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C^*-EMBEDDED DENSE SUBSETS OF z-NEIGHBORHOOD-SUBLINEAR SPACES ARE P-EMBEDDED

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Abstract. We define a concept of z-neighborhood-sublinear space and point out that

- every first-countable Tychonoff space and every generalized ordered space is z-neighborhood-sublinear,
- subspaces and finite products of z-neighborhood-sublinear spaces are z-neighborhood-sublinear.

As a main theorem, we prove that every C^*-embedded dense subset of a z-neighborhood-sublinear space is P-embedded.

In [7], the first author and Y. Yajima proved that for all subspaces A, B of an ordinal, if a closed subset F of $A \times B$ is C^*-embedded in $A \times B$, then F is P-embedded in $A \times B$. We can remove closedness from the assumption by applying the main theorem.

1. Introdution

Let $\mathbb{R}$ be the real line, and $\mathbb{I}$ the unit interval, i.e. $\mathbb{I} = [0,1] \subset \mathbb{R}$. A subset E in a space X is said to be *C^*-embedded* (respectively, C-embedded) in X if every continuous function from E into $\mathbb{I}$ (resp. $\mathbb{R}$) is continuously extended over X. A subset E in X is said to be *P-embedded* in X if every continuous function from E into Z is continuously

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extended over X whenever Z is a Banach space. Obviously, the following implication holds.

$$P\text{-embedding} \Rightarrow C\text{-embedding} \Rightarrow C^*\text{-embedding}$$

Let W be a subset of a linearly ordered set $X = (X, <)$. We say that W is *left-closed* if $(\leftarrow, b] = \{x \in X : x \leq b\} \subset W$ for every $b \in W$. We say that W is *right-closed* if $[a, \rightarrow) = \{x \in X : a \leq x\} \subset W$ for every $a \in W$. We call W a *convex* set if $(a, b) = \{x \in X : a < x < b\} \subset W$ for every $a, b \in W$ with $a < b$.

A *linearly ordered topological space* (abbreviated *LOTS*) is a linearly ordered set $(X, <)$ with the open interval topology, i.e.

$$\{(a, b) : a \in X \cup \{\leftarrow\}, b \in X \cup \{\rightarrow\}, a < b\}$$

is a base. A *generalized ordered space* (abbreviated *GO-space*) is a triple $X = (X, <, \tau)$ such that $(X, <)$ is a linearly ordered set and τ is a Hausdorff topology on X that has a base of convex sets.

With no mention, each ordinal μ is considered to be the LOTS with respect to the usual order $<$ of ordinals, where μ is identified with the set $\{\alpha : \alpha \text{ is an ordinal}, \alpha < \mu\}$. Such space μ is called an ordinal space.

Obviously, every subspace of a GO-space is also a GO-space. The following implication holds.

$$\text{ordinal space} \Rightarrow \text{LOTS} \Rightarrow \text{GO-space} \Rightarrow \text{monotonically normal}$$
$$\Rightarrow \text{collectionwise normal} \Rightarrow \text{normal}$$

In the present paper, we will define the concepts of *neighborhood-linear* space, *neighborhood-sublinear* space, and *z-neighborhood-sublinear* space. For them, the following implication holds.

				first-countable
				$\Downarrow$
ordinal space	$\Rightarrow$	Tychonoff + neighborhood-linear	$\Rightarrow$	neighborhood-linear
$\Downarrow$		$\Downarrow$		$\Downarrow$
GO-space	$\Rightarrow$	z-neighborhood -sublinear	$\Rightarrow$	neighborhood -sublinear

The class of (z-)neighborhood-sublinear spaces has the following nice properties.

- Every subspace of a (z-)neighborhood-sublinear space is also (z-)neighborhood-sublinear.
- Every finite product of (z-)neighborhood-sublinear spaces is also (z-)neighborhood-sublinear.

As a main theorem, we will prove that every C^*-embedded dense subset of a z-neighborhood-sublinear space is P-embedded.

All spaces are assumed to be Hausdorff topological spaces.

2. Neighborhood-sublinear spaces

Definition 2.1. We say that a family $\mathcal{D}$ of sets is *linear* if $D \subset D'$ or $D' \subset D$ holds for each $D, D' \in \mathcal{D}$. For a family $\mathcal{V}$ of sets, we call a subfamily $\mathcal{D}$ of $\mathcal{V}$ a *generator* of $\mathcal{V}$ if for each $V \in \mathcal{V}$, there is a $D \in \mathcal{D}$ with $D \subset V$.

Let X be a space with $p \in X$. A generator of the neighborhood filter $\mathrm{Nbd}_X(p) = \{V \subset X : p \in \mathrm{Int}_X V\}$ at p is called a *neighborhood base*. We say that X is *neighborhood-linear at p* if there is a linear neighborhood base at p. We say that a space X is *neighborhood-linear* if it is neighborhood-linear at each point $p \in X$. The same property has already been defined and studied by Sheldon Davis in 1978, and such a space is called a *lob-space* there [2].

The following fact is trivial.

Fact 2.2. *Every first-countable space is neighborhood-linear. Every ordinal space is neighborhood-linear.*

Example 2.3. *A LOTS which is not neighborhood-linear: Let $\prec$ be the linear order on the set $X = \omega_1 + \omega$ such that $\alpha \prec \beta$ holds if either $\alpha < \beta < \omega_1 + \omega$ with $\alpha \leq \omega_1$ or $\omega_1 < \beta < \alpha$, where $<$ is the usual order of ordinals. Then the LOTS $X = (X, \prec)$ is not neighborhood-linear at ω_1.*

Proof. Assume that there is a linear neighborhood base $\mathcal{D}$ at ω_1 for X. For each $n \in \omega \setminus \{0\}$, we can take a $D_n \in \mathcal{D}$ with $D_n \subset (\leftarrow, \omega_1 + n)_\prec$. There are $\gamma_n \in \omega_1$ and $l_n \in \omega \setminus \{0\}$ with $(\gamma_n, \omega_1 + l_n)_\prec \subset D_n$. Take an $\alpha \in \omega_1$ such that $\gamma_n < \alpha$ for every $n \in \omega \setminus \{0\}$. We can take a $D \in \mathcal{D}$ with $D \subset (\alpha, \rightarrow)_\prec$. Take $\gamma \in \omega_1$ and $l \in \omega \setminus \{0\}$ with $(\gamma, \omega_1 + l)_\prec \subset D$. And take an $n \in \omega$ with $l < n$. Since $\mathcal{D}$ is linear, $D \subset D_n$ or $D_n \subset D$ must hold. But $D \not\subset D_n$ by $\omega_1 + n \in (\gamma, \omega_1 + l)_\prec \subset D$ and $\omega_1 + n \notin (\leftarrow, \omega_1 + n)_\prec \supset D_n$. And $D_n \not\subset D$ by $\alpha \in (\gamma_n, \omega_1 + l_n)_\prec \subset D_n$ and $\alpha \notin (\alpha, \rightarrow)_\prec \supset D$. It is contradiction. □

Fact 2.4. *If Y is a neighborhood-linear space, then each subspace X of Y is neighborhood-linear.*

Proof. Let $p \in X \subset Y$ and take a linear neighborhood base $\mathcal{D}$ at p for Y. Then $\mathcal{D} \restriction X = \{D \cap X : D \in \mathcal{D}\}$ is a linear neighborhood base at p for X. □

The following example is mentioned in Brian Scott's paper [14].

Example 2.5 ([14]). *The product of neighborhood-linear spaces which is not neighborhood-linear: The ordinal spaces* $\omega_1 + 1$ *and* $\omega + 1$ *are neighborhood-linear, but the product* $(\omega_1+1)\times(\omega+1)$ *is not neighborhood-linear at* $\langle\omega_1, \omega\rangle$.

Proof. Let X be the LOTS in Example 2.3. We obtain a topological embedding e of X into $Y = (\omega_1 + 1) \times (\omega + 1)$ by letting $e(\alpha) = \langle\alpha, \omega\rangle$ for each $\alpha \in \omega_1 + 1$ and $e(\omega_1 + n) = \langle\omega_1, n\rangle$ for each $n \in \omega \setminus \{0\}$. By the proof of Fact 2.4, Y is not neighborhood-linear at $\langle\omega_1, \omega\rangle = e(\omega_1)$. □

Definition 2.6. We say that a family $\mathcal{D}$ of sets is *sublinear* if $\mathcal{D} = \bigcup_{k\in S} \mathcal{D}_k$ for some finite collection $\{\mathcal{D}_k : k \in S\}$ of linear subfamilies of $\mathcal{D}$. For a family $\mathcal{V}$ of subsets of a set X, we call a subfamily $\mathcal{D}$ of $\mathcal{V}$ a *subgenerator* of $\mathcal{V}$ (on X) if for each $V \in \mathcal{V}$, there is a finite subfamily $\mathcal{D}'$ of $\mathcal{D}$ such that $\bigcap \mathcal{D}' \subset V$. Here we consider that $\bigcap \emptyset = X$.

Let X be a space with $p \in X$. A subgenerator of the neighborhood fliter at p is called a *neighborhood subbase.* We say that X is *neighborhood-sublinear at* p if there is a sublinear neighborhood subbase at p. We say that a space X is *neighborhood-sublinear* if it is neighborhood-sublinear at each point $p \in X$.

These spaces are closely related to the *globular* spaces which are defined by Brian Scott. See [14] for details.

Fact 2.7. *Every GO-space is neighborhood-sublinear.*

Proof. Let X be a GO-space with $p \in X$. Let $\mathrm{Nbd}^{\leftarrow}_X(p)$ and $\mathrm{Nbd}^{\rightarrow}_X(p)$ be the neighborhood filters of $(\leftarrow, p]$ and $[p, \rightarrow)$, respectively, i.e.

$$\mathrm{Nbd}^{\leftarrow}_X(p) = \{V \subset X : (\leftarrow, p] \subset \mathrm{Int}_X V\},$$

$$\mathrm{Nbd}^{\rightarrow}_X(p) = \{V \subset X : [p, \rightarrow) \subset \mathrm{Int}_X V\}.$$

Let $\mathcal{D}^{\leftarrow} = \{D \in \mathrm{Nbd}^{\leftarrow}_X(p) : D \text{ is left-closed}\}$ and $\mathcal{D}^{\rightarrow} = \{D \in \mathrm{Nbd}^{\rightarrow}_X(p) : D \text{ is right-closed}\}$. It is routine to check that the family of left-closed (resp. right-closed) subsets of X is linear. In particular, $\mathcal{D}^{\leftarrow}$ and $\mathcal{D}^{\rightarrow}$ are linear.

Let $\mathcal{D} = \mathcal{D}^{\leftarrow} \cup \mathcal{D}^{\rightarrow}$. Obviously, $\mathcal{D}$ is a sublinear family of neighborhoods of p in X. Let V be a neighborhood of p in X. Then there is an open convex set W with $p \in W \subset V$. Let $D^{\leftarrow} = \bigcup_{b\in W} (\leftarrow, b]$ and $D^{\rightarrow} = \bigcup_{a\in W} [a, \rightarrow)$. Then they are open in X, and we have $D^{\leftarrow} \in \mathcal{D}^{\leftarrow}$ and $D^{\rightarrow} \in \mathcal{D}^{\rightarrow}$. If $x \in D^{\leftarrow} \cap D^{\rightarrow}$, then there are $a, b \in W$ with $a \leq x \leq b$. It follows that $x \in [a, b] \subset W \subset V$ since W is a convex set, so $D^{\leftarrow} \cap D^{\rightarrow} \subset V$ holds. Hence, $\mathcal{D}$ is a neighborhood subbase at p, and so it witnesses that X is neighborhood-sublinear at p. □

Similarly to Fact 2.4, we see:

Fact 2.8. *If Y is a neighborhood-sublinear space, then each subspace X of Y is neighborhood-sublinear.*

Fact 2.9. *Every finite product of neighborhood-sublinear spaces is also neighborhood-sublinear.*

Proof. A proof is straightforward. Let $\{X(j) : j \in J\}$ be a finite collection of spaces, and $X = \prod_{j\in J} X(j)$. Let $p \in X$. For each $j \in J$, let $\pi_j : X \to X(j)$ be the projection, and take a neighborhood subbase $\mathcal{D}(j)$ at $\pi_j(p)$ for $X(j)$. It is well-known that $\mathcal{D} = \{{\pi_j}^{-1}[D] : j \in J, D \in \mathcal{D}(j)\}$ is a neighborhood subbase at p for X.

We would like to show that X is neighborhod-sublinear at p by assuming that $X(j)$ is neighborhood-sublinear at $\pi_j(p)$ for each $j \in J$. In the argument above, we may assume that $\mathcal{D}(j)$ is sublinear. It suffices to show that $\mathcal{D}$ is sublinear. For each $j \in J$, take a finite collection $\{\mathcal{D}_k(j) : k \in S(j)\}$ of linear subfamilies of $\mathcal{D}(j)$ with $\mathcal{D}(j) = \bigcup_{k\in S(j)} \mathcal{D}_k(j)$. Let $S = \{\langle j, k\rangle : j \in J, k \in S(j)\}$, and $\mathcal{D}_{\langle j,k\rangle} = \{{\pi_j}^{-1}[D] : D \in \mathcal{D}_k(j)\}$ for each $\langle j, k\rangle \in S$. Obviously, S is a finite set, $\mathcal{D}_{\langle j,k\rangle}$ is a linear subfamily of $\mathcal{D}$ for each $\langle j, k\rangle \in S$, and $\mathcal{D} = \bigcup_{\langle j,k\rangle\in S} \mathcal{D}_{\langle j,k\rangle}$ holds. Hence, $\mathcal{D}$ is sublinear. □

From three facts above, we have:

Corollary 2.10. *Every subspace of a finite product of GO-spaces is neighborhood-sublinear.*

3. z-NEIGHBORHOOD-SUBLINEAR SPACES

A space X is said to be *monotonically normal* [6] if there is a function G which assigns to each ordered pair (A, U) of a closed set A and an open set U with $A \subset U$, an open set $G(A, U)$ such that

(a) $A \subset G(A, U) \subset \mathrm{Cl}_X G(A, U) \subset U$,
(b) if $A \subset B$ and $U \subset V$, then $G(A, U) \subset G(B, V)$.

Lemma 3.1 ([15, Lemma 2.1])**.** *If X is a monotonically normal space, then to each ordered pair (A, U), where A is a closed set and U is an open set in X with $A \subset U$, we can assign a continuous function $f_{A,U} : X \to \mathbb{I}$ such that $f_{A,U}(x) = 0$ for every $x \in A$, $f_{A,U} = 1$ for every $x \in X \setminus U$, and such that if $A \subset B$ and $U \subset V$, then $f_{A,U}(x) \geq f_{B,V}(x)$ for all $x \in X$.*

Definition 3.2. Let $\mathcal{V}$ be a family of sets. We call the cardinal

$$\lambda = \min\{|\mathcal{D}| : \mathcal{D} \text{ is a generator of } \mathcal{V}\}$$

the *downward cofinality* of $\mathcal{V}$. If $\mathcal{D} = \{D_\xi : \xi \in \lambda\}$ is a generator of $\mathcal{V}$ and $D_\xi \subsetneq D_\zeta$ for each $\zeta < \xi < \lambda$, then we call $\mathcal{D}$ a *strictly descending generator* of $\mathcal{V}$.

Lemma 3.3 (folklore). *Let λ be the downward cofinality of a family $\mathcal{V}$ of sets. Assume that $\mathcal{V}$ has a linear generator. Then*

(1) $\lambda = 0$, $\lambda = 1$ *or* λ *is a regular infinite cardinal.*
(2) *For each generator $\mathcal{D}'$ of $\mathcal{V}$, there is a strictly descending generator $\mathcal{D} = \{D_\xi : \xi \in \lambda\}$ of $\mathcal{V}$ such that $\mathcal{D} \subset \mathcal{D}'$.*

We give here a sketch of a proof for readers convenience.

Proof. Take generators $\mathcal{L}$ and $\mathcal{M}$ of $\mathcal{V}$ such that $\mathcal{L}$ is linear and $\mathcal{M} = \{M_\alpha : \alpha \in \lambda\}$. Let $\mathcal{D}'$ be an arbitrary generator of $\mathcal{V}$. By induction on $\alpha \in \lambda$, we can take $D_\alpha \in \mathcal{D}'$ such that $D_\alpha \subset M_\alpha$ and $D_\alpha \subsetneq D_\beta$ for each $\beta < \alpha < \lambda$. Let $\alpha \in \lambda$ and assume that $D_\beta \in \mathcal{D}'$ is defined for every $\beta < \alpha$. We can take a required D_α in this way: Take an $L_\beta \in \mathcal{L}$ with $L_\beta \subset D_\beta$ for each $\beta < \alpha$. Take a $V_\alpha \in \mathcal{V}$ such that $L_\beta \not\subset V_\alpha$ for all $\beta < \alpha$. Take $L'_\alpha, L''_\alpha \in \mathcal{L}$ with $L'_\alpha \subset V_\alpha$ and $L''_\alpha \subset M_\alpha$. And take $D_\alpha \in \mathcal{D}'$ such that $D_\alpha \subset L'_\alpha$ in case $L'_\alpha \subset L''_\alpha$, and $D_\alpha \subset L''_\alpha$ in case $L''_\alpha \subset L'_\alpha$. After finishing induction, we obtain a strictly descending generator $\mathcal{D} = \{D_\xi : \xi \in \lambda\}$ of $\mathcal{V}$ such that $\mathcal{D} \subset \mathcal{D}'$. And (1) is obtained by the minimality of λ. □

By Lemma 3.3, it is easily seen that if a countable space is neighborhood-sublinear, then it has a countable base. We cannot replace 'GO-space' in Fact 2.7 with 'monotonically normal space'. Let $\beta\omega$ denote the Čech-Stone compactification of a countably infinite discrete space ω.

Example 3.4. *A monotonically normal space which is not neighborhood-sublinear: Let $Y = \omega \cup \{p\}$ by taking a $p \in \beta\omega \setminus \omega$. Then it is monotonically normal, hereditarily paracompact, but not neighborhood-sublinear at p.*

Proof. Since Y does not have a countable neighborhood base at p, it is not neighborhood-sublinear at p. □

A subset V (respectively, D) of a space X is called a *cozero-set* (resp. *zero-set*) if there is a continuous function $f : X \to \mathbb{I}$ such that

$$V = \{x \in X : f(x) > 0\} \text{ (resp. } D = \{x \in X : f(x) = 0\}).$$

For a Tychonoff space, there is a neighborhood base by zero-sets at each point, so the following fact is obtained by using Lemma 3.3 (2).

Fact 3.5. *A Tychonoff space is neighborhood-linear if and only if it has a linear neighborhood base by zero-sets at each point.*

It does not seem so easy to obtain a fact like Fact 3.5 for neighborhood-sublinear spaces without additional assumptions.

Definition 3.6. Let X be a space with $p \in X$. We say that X is *z-neighborhood-sublinear at p* if there is a sublinear neighborhood subbase at p by zero-sets. We say that a space X is *z-neighborhood-sublinear* if it is z-neighborhood-sublinear at each point $p \in X$.

Problem 3.7. Is there a Tychonoff space which is neighborhood-sublinear, but not z-neighborhood-sublinear?

The following easy fact will be used with no mention.

Fact 3.8. *Let X be a space with $p \in X$.*

(1) *For each finite collection $\{\mathcal{D}_k : k \in S\}$ of non-empty linear families of neighborhoods of p in X, the union $\bigcup_{k\in S} \mathcal{D}_k$ is a neighborhood subbase at p if and only if for each neighborhood V of p in X, there is a $D_k \in \mathcal{D}_k$, for each $k \in S$, such that $\bigcap_{k\in S} D_k \subset V$.*

(2) *A space X is (z-)neighborhood-sublinear at p if and only if there is a finite collection $\{\mathcal{D}_k : k \in S\}$ of non-empty linear families such that $\bigcup_{k\in S} \mathcal{D}_k$ is a neighborhood subbase at p (by zero-sets).*

Proof. The 'if' parts are trivial. We show the 'only if' parts.

(1) Assume that $\mathcal{D} := \bigcup_{k\in S} \mathcal{D}_k$ is a neighborhood subbase at p. Let V be a neighborhood of p in X. Then there is a finite subfamily $\mathcal{D}'$ of $\mathcal{D}$ such that $\bigcap \mathcal{D}' \subset V$. Since $\mathcal{D}'_k := \mathcal{D}' \cap \mathcal{D}_k$ is a finite subfamily of a non-empty linear family $\mathcal{D}_k$, for each $k \in S$, we can take a $D_k \in \mathcal{D}_k$ such that $D_k \subset D'$ for all $D' \in \mathcal{D}'_k$. If $D' \in \mathcal{D}'$, then $D' \in \mathcal{D} = \bigcup_{k\in S} \mathcal{D}_k$, so $D' \in \mathcal{D}' \cap \mathcal{D}_{k'} = \mathcal{D}'_{k'}$ for some $k' \in S$, and so $\bigcap_{k\in S} D_k \subset D_{k'} \subset D'$. Hence, $\bigcap_{k\in S} D_k \subset \bigcap \mathcal{D}' \subset V$.

(2) Assume that X is (z-)neighborhood-sublinear at p. By the definion, there is a sublinear neighborhood subbase $\mathcal{D}$ at p (by zero-sets). And $\mathcal{D}$ is expressed as a finite union $\mathcal{D} = \bigcup_{k\in S'} \mathcal{D}_k$ of linear families $\mathcal{D}_k$. By letting $S = \{k \in S' : \mathcal{D}_k \neq \emptyset\}$, we can express $\mathcal{D}$ as a finite union $\mathcal{D} = \bigcup_{k\in S} \mathcal{D}_k$ of non-empty linear families. □

Fact 3.9. *A monotonically normal space is neighborhood-sublinear if and only if it is z-neighborhood-sublinear.*

Proof. Let X be a monotonically normal space with $p \in X$. Let the continuous functions $f_{A,U} : X \to \mathbb{I}$ be as in Lemma 3.1. Assume that X is neighborhood-sublinear at p. Then there is a finite collection $\{\mathcal{D}_k : k \in S\}$ of non-empty linear families such that $\mathcal{D} = \bigcup_{k\in S} \mathcal{D}_k$ is a neighborhood subbase at p. Define a zero-set $z(D)$ in X for each $D \in \mathcal{D}$ by putting $z(D) = \{x \in X : f_{\{p\},\mathrm{Int}_X D}(x) \leq 1/2\}$. Then $z(D)$ is a neighborhood of p in X with $z(D) \subset \mathrm{Int}_X D \subset D$ since $f_{\{p\},\mathrm{Int}_X D}(p) = 0$ and $f_{\{p\},\mathrm{Int}_X D}(x) = 1$ for every $x \in X \setminus \mathrm{Int}_X D$. If $D, D' \in \mathcal{D}$ with $D \subset D'$, then $f_{\{p\},\mathrm{Int}_X D}(x) \geq f_{\{p\},\mathrm{Int}_X D'}(x)$ holds for every $x \in X$, so we have

$z(D) \subset z(D')$. Therefore $z[\mathcal{D}_k] := \{z(D) : D \in \mathcal{D}_k\}$ is linear for each $k \in S$, and so $\bigcup_{k \in S} z[\mathcal{D}_k]$ is a sublinear neighborhood subbase at p by zero-sets. Hence, X is z-neighborhood-sublinear at p. □

Since GO-spaces are monotonically normal, the following fact is immediately obtained from Fact 2.7 and 3.9.

Fact 3.10. *Every GO-space is z-neighborhood-sublinear.*

Similarly to Fact 2.8, 2.9 and Corollary 2.10, we obtain the following two facts and a corollary.

Fact 3.11. *If Y is a z-neighborhood-sublinear space, then each subspace X of Y is z-neighborhood-sublinear.*

Fact 3.12. *Every finite product of z-neighborhood-sublinear spaces is z-neighborhood-sublinear.*

Corollary 3.13. *Every subspace of a finite product of GO-spaces is z-neighborhood-sublinear.*

Example 3.14. *Let X be Rudin's ZFC Dowker space* [13], *i.e.*

$$X = \{x = \langle x_n : 0 < n < \omega \rangle \in \Box_{0<n<\omega}(\omega_n + 1) :$$

$$\exists m < \omega \ \forall n (0 < n < \omega \rightarrow \omega < \mathrm{cf}\, x_n \leq \omega_m)\}.$$

Then X is a z-neighborhood-sublinear space.

Proof. Let $p = \langle p_n : 0 < n < \omega \rangle \in X$, and take an $m < \omega$ such that $\omega < \mathrm{cf}\, p_n \leq \omega_m$ for all n with $0 < n < \omega$. For each k with $0 < k \leq m$, set $N_k = \{n \in \omega \setminus \{0\} : \mathrm{cf}\, p_n = \omega_k\}$, and take an increasing cofinal sequence $\{\alpha_{n,\xi} : \xi \in \omega_k\} \subset p_n$ for each $n \in N_k$. Define a linear family $\mathcal{D}_k = \{D_{k,\xi} : \xi \in \omega_k\}$ of clopen sets of X by putting

$$D_{k,\xi} = \{x = \langle x_n : 0 < n < \omega \rangle \in X : x_n \in (\alpha_{n,\xi}, p_n] \text{ for every } n \in N_k\}$$

for each $\xi \in \omega_k$. Then $\mathcal{D} = \bigcup_{0<k\leq m} \mathcal{D}_k$ is a sublinear neighborhood subbase at p for X by clopen sets. Hence, X is z-neighborhood-sublinear at p. □

4. Comparing C^*-embeddings with P-embeddings

For a space X, let us consider the following conditions.

(C*=P:subset): For each **subset** E in X,
if E is C^*-embedded in X, then E is P-embedded in X.
(C*=P:closed): For each **closed subset** F in X,
if F is C^*-embedded in X, then F is P-embedded in X.
(C*=P:dense): For each **dense subset** D in X,
if D is C^*-embedded in X, then D is P-embedded in X.
(C*=P:$\forall \to \forall$): If **every closed subset** of X is C^*-embedded in X,
then **every closed subset** of X is P-embedded in X.

Obviously, the following implication holds.

$$\begin{array}{ccccc} (\text{C*=P:subset}) & \Rightarrow & (\text{C*=P:closed}) & \Rightarrow & (\text{C*=P:}\forall \to \forall) \\ \Downarrow & & & & \\ (\text{C*=P:dense}) & & & & \end{array}$$

It is well-known as the Tietze-Urysohn Theorem that a space X is normal if and only if each closed subset of X is C^*-embedded (C-embedded) in X. It is said to be essentially proved by Dowker [3] that a space X is collectionwise normal if and only if each closed subset of X is P-embedded in X. Hence, (C*=P:$\forall \to \forall$) fails for a space X if and only if X is normal, but not collectionwise normal. Bing's example of such space is well-known, see [4, Example 5.1.23]. In [12], the third author proved that the Niemytzki plane has the property (C*=P:closed). In the same paper, he also gave some examples of spaces having C^*-embedded, but not C-embedded closed subsets. Obviously, such spaces refute the condition (C*=P:closed).

For a class $\mathcal{X}$ of spaces, let [C*=P:$\cdots$] denote the condition that every $X \in \mathcal{X}$ satisfies the condition (C*=P:$\cdots$), where '$\cdots$' is 'subset', 'closed', 'dense' or '$\forall \to \forall$'. Using the characterizations of normality and collectionwise normality by Tietze-Urysohn and Dowker which are mentioned above, we see that the condition [C*=P:$\forall \to \forall$] is equivalent to the following condition [N=CWN].

[N=CWN]: For each $X \in \mathcal{X}$,
if X is normal, then X is collectionwise normal.

Some classes of spaces are known to have the property [N=CWN]. It is natural to consider whether such classes also satisfy [C*=P:closed] or [C*=P:subset].

It is obvious that any class of collectionwise normal spaces satisfies [C*=P:closed], in particular [N=CWN]. However, some of such classes, for instance the class of monotonically normal spaces or the class of paracompact spaces, do not satisfy [C*=P:dense]. Actually, the countable discrete space ω is C^*-embedded, but not C-embedded in $\omega \cup \{p\}$ for any

$p \in \beta\omega \setminus \omega$, see [5, Problems 4M]. Obviously, $\omega \cup \{p\}$ is monotonically normal and hereditarily paracompact.

Let us consider the class $\{A \times B : A, B \text{ are subspaces of an ordinal}\}$. This class has non-normal members, for instance $\omega_1 \times (\omega_1 + 1)$. The second and third author and K. Tamano proved in 1992 [9] that this class has the property [N=CWN]. The first author and Y. Yajima proved in 2017 that this class also has the property [C*=P:closed].

Theorem 4.1 ([7])**.** *Let A and B be subspaces of an ordinal. Then for each closed subset F of $A \times B$, if F is C^*-embedded in $A \times B$, then F is P-embedded in $A \times B$.*

Removing closedness from the assumption, we will show the following generalization is true, as a corollary of the main theorem, which says that the class $\{A \times B : A, B \text{ are subspaces of an ordinal}\}$ has the property [C*=P:subset].

Corollary 4.2. *Let A and B be subspaces of an ordinal. Then for each subset E of $A \times B$, if E is C^*-embedded in $A \times B$, then E is P-embedded in $A \times B$.*

In fact, we will prove that the class of z-neighborhood-sublinear spaces has the property [C*=P:dense] as a main theorem. By Corollary 3.13, the class of subspaces of finite products of GO-spaces has the property [C*=P:dense]. Then we see that the conditions (C*=P:closed) and (C*=P:subset) are equivalent for each subspace of a finite product of GO-spaces, see Corollary 5.3 in the next section. Obviously, this fact witnesses that Corollary 4.2 is immediately obtained from Theorem 4.1.

The second author, T. Nogura, K. D. Smith and Y. Yajima proved in [8] that the class $\{X : X \subset \mu \times \mu \text{ for some ordinal } \mu\}$ has the property [N=CWN]. However, the second author and T. Usuba recently proved that the property [C*=P:closed] fails for this class in some consistent model of ZFC [10]. The authors do not know whether this class can have the condition [C*=P:closed] in another consistent model of ZFC, but if it is possible, then it must also have the property [C*=P:subset] in the same model. Similarly, we can say that if the class $\{X \times Y : X \text{ and } Y \text{ are GO-spaces}\}$ has the property [C*=P:closed], then it must have the property [C*=P:subset] by our result though the authors do not know whether this class has the property [N=CWN].

5. Basic facts for embeddings

Lemmas and corollaries stated in this section are already known or easily obtained from known facts, so the readers who are familiar to C^*-embeddings and P-embeddings may skip this section.

Let X and Z be spaces, and $E \subset X$. The restriction of a function $g : X \to Z$ on E is denoted by $g \restriction E$, i.e. $g \restriction E : E \to Z$ is the function such that $(g \restriction E)(x) = g(x)$ for every $x \in E$. When we said, in the begining of introduction, that "every continuous function from E into Z is *continuously extended* over X", it means:

> For each continuous function $f : E \to Z$, there is a continuous function $g : X \to Z$ such that $g \restriction E = f$.

It is well-known that for all continuous mappings $f_0 : F \to Z$ and $f_1 : F \to Z$ from a space F into a Hausdorff space Z, if $f_0 \restriction E = f_1 \restriction E$ for some dense subset E of F, then $f_0 = f_1$. From this fact, the following lemma is easily obtained.

Lemma 5.1 (folklore)**.** *If a subset E of a space X is C^*-embedded in X, then $F = \mathrm{Cl}_X E$ is C^*-embedded in X.*

Recall notation (C*=P:$\cdots$) and [C*=P:$\cdots$] defined in the previous section. The following lemma is easily obtained from Lemma 5.1.

Lemma 5.2. *A space X has the property* (C*=P:subset) *if and only if X has the property* (C*=P:closed) *and each closed subspace F of X which is C^*-embedded in X satisfies the condition* (C*=P:dense).

From this lemma, the following corollary is immediately obtained.

Corollary 5.3. *Let $\mathcal{X}$ be a class of spaces such that*

> *$F \in \mathcal{X}$ holds for each $X \in \mathcal{X}$ and for each closed subset F of X.*

If $\mathcal{X}$ satisfies the condition [C*=P:dense], *then the conditions* (C*=P:closed) *and* (C*=P:subset) *are equivalent for each $X \in \mathcal{X}$.*

The following two lemmas are known.

Lemma 5.4 (folklore)**.** *Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a locally finite collection of cozero-sets in a space E. Then,*

(1) *the union $\bigcup_{\lambda\in\Lambda} U_\lambda$ is a cozero-set in E.*
(2) *For each collection $\{Z_\lambda : \lambda \in \Lambda\}$ of zero-sets in E, if $Z_\lambda \subset U_\lambda$ for every $\lambda \in \Lambda$, then the union $\bigcup_{\lambda\in\Lambda} Z_\lambda$ is a zero-set in E.*
(3) *If $\mathcal{U}$ covers E, then there are covers $\{W_\lambda : \lambda \in \Lambda\}$ and $\{Z_\lambda : \lambda \in \Lambda\}$ of E by cozero-sets and zero-sets, respectively, such that $W_\lambda \subset Z_\lambda \subset U_\lambda$ for every $\lambda \in \Lambda$.*

Lemma 5.5 (folklore)**.** *Let E be a subset of a space F, let Y be a cozero-set in F, and let $X = Y \cap E$. If E is C^*-embedded in F, then X is C^*-embedded in Y.*

For a family $\mathcal{V}$ of subsets of a space Y and for $X \subset Y$, let $\mathcal{V} \restriction X$ denote the restriction of $\mathcal{V}$ on X, i.e. $\mathcal{V} \restriction X = \{V \cap X : V \in \mathcal{V}\}$. By using cozero covers, C^*-embedding and P-embedding are characterized as below.

Lemma 5.6 ([11, Lemma 2.1]). *A non-empty subset X of a space Y is C^*-embedded in Y if and only if for each finite cozero cover $\mathcal{U}$ of X, there is a finite cozero cover $\mathcal{V}$ of Y such that $\mathcal{V} \restriction X$ refines $\mathcal{U}$.*

Lemma 5.7 (See [1, Theorem 14.7]). *A non-empty subset X of a space Y is P-embedded in Y if and only if for each locally finite cozero cover $\mathcal{U}$ of X, there is a locally finite cozero cover $\mathcal{V}$ of Y such that $\mathcal{V} \restriction X$ refines $\mathcal{U}$.*

The following lemma is useful when we would like to know whether a space F has the property (C*=P:dense). The result of Lemme 5.8 and 6.2 seem to be known but since the authors could not find proofs in the literature, the result will be given with the proofs.

Lemma 5.8. *A dense subset E of a space F is P-embedded in F if and only if E is C^*-embedded in F and $\{\mathrm{Cl}_F U : U \in \mathcal{U}\}$ covers F for each locally finite cozero cover $\mathcal{U}$ of E.*

Proof. The 'only if' part: Let $\mathcal{U}$ be a locally finite cozero cover of E. Take a cozero cover $\mathcal{V}$ of F which is obtained by Lemma 5.7. For each $V \in \mathcal{V}$, take a $U \in \mathcal{U}$ with $V \cap E \subset U$, then we have $V \subset \mathrm{Cl}_F U$ since E is dense in F. Hence, $\{\mathrm{Cl}_F(U) : U \in \mathcal{U}\}$ covers F.

The 'if' part: Let $\mathcal{U}$ be a locally finite cozero cover of E. Applying Lemma 5.4 (3) twice, take covers $\mathcal{W}_i = \{W_i(U) : U \in \mathcal{U}\}$ and $\mathcal{Z}_i = \{Z_i(U) : U \in \mathcal{U}\}$ of E by cozero-sets and zero-sets, respectively, for each $i \in 2$ such that $W_1(U) \subset Z_1(U) \subset W_0(U) \subset Z_0(U) \subset U$ for every $U \in \mathcal{U}$. Applying Lemma 5.6, we obtain a binary cozero cover $\{V(U), V'(U)\}$ of F for each $U \in \mathcal{U}$ such that $V(U) \cap E \subset W_0(U)$ and $V'(U) \cap E \subset E \setminus Z_1(U)$.

For each finite subfamily $\mathcal{R}$ of $\mathcal{U}$, put $P(\mathcal{R}) = (\bigcap \mathcal{R}) \setminus \bigcup\{Z_0(U) : U \in \mathcal{U} \setminus \mathcal{R}\}$. By Lemma 5.4 (2), $\mathcal{P} = \{P(\mathcal{R}) : \mathcal{R} \subset \mathcal{U}, |\mathcal{R}| < \omega\}$ is a locally finite cozero cover of E. Applying Lemma 5.4 (3), take covers $\mathcal{Q} = \{Q(P) : P \in \mathcal{P}\}$ and $\mathcal{Y} = \{Y(P) : P \in \mathcal{P}\}$ of E by cozero-sets and zero-sets, respectively, such that $Q(P) \subset Y(P) \subset P$ for every $P \in \mathcal{P}$. Applying Lemma 5.6, we obtain a binary cozero cover $\{L(P), L'(P)\}$ of F for each $P \in \mathcal{P}$ such that $L(P) \cap E \subset P$ and $L'(P) \cap E \subset E \setminus Y(P)$.

By the assumption, both $\{\mathrm{Cl}_F W_1 : W_1 \in \mathcal{W}_1\}$ and $\{\mathrm{Cl}_F Q : Q \in \mathcal{Q}\}$ cover F. If $W_1 \in \mathcal{W}_1$, then by taking $U \in \mathcal{U}$ with $W_1 = W_1(U)$, we have $\mathrm{Cl}_F W_1 \subset V(U)$ since $V'(U) \cap W_1 \subset V'(U) \cap Z_1(U) = \emptyset$. If $Q \in \mathcal{Q}$, then by taking $P \in \mathcal{P}$ with $Q = Q(P)$, we have $\mathrm{Cl}_F Q \subset L(P)$ since $L'(P) \cap Q \subset L'(P) \cap Y(P) = \emptyset$. Hence, both $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ and $\mathcal{L} = \{L(P) : P \in \mathcal{P}\}$ are cozero covers of F.

Let $L \in \mathcal{L}$. Then there are a $P \in \mathcal{P}$ with $L = L(P)$ and a finite subfamily $\mathcal{R}$ of $\mathcal{U}$ with $P = P(\mathcal{R})$. Let $V \in \mathcal{V}$. Then there is a $U \in \mathcal{U}$ with $V = V(U)$. Assume that $L \cap V \neq \emptyset$. Then we can take an $x \in L \cap V \cap E$ since E is dense in F. By $x \in V(U) \cap E \subset W_0(U) \subset Z_0(U)$ and $x \in L(P) \cap E \subset P = P(\mathcal{R})$, we have $U \in \mathcal{R}$. Hence, $\{V \in \mathcal{V} : L \cap V \neq \emptyset\} \subset \{V(U) : U \in \mathcal{R}\}$ is a finite family. Since $\mathcal{L}$ is an open cover of F, we see that $\mathcal{V}$ is a locally finite cozero cover of F such that $\mathcal{V} \restriction E$ refines $\mathcal{U}$. By Lemma 5.7, it follows that E is P-embedded in F.

□

6. Refuting C^*-embeddedness in neighborhood-sublinear spaces

Definition 6.1. Let $\{\lambda_k : k \in S\}$ be a finite collection such that $\lambda_k = 1$ or λ_k is a regular infinite cardinal for each $k \in S$. For $d = \langle d_k : k \in S\rangle \in \prod_{k\in S} \lambda_k$ and $t = \langle t_k : k \in S\rangle \in \prod_{k\in S} \lambda_k$, let $d \leq t$ denote that $d_k \leq t_k$ for every $k \in S$. We say that a subset T of $\prod_{k\in S} \lambda_k$ is *cofinal* in $\prod_{k\in S} \lambda_k$ if for each $d \in \prod_{k\in S} \lambda_k$, there is a $t \in T$ with $d \leq t$.

Lemma 6.2. *Let λ be a regular infinite cardinal, and $\{\lambda_k : k \in S\}$ a finite collection such that $\lambda_k = 1$ or λ_k is a regular infinite cardinal for each $k \in S$. Assume that $\lambda_k \neq \lambda$ for every $k \in S$. Then for each cofinal subset T of $\prod_{k\in S} \lambda_k$ and for each function $h : T \to \lambda$, there is an $\eta < \lambda$ such that $\{t \in T : h(t) \leq \eta\}$ is cofinal in $\prod_{k\in S} \lambda_k$.*

Proof. First we consider the case that $\lambda_k > \lambda$ for every $k \in S$. Assume that $T[\eta] = \{t \in T : h(t) \leq \eta\}$ is not cofinal in $\prod_{k\in S} \lambda_k$ for any $\eta < \lambda$. Then we can take $d[\eta] \in \prod_{k\in S} \lambda_k$ for each $\eta < \lambda$ satisfying:

For each $t \in T$, if $d[\eta] \leq t$, then $h(t) > \eta$.

Since $\lambda < \lambda_k$ for every $k \in S$, we can take a $t \in T$ such that $d[\eta] \leq t$ for all $\eta < \lambda$. In particular, $d[\eta] \leq t$ holds for the $\eta = h(t)$. By the definition of $d[\eta]$, it follows that $h(t) > \eta$. It is contradiction. Hence, $T[\eta]$ is cofinal in $\prod_{k\in\Lambda} \lambda_k$ for some $\eta < \lambda$.

Next we consider the general case. Let $S[-] = \{k \in S : \lambda_k < \lambda\}$ and $S[+] = \{k \in S : \lambda_k > \lambda\}$. Then $S = S[-] \cup S[+]$ is a disjoint union. For each $d_- \in \prod_{k\in S[-]} \lambda_k$, let

$$T[d_-] = \{t \restriction S[+] : t \in T, d_- \leq t \restriction S[-]\}.$$

Then $T[d_-]$ is a cofinal subset of $\prod_{k\in S[+]} \lambda_k$. We can take a function $h[d_-] : T[d_-] \to \lambda$ such that

for each $t_+ \in T[d_-]$, there is a $t \in T$ with $d_- \leq t \restriction S[-]$ and $t \restriction S[+] = t_+$ such that $h[d_-](t_+) = h(t)$.

By the argument above, there is an $\eta[d_-] < \lambda$ such that

$$\{t_+ \in T[d_-] : h[d_-](t_+) \leq \eta[d_-]\} \text{ is cofinal in } \prod_{k \in S[+]} \lambda_k.$$

Since $|\prod_{k \in S[-]} \lambda_k| < \lambda$, we can take an $\eta < \lambda$ such that $\eta[d_-] \leq \eta$ for all $d_- \in \prod_{k \in S[-]} \lambda_k$.

Let $d \in \prod_{k \in S} \lambda_k$. Put $d_- = d \restriction S[-]$ and $d_+ = d \restriction S[+]$. We obtain a $t_+ \in T[d_-]$ such that $d_+ \leq t_+$ and $h[d_-](t_+) \leq \eta[d_-] \leq \eta$. There is a $t \in T$ with $d_- \leq t \restriction S[-]$ and $t \restriction S[+] = t_+$ such that $h(t) = h[d_-](t_+) \leq \eta$. For such t, we have $d \leq t$. Hence, $\{t \in T : h(t) \leq \eta\}$ is cofinal in $\prod_{k \in S} \lambda_k$. $\square$

Lemma 6.3. *Let λ be a regular infinite cardinal, and $\{\lambda_k : k \in S\}$ a finite collection such that $\lambda_k = 1$ or λ_k is a regular infinite cardinal for each $k \in S$. Then for each cofinal subset T of $\prod_{k \in S} \lambda_k$ and for each function $h : T \times \lambda \to \lambda$, there is a strictly increasing sequence $\{\xi(\alpha) : \alpha \in \lambda\} \subset \lambda$ such that*

- *$\xi(0) = 0$ and $\xi(\alpha) = \sup\{\xi(\beta) : \beta < \alpha\}$ for each limit ordinal $\alpha < \lambda$,*
- *for each $d = \langle d_k : k \in S\rangle \in \prod_{k \in S} \lambda_k$ and $\alpha \in \lambda$, if $d_k \leq \alpha$ for every $k \in S$ with $\lambda_k = \lambda$, then there is a $t \in T$ such that $d \leq t$ and $h(t, \xi(\alpha)) \leq \xi(\alpha + 1)$.*

Proof. Let $S[=] = \{k \in S : \lambda_k = \lambda\}$ and $S[\neq] = \{k \in S : \lambda_k \neq \lambda\}$. Then $S = S[=] \cup S[\neq]$ is a disjoint union. By induction on $\alpha \in \lambda$, we will define a strictly increasing sequence $\{\xi(\alpha) : \alpha \in \lambda\} \subset \lambda$.

Define $\xi(0) \in \lambda$ by putting $\xi(0) = 0$. If $\alpha \in \lambda$ is a limit ordinal and $\xi(\beta) \in \lambda$ is defined for each $\beta < \alpha$, then define $\xi(\alpha) \in \lambda$ by putting $\xi(\alpha) = \sup\{\xi(\beta) : \beta < \alpha\}$.

Let $\alpha \in \lambda$ and assume that $\xi(\alpha) \in \lambda$ is defined. We would like to define $\xi(\alpha + 1) \in \lambda$ with $\xi(\alpha) < \xi(\alpha + 1)$. Set

$$T[\alpha] = \{t \restriction S[\neq] : t = \langle t_k : k \in S\rangle \in T, \alpha \leq t_k \text{ for every } k \in S[=]\}.$$

Then $T[\alpha]$ is a cofinal subset of $\prod_{k \in S[\neq]} \lambda_k$.

We can take a function $h[\alpha] : T[\alpha] \to \lambda$ such that

for each $t[\neq] \in T[\alpha]$, there is a $t = \langle t_k : k \in S\rangle \in T$ such that $\alpha \leq t_k$ for every $k \in S[=]$, $t \restriction S[\neq] = t[\neq]$ and $h[\alpha](t[\neq]) = h(t, \xi(\alpha))$.

By applying Lemma 6.2, we can take a $\xi(\alpha{+}1) \in \lambda$ with $\xi(\alpha) < \xi(\alpha{+}1)$ such that

$$\{t[\neq] \in T[\alpha] : h[\alpha](t[\neq]) \leq \xi(\alpha{+}1)\} \text{ is cofinal in } \prod_{k \in S[\neq]} \lambda_k.$$

After finishing induction, we obtain a strictly increasing sequence $\{\xi(\alpha) : \alpha \in \lambda\} \subset \lambda$. Let $d = \langle d_k : k \in S\rangle \in \prod_{k \in S} \lambda_k$ and $\alpha \in \lambda$ such that $d_k \leq \alpha$ for every $k \in S[=]$. Then there is a $t[\neq] \in T[\alpha]$ such that $d \restriction S[\neq] \leq t[\neq]$ and $h[\alpha](t[\neq]) \leq \xi(\alpha + 1)$. There is a

$t = \langle t_k : k \in S\rangle \in \prod_{k\in S}\lambda_k$ such that $\alpha \le t_k$ for every $\alpha \in S[=]$, $t \restriction S[\ne] = t[\ne]$ and $h(t,\xi(\alpha)) = h[\alpha](t[\ne]) \le \xi(\alpha+1)$. Then we have $d \le t$ since $d_k \le \alpha \le t_k$ for every $k \in S[=]$ and $d \restriction S[\ne] \le t[\ne] = t \restriction S[\ne]$. □

Lemma 6.4. *Let X be a subset of a neighborhood-sublinear space Y, and $p \in \mathrm{Cl}_Y(X)$. Assume that there are a linear family $\mathcal{D}$ of neighborhoods of p in Y, and a locally finite cozero cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$, there is a $D \in \mathcal{D}$ with $D \cap U = \emptyset$. Then X is not C^*-embedded in Y.*

Proof. Take a finite collection $\{\mathcal{D}_k : k \in S\}$ of non-empty linear families such that $\bigcup_{k\in S}\mathcal{D}_k$ is a neighborhood subbase at p for Y. Let λ_k and λ be the downward cofinality of $\mathcal{D}_k$, for each $k \in S$, and of $\mathcal{D}$, respectively.

Since $p \in \mathrm{Cl}_Y(X)$, we see that $X \ne \emptyset$, $\mathcal{U} \ne \emptyset$, and so $\mathcal{D} \ne \emptyset$. We have $\lambda \ne 0$. Each $D \in \mathcal{D}$ is a neighborhood of p in Y, so $D \cap X \ne \emptyset$, $D \cap U \ne \emptyset$ for some $U \in \mathcal{U}$, $D' \cap U = \emptyset$ for some $D' \in \mathcal{D}$, $D \not\subset D'$ for such D', and so $\{D\}$ is not a generator of $\mathcal{D}$. We have $\lambda \ne 1$. Hence, λ must be an infinite regular cardinal.

Take strictly descending generators $\{D_{k,\xi} : \xi \in \lambda_k\}$ of $\mathcal{D}_k$, for each $k \in S$, and $\{D_\xi : \xi \in \lambda\}$ of $\mathcal{D}$, respectively.

Let $t = \langle t_k : k \in S\rangle \in \prod_{k\in S}\lambda_k$ and $\xi \in \lambda$. Then D_{k,t_k}, for each $k \in S$, and D_ξ are neighborhoods of $p \in \mathrm{Cl}_Y(X)$ in Y, so we can take an $x(t,\xi) \in X \cap (\bigcap_{k\in S} D_{k,t_k}) \cap D_\xi$. Let $\mathcal{U}(t,\xi) = \{U \in \mathcal{U} : x(t,\xi) \in U\}$, then it is a non-empty finite subfamily of $\mathcal{U}$. Define a function $h : (\prod_{k\in S}\lambda_k) \times \lambda \to \lambda$ by taking $h(t,\xi) < \lambda$, for each $\langle t,\xi\rangle \in (\prod_{k\in S}\lambda_k) \times \lambda$, such that $D_{h(t,\xi)} \cap U = \emptyset$ for all $U \in \mathcal{U}(t,\xi)$. Applying Lemma 6.3 for $T = \prod_{k\in S}\lambda_k$, we obtain a strictly increasing sequence $\{\xi(\alpha) : \alpha \in \lambda\} \subset \lambda$ satisfying:

> For each $d = \langle d_k : k \in S\rangle \in \prod_{k\in S}\lambda_k$ and $\alpha \in \lambda$, if $d_k \le \alpha$ for every $k \in S$ with $\lambda_k = \lambda$, then there is a $t \in \prod_{k\in S}\lambda_k$ such that $d \le t$ and $h(t,\xi(\alpha)) \le \xi(\alpha+1)$.

Define a function $\rho : \mathcal{U} \to \lambda$ by putting

$$\rho(U) = \min\{\alpha \in \lambda : D_{\xi(\alpha+1)} \cap U = \emptyset\}$$

for each $U \in \mathcal{U}$. It follows that

> for each $t \in \prod_{k\in S}\lambda_k$ and $\alpha \in \lambda$, if $h(t,\xi(\alpha)) \le \xi(\alpha+1)$, then $\rho(U) = \alpha$ holds for all $U \in \mathcal{U}(t,\xi(\alpha))$.

Actually, $D_{\xi(\beta+1)} \cap U \ne \emptyset$ for every $\beta < \alpha$ since $x(t,\xi(\alpha)) \in D_{\xi(\alpha)} \cap U \subset D_{\xi(\beta+1)} \cap U$ by $\xi(\beta+1) \le \xi(\alpha)$. And $D_{\xi(\alpha+1)} \cap U \subset D_{h(t,\xi(\alpha))} \cap U = \emptyset$ holds by the definition of $h(t,\xi(\alpha))$ since $h(t,\xi(\alpha)) \le \xi(\alpha+1)$ is assumed. Hence, $\rho(U) = \alpha$ holds.

Take disjoint unbounded subsets A_0, A_1 of λ with $\lambda = A_0 \cup A_1$. And let $U_i = \bigcup\{U \in \mathcal{U} : \rho(U) \in A_i\}$ for each $i \in 2$. It follows from Lemma 5.4 (1) that $\{U_0, U_1\}$ is a binary cozero cover of X.

Assume that X is C^*-embedded in Y. Then by Lemma 5.6, there is a binary cozero cover $\{V_0, V_1\}$ of Y such that $V_i \cap X \subset U_i$ for each $i \in 2$. Take an $i \in 2$ with $p \in V_i$. Since $\bigcup_{k \in S} \mathcal{D}_k$ is a neighborhood subbase at p for Y, we can take a $d = \langle d_k : k \in S \rangle \in \prod_{k \in S} \lambda_k$ such that $\bigcap_{k \in S} D_{k,d_k} \subset V_i$. Take $j \in 2$ with $j \neq i$ and $\alpha \in A_j$ such that $d_k \leq \alpha$ for every $k \in S$ with $\lambda_k = \lambda$. Then there is a $t = \langle t_k : k \in S \rangle \in \prod_{k \in S} \lambda_k$ such that $d \leq t$, i.e. $d_k \leq t_k$ for every $k \in S$, and $h(t, \xi(\alpha)) \leq \xi(\alpha + 1)$. We have $x(t, \xi(\alpha)) \in X \cap \bigcap_{k \in S} D_{k,t_k} \subset X \cap \bigcap_{k \in S} D_{k,d_k} \subset V_i \cap X \subset U_i$, so $x(t, \xi(\alpha)) \in U$ for some $U \in \mathcal{U}$ with $\rho(U) \in A_i$. Since $U \in \mathcal{U}(t, \xi(\alpha))$ and $h(t, \xi(\alpha)) \leq \xi(\alpha + 1)$, we have $\rho(U) = \alpha \in A_j$. It contradicts that $A_i \cap A_j = A_0 \cap A_1 = \emptyset$. Hence, X is not C^*-embedded in Y. □

Lemma 6.4 does not remain valid if the space Y is not assumed to be neighborhood-sublinear. Actually, let Y and p be the space and the point of Example 3.4. Then $X = \omega$, $\mathcal{D} = \{\{p\} \cup \omega \setminus n : n \in \omega\}$ and $\mathcal{U} = \{\{k\} : k \in \omega\}$ satisfy the assumptions of the lemma, but X is C^*-embedded in Y.

7. Main theorem

Let us prove the main theorem: the class of z-neighborhood-sublinear spaces has the property [C*=P:dense].

Theorem 7.1 (main). *Let E be a dense subset of a z-neighborhood-sublinear space F. If E is C^*-embedded in F, then it is P-embedded in F.*

Proof. Let $\mathcal{U}$ be a locally finite cozero cover of E. By Lemma 5.8, it suffices to show that $\{\mathrm{Cl}_F U : U \in \mathcal{U}\}$ covers F. Assume that it does not cover F. Then there is a $p \in F \setminus \bigcup\{\mathrm{Cl}_F U : U \in \mathcal{U}\}$. Take a finite collection $\{\mathcal{D}_k : k \in S\}$ of non-empty linear families such that $\bigcup_{k \in S} \mathcal{D}_k$ is a neighborhood subbase at p for F by zero-sets. For each $U \in \mathcal{U}$, since $F \setminus \mathrm{Cl}_F U$ is a neighborhood of p in F, we can take a $\langle D_k(U) : k \in S \rangle \in \prod_{k \in S} \mathcal{D}_k$ with $(\bigcap_{k \in S} D_k(U)) \cap U = \emptyset$. By Lemma 5.4 (1), we see that $\{\bigcup\{U \setminus D_k(U) : U \in \mathcal{U}\} : k \in S\}$ is a finite cozero cover of E. Since E is assumed to be C^*-embedded in F, we obtain by using Lemma 5.6 a finite cozero cover $\{V_k : k \in S\}$ of F such that $V_k \cap E \subset \bigcup\{U \setminus D_k(U) : U \in \mathcal{U}\}$ for each $k \in S$. Take a $k \in S$ with $p \in V_k$ and set $Y = V_k$, $X = Y \cap E = V_k \cap E$ and $\mathcal{U}_X = \{(U \setminus D_k(U)) \cap X : U \in \mathcal{U}\}$. Then X is a subset of a neighborhood-sublinear space Y, and $\mathcal{U}_X$ is a locally finite cozero cover of X. Since E is assumed to be dense in F, we have $p \in \mathrm{Cl}_Y(X)$. Let $\mathcal{D} = \{D \cap Y : D \in \mathcal{D}_k\}$. Then it is a linear family of neighborhoods of p in Y. Obviously, $\mathcal{D}$ and $\mathcal{U}_X$ satisfy that for each $U \in \mathcal{U}_X$, there is a $D \in \mathcal{D}$ with $D \cap U = \emptyset$. By Lemma 6.4, X is not C^*-embedded in Y. On the other hand, X is C^*-embedded in Y by Lemma 5.5. It is contradiction. Hence, $\{\mathrm{Cl}_F U : U \in \mathcal{U}\}$ covers F, and so E is P-embedded in F. □

By Fact 3.11, Theorem 7.1 and Corollary 5.3, we obtain the following corollary.

Corollary 7.2. *Let X be a z-neighborhood-sublinear-space. Then the following conditions are equivalent.*

(C*=P:closed): For each closed subset F in X,
if F is C^*-embedded in X, then F is P-embedded in X.

(C*=P:subset): For each subset E in X,
if E is C^*-embedded in X, then E is P-embedded in X.

By Corollary 3.13, Theorem 7.1 and Corollary 7.2, we obtain the following corollary.

Corollary 7.3. *Let X be a subspace of a finite product of GO-spaces. Then,*

(1) *X has the following property.*

(C*=P:dense): For each dense subset D in X,
if D is C^*-embedded in X, then D is P-embedded in X.

(2) *The following conditions are equivalent.*

(C*=P:closed): For each closed subset F in X,
if F is C^*-embedded in X, then F is P-embedded in X.

(C*=P:subset): For each subset E in X,
if E is C^*-embedded in X, then E is P-embedded in X.

Now we see that Corollary 4.2 is obtained from Theorem 4.1.

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