THE HEREDITARILY COLLECTIONWISE HAUSDORFF PROPERTY IN PRODUCTS OF ω_1

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ABSTRACT. In this paper, we will show that

- All subspaces of finite powers of ω_1 are collectionwise Hausdorff and α-normal.
- There is a subspace of an infinite power of ω_1 which is neither collectionwise Hausdorff nor α-normal.

Spaces are assumed to be regular T_1. A space X is Collectionwise Hausdorff (CWH) if every closed discrete subspace D of X is separated, that is, there is a pairwise disjoint collecion $\{U(x) : x \in D\}$ of open sets of X such that $x \in U(x)$ for each $x \in D$. Disjoint closed sets F_0 and F_1 are separated if there are open sets U_0 and U_1 such that $F_i \subset U_i$, $i \in 2 = \{0, 1\}$, and $U_0 \cap U_1 = \emptyset$. Disjoint closed sets F_0 and F_1 are α-separated (β-separated) if there are open sets U_0 and U_1 such that $\mathrm{Cl}(F_i \cap U_i) = F_i$, $i \in 2 = \{0, 1\}$, and $U_0 \cap U_1 = \emptyset$ ($\mathrm{Cl}U_0 \cap \mathrm{Cl}U_1 = \emptyset$, respectively). A space is normal (α-normal, β-normal) if disjoint closed sets are separated (α-separated, β-separated). α-normality and β-normality are defined by Arkhangel'skiĭ and studied in [1, 4].

It is well known that the Tychonoff plank $(\omega + 1) \times (\omega_1 + 1) \setminus \{\langle\omega, \omega_1\rangle\}$ and $\omega_1 \times (\omega_1 + 1)$ are not α-normal, see [4, 2]. On the other hand, it is not difficult to show that these spaces are CWH. Moreover it is known that if A and B are disjoint stationary sets in ω_1, then $A \times B$ is not normal, see [3]. In this paper, we prove the results written in the abstract above.

As usual, an ordinal is equal to the set of all smaller ordinals, for example, $n = \{0, 1, 2, \cdots, n-1\}$ for each natural number n. The symbols ω and ω_1 stand for the set of all finite and respectively all countable ordinals. For notational conveniences, we consider -1 as the immediate predecessor of the smallest ordinal 0. As usual, we consider that ordinals have the usual order topology and product spaces have the usual Tychonoff product topology.

For $A \subset \omega_1$, put $\mathrm{Lim}(A) = \{\alpha < \omega_1 : \sup(A \cap \alpha) = \alpha\}$, $\mathrm{Succ}(A) = A \setminus \mathrm{Lim}(A)$, $\mathrm{Lim} = \mathrm{Lim}(\omega_1)$ and $\mathrm{Succ} = \mathrm{Succ}(\omega_1)$, where $\sup \emptyset = -1$. Observe that $\mathrm{Lim}(A)$ is closed and unbounded (club) in ω_1 whenever A is cofinal in ω_1. For a club set $C \subset \omega_1$ and $\alpha < \omega_1$, set $p_C(\alpha) = \sup(C \cap \alpha)$. Observe that $p_C(\alpha) \in C \cup \{-1\}$ holds and $p_C(\alpha) = \alpha$ iff $\alpha \in \mathrm{Lim}(C)$ is also holds. Note that if $\alpha \in \mathrm{Succ}(C)$, then $p_C(\alpha)$ is considered as the immediate predecessor of α in C. It is easy to show $\omega_1 \setminus C = \bigcup_{\alpha \in \mathrm{Succ}(C)} (p_C(\alpha), \alpha)$, where (α, β) denotes the usual open interval.

For sets A and B, B^A denotes the set of all functions on A to B and $\mathcal{P}(A)$ denotes the set of all subsets of A.

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Remark. $X = \omega \times \omega_1 \cup \{\omega\} \times \mathrm{Succ}$ is known to be non-normal by showing that $F_0 = \{\omega\} \times \mathrm{Succ}$ and $F_1 = \omega \times \mathrm{Lim}$ cannot be separated.

If X were β-normal, then F_0 and F_1 would be β-separated by open sets U_0 and U_1 respectively. Then, since F_0 is closed discrete and $\mathrm{Cl}(F_0 \cap U_0) = F_0$, actually U_0 includes F_0. Since $\mathrm{Cl}U_0 \cap F_1 \subset \mathrm{Cl}U_0 \cap \mathrm{Cl}U_1 = \emptyset$, F_0 and F_1 are separated by U_0 and $X \setminus \mathrm{Cl}U_0$, a contradiction. Thus X is not β-normal. However Theorem 1 shows that X is α-normal.

This example answers negatively, in ZFC, a problem posed by Arkhangel'skiĭ (see also Problem 7.39 of [4]):

Is a first countable, Tychonoff, α-normal space normal?

Theorem 1. *All subspaces of finite powers of ω_1 are CWH and α-normal.*

Proof. First we give a direct proof of the α-normality. Let F_0 and F_1 be disjoint closed sets in $X \subset \omega_1^n$, where $1 \leq n \in \omega$. For each $K \in \mathcal{P}(n) \setminus \{\emptyset\}$, $s \in \omega_1^{n \setminus K}$ and $\alpha < \omega_1$, define $x_{\alpha s} \in \omega_1^n$ by

$$x_{\alpha s}(k) = \begin{cases} s(k), & \text{if } k \in n \setminus K, \\ \alpha, & \text{if } k \in K. \end{cases}$$

Moreover for each $Y \subset \omega_1^n$, set

$$V_s(Y) = \{\alpha < \omega_1 : x_{\alpha s} \in Y\}.$$

For each $K \in \mathcal{P}(n) \setminus \{\emptyset\}$ and $s \in \omega_1^{n \setminus K}$, take a club set C_s in ω_1 such that

- if $i \in 2$ and $V_s(F_i)$ is cofinal, then $C_s \subset \mathrm{Lim}(V_s(F_i))$,
- if $i \in 2$ and $V_s(F_i)$ is non-stationary in ω_1, then $V_s(F_i) \cap C_s = \emptyset$.

Then it is straightforward to show that

$$C = \{\alpha < \omega_1 : \forall K \in \mathcal{P}(n) \setminus \{\emptyset\} \forall s \in \alpha^{n \setminus K} (\alpha \in C_s)\}$$

is a club set in ω_1.

Now for each $\alpha \in \mathrm{Succ}(C)$ and $K \in \mathcal{P}(n) \setminus \{\emptyset\}$, let

$$X(\alpha, K) = \{x \in X : \forall k \in n \setminus K (x(k) \leq p_C(\alpha)), \forall k \in K (p_C(\alpha) < x(k) \leq \alpha)\}.$$

Then $\{X(\alpha, K) : \alpha \in \mathrm{Succ}(C), K \in \mathcal{P}(n) \setminus \{\emptyset\}\}$ is a pairwise disjoint collection of open sets in X. Since each $X(\alpha, K)$ is countable (hence normal), there are disjoint open sets $U_0(\alpha, K)$ and $U_1(\alpha, K)$ in X such that $X(\alpha, K) \cap F_i \subset U_i(\alpha, K) \subset X(\alpha, K)$, $i \in 2$. Set for each $i \in 2$,

$$U_i = \bigcup \{U_i(\alpha, K) : \alpha \in \mathrm{Succ}(C), K \in \mathcal{P}(n) \setminus \{\emptyset\}\}.$$

Since U_0 and U_1 are disjoint, it suffices to show that $F_i \subset \mathrm{Cl}(F_i \cap U_i)$. To do this, let $x \in F_i$ and $\alpha = \min\{\alpha \in C : \max\{x(k) : k \in n\} \leq \alpha\}$.

Case 1. $\alpha \in \mathrm{Succ}(C)$.

Let $K = \{k \in n : p_C(\alpha) < x(k) \leq \alpha\}$. Then $K \neq \emptyset$ and $x \in X(\alpha, K) \cap F_i \subset F_i \cap U_i \subset \mathrm{Cl}(F_i \cap U_i)$.

Case 2. $\alpha \in \mathrm{Lim}(C)$.

Since $x(k) = \alpha$ for some $k \in n$, $K = \{k \in n : x(k) = \alpha\}$ is not empty. Set $s = x \restriction (n \setminus K)$. Then $s \in \alpha^{n \setminus K}$ and by the definition of C, we have $\alpha \in C_s \cap V_s(F_i)$. Moreover by the definition of C_s, $V_s(F_i)$ is stationary hence cofinal in ω_1, thus we have $C_s \subset \mathrm{Lim}(V_s(F_i))$. Let W be a neighborhood of x, then there is $\gamma < \alpha$ such that $s \in \gamma^{n\setminus K}$ and $X \cap \{x_{\beta s} : \beta \in (\gamma, \alpha]\} \subset W$. Since $\gamma < \alpha \in \mathrm{Lim}(C)$, there is $\delta \in \mathrm{Succ}(C)$ such that $\gamma < p_C(\delta) < \delta < \alpha$. Then it follows from $\delta \in C \cap \alpha$ and $s \in \delta^{n\setminus K}$ that $\gamma < p_C(\delta) < \delta \in C_s \subset \mathrm{Lim}(V_s(F_i))$. Pick $\beta \in V_s(F_i)$ with $p_C(\delta) < \beta < \delta$. Then

$$x_{\beta s} \in W \cap F_i \cap X(\delta, K) \subset$$
$$W \cap F_i \cap U_i(\delta, K) \subset W \cap F_i \cap U_i.$$

Thus $x \in \mathrm{Cl}(F_i \cap U_i)$.

Next we show the CWH-ness. This proof is almost similar to the proof above, so we only give its abstract proof. Let D be a closed discrete subspace of $X \subset \omega_1^n$. For each $K \in \mathcal{P}(n) \setminus \{\emptyset\}$ and $s \in \omega_1^{n\setminus K}$, since $V_s(D)$ is not stationary, take a club set C_s in ω_1 which is disjoint from $V_s(D)$. Similarly as above let

$$C = \{\alpha < \omega_1 : \forall K \in \mathcal{P}(n) \setminus \{\emptyset\} \forall s \in \alpha^{n\setminus K}(\alpha \in C_s)\},$$

$$X(\alpha, K) = \{x \in X : \forall k \in n \setminus K(x(k) \le p_C(\alpha)), \forall k \in K(p_C(\alpha) < x(k) \le \alpha)\}$$

for each $\alpha \in \mathrm{Succ}(C)$ and $K \in \mathcal{P}(n) \setminus \{\emptyset\}$. Moreover let

$$U = \bigcup \{X(\alpha, K) : \alpha \in \mathrm{Succ}(C), K \in \mathcal{P}(n) \setminus \{\emptyset\}\}.$$

Then as in the proof of $F_i \subset \mathrm{Cl}(F_i \cap U_i)$ above, we can also show $D \subset U$. Since $\{X(\alpha, K) : \alpha \in \mathrm{Succ}(C), K \in \mathcal{P}(n) \setminus \{\emptyset\}\}$ is a pairwise disjoint collection of countable regular (hence CWH) open subspaces of X, U is also CWH. Since D is separated in U, it is in fact separated in X.

J. E. Porter [6] informed us that scattered hereditarily CWH spaces are hereditarily α-normal, see [5], so we can also get hereditarily α-normality of finite powers of ω_1 from its hereditarily CWH-ness. □

Remark. Note that the Tychonoff Plank and $\omega_1 \times (\omega_1 + 1)$ are scattered CWH but not α-normal spaces.

Theorem 2. *There is a subspace of the product of denumerably many copies of ω_1 which is neither CWH nor α-normal.*

Proof. Let $\{S_n : n \in \omega\}$ be a decreasing sequence of stationary sets in ω_1 with the empty intersection. We will show that the closed subspace

$$X = \{x \in \prod_{n\in\omega} S_n : \forall n \in \omega(x(n) \le x(n+1))\}$$

of $\prod_{n\in\omega} S_n$ is neither CWH nor α-normal.

For each $n \in \omega$ and $\alpha \in S_n$, fix $x_{\alpha n} \in X$ such that

$$\alpha = x_{\alpha n}(0) = \cdots = x_{\alpha n}(n) < x_{\alpha n}(n+1)$$

and let

$$\gamma(\alpha, n) = \sup\{x_{\alpha n}(k) : k \in \omega\}.$$

Then

$$C = \mathrm{Lim}(S_0) \cap \bigcap_{n\in\omega} \{\beta < \omega_1 : \forall \alpha \in S_n \cap \beta(\gamma(\alpha, n) < \beta)\}$$

is a club set in ω_1. Since $C \subset \mathrm{Lim}(S_0)$, $|S_0 \setminus C| = \omega_1$ holds. So take an uncountable subset $E \subset S_0 \setminus C$ such that $p_C(\alpha) \ne p_C(\alpha')$ whenever $\alpha \ne \alpha' \in E$.

Let
$$F_0 = \{x_{\alpha 0} : \alpha \in E\}, F_1 = \bigcup_{n\in\omega} \{x_{\alpha n} : \alpha \in S_n \cap C\}.$$
Obviously F_0 and F_1 are disjoint. We show:

Claim. Both F_0 and F_1 are closed discrete.

Proof. Let $x \in X$, $\beta = x(0)$ and $n \in \omega$ be the smallest n such that $x(n) < x(n+1)$.

Case 1. $\beta \in C$.

Consider the open neighborhood
$$U = \{y \in X : \forall k \leq n(y(k) \leq \beta), y(n+1) \in (\beta, x(n+1)]\}$$
of x. If $\beta < \alpha \in S_m$ and $m \in \omega$, then obviously $x_{\alpha m} \notin U$ holds. If $\beta > \alpha \in S_m$ and $m \in \omega$, then by $\alpha < \beta \in C$, we have $\gamma(\alpha, m) < \beta$. Therefore $x_{\alpha m}(n+1) < \beta$ so $x_{\alpha m} \notin U$ holds. If $\beta = \alpha \in S_m$ and $n \neq m \in \omega$, then obviously $x_{\alpha m} \notin U$ holds. This argument shows $|U \cap F_1| \leq 1$. Similarly we can show $U \cap F_0 = \emptyset$.

Case 2. $\beta \notin C$.

In this case, the open neighborhood
$$U = \{y \in X : y(0) \in (p_C(\beta), \beta]\}$$
of x misses F_1 and satisfies $|U \cap F_0| \leq 1$.

Assume that F_0 and F_1 are α-separated by U_0 and U_1 respectively. By the Claim, in fact, F_0 and F_1 are separated by U_0 and U_1. For each $\alpha \in E$, by $x_{\alpha 0} \in U_0$, there is $n(\alpha) \in \omega$ such that
$$V_\alpha = \{y \in X : \forall k \leq n(\alpha)(x_{\alpha 0}(k) = y(k))\} \subset U_0.$$
Then there are an uncountable subset $E' \subset E$ and $n \in \omega$ such that $n(\alpha) = n$ for all $\alpha \in E'$. Pick $\beta \in \mathrm{Lim}(E') \cap C \cap S_n$, then $x_{\beta n} \in F_1 \subset U_1$.

Since U_1 is open, there are $i > n$ and $\gamma < \beta$ such that
$$V = \{y \in X : \forall k \leq n(y(k) \in (\gamma, \beta]), n < \forall k \leq i(y(k) = x_{\beta n}(k))\} \subset U_1.$$
By $\gamma < \beta \in \mathrm{Lim}(E')$, fix $\alpha \in E'$ with $\gamma < \alpha < \beta$. Let
$$x(k) = \begin{cases} x_{\alpha 0}(k), & \text{if } k \leq n, \\ x_{\beta n}(k), & \text{if } k > n. \end{cases}$$
Then $x \in V_\alpha \cap V \subset U_0 \cap U_1$, a contradiction. This shows that X is not α-normal.

Now, consider the closed discrete subspace $D = F_0 \cup F_1$. If D were separated by $\{U(x) : x \in D\}$, then F_0 and F_1 would be separated by $\bigcup\{U(x) : x \in F_0\}$ and $\bigcup\{U(x) : x \in F_1\}$, a contradiction. This shows that X is not CWH. This completes the proof. □

By Theorem 1 and a result in [3], if A and B are disjoint stationary sets in ω_1, then $A \times B$ is CWH and α-normal but not normal. In connection with Theorem 2, it is natural to ask:

Problem. If $\{S_n : n \in \omega\}$ is a pairwise disjoint sequence of stationary sets in ω_1, then is the product $\prod_{n\in\omega} S_n$ neither CWH nor α-normal?

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