CERTAIN SEQUENCES WITH COMPACT CLOSURE

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Dedicated to the Memory of Professor Jan Pelant

Abstract. This paper deals with a question which is stated by quite simple definitions. A sequence $\{x_n\}$ in a space X is called a β-*sequence* if every subsequence of it has a cluster point in X. The *closure* of the sequence $\{x_n\}$ means the closure of $\{x_n : n \in \omega\}$ in X. Here we consider the question when a β-sequence has compact closure. We give several answers to this question.

1. Introduction

Throughout this paper, all spaces are assumed to be *Hausdorff*.

A *sequence* in a space X is a function φ from ω into X, which is denoted by $\{x_n\}$ if $\varphi(n) = x_n$ for each $n \in \omega$, where ω is the first infinite ordinal. For a subspace A in X, we denote by $\mathrm{Cl}\, A$ the closure of A in X.

For a sequence $\{x_n\}$ in a space X, the *closure* of $\{x_n\}$ means the closure of its range in X, that is, $\mathrm{Cl}\{x_n : n \in \omega\}$.

Let us begin with the following simple definitions, which is a key of this paper.

Definition 1. A sequence $\{x_n\}$ in a space X is called a β-*sequence* if every subsequence of it has a cluster point in X.

Remark. Recall that a space X is called *e-countably compact* with respect to a dense subset D if every sequence in D has a cluster point in X (see [S]). Using this term, a sequence is a β-sequence iff its closure is e-countably compact with respect to its range.

It is well known that a space X is countably compact iff every sequence in X is a β-sequence. However, the concept of β-sequences is rather motivated by the definitions of many generalized metric spaces such as M-spaces, $w\Delta$-spaces, Σ-spaces, β-spaces, q-spaces and so on. In fact, they are defined by the following form:

(∗) If there is $y \in X$ such that x_n and y have some relation $\sim_n$ (depending on n) for each $n \in \omega$, then $\{x_n\}$ has a cluster point in X.

As one example, recall that a space X is called a $w\Delta$-*space* if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that if $x_n \in \mathrm{St}(y, \mathcal{U}_n)$ for each $n \in \omega$, then $\{x_n\}$ has a cluster point in X. Where $x_n \in \mathrm{St}(y, \mathcal{U}_n)$ is an example of the relation $\sim_n$ of x_n and y. In this case, assuming that each $\mathcal{U}_n$ refines $\mathcal{U}_{n-1}$, the sequence $\{x_n\}$ is a β-sequence. In fact,

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in almost all such generalized metric spaces, we may assume without loss of generality that $\{x_n\}$ is a β-sequence in X (also see Proposition 4.1 below for the case of β-spaces).

Obviously, every sequence $\{x_n\}$ with (countably) compact closure is a β-sequence in a space X. Then it is natural to consider when the converse is true. In fact, we come up with the following question.

Question. When does a β-sequence $\{x_n\}$ in a space X have compact closure?

Of course, it is not true without any condition. Because the space $X = \beta\omega \setminus \{p\}$, where $p \in \beta\omega \setminus \omega$, is countably compact, separable and non-compact (see [E, Example 3.10.18]). On the other hand, it is not difficult to see that it is true under the paracompactness of X (see [Y1] or Corollary 2.9 below).

In Section 2, we observe that every β-sequences has pseudocompact closure. In Section 3, we give some equivalent conditions to compact closure of β-sequences. In Section 4, we immediately apply the previous results to the argument when β-spaces are strong β-spaces, as dealt with in [Y1]. In Section 5, we show that β-sequences with countable closure are expressed by fairly concrete forms. In Sections 6, we give some examples in the negative aspects for β-sequences.

2. Closure of β-sequences

Recall that a space X is *feebly compact* if every locally finite collection of open sets in X is at most finite.

The following is pointed out by the referee.

Proposition 2.1. *Every β-sequence in a space X has feebly compact closure.*

Proof. Assume the contrary. Let $\{x_n\}$ be a β-sequence with not feebly compact closure. There is an infinite countable locally finite collection $\{V_i : i \in \omega\}$ of open sets in $\mathrm{Cl}\{x_n : n \in \omega\}$. One can take an $x_{n_i} \in V_i$ for each $i \in \omega$. Then $\{x_{n_i} : i \in \omega\}$ is a discrete closed set in X. This contradicts that $\{x_{n_i}\}$ has a cluster point in X. □

Since every feebly compact space is pseudocompact (the converse is also true if the space is Tychonoff, as seen in [E, Theorem 3.10.22]), the above immediately yields

Corollary 2.2. *Every β-sequence in a space X has pseudocompact closure.*

Since a space is countably compact if it is normal and pseudocompact or if it is countably paracompact and feebly compact, we also have

Corollary 2.3. *If a space X is either normal or countably paracompact, then every β-sequence in a space X has countably compact closure*

Weiss [W] proved that every countably compact, perfectly normal space is compact under Martin's Axiom (MA) and $2^{\aleph_0} > \aleph_1$. The combination of Corollary 2.3 and this result immediately yield

Corollary 2.4. *Assume MA and $2^{\aleph_0} > \aleph_1$. Every β-sequence in a perfectly normal space X has compact closure.*

3. β-SEQUENCES WITH COMPACT CLOSURE

First, we give an auxiliary concept for closure of sequences.

Definition 2. For a sequence $\{x_n\}$ in a space X, the set of all cluster points of $\{x_n\}$, that is, $\bigcap_{k\in\omega} \mathrm{Cl}\{x_n : n \geq k\}$ is called the *cluster* of $\{x_n\}$.

Proposition 3.1. *Let $\{x_n\}$ be a sequence in a space X.*

(1) *If E and F be the closure and the cluster of $\{x_n\}$, respectively, then $E = \{x_n : n \in \omega\} \cup F$ holds.*
(2) *A β-sequence $\{x_n\}$ in X has compact closure if and only if it has compact cluster.*

Proof. (1): This is easily seen.

(2): Let E and F be the closure and the cluster of $\{x_n\}$, respectively. Since F is a closed subset of E, if E is compact, then so is F. Assume that F is compact. Let $\mathcal{U}$ be an open cover of E. There is a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ with $F \subset \bigcup \mathcal{V}$. It suffices to show that $|E \setminus \bigcup \mathcal{V}| < \omega$. Assume the contrary. We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \notin \bigcup \mathcal{V}$ for each $i \in \omega$. Since $\{x_n\}$ is a β-sequence in X, $\{x_{n_i}\}$ has a cluster point y in F. However, by the choice of x_{n_i}'s, y is not in $\bigcup \mathcal{V}$. This contradicts $F \subset \bigcup \mathcal{V}$. □

Proposition 3.2. *For a β-sequence $\{x_n\}$ in a regular space X, the following are equivalent.*

(a) *$\{x_n\}$ has compact closure in X.*
(b) *$\{x_n\}$ has Lindelöf closure in X.*
(c) *$\{x_n\}$ has paracompact closure in X.*
(d) *$\{x_n\}$ has metaLindelöf closure in X.*

Proof. The implications (a) → (b) → (c) → (d) are obvious.

(d) → (a): Without loss of generality, we may assume that the closure of $\{x_n\}$ is the whole space X. Then X is separable. Recall the facts that every separable metaLindelöf space is Lindelöf and that every regular Lindelöf space is normal. It follows Corollary 2.2 that X is pseudocompact. Recall the facts that every normal pseudocompact space is countably compact and that every countably compact and Lindelöf space is compact. Hence X is compact. □

This immediately yields the following special case, which will be discussed latter.

Corollary 3.3. *If a β-sequence in a regular space X has countable closure, then it has compact closure.*

For a Tychonoff space X, we denote by βX the Stone-Čech compactification of X.

Theorem 3.4. *For a β-sequence $\{x_n\}$ in a Tychonoff space X, the following are equivalent.*

(a) *$\{x_n\}$ has compact closure in X.*
(b) *$\{x_n\}$ has realcompact closure in X.*
(c) *$\{x_n\}$ has realcompact cluster which is C^*-embedded in the closure.*

Proof. (a) → (b) and (a) → (c) are obvious. Since every realcompact pseudocompact space is compact (see [E, Theorem 3.11.1]), (b) → (a) is obvious from Corollary 2.2.

(c) → (a): We may assume that the closure of $\{x_n\}$ is the whole space X. Let F be the cluster of $\{x_n\}$. It suffices to show from Proposition 3.1(2) that F is compact. Assume the contrary. There is a point $y \in \beta F \setminus F$. It follows from [E, Theorem 3.11.10] that there is a continuous function $f\colon \beta F \to [0,1]$ such that $f(y) = 0$ and $f(x) > 0$ for each $x \in F$. Since F is C^*-embedded in X, it follows from [E, Corollary 3.6.7] that $\beta F = \mathrm{Cl}_{\beta X} F \subset \beta X$. There is a continuous extension $g\colon \beta X \to [0,1]$ of f. Then $g(y) = 0$ and $g(x) > 0$ for each $x \in F$. Let $V_i = \{x \in \beta X : g(x) < 1/(i+1)\}$ for each $i \in \omega$. Since V_i is an open neighborhood of y in βX and $\{x_n : n \in \omega\}$ is dense in βX, each V_i contains infinitely many x_n's. So we can inductively choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \in V_i$ for each $i \in \omega$. Since $\{x_n\}$ is a β-sequence in X, $\{x_{n_i}\}$ has a cluster point z in F. On the other hand, we have that

$$z \in \bigcap_{k\in\omega} \mathrm{Cl}\,\{x_{n_i} : i \geq k\} \subset \bigcap_{k\in\omega} \mathrm{Cl}_{\beta X}\,\{x_{n_i} : i \geq k\} \subset \bigcap_{k\in\omega} \mathrm{Cl}_{\beta X} V_k = \bigcap_{k\in\omega} V_k = g^{-1}(0).$$

Hence we obtain $z \in F \cap g^{-1}(0) = \emptyset$. This is a contradiction. □

Theorem 3.5. *If a space X is monotonically normal, then every β-sequence in X has compact closure.*

Proof. Assuming the separability of X, we show that X is compact. It suffices from Proposition 3.2 that X is paracompact. Assume the contrary. It follows from Balogh-Rudin's result [BR, Theorem I] that there is a closed subspace F homeomorphic to a stationary subset of an uncountable regular cardinal κ. Let D be the set of all isolated points of F. Then D is a discrete subspace of X with cardinality κ. Note that monotonical normality is hereditary with respect to any subspaces and that it implies collectionwise normality. So X is hereditarily collectionwise normal. Since $(X \setminus \mathrm{Cl}\, D) \cup D$ is collectionwise Hausdorff and $X \setminus \mathrm{Cl}\, D$ is separable, the discrete subspace D of X is at most countable (generally, every discrete subspace of a separable monotonically normal space is countable). This contradicts $|D| = \kappa > \omega$. □

4. β-spaces and strong β-spaces

Recall that a space X is called a *β-space* [H] if there is a function $g\colon X \times \omega \to \mathrm{Top}(X)$, where $\mathrm{Top}(X)$ denotes the topology of X, satisfying

(i) $x \in g(x, n+1) \subset g(x, n)$ for each $x \in X$ and each $n \in \omega$,
(ii) if $y \in \bigcap_{n\in\omega} g(x_n, n)$ for some $y \in X$, then $\{x_n\}$ has a cluster point.

Then we have the following after which we have named β-sequences.

Proposition 4.1 [Y1]. *Let X be a β-space with a function g described above. If $\{x_n\}$ is a sequence in X such that $y \in \bigcap_{n\in\omega} g(x_n, n)$ for some $y \in X$, then $\{x_n\}$ is a β-sequence in X.*

A space X is called a *strong β-space* [Y1] if there is a function $g\colon X \times \omega \to \mathrm{Top}(X)$, satisfying

(i) $x \in g(x, n+1) \subset g(x, n)$ for each $x \in X$ and each $n \in \omega$,
(ii) if $y \in \bigcap_{n\in\omega} g(x_n, n)$ for some $y \in X$, then $\{x_n\}$ has compact closure.

Remark. The original definition of strong β-spaces is in terms of compact cluster of $\{x_n\}$ in (ii). The equivalence is assured by Proposition 3.1(2).

The class of strong β-spaces is well behaved than that of β-spaces, for the former is countably productive, and these two classes coincide under the assumption of paracompactness (see [Y1]). Moreover, as is seen in [Y2], the property of strong β-spaces plays an important role in the study of infinite products and Σ-products of paracompact β-spaces.

Between β-spaces and strong β-spaces, it is natural to ask

Question*. When is a β-space strong β-space?

To tell the truth, the Question* is a background of Definitions 1 and the Question in the Introduction. From our results mentioned above, we immediately obtain several answers to the Question*.

Corollary 4.2. *A β-space X is a strong β-space in each of the following cases:*

(1) *X is regular metaLindelöf.*
(2) *X is realcompact.*
(3) *X is normal and isocompact* [Y1].
(4) *X is countably paracompact and isocompact.*
(5) *X is monotonically normal.*

However, we have not solved the following problem.

Problem. Is every regular submetacompact β-space a strong β-space?

Remark. Since every submetacompact space is isocompact, it follows from Corollary 4.2 (3) and (4) that the Problem is affirmative under the normality or countable paracompactness of X.

5. β-Sequences with countable closure

By Corollary 3.3, we see that countable closure of β-sequences is a special case of compact closure. Here we show such β-sequences can be expressed by fairly concrete form. First, let us begin from β-sequences with finite cluster.

Proposition 5.1. *Let $m \in \omega$. A β-sequence $\{x_n\}$ in a space X has cluster consisting of $m+1$ points if and only if it is decomposed into $m+1$ convergent subsequences with different limit points.*

Proof. Let $\{y_0, \cdots, y_m\}$ be the cluster of $\{x_n\}$, where $y_i \neq y_j$ if $i \neq j$. Choose pairwise disjoint open sets $U_0, \cdots, U_m$ in X with $y_i \in U_i$ for each $i \leq m$. Since $\{x_n\}$ is a β-sequence, note that $\{n \in \omega \colon x_n \notin \bigcup_{i \leq m} U_i\}$ is at most finite. Let $N_0 = \{n \in \omega \colon x_n \in U_0 \cup (X \setminus \bigcup_{i \leq m} U_i)\}$ and let $N_i = \{n \in \omega \colon x_n \in U_i\}$ for $1 \leq i \leq m$. Then each subsequence of $\{x_n \colon n \in N_i\}$ has the unique cluster point y_i. Hence we have that $\{x_n \colon n \in \omega\} = \bigoplus_{i \leq m} \{x_n \colon n \in N_i\}$ and that $\{x_n \colon n \in N_i\}$ converges to y_i for each $i \leq m$. The converse is easy to check. $\square$

Lemma 5.2. *If $\{x_n\}$ is a β-sequence in a space X with the countable cluster F, then $\{x_n\}$ has a subsequence converging to z for each $z \in F$.*

Proof. Let $F = \{z, y_0, y_1, \cdots\}$. For each $i \in \omega$, there is an open neighborhood V_i of z in X such that $y_j \notin \operatorname{Cl} V_i$ and $V_i \subset V_j$ if $j < i$. By the choice of z, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \in V_i$ for each $i \in \omega$. Since $x_{n_i} \in V_i \subset V_k$ if $i \geq k$, it follows that

$$\emptyset \neq \bigcap_{k \in \omega} \operatorname{Cl} \{x_{n_i} : i \geq k\} \subset \big(\bigcap_{k \in \omega} \operatorname{Cl} V_k\big) \cap F = \{z\}.$$

Hence the subsequence $\{x_{n_i}\}$ has the unique cluster point z. Since every subsequence of $\{x_{n_i}\}$ has the same cluster point z, it follows that $\{x_{n_i}\}$ converges to z. □

Theorem 5.3. *Every non-trivial β-sequence with countable closure in a space X is decomposed into non-trivial convergent subsequences with different limit points.*

Proof. Let $\{x_n\}$ be a non-trivial β-sequence with the countable cluster F in X. By Proposition 5.1, we may assume $|F| = \omega$. Let $F = \{y_i : i \in \omega\}$, where $y_i \neq y_j$ if $i \neq j$. We construct a sequence $\{N_i\}$ of infinite subsets in ω, satisfying for each $i \in \omega$,

(i) $N_i \cap N_j = \emptyset$ if $i \neq j$,
(ii) $\{0, \cdots, i\} \subset \bigcup_{j \leq i} N_j$,
(iii) $\{x_n : n \in N_i\}$ converges to y_i.

Assume that $\{N_j : j < i\}$ has been constructed. It follows from Proposition 5.1 that the set of all cluster points of $\{x_n : n \in \bigoplus_{j<i} N_i\}$ is exactly $\{y_j : j < i\}$. So y_i is a cluster point of $\{x_n : \omega \setminus \bigcup_{j<i} N_j\}$. By Lemma 5.2, there is $N_i \subset \omega \setminus \bigcup_{i<j} N_j$ with $|N_i| = \omega$ and $\{x_n : n \in N_i\}$ converges to y_i. Moreover, we may let $i \in N_i$ if $i \notin \bigcup_{j<i} N_j$. Then $\{N_j : j \leq i\}$ satisfies (i)–(iii) above. This implies that $\bigoplus_{i \in \omega} \{x_n : n \in N_i\}$ is a desired decomposition of $\{x_n\}$. □

6. Examples

The following shows that the converse of Corollary 2.2 is not true.

Example 6.1. There is a Tychonoff space X which has a sequence $\{x_n\}$ with pseudocompact closure, but it is not a β-sequence.

Proof. There is a separable, pseudocompact, Tychonoff space X which is not countably compact (see [E, Example 3.10.29]). Let D be a countable dense subset in X. Let E be an infinite countable closed discrete subset in X. Let $\{x_n\}$ be the sequence obtained by the enumeration of all points of $D \cup E$. Since $\{x_n : n \in \omega\}$ is also dense in X, the closure of $\{x_n\}$ is the space X which is pseudocompact. Since $\{x_n : n \in \omega\}$ contains E, $\{x_n\}$ has a subsequence which has no cluster point in X. □

The following example shows that "paracompact closure" in Proposition 3.2 can be replaced by neither "subparacompact closure" nor "paracompact cluster", and that "C^*-embedded" in Theorem 3.4(c) cannot be excluded.

Example 6.2. There is a locally compact, Moore space Ψ which has a β-sequence $\{x_n\}$ with non-compact closure such that the cluster of $\{x_n\}$ is discrete and realcompact.

Proof. Let $\mathcal{A}$ be a maximal almost disjoint (mad) family of infinite subsets in ω with $|\mathcal{A}| = 2^\omega$ (In fact, assign a convergent sequence $\{q_n^r\}$ of rational numbers with the limit r for each irrational number r, let $\mathcal{A}'$ be the set of all such $\{q_n^r\}$'s, and take an mad family $\mathcal{A}$ containing $\mathcal{A}'$). Let $\Psi = \omega \cup \{p_A \colon A \in \mathcal{A}\}$ as a set. We introduce a topology of Ψ as follows;

(i) each point of ω is isolated in Ψ,
(ii) $\big\{\{p_A\} \cup (A \setminus F) \colon F$ is a finite subset of $A\big\}$ is a neighborhood base of p_A for each $A \in \mathcal{A}$.

For example, the space Ψ is found in [vD, Section 11]. It is known that Ψ is a locally compact Moore space (hence it is subparacompact).

Note that ω is a β-sequence in Ψ. In fact, take an infinite subset B in ω. In case of $B \in \mathcal{A}$: Then $p_B \in \Psi$ is a cluster point of B. In case of $B \notin \mathcal{A}$: Since $\mathcal{A}$ is maximal, there is $A_0 \in \mathcal{A}$ with $|A_0 \cap B| = \omega$. Then p_{A_0} is a cluster point of B. Identify $\{x_n\}$ with ω. Since ω is dense in Ψ, the space Ψ is the closure of $\{x_n\}$. Of course, Ψ is not countably compact. Let $D = \{p_A \colon A \in \mathcal{A}\}$. By (i) and (ii), D is the cluster of $\{x_n\}$, which is closed discrete in Ψ. Since $|D| = 2^\omega$ and 2^ω is a non-measurable cardinal, it follows from [E, Exercise 3.11.D] that D is realcompact. □

It is natural from and Corollaries 2.2 and 2.3 and Theorem 3.5 to ask in what kinds of normal spaces every β-sequence has compact closure. However, the following example shows that some β-sequences do not have compact closure in even normal spaces with fairly strong conditions.

Example 6.3. There is a sequentially compact, locally compact and normal space Y which has a β-sequence with non-compact closure.

Proof. Let $Y = \omega \cup \mathcal{T}$ be the space described in [vD, Example 7.1]. Then it is shown in there that Y is a sequentially compact, locally compact and normal space which contains ω as a dense subset. Since Y is countably compact, ω is a β-sequence in Y. Since ω is dense in Y, Y is the closure of ω. However, since the closed set $\mathcal{T}$ in Y is homeomorphic to some regular cardinal $\mathfrak{t} \geq \omega_1$ (see [vD, Theorem 3.1]), Y is not compact. □

Remark. The above space $Y = \omega \cup \mathcal{T}$ is a strong β-space. In fact, we define the function $g \colon Y \times \omega \to \mathrm{Top}(Y)$ such that $g(m, n) = \{m\}$ for each $(m, n) \in \omega \times \omega$ and $g(T, n) = \mathcal{T} \setminus \{0, \cdots, n\}$ for each $(T, n) \in \mathcal{T} \times \omega$. Then it is easily seen that g is a desired function.

Without the normality of Y, we have a more extreme example.

Example 6.4. There is a countably compact subspace X of $\omega^*(= \beta\omega \setminus \omega)$ which has no infinite sequence with compact closure.

Proof. Let X be a countably compact subspace of ω^* with $|X| = \mathfrak{c}$ (for example, consider the Novák's example in [E, Example 3.10.19]). Let $\{x_n\}$ be an infinite sequence with the compact closure E in X. Since E is infinite, it follows from [E, Theorem 3.1.14] that $|E| = 2^{\mathfrak{c}}$. This contradicts $|E| \leq |X| = \mathfrak{c}$. □

Remark. If the above space X is dense in ω^* such as [Y1, Example 3.6], it follows from [Y1, Lemma 3.5] that X is not a strong β-space.

By Corollary 2.4 and the following, we can see that it is independent from ZFC whether every β-sequence has compact closure in a perfectly normal space.

Example 6.5. Under $\diamondsuit$, there is a perfect normal space X which has a β-sequence with non-compact closure.

Proof. Consider the Ostaszewski line X in [O]. The space X can be constructed under $\diamondsuit$. Then X is a perfectly normal, countably compact S-space which is not compact (also see [R, p.312]). Since X is separable, let $\{x_n : n \in \omega\}$ be a countable dense set in X. Then the sequence $\{x_n\}$ is a β-sequence in X with the non-compact closure X. □

The following shows that the converse of Theorem 5.3 is not true.

Example 6.6. There is a β-sequence $\{x_n\}$ in $[0,1]$ such that $\{x_n\}$ is decomposed into countably many non-trivial convergent subsequences, but the closure of it is $[0,1]$.

Proof. Let $\{x_n\}$ be the sequence consisting of all rational numbers in $[0,1]$. Since $[0,1]$ is compact, $\{x_n\}$ is clearly a β-sequence in $[0,1]$. Since $\{x_n : n \in \omega\}$ is dense in $[0,1]$, the closure of $\{x_n\}$ is $[0,1]$. In a similar way as in the proof of Theorem 5.3, we can inductively choose a sequence $\{N_i\}_{i=1}^{\infty}$ of infinite subsets in ω, satisfying for each $i \geq 1$, $N_i \cap N_j = \emptyset$ if $i \neq j$, $\{0, \cdots, i\} \subset \bigcup_{j \leq i} N_j$, $\{x_n : n \in N_i\}$ converges to $1/i$. Then we have $\{x_n : n \in \omega\} = \bigoplus_{i=1}^{\infty} \{x_n : n \in N_i\}$. □

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