0(\alpha) \in \mathcal{U}$, $f_0(\alpha) < \alpha$, and $g_0(\alpha) < \mu$ such that

$$\{\beta \in T \cap \operatorname{Lim}(\mu) : U(\alpha, \beta) = U_0(\alpha), f(\alpha, \beta) = f_0(\alpha), g(\alpha, \beta) = g_0(\alpha)\}$$

is stationary in μ. By PDL again, we can take $U_0^* \in \mathcal{U}$ and $\alpha^* < \lambda$ such that

$$S^* := \{\alpha \in S \cap \operatorname{Lim}(\lambda) \cap (\alpha^*, \lambda) : U_0(\alpha) = U_0^*, f_0(\alpha) = \alpha^*\}$$

is stationary in λ. By putting $a^* = a(\alpha^*)$ and $g^* = g_0 \restriction S^*$, we obtain required U_0^*, a^*, S^* and g^*. $\square$

Similarly, we obtain the claim below.

Claim 2-2. If $\lambda \geq \mu$, then there are $U_1^* \in \mathcal{U}$, $b^* \in (\leftarrow, b)_{lY}$, $T^* \subset T \cap \{\beta \in \mu : b^* < b(\beta)\}$ which is stationary in μ, and a function $f^* : T^* \to \lambda$ such that $Z_1^* \subset U_1^*$, where

$$Z_1^* = \bigcup_{\beta \in T^*} (((a(f^*(\beta)), a) \cap X) \times ((b^*, b(\beta)] \cap Y)).$$

Claim 2-3. If $\lambda = \mu$ and $S \cap T$ is stationary in λ, then there are $U^*_{0,1} \in \mathcal{U}$, $a^* \in (\leftarrow, a)_{lX}$, and $b^* \in (\leftarrow, b)_{lY}$ such that $Z^*_{0,1} \subset U^*_{0,1}$, where

$$Z^*_{0,1} = (((a^*, a) \cap X) \times ((b^*, b) \cap Y)).$$

Proof. By PDL, we can take $U^*_{0,1} \in \mathcal{U}$ and $\alpha^*, \beta^* < \lambda$ such that

$$\{\xi \in S \cap T \cap \mathrm{Lim}(\lambda) : U(\xi, \xi) = U^*_{0,1}, f(\xi, \xi) = \alpha^*, g(\xi, \xi) = \beta^*\}$$

is stationary in λ. By putting $a^* = a(\alpha^*)$ and $b^* = b(\beta^*)$, we obtain required $U^*_{0,1}$, a^* and b^*. □

We return to the proof of Claim 2. There are four cases.

Case 2-1. $\lambda < \mu$.

In this case, let $b^* = \sup_{lY}\{b(g^*(\alpha)) : \alpha \in S^*\}$. Then we have $b^* < b$ because of $|S^*| \leq \lambda < \mu$. And $\mathcal{V} = \{U^*_0\}$ is a finite partial open refinement of $\mathcal{U}$ covering $((a^*, a) \cap X) \times ((b^*, b) \cap Y)$.

Case 2-2. $\lambda > \mu$.

In this case, let $a^* = \sup_{lX}\{a(f^*(\beta)) : \beta \in T^*\}$. Then similarly $\mathcal{V} = \{U^*_1\}$ is a finite partial open refinement of $\mathcal{U}$ covering $((a^*, a) \cap X) \times ((b^*, b) \cap Y)$.

Case 2-3. $\lambda = \mu$ and $S \cap T$ is stationary in λ.

In this case, $\mathcal{V} = \{U^*_{0,1}\}$ is a finite partial open refinement of $\mathcal{U}$ covering $((a^*, a) \cap X) \times ((b^*, b) \cap Y)$.

Case 2-4. $\lambda = \mu$ and $S \cap T$ is not stationary in λ.

In this case, take a club set $C \subset \mathrm{Lim}(S^*) \cap \mathrm{Lim}(T^*) \setminus (S \cap T)$ in λ such that

- $g^*(\alpha') < \beta$ holds for every $\beta \in C$ and $\alpha' \in S^* \cap \beta$,
- $f^*(\beta') < \alpha$ holds for every $\alpha \in C$ and $\beta' \in T^* \cap \alpha$.

For each $\alpha \in \mathrm{Succ}(C)$, let

$$\begin{aligned} Z^{**}_{0,\alpha} &= ((a(p_C(\alpha)), a(\alpha)) \cap X) \times ((\leftarrow, b(\alpha)) \cap Y), \\ Z^{**}_{1,\alpha} &= ((\leftarrow, a(\alpha)) \cap X) \times ((b(p_C(\alpha)), b(\alpha)) \cap Y). \end{aligned}$$

For each $\alpha \in \mathrm{Succ}(C)$, it follows from $a(\alpha) \in (\leftarrow, a)_{lX} \subset A$ that there are point finite partial open refinements $\mathcal{V}_{0,\alpha}$ and $\mathcal{V}_{1,\alpha}$ of $\mathcal{U}$ with $Z^{**}_{0,\alpha} = \bigcup \mathcal{V}_{0,\alpha}$ and $Z^{**}_{1,\alpha} = \bigcup \mathcal{V}_{1,\alpha}$. Since for $i \in 2 = \{0, 1\}$, $\mathcal{Z}^{**}_i = \{Z^{**}_{i,\alpha} : \alpha \in \mathrm{Succ}(C)\}$ is a pairwise disjoint collection of open sets in Z, $\mathcal{V}_i = \bigcup_{\alpha \in \mathrm{Succ}(C)} \mathcal{V}_{i,\alpha}$ is a point finite partial open refinement of $\mathcal{U}$ with $Z^{**}_i = \bigcup \mathcal{V}_i$, where $Z^{**}_i = \bigcup \mathcal{Z}^{**}_i$. Therefore $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \{U^*_0, U^*_1\}$ is a point finite partial open refinement of $\mathcal{U}$ which covers $Z^{**}_0 \cup Z^{**}_1 \cup Z^*_0 \cup Z^*_1$.

To see that Claim 2 holds in this case, it suffices to show that

$$((a^*,a)\cap X)\times((b^*,b)\cap Y)\subset Z_0^{**}\cup Z_1^{**}\cup Z_0^*\cup Z_1^*.$$

Let $\langle x,y\rangle\in((a^*,a)\cap X)\times((b^*,b)\cap Y)$. Take the smallest $\alpha\in C$ with $x\leq a(\alpha)$ and the smallest $\beta\in C$ with $y\leq b(\beta)$.

In case $x<a(\beta)$: If $y<b(\beta)$, then $\beta\in\mathrm{Succ}(C)$, $b(p_C(\beta))<y<b(\beta)$ and $x<a(\beta)$ hold. Therefore

$$\langle x,y\rangle\in((\leftarrow,a(\beta))\cap X)\times(b(p_C(\beta)),b(\beta))\cap Y)=Z_{1,\beta}^{**}\subset Z_1^{**}.$$

If $y=b(\beta)$, then because of $\beta\in C\subset\mathrm{Lim}(S^*)$ and $x<a(\beta)=\sup_{lX}\{a(\gamma):\gamma<\beta\}$, we find $\alpha'\in S^*\cap\beta$ with $x<a(\alpha')$. It follows from $\beta\in C$ that $g^*(\alpha')<\beta$. By $a^*<x<a(\alpha')$ and $b(g^*(\alpha'))<b(\beta)=y<b$, we have

$$\langle x,y\rangle\in((a^*,a(\alpha')]\cap X)\times(b(g^*(\alpha')),b)\cap Y)\subset Z_0^*.$$

In case $y<b(\alpha)$: In a similar way, we see that $\langle x,y\rangle\in Z_{0,\alpha}^{**}\subset Z_0^{**}$ holds if $x<a(\alpha)$, and $\langle x,y\rangle\in Z_1^*$ holds if $x=a(\alpha)$.

Another case (that is, $a(\beta)\leq x$ and $b(\alpha)\leq y$) does not happen. Otherwise, we have $\alpha\geq\beta$ by $a(\alpha)\geq x\geq a(\beta)$, moreover we have $\beta\geq\alpha$ by $b(\beta)\geq y\geq b(\alpha)$, therefore we have $\alpha=\beta$. It follows from $a(\alpha)=a(\beta)=x\in X$ and $b(\alpha)=b(\beta)=y\in Y$ that $\alpha\in S$ and $\beta\in T$. Because of $\alpha=\beta\in C$, we have $S\cap T\cap C\neq\emptyset$, a contradiction.

We see that $\mathcal{V}_0\cup\mathcal{V}_1\cup\{U_0^*,U_1^*\}$ is a point finite partial open refinement of $\mathcal{U}$ which covers $((a^*,a)\cap X)\times((b^*,b)\cap Y)$. The proof of Claim 2 is complete. □

Before now, the both assumptions (1) and (2) of Theorem 7 have not been required. The assumption (1) is only used in the following Claim.

Claim 3. Let $a\in lX$ with $(\leftarrow,a)_{lX}\subset A$. Then $\mathcal{U}$ has a point finite partial open refinement which covers $((\leftarrow,a)\cap X)\times Y$, consequently by Claim 1 (1), $a\in A$.

Proof. Let $a\in lX$ with $(\leftarrow,a)_{lX}\subset A$. And let $\lambda=0\text{-}\operatorname{cf}a$. Take a 0-normal sequence $\{a(\alpha):\alpha<\lambda\}$ for a in lX, and let $S=\{\alpha\in\lambda:a(\alpha)\in X\}$. Note that $\{a(\alpha):\alpha<\lambda\}\subset(\leftarrow,a)\subset A$. There are six Cases (Case 3-1 — Case 3-6) in the proof of Claim 3. Case 3-6 will not happen under the assumption (1A).

Case 3-1. $\lambda=0$.

In this case, we have $(\leftarrow,a)=\emptyset$, so the empty family is a point finite partial open refinement of $\mathcal{U}$ which covers $((\leftarrow,a)\cap X)\times Y=\emptyset$.

Case 3-2. $\lambda=1$.

In this case, $a' = \max(\leftarrow, a)_{lX}$ exists. It follows from $a' \in (\leftarrow, a) \subset A$ that $\mathcal{U}$ has a point finite partial open refinement which covers $((\leftarrow, a) \cap X) \times Y = ((\leftarrow, a'] \cap X) \times Y$.

Case 3-3. $\lambda = \omega$.

Let $X_n = (a(n-1), a(n+1)) \cap X$ for every $n \in \omega$, where $a(-1) = \leftarrow$. Note that $\{X_n \times Y : n \in \omega\}$ is a point finite collection of open sets in Z with $((\leftarrow, a) \cap X) \times Y = \bigcup_{n\in\omega}(X_n \times Y)$. By $a(n+1) \in A$, we can find a point finite partial open refinement $\mathcal{V}_n$ of $\mathcal{U}$ with $X_n \times Y = \bigcup \mathcal{V}_n$. Then $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ is a point finite partial open refinement of $\mathcal{U}$ with $((\leftarrow, a) \cap X) \times Y \subset \bigcup \mathcal{V}$.

Case 3-4. $\lambda > \omega$ and S is not stationary in λ.

Let C be a club set disjoint from S. For every $\alpha \in \mathrm{Succ}(C)$, let $X_\alpha = (a(p_C(\alpha)), a(\alpha)) \cap X$. By $a(\alpha) \in A$, one can take a point finite partial open refinement $\mathcal{V}_\alpha$ of $\mathcal{U}$ with $X_\alpha \times Y = \bigcup \mathcal{V}_\alpha$. Then $\mathcal{V} = \bigcup_{\alpha\in\mathrm{Succ}(C)} \mathcal{V}_\alpha$ is a point finite partial open refinement of $\mathcal{U}$ with $((\leftarrow, a) \cap X) \times Y \subset \bigcup \mathcal{V}$.

Case 3-5. $\lambda > \omega$, S is stationary in λ and $a \in X$.

In this case, for $b, b' \in lY$, define $b \sim b'$ by either one of the following:

(i) $b = b'$,

(ii) $b < b'$ and there are $\alpha_0 < \lambda$ and a point finite partial open refinement $\mathcal{V}$ of $\mathcal{U}$ covering $((a(\alpha_0), a) \cap X) \times ([b, b'] \cap Y)$,

(iii) $b' < b$ and there are $\alpha_0 < \lambda$ and a point finite partial open refinement $\mathcal{V}$ of $\mathcal{U}$ covering $((a(\alpha_0), a) \cap X) \times ([b', b] \cap Y)$,

Obviously $\sim$ is an equivalence relation on lY and each equivalence class is convex in lY. Let $\mathcal{E}$ be the collection of all equivalence classes meeting Y, that is, $\mathcal{E} = \{E \in lY/_\sim : E \cap Y \neq \emptyset\}$.

Claim 3-1. $E \cap Y$ is open in Y for every $E \in \mathcal{E}$.

Proof. Let $b \in E \cap Y$. It follows from $\langle a, b\rangle \in X \times Y$ that $(X \cap (a(\alpha_0), a]) \times J \subset U_0$ holds for some $U_0 \in \mathcal{U}$, $\alpha_0 < \lambda$ and an open convex set J in Y with $b \in J$. For each $y \in J$, α_0 and $\mathcal{V} = \{U_0\}$ witness $b \sim y$. Therefore J is a neighborhood of b contained in $E \cap Y$. □

Claim 3-2. For every $E \in \mathcal{E}$, there is a point finite partial open refinement $\mathcal{V}_E$ of $\mathcal{U}$ covering $((\leftarrow, a) \cap X) \times (E \cap Y)$.

Proof. Let $E \in \mathcal{E}, y \in E \cap Y$ and $b = \sup_{lY}(E \cap Y)$. Fix, for each $\alpha < \lambda$, a point finite partial open refinement $\mathcal{V}_\alpha$ of $\mathcal{U}$ covering $((\leftarrow, a(\alpha)] \cap X) \times Y$.

Claim 3-2-1. There is a point finite partial open refinement $\mathcal{V}'$ of $\mathcal{U}$ covering $((\leftarrow, a) \cap X) \times ([y, b] \cap Y)$.

Proof. First assume $b \in Y$. Since $b \in \mathrm{Cl}_{lY}(E \cap Y)$ and $E \cap Y$ is clopen in Y, we have $b \in E \cap Y$ thus $b \sim y$. Hence $\mathcal{U}$ has a point finite partial open refinement $\mathcal{W}$ of $\mathcal{U}$ covering $((a(\alpha_0), a) \cap X) \times ([y, b] \cap Y)$ for some $\alpha_0 < \lambda$. Then $\mathcal{V}' = \mathcal{V}_{\alpha_0} \cup \mathcal{W}$ is as desired.

Next assume $b \notin Y$. Then $0\text{-}\mathrm{cf}\, b \geq \omega$. Let $\mu = 0\text{-}\mathrm{cf}\, b$ and $T = \{\beta \in \mu : b(\beta) \in Y\}$, where $\{b(\beta) : \beta \in \mu\}$ is a 0-normal sequence for b in lY. Take $\beta^* < \mu$ with $y < b(\beta^*)$. For every $\beta \in \mu$ with $\beta^* \leq \beta$, using $y \sim b(\beta)$, fix $\alpha(\beta) < \lambda$ and a point finite partial open refinement $\mathcal{W}_\beta$ of $\mathcal{U}$ covering $((a(\alpha(\beta)), a) \cap X) \times ([y, b(\beta)] \cap Y)$. When $\mu = \omega$, $\mathcal{V}' = \{V \cap [X \times ((\leftarrow, b(\beta^* + 1)) \cap Y)] : V \in \mathcal{V}_{\alpha(\beta^*+1)} \cup \mathcal{W}_{\beta^*+1}\} \cup \bigcup_{\beta^* < \beta \in \omega} \{V \cap [X \times ((b(\beta - 1), b(\beta + 1)) \cap Y)] : V \in \mathcal{V}_{\alpha(\beta+1)} \cup \mathcal{W}_{\beta+1}\}$ is as desired. When $\mu > \omega$ and T is not stationary in μ, take a club set $D \subset (\beta^*, \mu)$ in μ disjoint from T. Then $\mathcal{V}' = \bigcup_{\beta \in \mathrm{Succ}(D)} \{V \cap [X \times ((b(p_D(\beta)), b(\beta)) \cap Y)] : V \in \mathcal{V}_{\alpha(\beta)} \cup \mathcal{W}_\beta\}$ is as desired. When $\mu > \omega$ and T is stationary in μ, by Claim 2, $\mathcal{U}$ has a point finite partial open refinement $\mathcal{V}^*$ covering $((a(\alpha_0), a) \cap X) \times ((b(\beta_0), b) \cap Y)$ for some $\alpha_0 \in \lambda$ and $\beta_0 \in \mu$. We may take α_0 as $\alpha(\beta_0) < \alpha_0$. Then $\mathcal{V}' = \mathcal{V}_{\alpha_0} \cup \mathcal{W}_{\beta_0} \cup \mathcal{V}^*$ is as desired. □

Let $b' = \inf_{lY}(E \cap Y)$. By a similar argument, we have:

Claim 3-2-2. There is a point finite partial open refinement $\mathcal{V}''$ of $\mathcal{U}$ covering $((\leftarrow, a) \cap X) \times ([b', y] \cap Y)$.

Putting $\mathcal{V}_E = \mathcal{V}' \cup \mathcal{V}''$, we see Claim 3-2. □

Now $\bigcup_{E \in \mathcal{E}} \{V \cap [X \times (E \cap Y)] : V \in \mathcal{V}_E\}$ is a point finite partial open refinement of $\mathcal{U}$ covering $((\leftarrow, a) \cap X) \times Y$. The Case 3-5 is complete.

Note that before now, we have not used the both assumptions (1) and (2). If X is paracompact (=(1A)), in particular, X is not 0-stationary at c for every $c \in lX \setminus X$, then the following remaining case does not happen. Therefore the proof of Claim 3 is complete in the case (1A). We continue the proof of Claim 3 in the case (1B) that Y is countably 1-compact.

Case 3-6. $\lambda > \omega$, S is stationary in λ and $a \notin X$.

In this case, set

$$B = \{b' \in lY : \text{ there is a point finite partial open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ with } ((\leftarrow, a) \cap X) \times ((\leftarrow, b'] \cap Y) \subset \bigcup \mathcal{V}\}.$$

Then B is also an initial segment of lY.

Claim 3-3. Let $b \in lY$ with $(\leftarrow, b)_{lY} \subset B$. Then $\mathcal{U}$ has a point finite partial open refinement which covers $((\leftarrow, a) \cap X) \times ((\leftarrow, b) \cap Y)$, consequently by Claim 1 (2), $b \in B$.

Proof. Let $b \in lY$ with $(\leftarrow, b)_{lY} \subset B$. And let $\mu = 0\text{-}\operatorname{cf} b$. Take a 0-normal sequence $\{b(\beta) : \beta < \mu\}$ for b in lY, and let $T = \{\beta \in \mu : b(\beta) \in Y\}$. Note that $\{b(\beta) : \beta < \mu\} \subset (\leftarrow, b) \subset B$. There are five Cases (Case 3-3-1 — Case 3-3-5) in the proof of Claim 3-3.

Case 3-3-1. $\mu = 0$.
Case 3-3-2. $\mu = 1$.
Case 3-3-3. $\mu = \omega$.
Case 3-3-4. $\mu > \omega$ and T is not stationary in μ.

In the four cases above, we can take a point finite partial open refinement of $\mathcal{U}$ which covers $((\leftarrow, a) \cap X) \times ((\leftarrow, b) \cap Y)$ in a similar way to Cases 3-1, 3-2, 3-3, 3-4 respectively.

Case 3-3-5. $\mu > \omega$ and T is stationary in μ.

Take $a^* \in (\leftarrow, a)_{lX}$ and $b^* \in (\leftarrow, b)_{lY}$ as in Claim 2. Then $\mathcal{U}$ has point finite partial open refinements

- covering $((\leftarrow, a^*] \cap X) \times Y$ by $a^* \in (\leftarrow, a)_{lX} \subset A$,
- covering $((\leftarrow, a) \cap X) \times ((\leftarrow, b^*] \cap Y)$ by $b^* \in (\leftarrow, b)_{lY} \subset B$,
- and covering $((a^*, a) \cap X) \times ((b^*, b) \cap Y)$ by Claim 2.

Hence, $\mathcal{U}$ has a point finite partial open refinement which covers $((\leftarrow, a) \cap X) \times ((\leftarrow, b) \cap Y)$. The proof of Claim 3-3 is complete. □

Still in Case 3-6, we prove:

Claim 3-4. If $b \in B$ and $b < \max lY$ hold, then there is $b^* \in B$ with $b < b^*$.

Proof. Let $b \in B$ with $b < \max lY$. We show the only one claim below in the proof of Claim 3-4, where we first require the assumption (1B) that Y is countably 1-compact.

Claim 3-4-1. $\mathcal{U}$ has a point finite partial open refinement which covers $((a^*, a) \cap X) \times ((b, b^*) \cap Y)$ for some $a^* \in (\leftarrow, a)_{lX}$ and $b^* \in (b, \rightarrow)_{lY}$.

Proof. Let $\mu = 1\text{-}\operatorname{cf} b$. Take a 1-normal sequence $\{b(\beta) : \beta < \mu\}$ for b in lY, and let $T = \{\beta \in \mu : b(\beta) \in Y\}$. By $b < \max lY$, we have $\mu \neq 0$. There are three cases in the proof of Claim 3-4-1.

Case 3-4-1-1. $\mu = 1$.

In this case, $b^* = \min(b, \rightarrow)_{lY}$ exists, and we have $(b, b^*) = \emptyset$. The empty family is a point finite partial open refinement of $\mathcal{U}$ which covers $((a^*, a) \cap X) \times ((b, b^*) \cap Y) = \emptyset$ for any $a^* \in (\leftarrow, a)_{lX}$.

Case 3-4-1-2. $\mu = \omega$.

In this case, by the assumption (1B), we see $b \in Y$. For every $\alpha \in S \cap \mathrm{Lim}(\lambda)$, by $\langle a(\alpha), b\rangle \in Z$, take $U(\alpha) \in \mathcal{U}$, $f(\alpha) < \alpha$ and $g(\alpha) < \mu$ such that

$$((a(f(\alpha)), a(\alpha)] \cap X) \times ([b, b(g(\alpha))) \cap Y) \subset U(\alpha).$$

Applying PDL, we can find $U \in \mathcal{U}$, $\alpha_0 < \lambda$ and $\beta_0 < \mu$ such that

$$\{\alpha \in S \cap \mathrm{Lim}(\lambda) : U(\alpha) = U, f(\alpha) = \alpha_0, g(\alpha) = \beta_0\}$$

is stationary in λ. By putting $a^* = a(\alpha_0)$ and $b^* = b(\beta_0)$, $\{U\}$ is a point finite partial open refinement of $\mathcal{U}$ covering $((a^*, a) \cap X) \times ((b, b^*) \cap Y)$.

Case 3-4-1-3. $\mu > \omega$.

In this case, for each $\beta \in \mu$ with $\mathrm{cf}\,\beta = \omega$, since 1-$\mathrm{cf}\, b(\beta) = \omega$, we have $b(\beta) \in Y$ by the assumption (1B). Hence, $\{\beta \in \mu : \mathrm{cf}\,\beta = \omega\}$ is contained in T thus T is stationary in μ. By an analogous result of Claim 2, we see that $\mathcal{U}$ has a point finite partial open refinement which covers $((a^*, a) \cap X) \times ((b, b^*) \cap Y)$ for some $a^* \in (\leftarrow, a)_{lX}$ and $b^* \in (b, \rightarrow)_{lY}$.

The proof of Claim 3-4-1 is complete. □

We return to the proof of Claim 3-4. Take $a^* \in (\leftarrow, a)_{lX}$ and $b^* \in (b, \rightarrow)_{lY}$ as in Claim 3-4-1. Then $\mathcal{U}$ has point finite partial open refinements

- covering $((\leftarrow, a) \cap X) \times ((\leftarrow, b] \cap Y)$ by $b \in B$,
- covering $((\leftarrow, a^*] \cap X) \times Y$ by $a^* \in (\leftarrow, a)_{lX} \subset A$,
- covering $((a^*, a) \cap X) \times ((b, b^*) \cap Y)$ by Claim 3-4-1,
- and covering $X \times \{b^*\}$, if $b^* \in Y$, by Claim 1 (2).

Therefore $\mathcal{U}$ has a point finite partial open refinement which covers $((\leftarrow, a) \cap X) \times ((\leftarrow, b^*] \cap Y)$. That is, $b^* \in B$ holds. And we have $b < b^*$. The proof of Claim 3-4 is complete. □

Claim 3-5. $\max lY \in B$.

Proof. Let $b = \sup_{lY} B$. Then $(\leftarrow, b)_{lY} \subset B$. By Claim 3-3, we have $b \in B$. If $b < \max lY$, then by Claim 3-4, there is a $b^* \in B$ with $b < b^*$. This contradicts that $b = \sup_{lY} B$. Hence, $\max lY = b \in B$ holds. □

Now by Claim 3-5, we have $\max lY \in B$. It follows from $Y = (\leftarrow, \max lY] \cap Y$ that $\mathcal{U}$ has a point finite partial open refinement which covers

$$((\leftarrow, a) \cap X) \times ((\leftarrow, \max lY] \cap Y) = ((\leftarrow, a) \cap X) \times Y.$$

Case 3-6 is finished and the proof of Claim 3 is complete. □

Note that still now, we have not used the assumption (2). This assumption is only used in the following claim.

Claim 4. If $a \in A$ and $a < \max lX$ hold, then there is $a^* \in A$ with $a < a^*$.

Proof. Let $a \in A$ and $a < \max lX$. We show the claim below.

Claim 4-1. $\mathcal{U}$ has a point finite partial open refinement which covers $((a, a^*) \cap X) \times Y$ for some $a^* \in (a, \rightarrow)_{lX}$.

Proof. Let $\lambda = 1\text{-}\mathrm{cf}\, a$. Take a 1-normal sequence $\{a(\alpha) : \alpha < \lambda\}$ for a in lX, and let $S = \{\alpha \in \lambda : a(\alpha) \in X\}$. By $a < \max lX$, we have $\lambda \neq 0$. We consider some cases in the proof of Claim 4-1.

Case 4-1-1. $\lambda = 1$.

In this case, $a^* = \min(a, \rightarrow)_{lX}$ exists, thus $(a, a^*) = \emptyset$. The empty family is a point finite partial open refinement of $\mathcal{U}$ which covers $((a, a^*) \cap X) \times Y$.

If X is well-ordered, then the remaining cases do not happen. Therefore the proof of Claim 4-1 is complete in the case (2A). We continue the proof of Claim 4-1 in the case (2B) that Y is countably compact. Assuming countable compactness of Y, we generally show:

Claim 4-1-1. If $c \in lX$ and $1\text{-}\mathrm{cf}\, c = \omega$, then there are $a^* \in (c, \rightarrow)_{lX}$ and a finite subfamily $\mathcal{U}^*$ of $\mathcal{U}$ such that $([c, a^*) \cap X) \times Y \subset \bigcup \mathcal{U}^*$.

Proof. Let $c \in lX$ with $1\text{-}\mathrm{cf}\, c = \omega$. By countable 1-compactness of X, we have $c \in X$. Take a 1-normal sequence $\{c(k) : k \in \omega\}$ for c in lX. For each $k \in \omega$ and $U \in \mathcal{U}$, let

$$V_k(U) = \bigcup \{V \subset Y : V \text{ is open in } Y,\ ([c, c(k)) \cap X) \times V \subset U\}.$$

Then $\{V_k(U) : k \in \omega\}$ is increasing for every $U \in \mathcal{U}$ and $\{V_k(U) : k \in \omega, U \in \mathcal{U}\}$ is a countable open cover of Y. Since Y is countably compact(=(2B)), there are $m \in \omega$ and a finite subfamily $\mathcal{U}^*$ of $\mathcal{U}$ such that $Y = \bigcup_{U \in \mathcal{U}^*} V_m(U)$. Then $a^* = c(m)$ and $\mathcal{U}^*$ satisfy the required condition. Actually, let $\langle x, y \rangle \in ([c, a^*) \cap X) \times Y$. Then there is $U \in \mathcal{U}^*$ with $y \in V_m(U)$. Therefore $y \in V$ for some open set V in Y with $([c, c(m)) \cap X) \times V \subset U$. By $c \leq x < a^* = c(m)$, we have

$\langle x, y\rangle \in ([c, c(m))\cap X)\times V \subset U \subset \bigcup\mathcal{U}^*$. Hence, $([c, a^*)\cap X)\times Y \subset \bigcup\mathcal{U}^*$ holds. □

We continue the remaining two cases assuming (2B).

Case 4-1-2. $\lambda = \omega$.

In this case, by applying Claim 4-1-1 for a, we obtain $a^* \in (a, \rightarrow)_{lX}$ and a finite subfamily $\mathcal{U}^*$ of $\mathcal{U}$ such that $([a, a^*)\cap X)\times Y \subset \bigcup\mathcal{U}^*$ holds. In particular, $\mathcal{U}^*$ is a point finite partial open refinement of $\mathcal{U}$.

Case 4-1-3. $\lambda > \omega$.

For each $\alpha \in \lambda$ with $\mathrm{cf}\,\alpha = \omega$, because of $1\text{-}\mathrm{cf}\,a(\alpha) = \omega$ and Claim 4-1-1, there are $a^*(\alpha) \in (a(\alpha), \rightarrow)_{lX}$ and a finite subfamily $\mathcal{U}^*(\alpha)$ such that $([a(\alpha), a^*(\alpha))\cap X)\times Y \subset \bigcup\mathcal{U}^*(\alpha)$. Take $f(\alpha) < \alpha$ with $a(f(\alpha)) \leq a^*(\alpha)$. By PDL, we can take $\alpha_0 < \lambda$ and a finite subfamily $\mathcal{U}^*$ of $\mathcal{U}$ such that

$$\{\alpha \in \lambda : \mathrm{cf}(\alpha) = \omega, \mathcal{U}^*(\alpha) = \mathcal{U}^*, f(\alpha) = \alpha_0\}$$

is stationary in λ. Put $a^* = a(\alpha_0)$. Then $\mathcal{U}^*$ is a point finite partial open refinement of $\mathcal{U}$ which covers $((a, a^*) \cap X) \times Y$.

The proof of Claim 4-1 is complete. □

We return to the proof of Claim 4. Let $a^* \in (a, \rightarrow)_{lX}$ be as in Claim 4-1. Then $\mathcal{U}$ has point finite partial open refinements

- covering $((\leftarrow, a] \cap X) \times Y$ by $a \in A$,
- covering $((a, a^*) \cap X) \times Y$ by Claim 4-1,
- and covering $\{a^*\} \times Y$, if $a^* \in X$, by Claim 1 (1).

Therefore $\mathcal{U}$ has a point finite partial open refinement which covers $((\leftarrow, a^*] \cap X) \times Y$. That is $a^* \in A$ holds. And we have $a < a^*$. The proof of Claim 4 is complete. □

In the rest of the proof, we only use Claims 3 and 4 as well as Claims 1 and 2, but not assumptions (1) and (2) themselves.

Claim 5. $\max lX \in A$.

Proof. Let $a = \sup_{lX} A$. Then $(\leftarrow, a)_{lX} \subset A$. By Claim 3, we have $a \in A$. If $a < \max lX$, then by Claim 4, there is $a^* \in A$ with $a < a^*$. This contradicts that $a = \sup_{lX} A$. Hence, $\max lX = a \in A$ holds. □

We return to the theorem. By Claim 5, we have $\max lX \in A$. It follows from $X = (\leftarrow, \max lX] \cap X$ that $\mathcal{U}$ has a point finite (partial) open refinement which covers

$$((\leftarrow, \max lX] \cap X) \times Y = X \times Y.$$

Hence, $X \times Y$ is countably metacompact. The proof of the theorem is complete. □

Applying some claims in the proof of Theorem 7, we obtain the corollary below.

Corollary 22. *Let X be a GO-space, and Y a space. And assume that X and Y are countably compact. Then $X \times Y$ is countably metacompact.*

Proof. Let $\mathcal{U}$ be a countable open cover of $X \times Y$. Let $x \in lX$. Since Y is countably metacompact, we can apply Claim 1 (1), and obtain a point finite partial open refinement $\mathcal{V}_2(x)$ of $\mathcal{U}$ such that $(X \cap \{x\}) \times Y \subset \bigcup \mathcal{V}_2(x)$. Since X is countably 1-compact and Y is countably compact, we can apply Claim 4-1, and obtain a point finite partial open refinement $\mathcal{V}_1(x)$ of $\mathcal{U}$ such that $(X \cap (x, x^*)) \times Y \subset \bigcup \mathcal{V}_1(x)$ for some $x^* \in (x, \rightarrow]$. Since X is also countably 0-compact and Y is countably compact, and by symmetry, we obtain a point finite partial open refinement $\mathcal{V}_0(x)$ of $\mathcal{U}$ such that $(X \cap (x_*, x)) \times Y \subset \bigcup \mathcal{V}_0(x)$ for some $x_* \in [\leftarrow, x)$. By putting $P(x) = (x_*, x^*)$ and $\mathcal{V}(x) = \bigcup_{i \in 3} \mathcal{V}_i(x)$, we obtain a neighborhood $P(x)$ of x in lX and a point finite partial open refinement $\mathcal{V}(x)$ of $\mathcal{U}$ such that $(X \cap P(x)) \times Y \subset \bigcup \mathcal{V}(x)$. By compactness of lX, we obtain a finite subset M of lX such that $lX = \bigcup_{x \in M} P(x)$. By putting $\mathcal{V} = \bigcup_{x \in M} \mathcal{V}(x)$, we obtain a point finite open refinement $\mathcal{V}$ of $\mathcal{U}$. Hence, $X \times Y$ is countably metacompact. □

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