RECTANGULAR PRODUCTS WITH ORDINAL FACTORS

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Abstract. Let A and B be subspaces of an ordinal. It is proved that the product $A \times B$ is countably paracompact if and only if it is rectangular. Before this main result, we discuss several covering properties of products with one ordinal factor. In particular, for every paracompact space X, it is proved that the product $X \times A$ is paracompact if so is A.

1. Introduction

All spaces are assumed to be T_1. However, all paracompact spaces are assumed to be Hausdorff. For a set S, we denote by $|S|$ the cardinality of S. The letter λ stands for an ordinal.

Let X be a space. Recall that U is a *cozero-set* in X if there is a continuous function $f\colon X \to [0,1]$ such that $U = \{x \in X : f(x) > 0\}$. We say that $\mathcal{G}$ is a *cozero cover* of X if $\mathcal{G}$ is a cover of X such that each member of $\mathcal{G}$ is a cozero-set in X.

Let $X \times Y$ be a product space. A subset of the form $U \times V$ in $X \times Y$ is called a *rectangle*. A cover $\mathcal{G}$ of $X \times Y$ is *rectangular* if each member of $\mathcal{G}$ is a rectangle in $X \times Y$. The product $X \times Y$ is said to be *rectangular* if every finite (or binary) cozero cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement. Pasynkov [9] proved a remarkable result as the product theorem in dimension theory:

(I) Let X and Y be Tychonoff spaces. If the product $X \times Y$ is rectangular, then $\dim X \times Y \le \dim X + \dim Y$.

Let A and B be subspaces of an ordinal. That is, A and B are subspaces of an infinite ordinal λ with the usual order topology. The study of the product $A \times B$ was essentially begun by Ohta, Tamano and the first author [3]. Subsequently, Fleissner, Terasawa and the first author [1] proved that

(II) $A \times B$ is strongly zero-dimensional, that is, $\dim A \times B = 0$.

From (I) and (II), it is natural to raise the question:

(Q*) Is $A \times B$ always rectangular?

If it was true, (II) would be an immediate consequence of (I). However, (Q*) has a negative answer when A and B are disjoint stationary subsets of ω_1 (see Remark 1 to Corollary 4.4). So we are led to refine the question:

(Q) What are equivalent conditions for $A \times B$ to be rectangular?

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The main purpose of this paper is to give a complete answer to this question. Namely, we prove that

(III) $A \times B$ is rectangular iff $A \times B$ is countably paracompact.

We also give topological characterizations for normality and countable paracompactness of $A \times B$. From these characterizations, we can see a quite delicate difference between normality and countable paracompactness of the products of ordinals (see Theorem 4.1 and Corollary 4.2). Moreover, from the difference, we can immeditately see that no $A \times B$ is a Dowker space (see Corollary 4.3).

The proof of our main result (III) might be not short and quite technical for the readers who are not familiar to these arguments. So, before we state this, we like to look into an intermediate world between those of general products and ordinal products. That is, we discuss some covering properties of products of a general space X and an ordinal factor A. Since the proofs here are rather short, they might make the readers be familiar to deal with subspaces of ordinals.

So the purpose of the next two sections is to generalize the results for covering properties of $A \times B$ to the products $X \times A$, where X is mainly a generalized paracompact space. In fact, in Section 2, one is to prove that $X \times A$ is paracompact iff X and A are paracompact. Another will be used in the proof of the main theorem. In Section 3, we show the equivalence of orthocompactness and weak suborthocompactness of $X \times A$.

2. Paracompactness of products with one ordinal factor

Let λ be a limit ordinal. A subset A of λ is *unbounded* (resp., *bounded*) in λ if for each $\alpha \in A$, there is $\beta \in A$ with $\beta > \alpha$ (resp., $A \subset \alpha$ for some $\alpha < \mu$). We denote by $\mathrm{cf}(\lambda)$ the cofinality of λ. Let $\mathrm{cf}(\lambda) \geq \omega_1$. A subset A of λ is *stationary* in λ if it intersects all closed and unbounded (abbreviated by *cub*) sets in λ.

Let us begin two fundamental lemmas, which will be frequently used in our proofs.

Lemma 2.1 (PDL). *Let $\mathrm{cf}(\lambda) > \omega$ and let S be a stationary subset in λ. If $f(\alpha) < \alpha$ for each $\alpha \in S$, then there are $T \subset S$ and $\alpha_0 \in S$ such that T is stationary in λ with $|T| = \mathrm{cf}(\lambda)$ and $f(\alpha) < \alpha_0$ for each $\alpha \in T$.*

Lemma 2.2. *Let $A \subset \lambda$. Assume that $\mathrm{cf}(\lambda) \leq \omega$ or that $\mathrm{cf}(\lambda) > \omega$ and A is non-stationary in λ. Then A is represented as the topological sum $\bigoplus_{\gamma \in \mathrm{cf}(\lambda)} B_\gamma$ such that each B_γ is bounded in λ.*

An open cover $\mathcal{U}$ of a space X is a *weak $\delta\theta$-cover* if we can represent as $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ such that for each $x \in X$, there is $m \in \omega$ with $0 < \mathrm{ord}(x, \mathcal{U}_m) \leq \omega$. where recall $\mathrm{ord}(x, \mathcal{U}_m) = |\{U \in \mathcal{U}_m : x \in U\}|$. In particular, every σ-point-finite open cover of X is a weak $\delta\theta$-cover.

Lemma 2.3. *Let $S \subset \lambda + 1$, where $\mathrm{cf}(\lambda) > \omega$. Let S be stationary in λ or $\lambda \in S$. If $\mathcal{U}$ is a weak $\delta\theta$-cover of S, then there are $U_0 \in \mathcal{U}$ and $\alpha_0 \in S \cap \lambda$ such that $(\alpha_0, \lambda] \cap S \subset U_0$.*

Proof. Since it is obvious in case of $\lambda \in S$, we may assume that S is a stationary subset in λ. Let $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ such that for each $\alpha \in S$, there is $n_\alpha \in \omega$ with $0 < \mathrm{ord}(\alpha, \mathcal{U}_{n_\alpha}) \leq \omega$. Let $S_n = \{\alpha \in S : 0 < \mathrm{ord}(\alpha, \mathcal{U}_n) \leq \omega\}$ for each $n \in \omega$. Since $S = \bigcup_{n \in \omega} S_n$, there is $m \in \omega$ such that S_m is stationary in λ. Take a $U_\alpha \in \mathcal{U}_m$ with $\alpha \in U_\alpha$ for each $\alpha \in S_m$.

Assume that U_α is not stationary in λ for each $\alpha \in S_m$. For each $\alpha \in S_m$, take a cub set F_α (in λ) disjoint from U_α, and find an $f(\alpha) < \alpha$ with $(f(\alpha), \alpha] \cap S \subset U_\alpha$. By PDL, there are a stationary set T in λ and an $\alpha^* \in S_m$ such that $T \subset S_m \cap (\alpha^*, \lambda)$ and $f(\alpha) < \alpha^*$ for each $\alpha \in T$. Let $\gamma < \omega_1$. Assume that we have already taken $\{\alpha_\beta \colon \beta < \gamma\} \subset T$ such that $\beta < \beta'$ implies $\alpha_\beta < \alpha_{\beta'}$. Since $\bigcap_{\beta<\gamma} F_{\alpha_\beta}$ is a cub set in λ, we can find an $\alpha_\gamma \in T \cap (\bigcap_{\beta<\gamma} F_{\alpha_\beta})$ with $\alpha_\gamma > \sup_{\beta<\gamma} \alpha_\beta$. Thus we have constructed $\{\alpha_\beta \colon \beta < \omega_1\} \subset T$. Let $\beta < \gamma < \omega_1$. Since $\alpha_\gamma \in F_{\alpha_\beta}$, we have $\alpha_\gamma \notin U_{\alpha_\beta}$. Since $\alpha_\gamma \in U_{\alpha_\gamma}$, U_{α_β} and U_{α_γ} are different. On the other hand, we have $\alpha^* \in \bigcap_{\beta<\omega_1} (f(\alpha_\beta), \alpha_\beta] \cap S \subset \bigcap_{\beta<\omega_1} U_{\alpha_\beta}$. Since $\alpha^* \in S_m$, observe $\mathrm{ord}(\alpha^*, \mathcal{U}_m) \le \omega$. This is a contradiction. Hence there is some $U_0 \in \mathcal{U}_m$ which is stationary in λ. Since U_0 is open in S, it is easily seen by PDL again that there is $\alpha_0 \in S$ such that $(\alpha_0, \lambda) \cap S \subset U_0$. $\square$

Note that a product space $X \times Y$ is paracompact and rectangular iff every open cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement.

Theorem 2.4. *If X is a paracompact space and A is a paracompact subspace of an ordinal, then the product $X \times A$ is paracompact and rectangular.*

Proof. Let $A \subset \lambda + 1$. Assume the contrary. Let

$$\begin{aligned}\lambda' = \min\{\mu \le \lambda \colon & \text{ There is a paracompact } A' \subset \mu + 1 \text{ such that} \\ & X \times A' \text{ is not either paracompact or rectangular}\}.\end{aligned}$$

Replace these λ' and A' with λ and A, respectively, over again. Then we may assume without loss of generality that $X \times A$ is not either paracompact or rectangular, and that

(∗) $X \times B$ is paracompact and rectangular for each paracompact $B \subset A$ with $\sup B < \lambda$.

Case 1. Assume that $\lambda \notin A$ and $\mathrm{cf}(\lambda) \le \omega$, or that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is non-stationary in λ.

By Lemma 2.2, we can represent as $A = \bigoplus_{\gamma \in \mathrm{cf}(\lambda)} B_\gamma$, where $\sup B_\gamma < \lambda$ for each $\gamma \in \mathrm{cf}(\lambda)$. Then $\{X \times B_\gamma \colon \gamma \in \mathrm{cf}(\lambda)\}$ is a discrete rectangular clopen cover of $X \times A$. By (∗), each $X \times B_\gamma$ is paracompact and rectangular. Hence so is $X \times A$. This is a contradiction.

Case 2. Assume that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is stationary in λ, or that $\lambda \in A$.

There is an open cover $\mathcal{O}$ of $X \times A$ which has no σ-locally finite rectangular cozero refinement. We may let $\mathcal{O} = \{U_\xi \times V_\xi \colon \xi \in \Delta\}$, where each $U_\xi \times V_\xi$ denotes an open rectangle in $X \times A$. Let

$$\Xi = \{\xi \in \Delta \colon (\alpha_\xi, \lambda] \cap A \subset V_\xi \text{ for some } \alpha_\xi \in A \cap \lambda\}.$$

Claim. *$\{U_\xi \colon \xi \in \Xi\}$ is an open cover of X.*

Proof. Pick an $x \in X$. Let $\Delta_x = \{\xi \in \Delta \colon x \in U_\xi\}$. Then $\{V_\xi \colon \xi \in \Delta_x\}$ is an open cover of A. Since A is paracompact, there is a locally finite open cover $\{V^*_\xi \colon \xi \in \Delta_x\}$ of A such that $V^*_\xi \subset V_\xi$ for each $\xi \in \Delta_x$. It follows from Lemma 2.3 that there are

$\xi_0 \in \Delta_x$ and $\alpha_{\xi_0} \in A$ such that $(\alpha_{\xi_0}, \lambda] \cap A \subset V^*_{\xi_0} \subset V_{\xi_0}$. This means that $\xi_0 \in \Xi$ with $x \in U_{\xi_0}$.

Since X is paracompact, there is a locally finite cozero cover $\{W_\xi : \xi \in \Xi\}$ of X such that $W_\xi \subset U_\xi$ for each $\xi \in \Xi$. Let $B_\xi = [0, \alpha_\xi] \cap A$ for each $\xi \in \Xi$. It follows from $(*)$ that each $\mathcal{O} \restriction (X \times B_\xi)$ has a σ-locally finite rectangular cozero refinement $\mathcal{G}_\xi$. Now, we let

$$\mathcal{G} = \{W_\xi \times \big((\alpha_\xi, \lambda] \cap A\big) : \xi \in \Xi\} \cup \bigcup \{\mathcal{G}_\xi \restriction (W_\xi \times B_\xi) : \xi \in \Xi\}.$$

Then it is easily checked that $\mathcal{G}$ is a σ-locally finite rectangular cozero refinement of $\mathcal{O}$. This contradicts the choice of $\mathcal{O}$. $\square$

Remark. Pasynkov asked in [9, Question 1] whether a paracompact product $X \times Y$ is rectangular. By Theorem 2.4, one cannot find a negative answer in the class of all products with one ordinal factor.

Immediately, we have

Corollary 2.5 [4]**.** *Let A and B be two subspace of an ordinal. Then the product $A \times B$ is paracompact if and only if A and B are paracompact*

Recall that an open cover $\mathcal{O}$ of a space X is *normal* if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$ for each $n \in \omega$, where $\mathcal{U}_0 = \mathcal{O}$. It is well known that a Hausdorff space X is paracompact iff every open cover of X is normal.

Lemma 2.6. *Every σ-point-finite open cover of a collectionwise normal and countably paracompact space is normal.*

It was independently proved in [6] and [7] that every point-finite open cover of a collectionwise normal space is normal. Using this result, Lemma 2.6 is easily verified.

Theorem 2.7. *Let X be a collectionwise normal and countably paracompact space with* $\dim X = 0$*. Let A be a subspace of an ordinal. Then every σ-point-finite rectangular open cover of $X \times A$ has a discrete rectangular clopen refinement.*

Proof. Let $A \subset \lambda + 1$. Assume the contrary. For convenience, we denote by (S) the statement of our conclusion. Let

$$\lambda' = \min\{\mu \leq \lambda : X \times A' \text{ does not satisfy } (S) \text{ for some } A' \subset \mu + 1\}.$$

Replacing these λ' and A' with λ and A, respectively, we may assume that $X \times A$ does not satisfy (S), and that

$(*)$ $X \times B$ satisfies (S) for each $B \subset A$ with $\sup B < \lambda$.

Case 1. Assume that $\lambda \notin A$ and $\mathrm{cf}(\lambda) \leq \omega$, or that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is non-stationary in λ.

By Lemma 2.2, we can represent as $A = \bigoplus_{\gamma \in \mathrm{cf}(\lambda)} B_\gamma$, where $\sup B_\gamma < \lambda$ for each $\gamma \in \mathrm{cf}(\lambda)$. It follows from $(*)$ that $\{X \times B_\gamma : \gamma \in \mathrm{cf}(\lambda)\}$ is a discrete rectangular clopen cover of $X \times A$ such that each $X \times B_\gamma$ satisfies (S). Then it is easy to see that $X \times A$ satisfies (S). This is a contradiction.

Case 2. Assume that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is stationary in λ, or that $\lambda \in A$.

There is a σ-point-finite rectangular open cover $\mathcal{O} = \{U_\xi \times V_\xi \colon \xi \in \bigcup_{n\in\omega} \Delta_n\}$ of $X \times A$ which has no discrete rectangular clopen refinement, where $\{U_\xi \times V_\xi \colon \xi \in \Delta_n\}$ is point-finite in $X \times A$ for each $n \in \omega$. Let

$$\Xi_n = \{\xi \in \Delta_n \colon (\alpha_\xi, \lambda] \cap A \subset V_\xi \text{ for some } \alpha_\xi \in A \cap \lambda\}$$

for each $n \in \omega$, and let $\mathcal{U} = \{U_\xi \colon \xi \in \Xi_n \text{ and } n \in \omega\}$. For each $x \in X$, since $\mathcal{O} \restriction (\{x\} \times A)$ is a σ-point-finite open cover of $\{x\} \times A$, it follows from Lemma 2.3 that there are $\alpha_x \in A \cap \lambda$ and $\xi_x \in \bigcup_{n\in\omega} \Delta_n$ such that $x \in U_{\xi_x}$ and $(\alpha_x, \lambda] \cap A \subset V_{\xi_x}$. Hence $\mathcal{U}$ covers X.

Claim. *$\{U_\xi \colon \xi \in \Xi_n\}$ is point-finite in X for each $n \in \omega$.*

Proof. Assume that some $\{U_\xi \colon \xi \in \Xi_n\}$ is not point-finite at $p \in X$. There are an infinite sequence $\{\xi_i\}$ of distinct members of Ξ_n with $p \in \bigcap_{i\in\omega} U_{\xi_i}$. For each $i \in \omega$, there is $\alpha_i \in A \cap \lambda$ such that $(\alpha_i, \lambda] \cap A \subset V_{\xi_i}$. Take $\beta \in A$ with $\beta > \sup_{i\in\omega} \alpha_i$. Then we have $\langle p, \beta\rangle \in \bigcap_{i\in\omega} U_{\xi_i} \times V_{\xi_i}$. This contradicts the point-finiteness of $\{U_\xi \times V_\xi \colon \xi \in \Xi_n\}$ in $X \times A$.

By the Claim, $\mathcal{U}$ is a σ-point-finite open cover of X. It follows from Lemma 2.6 that $\mathcal{U}$ is a normal cover of X. Since $\dim X = 0$, there is a discrete clopen refinement $\{L_\eta \colon \eta \in \Omega\}$ of $\mathcal{U}$. For each $\eta \in \Omega$, choose a $\xi(\eta) \in \bigcup_{n\in\omega} \Xi_n$ with $L_\eta \subset U_{\xi(\eta)}$. For each $\eta \in \Omega$, let $B_\eta = [0, \alpha_{\xi(\eta)}] \cap A$. By $(*)$, $X \times B_\eta$ is a clopen rectangle in $X \times A$, satisfying (S). Since $\mathcal{O} \restriction (X \times B_\eta)$ is a σ-point-finite rectangular open cover of $X \times B_\eta$, there is a discrete rectangular clopen refinement $\mathcal{D}_\eta$ of $\mathcal{O} \restriction (X \times B_\eta)$. Now, we put

$$\mathcal{D} = \{L_\eta \times \big((\alpha_{\xi(\eta)}, \lambda] \cap A\big) \colon \eta \in \Omega\} \cup \bigcup \{\mathcal{D}_\eta \restriction (L_\eta \times B_\eta) \colon \eta \in \Omega\}.$$

Then it is easily verified that $\mathcal{D}$ is a discrete rectangular clopen refinement of $\mathcal{O}$. This contradicts the choice of $\mathcal{O}$. $\square$

Since a subspace A of an ordinal is a GO-space, it is collectionwise normal and countably paracompact. Moreover, A is strongly zero-dimensional. So the following is an immediate consequence of Theorem 2.7.

Corollary 2.8. *Let A and B be two subspaces of an ordinal. Then every σ-point-finite rectangular open cover of $A \times B$ has a discrete rectangular clopen refinement.*

We will use Corollary 2.8 in the proof of our main theorem later.

3. Orthocompactness of products of one ordinal factor

A space X is *weakly suborthocompact* [4] if every open cover $\mathcal{G}$ of X has an open refinement $\bigcup_{n\in\omega} \mathcal{H}_n$, satisfying for each $x \in X$, there is $n_x \in \omega$ such that $\bigcap\{H \in \mathcal{H}_{n_x} \colon x \in H\}$ is open in X.

Let κ be an uncountable regular cardinal. A space X has *orthocaliber κ at $p \in X$* [5] if for any collection $\mathcal{U}$ of open neighborhood of p in X with $|\mathcal{U}| = \kappa$, there is a subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $|\mathcal{V}| = \kappa$ and $p \in \mathrm{Int}(\bigcap \mathcal{V})$.

Lemma 3.1. *Let $A \subset \lambda$ be stationary in λ, where $\mathrm{cf}(\lambda) > \omega$. If $X \times A$ is weakly suborthocompact, then X has orthocaliber $\mathrm{cf}(\lambda)$ at each $x \in X$.*

Proof. Note that there is a stationary subset A^* of $\mathrm{cf}(\lambda)$ such that A^* is homeomorphic to a closed subspace of A. So we may assume that A is a stationary subset of the regular cardinal $\mathrm{cf}(\lambda)$. Moreover, if $A = \bigcup_{n\in\omega} A_n$, then some A_m must be stationary in $\mathrm{cf}(\lambda)$. So the proof can be obtained by modifying that of [5, Lemma 1.1]. The detail is left to the readers. $\square$

Let $\mathcal{U}$ be a collection of open sets in a space X. We say that $\mathcal{U}$ is *interior-preserving* if $\bigcap \mathcal{V}$ is open for every $\mathcal{V} \subset \mathcal{U}$. The following is easily seen.

Lemma 3.2. *Let X be a space. Let $\mathcal{G}_\xi$ be an interior-preserving collection of open sets in X for each $\xi \in \Xi$. If $\{\bigcup \mathcal{G}_\xi \colon \xi \in \Xi\}$ is point-finite in X, then $\bigcup\{\mathcal{G}_\xi \colon \xi \in \Xi\}$ is interior-preserving in X.*

A space X is *orthocompact* (resp., *suborthocompact*) if every open cover $\mathcal{U}$ of X has an interior-preserving open refinement $\mathcal{V}$ (resp., a sequence $\{\mathcal{V}_n\}$ of open covers of X, satisfying for each $x \in X$, there is $n_x \in \omega$ such that $\bigcap\{V \in \mathcal{V}_{n_x} \colon x \in V\}$ is open in X).

Theorem 3.3. *Let X be a metacompact (resp., submetacompact) space and let A a subspace of an ordinal. Then $X \times A$ is orthocompact (resp., suborthocompact) if and only if it is weakly suborthocompact.*

Proof. Let $A \subset \lambda + 1$. Assume the contrary. Let

$$\lambda' = \min\{\mu \le \lambda : \text{ There is } A' \subset \mu + 1 \text{ such that } X \times A' \\ \text{is weakly suborthocompact, but not orthocompact}\}.$$

Replace these λ' and A' with λ and A, respectively. We may assume that $X \times A$ is weakly suborthocompact but not orthocompact, and that

- $(*)$ $X \times B$ is orthocompact if it is weakly suborthocompact for each $B \subset A$ with $\sup B < \lambda$.

Case 1. Assume that $\lambda \notin A$ and $\mathrm{cf}(\lambda) \le \omega$, or that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is non-stationary in λ.

By Lemma 2.2, we can easily get a contradiction in the similar way as the above.

Case 2. Assume that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is stationary in λ, or that $\lambda \in A$.

There is an open cover $\mathcal{O}$ of $X \times A$ such that $\mathcal{O}$ has no interior-preserving open refinement. Pick a point $p \in X$. In the case of $\lambda \notin A$: For each $\alpha \in A$, there are an open neighborhood U_p of p in X and an $f(\alpha) < \alpha$ such that $U_\alpha \times \big((f(\alpha), \alpha] \cap A\big) \subset O_\alpha$ for some $O_\alpha \in \mathcal{O}$. By PDL, there are $S_p \subset A$ and $\alpha_p \in A$ such that S_p is stationary in $\lambda, |S_p| = \mathrm{cf}(\lambda), S_p \cap [0, \alpha_p] = \emptyset$ and $f(\alpha) < \alpha_p$ for each $\alpha \in S_p$. It follows Lemma 3.1 that there are an open neighborhood V_p of p in X and a $T_p \subset S_p$ such that $|T_p| = \mathrm{cf}(\lambda)$ and $V_p \subset \mathrm{Int}\big(\bigcap_{\alpha\in T_p} U_\alpha\big)$. Then we have that $V_p \times \big((\alpha_p, \alpha] \cap A\big) \subset O_\alpha \in \mathcal{O}$ for each $\alpha \in T_p$ and that $\bigcup\{V_p \times \big((\alpha_p, \alpha] \cap A\big) \colon \alpha \in T_p\} = V_p \times \big((\alpha_p, \lambda] \cap A\big)$. In the case of $\lambda \in A$: Since $\langle p, \lambda\rangle \in X \times A$, there are an open neighborhood V_p of p in X and an $\alpha_p < \lambda$ such that $V_p \times \big((\alpha_p, \lambda] \cap A\big) \subset O_p$ for some $O_p \in \mathcal{O}$.

Since X is metacompact, there is a point-finite open cover $\{W_p \colon p \in X\}$ of X such that $W_p \subset V_p$ for each $p \in X$. Let $B_p = [0, \alpha_p] \cap A$ for each $p \in X$. It follows from $(*)$ that $X \times B_p$ is orthocompact. So there is an interior-preserving open refinement $\mathcal{G}_p$ of $\mathcal{O} \restriction (X \times B_p)$. Here, we let

$$\mathcal{G} = \{W_p \times \big((\alpha_p, \alpha] \cap A\big) \colon \alpha \in T_p \text{ and } p \in X\} \cup \bigcup \{\mathcal{G}_p \restriction (W_p \times B_p) \colon p \in X\}.$$

Then $\mathcal{G}$ is an interior-preserving open refinement of $\mathcal{O}$. This is a contradiction.

The argument of parenthetic part is similar to the above. The detail is left to the readers. □

Remark. For two subspace A and B of an ordinal, it was shown in [4] that the product $A \times B$ is orthocompact iff it is weakly suborthocompact. However, we do not know whether the metacompactness of X in Theorem 3.3 can be replaced by the orthocompactness of X.

Lemma 3.4. *Let $A \subset \lambda + 1$, where $\mathrm{cf}(\lambda) > \omega$. Let A be stationary in λ or $\lambda \in A$. Assume that a space X has orthocaliber $\mathrm{cf}(\lambda)$ at $p \in X$. If O is an open set in $X \times A$ with $\{p\} \times A \subset O$, then there are an open neighborhood V of p in X and a $\beta \in A$ such that $V \times \big((\beta, \lambda] \cap A\big) \subset O$.*

Proof. The case of $\lambda \in A$ is obvious. We may assume that A is a stationary subset of λ. For each $\alpha \in A$, there are an open neighborhood U_α of p in X and an $f(\alpha) < \alpha$ such that $U_\alpha \times \big((f(\alpha), \alpha] \cap A\big) \subset O$. By PDL, there are $S \subset A$ and $\beta \in A$ such that S is stationary in $\lambda, |S| = \mathrm{cf}(\lambda), S \cap [0, \beta] = \emptyset$ and $f(\alpha) < \beta$ for each $\alpha \in S$. Then $U_\alpha \times \big((\beta, \alpha] \cap A\big) \subset O$ for each $\alpha \in S$. By the assumption of X, there are $T \subset S$ and an open neighborhood V of p in X such that $|T| = \mathrm{cf}(\lambda)$ and $V \subset \bigcap_{\alpha \in T} U_\alpha$. Hence we have $V \times \big((\beta, \lambda] \cap A\big) \subset O$. □

Note that a product space $X \times Y$ is normal and rectangular iff every binary open cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement.

Theorem 3.5. *Let X be a paracompact space and A a subspace of an ordinal. If $X \times A$ is orthocompact, then it is normal and rectangular.*

Proof. Let $A \subset \lambda + 1$. Assume the contrary. Let

$$\begin{aligned} \lambda' = \min\{\mu \le \lambda \colon & \text{ There is } A' \subset \mu + 1 \text{ such that } X \times A' \\ & \text{is orthocompact, but not either normal or rectangular}\}. \end{aligned}$$

Replace these λ' and A' with λ and A, respectively. We may assume that $X \times A$ is orthocompact but not either normal or rectangular, and that

$(*)$ $X \times B$ is normal and rectangular if it is orthocompact for each $B \subset A$ with $\sup B < \lambda$.

Case 1. Assume that $\lambda \notin A$ and $\mathrm{cf}(\lambda) \le \omega$, or that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is non-stationary in λ.

This case is similar to the above.

Case 2. Assume that $\lambda \notin A$, $\mathrm{cf}(\lambda) > \omega$ and A is stationary in λ, or that $\lambda \in A$.

Let $\mathcal{O} = \{O_0, O_1\}$ be any binary open cover of $X \times A$. Pick a point $p \in X$. In the case of $\lambda \notin A$: By PDL, there are $\delta \in A$ and $k_p \in 2$ such that $\{p\} \times \big((\delta, \lambda] \cap A\big) \subset O_{k_p}$. Since $X \times \big((\delta, \lambda] \cap A\big)$ is orthocompact and $(\delta, \lambda] \cap A$ is stationary in λ, it follows from Lemmas 3.1 and 3.4 that there are an open neighborhood V_p of p in X and a $\beta_p \in A$ such that $V_p \times \big((\beta_p, \lambda] \cap A\big) \subset O_{k_p}$. In the case of $\lambda \in A$: Obviously, there are such V_p and β_p. Since X is paracompact, there is a locally finite cozero cover $\{W_p : p \in X\}$ of X such that $W_p \subset V_p$ for each $p \in X$. Let $B_p = [0, \beta_p] \cap A$ for each $p \in X$. It follows from $(*)$ that each $X \times B_p$ is normal and rectangular. So there is a σ-locally finite rectangular cozero refinement $\mathcal{G}_p$ of $\mathcal{O} \restriction (X \times B_p)$. Now, we let

$$\mathcal{G} = \{W_p \times \big((\beta_p, \lambda] \cap A\big) : p \in X\} \cup \bigcup \{\mathcal{G}_p \restriction (W_p \times B_p) : p \in X\}.$$

Then $\mathcal{G}$ is a locally finite rectangular cozero refinement of $\mathcal{O}$. Hence $X \times A$ is normal and rectangular. This is a contradiction. $\square$

Immediately, we have

Theorem 3.6 [5]**.** *Let X be a paracompact space and κ a uncountable regular cardinal. If $X \times \kappa$ is orthocompact, then it is normal and rectangular.*

4. A main theorem and corollaries

In this section, we deal with the product $A \times B$ of two subspaces of an ordinal instead of the product $X \times A$ in the previous sections. More special situations may yield more curious results. The following main theorem of this paper illustrates this phenomena.

Theorem 4.1 (Main). *Let A and B be two subspaces of an ordinal. Then the following are equivalent.*

(a) *$A \times B$ is countably paracompact.*
(b) *$A \times B$ is rectangular.*
(c) *Every binary cozero cover of $A \times B$ has a discrete rectangular clopen refinement.*

Remark. There is no implication, in general, between rectangularity and countable paracompactness of a product space $X \times Y$. In fact, for a Tychonoff space X which is not countably paracompact, the product of the form $X \times \{p\}$ is rectangular but not countably paracompact. On the other hand, it follows from [8, Theorem 1] that, for a countably paracompact (and normal) space X which is not paracompact, there is a paracompact space Y such that $X \times Y$ is countably paracompact (and normal) but not rectangular.

Corollary 4.2. *Let A and B be two subspaces of an ordinal. Then the following are equivalent.*

(a) *$A \times B$ is normal.*
(b) *Every binary open cover of $A \times B$ has a σ-locally finite rectangular open refinement.*
(c) *Every binary open cover of $A \times B$ has a discrete rectangular clopen refinement.*

Proof. (a) → (b): Since $A \times B$ is normal, every binary open cover $\mathcal{O}$ of $A \times B$ has a binary cozero refinement (shrinking) $\mathcal{G}$. It follows from [3, Theorems A and B] that $A \times B$ is countably paracompact. By Theorem 4.1, $A \times B$ is rectangular. Hence $\mathcal{G}$ has a desired refinement.

(b) → (c): This follows from Corollary 2.8.

(c) → (a): This is obvious. □

Comparing Theorem 4.1 and Corollary 4.2, we can see a delicate difference between countable paracompactness and normality of $A \times B$ from the topological aspect. The both results immediately yields

Corollary 4.3 [3]**.** *Any product of two subspaces of an ordinal is not a Dowker space.*

This is originally a consequence of [3, Theorems A and B]. However, the set-theoretic conditions stated there seem to be too complicated.

As a particular case of them, the following is an immediate consequence of [3, Theorem B and Corollary 3.3] and our Theorem 4.1.

Corollary 4.4. *Let $A, B \subset \omega_1$. Then the following are equivalent.*

(a) *$A \times B$ is normal.*
(b) *$A \times B$ is countably paracompact.*
(c) *$A \times B$ is rectangular.*
(d) *A is non-stationary in ω_1 or B is non-stationary in ω_1 or $A \cap B$ is stationary in ω_1.*

Remark 1. Let A and B be disjoint stationary subsets in ω_1. It follows from Corollary 4.4 that $A \times B$ is not rectangular. The referee pointed out that the fact is also directly shown by PDL. Since the verification is not difficult, it is left to the reader.

Remark 2. It follows Theorem 4.1 and Corollary 4.2 that normality of $A \times B$ implies its rectangularity. However, the converse is not true. In fact, it is well known that $\omega_1 \times (\omega_1 + 1)$ is not normal. On the other hand, since $\omega_1 \times (\omega_1 + 1)$ is countably compact, it follows from Theorem 4.1 that $\omega_1 \times (\omega_1 + 1)$ is rectangular.

5. Preliminaries for the proofs

The letters μ and ν stand for limit ordinals with $\mu, \nu \leq \lambda$ for a sufficiently large ordinal λ. For $P,\ Q \subset \lambda + 1 = [0, \lambda]$ and $X \subset (\lambda + 1)^2 = [0, \lambda]^2$, we put $X_P = \big(P \times (\lambda + 1)\big) \cap X$, $X^Q = \big((\lambda + 1) \times Q\big) \cap X$ and $X_P^Q = X_P \cap X^Q = (P \times Q) \cap X$.

Here, we always put $X = A \times B$ for two subspaces A and B in $\lambda + 1$. According to this notation, for each $P,\ Q \subset \lambda + 1$, we define $X_P = (P \cap A) \times B$, $X^Q = A \times (Q \cap B)$ and $X_P^Q = (P \cap A) \times (Q \cap B)$.

For each $A \subset \mu$, $\mathrm{Lim}_\mu(A)$ is the set $\{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of A in μ. For convenience, we let $\sup \emptyset = -1$, where -1 is the immediate predecessor of the ordinal 0. Obviously, $\mathrm{Lim}_\mu(A)$ is cub in μ whenever A is unbounded in μ. We use $\mathrm{Lim}(A)$ instead of $\mathrm{Lim}_\mu(A)$ without the confusion.

Let C be a cub set in μ, where $\mathrm{cf}(\mu) \geq \omega_1$. Clearly, $\mathrm{Lim}(C) \subset C$. We put $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}(C)$, that is, $\mathrm{Succ}(C)$ means the set of all successors in C. Next, we put

$p_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Note that, for each $\alpha \in C$, $p_C(\alpha) \in C \cup \{-1\}$ and that $p_C(\alpha) < \alpha$ iff $\alpha \in \mathrm{Succ}(C)$. So, for each $\alpha \in \mathrm{Succ}(C)$, $p_C(\alpha)$ is the immediate predecessor of α in $C \cup \{-1\}$.

A strictly increasing function $M : \mathrm{cf}(\mu)+1 \to \mu+1$ is said to be a *normal function for* μ if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma \leq \mathrm{cf}(\mu)$ and $M(\mathrm{cf}(\mu)) = \mu$. For convenience, we may define $M(0) = 0$. Note that we can always take a normal function M for μ whenever $\mathrm{cf}(\mu) \geq \omega$. In particular, if μ is a regular cardinal, then we can fix the identity map on $\mu + 1$ as the normal function.

For the function M, we have

Fact 5.1. *Let* $\mathrm{cf}(\mu) \geq \omega_1$. *A normal function* M *for* μ *satisfies*

(1) M *is a homeomorphism from* $\mathrm{cf}(\mu) + 1$ *into* $\mu + 1$,
(2) $M\big([0, \mathrm{cf}(\mu))\big)$ *is a cub set in* μ,
(3) *for two normal functions* M *and* M' *for* μ, *there is a cub set* C *in* μ *such that* $M \restriction C = M' \restriction C$.
(4) S *is a stationary set in* μ *iff* $M^{-1}(S)$ *is stationary set in* $\mathrm{cf}(\mu)$

Let μ and ν be limit ordinals with $\kappa = \mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$. Let $A \subset \mu$ and $B \subset \nu$. We take two normal functions M and N for μ and ν, respectively. By Fact 5.1, we can fix them. After this, we denote by M and N the fixed normal functions for μ and ν, respectively.

It follows from Fact 5.1 (3) that the stationarity of $M^{-1}(A) \cap N^{-1}(B)$ in κ does not depend on the choices of normal functions M and N. So we say that A and B have *stationary intersection* (resp., *non-stationary intersection*) if $M^{-1}(A) \cap N^{-1}(B)$ is stationary (resp., non-stationary) in κ for some (any) normal functions M and N.

Let M be a normal function for μ and let C be a cub set in $\mathrm{cf}(\mu)$. Then we define the map $m_C \colon \mu + 1 \to C \cup \{\mathrm{cf}(\mu)\}$ by

$$m_C(\alpha) = \min\{\gamma \in C \cup \{\mathrm{cf}(\mu)\} \colon \alpha \leq M(\gamma)\}$$

for each $\alpha \leq \mu$. For the M and the m_C, we have

Fact 5.2. *Let* C *be a cub set in* $\mathrm{cf}(\mu)$. *Let* m_C *be the map defined as above. Then*

(1) m_C *is a continuous map*,
(2) $m_C(\alpha) \in \mathrm{Lim}(C)$ *or* $\alpha \in M(C)$ *implies* $M(m_C(\alpha)) = \alpha$,
(3) $m_C(\alpha) \in \mathrm{Succ}(C)$ *implies* $M(p_C(m_C(\alpha))) < \alpha \leq M(m_C(\alpha))$,
(4) $\gamma \in C$ *implies* $m_C(M(\gamma)) = \gamma$,
(5) $m_C(\mu) = \mathrm{cf}(\mu)$.

For the normal function N for a limit ordinal ν and a cub set D in $\mathrm{cf}(\nu)$, we similarly define $n_D : \nu + 1 \to D \cup \{\mathrm{cf}(\nu)\}$ by $n_D(\beta) = \min\{\delta \in D \cup \{\mathrm{cf}(\nu)\} : \beta \leq N(\delta)\}$ for each $\beta \leq \nu$.

6. Proof of main theorem

Lemma 6.1. *Let* $\mathrm{cf}(\mu) \geq \omega_1$ *and* $\mathrm{cf}(\nu) \geq \omega_1$. *Let* $A \subset \mu$ *and* $B \subset \nu$ *such that* A *and* B *are stationary in* μ *and* ν, *respectively. If* $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$ *and* $\mathcal{G}$ *is a finite open cover* *of $X = A \times B$, then there are $\alpha_0 \in \mu$ and $\beta_0 \in \nu$ such that $X^{(\beta_0,\nu)}_{(\alpha_0,\mu)}$ is contained in some member of $\mathcal{G}$.*

Proof. We may assume $\mathrm{cf}(\mu) < \mathrm{cf}(\nu)$. For each $\gamma \in M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$ and each $\delta \in N^{-1}(B) \cap \mathrm{Lim}(\mathrm{cf}(\nu))$, there are $G(\gamma,\delta) \in \mathcal{G}$, $f(\gamma,\delta) < \gamma$ and $g(\gamma,\delta) < \delta$ such that

$$\langle M(\gamma), N(\delta)\rangle \in X^{(N(g(\gamma,\delta)),N(\delta)]}_{(M(f(\gamma,\delta)),M(\gamma)]} \subset G(\gamma,\delta).$$

Since $N^{-1}(B) \cap \mathrm{Lim}(\mathrm{cf}(\nu))$ is stationary in $\mathrm{cf}(\nu)$ and $\gamma < \mathrm{cf}(\mu) < \mathrm{cf}(\nu)$, it follows from PDL that there are $G(\gamma) \in \mathcal{G}$, $f(\gamma) < \gamma$ and $g(\gamma) < \mathrm{cf}(\nu)$ such that $X^{(N(g(\gamma)),\nu)}_{(M(f(\gamma)),M(\gamma)]} \subset G(\gamma)$. Since $M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$ is stationary in $\mathrm{cf}(\mu)$, it follows from PDL again that there are $S \subset M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$, $\gamma_0 < \mathrm{cf}(\mu)$ and $G_0 \in \mathcal{G}$ such that S is stationary in $\mathrm{cf}(\mu)$ with $\gamma > \gamma_0$, $f(\gamma) < \gamma_0$ and $G(\gamma) = G_0 \in \mathcal{G}$ for each $\gamma \in S$. By $|S| = \mathrm{cf}(\mu) < \mathrm{cf}(\nu)$, take a $\delta_0 \in \mathrm{cf}(\nu)$ with $\delta_0 > \sup\{g(\gamma) \colon \gamma \in S\}$. Then it is easily verified that $X^{(N(\delta_0),\nu)}_{(M(\gamma_0),\mu)} \subset G_0 \in \mathcal{G}$. $\square$

Lemma 6.2. *Let $\mathrm{cf}(\mu) \geq \omega_1$. Let $A \subset \mu$ and $\nu \in B \subset \nu + 1$ such that A is stationary in μ and $B \cap \nu$ is unbounded in ν. If $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$ and $\mathcal{G}$ is a finite open cover of $X = A \times B$, then there are $\alpha_0 \in \mu$ and $\beta_0 \in \nu$ such that $X^{(\beta_0,\nu]}_{(\alpha_0,\mu)}$ is contained in some member of $\mathcal{G}$.*

Proof. For each $\gamma \in M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$, by $\langle M(\gamma), \nu\rangle \in X$, there are $G(\gamma) \in \mathcal{G}$, $f(\gamma) < \gamma$ and $g(\gamma) < \mathrm{cf}(\nu)$ such that $X^{(N(g(\gamma)),\nu]}_{(M(f(\gamma)),M(\gamma)]} \subset G(\gamma)$. It follows from PDL that there are $S \subset M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$, $\gamma_0 < \mathrm{cf}(\mu)$ and $G_0 \in \mathcal{G}$ such that S is stationary in $\mathrm{cf}(\mu)$ with $\gamma > \gamma_0$, $f(\gamma) < \gamma_0$ and $G(\gamma) = G_0 \in \mathcal{G}$ for each $\gamma \in S$.

In the case of $\mathrm{cf}(\nu) < \mathrm{cf}(\mu)$; there are $T \subset S$ and $\delta_0 \in \mathrm{cf}(\nu)$ such that T is stationary in $\mathrm{cf}(\mu)$ with $g(\gamma) = \delta_0$ for each $\gamma \in T$. In the case of $\mathrm{cf}(\mu) < \mathrm{cf}(\nu)$; take $\delta_0 \in \mathrm{cf}(\nu)$ with $\delta_0 > \sup\{g(\gamma) \colon \gamma \in S\}$. In both cases, it is easily verified that $X^{(N(\delta_0),\nu]}_{(M(\gamma_0),\mu)} \subset G_0$. $\square$

Lemma 6.3. *Let $\kappa = \mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$. Let $A \subset \mu$ and $B \subset \nu + 1$. Let $\mathcal{G}$ be a finite cozero cover of $X = A \times B$. If A and $B \cap \nu$ have stationary intersection, then there is $\gamma_0 \in \kappa$ such that $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)}$ is contained in some member of $\mathcal{G}$.*

Proof. For each $\gamma \in M^{-1}(A) \cap N^{-1}(B \cap \nu) \cap \mathrm{Lim}(\kappa)$, we choose $G(\gamma) \in \mathcal{G}$ containing $\langle M(\gamma), N(\gamma)\rangle$. Since $G(\gamma)$ is open in X, there is $f(\gamma) < \gamma$ such that $X^{(N(f(\gamma)),N(\gamma)]}_{(M(f(\gamma)),M(\gamma)]} \subset G(\gamma)$. Since $\mathcal{G}$ is finite, it follows from PDL that there are $S \subset M^{-1}(A) \cap N^{-1}(B \cap \nu) \cap \mathrm{Lim}(\kappa)$, $\gamma_0 \in \kappa$ and $G_0 \in \mathcal{G}$ such that S is stationary in κ with $\gamma > \gamma_0$, $f(\gamma) < \gamma_0$ and $G(\gamma) = G_0$ for each $\gamma \in S$. Then we have $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \subset G_0 \in \mathcal{G}$. It is clear that $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} = X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \subset G_0$ whenever $\nu \notin B$. So we may let $\nu \in B$.

Now, assume that $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \not\subset G_0$. There is $\alpha_0 \in A$ such that $\alpha_0 > M(\gamma_0)$ and $\langle \alpha_0, \nu\rangle \notin G_0$. Since G_0 is a cozero-set in X, there is a sequence $\{F_n \colon n \in \omega\}$ of closed sets in X whose union is G_0. For each $n \in \omega$, by $\langle \alpha_0, \nu\rangle \notin F_n$, we find $\beta_n \in \nu$ with $X^{(\beta_n,\nu]}_{\{\alpha_0\}} \cap F_n = \emptyset$. By $\mathrm{cf}(\nu) \geq \omega_1$, we take $\beta_\omega \in B \cap \nu$ with $\beta_\omega > \sup\{\beta_n \colon n \in \omega\}$ and $\beta_\omega > N(\gamma_0)$. Then, since $\langle \alpha_0, \beta_\omega\rangle \notin \bigcup_{n\in\omega} F_n = G_0$, it follows that $\langle \alpha_0, \beta_\omega\rangle \in X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \setminus G_0 \neq \emptyset$. This contradicts $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \subset G_0$. $\square$

Lemma 6.4. *Let* $\mathrm{cf}(\mu) \geq \omega_1$ *and* $\mathrm{cf}(\nu) \geq \omega_1$. *Let* $A \subset \mu$ *and* $B \subset \nu$ *such that* A *and* B *are stationary in* μ *and* ν, *respectively. If* $\mathcal{G}$ *is a* σ*-locally finite rectangular open cover of* $X = A \times B$, *then there are* $\alpha_0 \in \mu$ *and* $\beta_0 \in \nu$ *such that* $X^{(\beta_0,\nu)}_{(\alpha_0,\mu)}$ *is contained in some member of* $\mathcal{G}$.

Proof. We may assume $\omega_1 \leq \mathrm{cf}(\mu) \leq \mathrm{cf}(\nu)$. Let $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$, where each $\mathcal{G}_n$ is locally finite in X. For each $G \in \mathcal{G}$ and $\alpha \in A$, let $V_\alpha(G) = \{\beta \in B \colon \langle \alpha, \beta\rangle \in G\}$.

Take a $\gamma \in M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$. Since $\{V_{M(\gamma)}(G) \colon G \in \mathcal{G}\}$ is a σ-locally finite open cover of B, it follows from Lemma 2.3 that there are $n(\gamma) \in \omega$ and $G(\gamma) \in \mathcal{G}_{n(\gamma)}$ such that $V_{M(\gamma)}(G(\gamma))$ is stationary in ν. Since $G(\gamma)$ is open in X and γ is a limit ordinal with $\gamma < \mathrm{cf}(\mu) \leq \mathrm{cf}(\nu)$, it follows from PDL that there are $f(\gamma) < \gamma$ and $g(\gamma) < \nu$ such that $X^{(N(g(\gamma)),\nu)}_{(M(f(\gamma)),M(\gamma)]} \subset G(\gamma) \in \mathcal{G}_{n(\gamma)}$. Since f is regressive, it follows from PDL again that there are $S \subset M^{-1}(A) \cap \mathrm{Lim}(\mathrm{cf}(\mu))$, $\gamma_0 \in \mathrm{cf}(\mu)$ and $n_0 \in \omega$ such that S is stationary in $\mathrm{cf}(\mu)$ with $\gamma > \gamma_0$, $f(\gamma) < \gamma_0$ and $n(\gamma) = n_0$ for each $\gamma \in S$. Then we have $X^{(N(g(\gamma)),\nu)}_{(M(\gamma_0),M(\gamma)]} \subset G(\gamma) \in \mathcal{G}_{n_0}$ for each $\gamma \in S$.

Now, we introduce the equivalence relation $\sim$ on S defined by $\gamma \sim \gamma'$ iff $G(\gamma) = G(\gamma')$. We denote by $S/\sim$ the set of $\sim$-equivalence classes of S. For each $E \in S/\sim$, let $G_E = G(\gamma_E)$ for some (any) $\gamma_E \in E$. Note that $G_E \neq G_{E'}$ for any $E, E' \in S/\sim$ with $E \neq E'$.

Claim. $S/\sim$ *is finite.*

Proof. Assume the contrary. There is a sequence $\{E_n \colon n \in \omega\}$ of distinct members of $S/\sim$. Pick a $\delta_n \in E_n$ for each $n \in \omega$. Let $\xi = \min\{\delta_n \colon n \in \omega\}$. Then we have $\gamma_0 < \xi \leq \delta_n$ for each $n \in \omega$. By $\mathrm{cf}(\nu) \geq \omega_1$, there is $\eta \in B$ with $\eta > \sup_{n\in\omega} g(\delta_n)$. Then we have

$$\langle M(\xi), N(\eta)\rangle \in X^{(N(g(\delta_n)),\nu)}_{(M(\gamma_0),M(\delta_n)]} \subset G(\delta_n) = G_{E_n} \in \mathcal{G}_{n_0}.$$

for each $n \in \omega$. However, since $G_{E_n} \neq G_{E_{n'}}$ if $n \neq n'$, this contradicts the local finiteness of $\mathcal{G}_{n_0}$ in X.

By the Claim, there is $E_0 \in S/\sim$ such that E_0 is stationary in $\mathrm{cf}(\mu)$. Then we have $\bigcup_{\gamma\in E_0} X^{(N(g(\gamma)),\nu)}_{(M(\gamma_0),M(\gamma)]} \subset \bigcup_{\gamma\in E_0} G(\gamma) = G_{E_0} \in \mathcal{G}_{n_0}$. Let $\rho = \min E_0$ $(\in E_0 \subset S)$. Since G_{E_0} is a rectangle, we can let $G_{E_0} = G'_{E_0} \times G''_{E_0}$. It suffices to show that $X^{(N(g(\rho)),\nu)}_{(M(\rho),\mu)} \subset G_{E_0} \in \mathcal{G}$. Pick any $\langle \alpha, \beta\rangle \in X^{(N(g(\rho)),\nu)}_{(M(\rho),\mu)}$. Take $\zeta \in E_0$ with $M(\zeta) > \alpha$, and take $\beta_1 \in B$ with $\beta_1 > N(g(\zeta))$. Since $\rho \leq \zeta$ and $M(\gamma_0) < M(\rho) < \alpha < M(\zeta)$, we have

$$\langle \alpha, \beta_1\rangle \in X^{(N(g(\zeta)),\nu)}_{(M(\rho),M(\zeta)]} \subset X^{(N(g(\zeta)),\nu)}_{(M(\gamma_0),M(\zeta)]} \subset G_{E_0} = G'_{E_0} \times G''_{E_0}.$$

Hence we have $\alpha \in G'_{E_0}$. On the other hand, we have

$$\langle M(\rho), \beta\rangle \in X^{(N(g(\rho)),\nu)}_{(M(\gamma_0),M(\rho)]} \subset G_{E_0} = G'_{E_0} \times G''_{E_0}.$$

Hence we have $\beta \in G''_{E_0}$. Therefore, we obtain $\langle \alpha, \beta\rangle \in G'_{E_0} \times G''_{E_0} = G_{E_0}$. $\square$

Lemma 6.5. *Let* $\mathrm{cf}(\mu) \geq \omega_1$*. Let* $A \subset \mu$ *and* $\nu \in B \subset \nu + 1$ *such that* A *is stationary in* μ *and* $B \cap \nu$ *is unbounded in* ν*. If* $\mathcal{G}$ *is a* σ*-locally finite rectangular open cover of* $X = A \times B$*, then there are* $\alpha_0 \in \mu$ *and* $\beta_0 \in \nu$ *such that* $X_{(\alpha_0,\mu)}^{(\beta_0,\nu]}$ *is contained in some member of* $\mathcal{G}$*.*

Proof. Let $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$, where each $\mathcal{G}_n$ is locally finite in X. Take an $\alpha \in A$. By $\langle \alpha, \nu \rangle \in X$, there are $n(\alpha) \in \omega$ and $G(\alpha) \in \mathcal{G}_{n(\alpha)}$ with $\langle \alpha, \nu \rangle \in G(\alpha)$. Since $G(\alpha)$ is open in X, there are $f(\alpha) < \alpha$ and $g(\alpha) < \nu$ such that $X_{(f(\alpha),\alpha]}^{(g(\alpha),\nu]} \subset G(\alpha) \in \mathcal{G}_{n(\alpha)}$. Since $\mathrm{cf}(\mu) \geq \omega_1$, it follows from PDL that there are $S \subset A$, $\alpha_0 \in \mu$ and $n_0 \in \omega$ such that S is stationary in μ with $\alpha > \alpha_0, f(\alpha) < \alpha_0$ and $n(\alpha) = n_0$ for each $\alpha \in S$. Then we have $X_{(\alpha_0,\alpha]}^{(g(\alpha),\nu]} \subset X_{(f(\alpha),\alpha]}^{(g(\alpha),\nu]} \subset G(\alpha) \in \mathcal{G}_{n_0}$ for each $\alpha \in S$.

Case 1. Assume $\mathrm{cf}(\nu) \geq \omega_1$. We introduce the same equivalence relation $\sim$ on S as above, that is, $\alpha \sim \alpha'$ iff $G(\alpha) = G(\alpha')$. We similarly denote by $S/\sim$ the set of $\sim$-equivalence classes of S. Since $B \cap \nu$ is unbounded in ν and $\mathrm{cf}(\nu) \geq \omega_1$, we can conclude that $S/\sim$ is finite (by the same argument as in the proof of the Claim above). Take a stationary subset E_0 in μ with $E_0 \in S/\sim$. Let $\alpha_1 = \min E_0$ and take a $\beta_1 \in B \cap \nu$ with $\beta_1 > g(\alpha_1)$. Let $G_{E_0} = G(\alpha_1)$. In the similar way as in the proof of Lemma 6.4, it is verified that $X_{(\alpha_1,\mu)}^{(\beta_1,\nu]} \subset G_{E_0} \in \mathcal{G}_{n_0}$.

Case 2. Assume $\mathrm{cf}(\nu) \leq \omega$. There is a cofinal sequence $\{\zeta_n : n \in \omega\}$ in $B \cap \nu$. There are $S_0 \subset S$ and $m \in \omega$ such that S_0 is stationary in μ with $g(\alpha) < \zeta_m$ for each $\alpha \in S_0$. Let $\alpha_2 = \min S_0$. Since $\alpha_2 \in S_0$ and $\alpha_2 > \alpha_0$, we have $\langle \alpha_2, \zeta_m \rangle \in X_{(\alpha_0,\alpha]}^{(g(\alpha),\nu]} \subset G(\alpha) \in \mathcal{G}_{n_0}$ for each $\alpha \in S_0$. Since $\mathcal{G}_{n_0}$ is locally finite in X, there are $T \subset S_0$ and $G_0 \in \mathcal{G}_{n_0}$ such that T is stationary in μ with $G(\alpha) = G_0$ and $\alpha > \alpha_2$ for each $\alpha \in T$. Then it is easily seen that $X_{(\alpha_2,\mu)}^{(\zeta_m,\nu]} \subset G_0 \in \mathcal{G}_{n_0}$. $\square$

Recall the two functions m_C and n_C in the previous section. Moreover, recall the following which is a key for these functions.

Lemma 6.6 [2]**.** *Let* $\kappa = \mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$*. Let* $A \subset \mu$ *and* $B \subset \nu + 1$*. If there is a cub set* C *in* κ *such that* $C \cap M^{-1}(A) \cap M^{-1}(B) = \emptyset$*, then*

$$Y = \{\langle \alpha, \beta \rangle \in A \times B : m_C(\alpha) < n_C(\beta)\}$$

is clopen in $A \times B$*.*

Making use of this, we obtain

Lemma 6.7. *Let* $\kappa = \mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$*. Let* $A \subset \mu$ *and* $B \subset \nu + 1$ *such that* A *and* $B \cap \nu$ *have non-stationary intersection. Assume one of the following cases:*

(a) *A is stationary in μ, $\nu \notin B$ and B is stationary in ν.*
(b) *A is stationary in μ, $\nu \in B$ and $B \cap \nu$ is unbounded in ν.*

Then $A \times B$ *is not rectangular.*

Proof. Let $X = A \times B$. Assume that X is rectangular. By the assumption, there is a cub set C in κ such that $C \cap M^{-1}(A) \cap N^{-1}(B) = \emptyset$. Let Y be the clopen set in X, described in Lemma 6.6. Since $\{Y, X \smallsetminus Y\}$ is a binary disjoint clopen cover of X, there is a σ-locally finite rectangular open cover $\mathcal{G}$ of X such that $G \subset Y$ or $G \cap Y = \emptyset$ for each $G \in \mathcal{G}$. It follows from Lemmas 6.4 and 6.5 that there are $\gamma_0 \in \kappa$ and $G_0 \in \mathcal{G}$ such that $X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \subset G_0$.

The case (a): Since $M^{-1}(A) \cap C$ and $N^{-1}(B) \cap C$ are stationary in κ, we can take some $\gamma_1 \in M^{-1}(A) \cap C$ and $\delta_1 \in N^{-1}(B) \cap C$ with $\gamma_0 < \gamma_1 < \delta_1$. Clearly, $M(\gamma_0) < M(\gamma_1)$ and $N(\gamma_0) < N(\delta_1)$. It follows from Fact 5.2 (4) that $m_C(M(\gamma_1)) = \gamma_1 < \delta_1 = n_C(N(\delta_1))$. Then we have

$$\langle M(\gamma_1), N(\delta_1)\rangle \in X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \cap Y \subset G_0 \cap Y \neq \emptyset\,.$$

On the other hand, we take $\delta_2 \in N^{-1}(B) \cap C$ and $\gamma_2 \in M^{-1}(A) \cap C$ with $\gamma_0 < \delta_2 < \gamma_2$. Similarly, we have $M(\gamma_0) < M(\gamma_2), N(\gamma_0) < N(\delta_2)$ and $n_C(N(\delta_2)) = \delta_2 < \gamma_2 = m_C(M(\gamma_2))$. Hence we have

$$\langle M(\gamma_2), N(\delta_2)\rangle \in X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \setminus Y \subset G_0 \setminus Y \neq \emptyset\,.$$

This is a contradiction.

The case (b): Since $M^{-1}(A) \cap C$ is stationary in κ, we take $\gamma_1 \in M^{-1}(A) \cap C$ with $\gamma_0 < \gamma_1$. It follows from $\gamma_1 \in C$ and Fact 5.2 (4),(5) that $m_C(M(\gamma_1)) = \gamma_1 < \mathrm{cf}(\mu) = \mathrm{cf}(\nu) = n_C(\nu)$. Hence we have

$$\langle M(\gamma_1), \nu\rangle \in X^{(N(\gamma_0),\nu]}_{(M(\gamma_0),\mu)} \cap Y \subset G_0 \cap Y \neq \emptyset\,.$$

On the other hand, since $B \cap \nu$ is unbounded in ν, we take $\beta_2 \in B \cap \nu$ with $N(\gamma_0) < \beta_2$ and take $\gamma_2 \in M^{-1}(A) \cap C$ with $n_C(\beta_2) < \gamma_2$. Since $\gamma_2 \in C$ and $\gamma_0 < n_C(\beta_2) < \gamma_2$, we have $M(\gamma_0) < M(\gamma_2)$ and $n_C(\beta_2) < \gamma_2 = m_C(M(\gamma_2))$. Hence we have

$$\langle M(\gamma_2), \beta_2\rangle \in X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)} \setminus Y \subset G_0 \setminus Y \neq \emptyset\,.$$

This is a contradiction. □

Theorem 6.8. *Let λ be an ordinal. Let A and B be two subspaces of $\lambda + 1$. Then $A \times B$ is rectangular if and only if, for each $\mu, \nu \leq \lambda$ with $\kappa = \mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$ such that $A \cap \mu$ and $B \cap \nu$ have non-stationary intersection, the following clauses hold:*

1. *If $\mu \notin A$ and $\nu \notin B$, then $A \cap \mu$ is non-stationary in μ or $B \cap \nu$ is non-stationary in ν.*
2. *If $\mu \notin A$ and $\nu \in B$, then $A \cap \mu$ is non-stationary in μ or $B \cap \nu$ is bounded in ν.*
3. *If $\mu \in A$ and $\nu \notin B$, then $A \cap \mu$ is bounded in μ or $B \cap \nu$ is non-stationary in ν.*

Proof. The "only if" part: Assume that $A \times B$ is rectangular. Take any $\mu, \nu \leq \lambda$ with $\kappa = \mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$ such that $A \cap \mu$ and $B \cap \nu$ have non-stationary intersection.

Let $\mu \notin A$ and $\nu \notin B$. Since $X = (A \cap [0, \mu]) \times (B \cap [0, \nu])$ is rectangular, by Lemma 6.7 (a), $A \cap \mu$ is non-stationary in μ or $B \cap \nu$ is non-stationary in ν. So the clause (1) is true. Similarly, it follows from Lemma 6.7 (b) that the clauses (2) and (3) are true. The "if" part: Assume that $A \times B$ is not rectangular. Let

$$\mu = \min\{\xi \leq \lambda : (A \cap [0, \xi]) \times B \text{ is not rectangular }\},$$

$$\nu = \min\{\eta \le \lambda : (A \cap [0,\mu]) \times (B \cap [0,\eta]) \text{ is not rectangular }\},$$

$$X = (A \cap [0,\mu]) \times (B \cap [0,\nu]).$$

Then X is not rectangular. However, by the minimality of μ and ν, $X_{[0,\alpha]}$ and $X^{[0,\beta]}$ are rectangular for each $\alpha < \mu$ and $\beta < \nu$. Obviously, μ and ν are limit ordinals.

Case 1. Assume that $\mu \notin A$ and $\mathrm{cf}(\mu) \le \omega$, or assume that $\mu \notin A$ with $\mathrm{cf}(\mu) \ge \omega_1$ and $A \cap \mu$ is non-stationary in μ. It follows from Lemma 2.2 and the minimality of μ that X can be represented as the topological sum $\bigoplus\{X_\gamma : \gamma \in \mathrm{cf}(\mu)\}$ of its clopen rectangles such that each X_γ is rectangular. This implies that X is rectangular, which is a contradiction.

Since X is not rectangular, there is a finite cozero cover $\mathcal{G}$ of X such that $\mathcal{G}$ has no σ-locally finite rectangular open refinement. It suffices to show the following statement:

($*$) There are $\alpha_0 \in \mu$, $\beta_0 \in \nu$ and $G_0 \in \mathcal{G}$ such that $X^{(\beta_0,\nu]}_{(\alpha_0,\mu]} \subset G_0$.

In fact, since $X_{[0,\alpha_0]}$ and $X^{[0,\beta_0]}$ are rectangular, it is easy to find a σ-locally finite rectangular open refinement $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{H}$ contains $X^{(\beta_0,\nu]}_{(\alpha_0,\mu]}$. This is a contradiction.

Now, we assume that the statement ($*$) is not true.

Case 2. Assume that $\langle \mu, \nu \rangle \in X$. Take $G_0 \in \mathcal{G}$ with $\langle \mu, \nu \rangle \in G_0$. The openness of G_0 in X gives us a contradiction.

Case 3. Assume that $\mu \notin A$ and $\nu \notin B$ with $\mathrm{cf}(\mu) \ge \omega_1$ and $\mathrm{cf}(\nu) \ge \omega_1$ and that $A \cap \mu$ and $B \cap \nu$ are stationary in μ and ν, respectively. It follows from Lemma 6.1 that $\mathrm{cf}(\mu) = \mathrm{cf}(\nu)$. By the clause (1), $A \cap \mu$ and $B \cap \nu$ have stationary intersection. However, Lemma 6.3 gives us a contradiction.

Case 4. Assume that $\mu \notin A$ with $\mathrm{cf}(\mu) \ge \omega_1$ and $\nu \in B$ and that $A \cap \mu$ is stationary in μ and $B \cap \nu$ is unbounded in ν. This case is similar to Case 3, using Lemma 6.2 and the clause (2) instead of Lemma 6.1 and the clause (1), respectively.

Case 5. Assume that $\mu \notin A$ with $\mathrm{cf}(\mu) \ge \omega_1$ and $\nu \in B$ and that $A \cap \mu$ is stationary in μ and $B \cap \nu$ is bounded in ν. By Lemma 2.3, there are $G_0 \in \mathcal{G}$ and $\alpha_0 \in \mu$ such that $((\alpha_0, \mu) \cap A) \times \{\nu\} = X^{\{\nu\}}_{(\alpha_0,\mu)} \subset G_0 \cap (A \times \{\nu\})$. Take $\beta_0 \in \nu$ with $B \cap \nu \subset \beta_0$. Then we have $X^{(\beta_0,\nu]}_{(\alpha_0,\mu)} = X^{(\beta_0,\nu]}_{(\alpha_0,\mu]} \subset G_0$.

Other cases are similar to one of the five cases above. □

Proof of Theorem 4.1. It should be noted that the equivalent condition in Theorem 6.8 is exactly the same as that of [3, Theorem B (i)]. This means that the implication (a) $\Longleftrightarrow$ (b) holds. Moreover, the implication (b) $\Rightarrow$ (c) follows from Corollary 2.8 and the converse is obvious. Thus Theorem 4.1 has been proved. □

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