ON C-EMBEDDEDNESS OF HYPERSPACES

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ABSTRACT. Let $\mathcal{CL}(X)$ and $\mathcal{K}(X)$ denote the hyperspaces of non-empty closed and non-empty compact subsets of X, respectively, with the Vietoris topology. In this paper we show that, given an ordinal number γ, the space $\mathcal{K}([0,\gamma))$ is C-embedded in $\mathcal{CL}([0,\gamma))$ if and only if $\mathrm{cof}(\gamma) \neq \omega$. Moreover we answer some problems posed by the first author and Jun Terasawa.

1. INTRODUCTION

Every space in this article is a Tychonoff space with more than one point. The letters ξ, α and γ represent ordinal numbers.; ω is the first infinite cardinal, ω_1 is the first uncountable cardinal and $\mathrm{cof}(\xi)$ is the cofinality of the ordinal ξ. If γ is an ordinal number, let $[0,\gamma)$ denote the ordinal space with the order topology. $\mathbb{R}$ is the space of real numbers with its usual topology, $\mathbb{I}$ is the interval $[0,1]$ as a subspace of $\mathbb{R}$ and $\mathbb{N}$ is the subspace of $\mathbb{R}$ constituted by the natural numbers. For given two spaces X and Y, $C(X,Y)$ denotes the set of continuous functions from X to Y; if $Y = \mathbb{R}$, we write $C(X)$ for the set of continuous functions with real values and $C^*(X)$ for the set of bounded continuous functions with real values. Recall that a subspace Y of X is C-embedded (C^*-embedded) in X if for every function $f \in C(Y)$ ($f \in C^*(Y)$) there is a function $F \in C(X)$ ($F \in C^*(X)$) with $F \restriction Y = f$. $\beta(X)$ denotes the *Stone-Čech compactification* of the space X. Remember that $\beta(X)$ is characterized as the compactification of X such that X is C^*-embedded in it.

For a topological space $(X,\mathcal{T})$, let $CL(X)$ and $K(X)$ denote the sets of non-empty closed and compact subsets of X, respectively. Let $\mathcal{CL}(X)$ denotes the hyperspace $CL(X)$ with the Vietoris topology. $\mathcal{K}(X)$ is the set $K(X)$ with the topology of subspace of $\mathcal{CL}(X)$ respectively. For a given subset V of X, let

$$V^+ = \{A \in CL(X) : A \subseteq V\} \quad \text{and} \quad V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}.$$

Remember that, the Vietoris topology has the sets of the form V^+ and V^- as a subbase, where V is an open subset of X. For given subsets $U_1,\ldots,U_n$ of X, we define

$$\langle U_1,\ldots,U_n\rangle = (\bigcup_{1\le k\le n} U_k)^+ \cap (\bigcap_{1\le k\le n} U_k^-).$$

Then, the collection

$$\{\langle U_1,\ldots,U_n\rangle : n \in \mathbb{N}, U_1,\ldots,U_n \in \mathcal{T}\}$$

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constitutes a base for the Vietoris topology in $\mathcal{CL}(X)$. When $U_1,\ldots,U_n$ are *non-empty* open, we say that $\langle U_1,\ldots,U_n\rangle$ is a *basic open set* in $\mathcal{CL}(X)$. Since the greatest part of this work is about is the hyperspace of compact subsets, when the context are clear we understand that the notations V^+, V^- and $\langle U_1,\ldots,U_n\rangle$ represent $(V^+)\cap\mathcal{K}(X)$, $(V^-)\cap\mathcal{K}(X)$ and $\langle U_1,\ldots,U_n\rangle\cap\mathcal{K}(X)$ respectively. For those concepts which appear in this article without definition consult [4, 9].

The class of hyperspaces has been widely studied and continues to generate significant results and problems. One of the beautiful and several problems studied by John Ginsburg was to find conditions under which $\beta\big(\mathcal{CL}(X)\big)=\mathcal{CL}\big(\beta(X)\big)$. In [2], Ginsburg obtained the following results:

Theorem 1.1. ([2]) *The following hold:*

(1) If $\beta\big(\mathcal{CL}(X)\big)=\mathcal{CL}\big(\beta(X)\big)$, then $\mathcal{CL}(X)$ is pseudocompact.
(2) Let $\mathcal{CL}(X)\times\mathcal{CL}(X)$ be pseudocompact. Then $\beta\big(\mathcal{CL}(X)\big)=\mathcal{CL}\big(\beta(X)\big)$.

By Ginsburg's results, the following questions arise naturally.

Questions 1.2. *Under which conditions of X, $\beta\big(\mathcal{K}(X)\big)=\beta\big(\mathcal{CL}(X)\big)$ holds (equivalently, $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$) ?*

In [1], the authors started the study of this problem. In particular they proved that $\mathcal{K}(X)$ is not C^*-embedded in $\mathcal{CL}(X)$ when X is a non-compact metric space or a σ-compact non-compact space. In particular, for every ordinal number γ, if $cof(\gamma)=\omega$ then $\mathcal{K}([0,\gamma))$ is not C^*-embedded in $\mathcal{CL}([0,\gamma))$. In the same paper we stated the following question.

Question 1.3. *For which ordinal numbers γ, $\mathcal{K}([0,\gamma))$ is C^*-embedded in $\mathcal{CL}([0,\gamma))$?*

Remark that in general, a pseudocompact subspace is C^*-embedded in a space iff it is C-embedded in that. Also remark that $\mathcal{K}([0,\gamma))$ is pseudocompact for each γ with $cof(\gamma)>\omega$. Our main result answers this last question.

Also remark the following results:

(a) ([6, Theorem 8]) If γ is a regular uncountable cardinal, then $\mathcal{K}([0,\gamma))$ is normal.
(b) ([8, Theorem 1]) $\mathcal{CL}(\omega)$ is strongly 0-dimensional.
(c) ([8, Theorem 2]) $\mathcal{K}([0,\gamma))$ is strongly 0-dimensional for every non-zero ordinal γ.

The proofs of (a) and (c) require elementary submodel techniques. In [6, 8] the following asked:

Questions 1.4. ([6, 8]) *Let X be a topological space.*

(1) Is X ω-bounded if $\mathcal{CL}(X)$ is countably paracompact?
(2) Give a proof of (a) above without using elementary submodel techniques.
(3) Is $\mathcal{CL}([0,\omega_1))$ strongly 0-dimensional?
(4) Give a proof of (c) above without using elementary submodels.

Here we remark that the following Ginsburg's result immediately answers (1) negatively.

Theorem 1.5. ([3]) *Let X be a space and $p\in\mathbb{N}^*$.*

(1) X is p-compact iff $\mathcal{CL}(X)$ is p-compact, and
(2) X is ω-bounded iff $\mathcal{CL}(X)$ is ω-bounded.

Because, a p-compact, non- ω-bounded space [1] is a required counter example. Also we will consider other questions later.

2. On C-embeddedness of $\mathcal{K}([0,\gamma))$ in $\mathcal{CL}([0,\gamma))$

The first lemma below may be found in somewhere, but for the reader's sake, we give its proof. The second lemma below is the main and complex part of the construction and requires elementary submodel techniques [2]. However we refer several definitions and claims in the proof of Theorem 2 in [8], because basic arguments are similar.

Lemma 2.1. *Let X be a dense subspace of Y and $g : X \longrightarrow Z$ a continuous function. Assume that for every $y \in Y \setminus X$, there is a function $g_y : X \cup \{y\} \longrightarrow Z$ extending g such that g_y is continuous at y in $X \cup \{y\}$. Then the function $G = \bigcup_{y \in Y \setminus X} g_y : Y \longrightarrow Z$ is continuous extending g.*

Proof. Of course, g_y is continuous on $X \cup \{y\}$ for every $y \in Y \setminus X$. To see that G is continuous, let $y \in Y$ and U an open set in Z with $G(y) \in U$. By the regularity of Z, there is an open set U' in Z with $G(y) \in U' \subseteq \mathrm{Cl}_Z(U') \subseteq U$. Now we have two cases to consider:

In the case $y \in X$, since $g(y) = G(y) \in U'$ and g is continuous on X, there is an open set V in Y such that $y \in V$ and $g[V \cap X] \subseteq U'$. Then $G[V \cap (X \cup \{y\})] = G[V \cap X] = g[V \cap X] \subseteq U'$.

In the case $y \notin X$, since $g_y(y) = G(y) \in U'$ and g_y is continuous on $X \cup \{y\}$, there is an open set V in Y such that $y \in V$ and $g_y[V \cap (X \cup \{y\})] \subseteq U'$. Then $G[V \cap (X \cup \{y\})] = g_y[V \cap (X \cup \{y\})] \subseteq U'$.

In any case, we have an open set V in Y such that $y \in V$ and $G[V \cap (X \cup \{y\})] \subseteq U'$. To prove that $G[V] \subseteq \mathrm{Cl}(U')$, assume on the contrary that there is $z \in V$ with $G(z) \notin \mathrm{Cl}(U')$. Then by $G[V \cap (X \cup \{y\})] \subseteq U'$ and $G(z) \notin U'$, we have $z \notin V \cap (X \cup \{y\})$ therefore $z \in Y \setminus X$. Since $g_z : X \cup \{z\} \longrightarrow Z$ is continuous at z, there is an open set V' in Y with $z \in V' \subseteq V$ such that $g_z[V' \cap (X \cup \{z\})] \subseteq Z \setminus \mathrm{Cl}(U')$. Since X is dense in Y, one can pick $x \in V' \cap X$. Then we have $G(x) = g(x) = g_z(x) \notin \mathrm{Cl}(U')$. On the other hand, by $x \in V' \cap X \subseteq V \cap X$, we have $G(x) \in G[V \cap (X \cup \{y\})] \subseteq U'$, a contradiction. This shows $G[V] \subseteq \mathrm{Cl}(U')$. Finally $\mathrm{Cl}(U') \subseteq U$ shows that G is continuous at y in Y. □

Lemma 2.2. *Let γ be an ordinal number with $\mathrm{cof}(\gamma) > \omega$ and let $f : \mathcal{K}([0,\gamma)) \longrightarrow \mathbb{I}$ be a continuous function. Then for each $T \in \mathcal{CL}([0,\gamma)) \setminus \mathcal{K}([0,\gamma))$, there is a function $f_T : \mathcal{K}([0,\gamma)) \cup \{T\} \longrightarrow \mathbb{I}$ extending f such that f_T is continuous at T in $\mathcal{K}([0,\gamma)) \cup \{T\}$.*

Proof. Let $X = \mathcal{K}([0,\gamma))$. Fix $T \in \mathcal{CL}([0,\gamma)) \setminus X$ and let $X_T = X \cup \{T\}$. First we follow the proof Theorem 2 in [8], where we can find the proof of Claims $1-4$. Let M be a countable elementary submodel of $H(\theta)$, where θ is large enough, such that $\gamma, f \in M$. For each $\beta < \gamma$, let

$$u(\beta) = \min([\beta,\gamma] \cap M).$$

Obviously we have:

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[1] The existence of such spaces is well-known.

[2] In a previous version of this paper we include a proof without elementary submodels. The referee note that both proofs are similar and recommend to leave just the elementary submodel's proof because it is more explicit. So, we do not ask for a proof without elementary submodels.

(a) for each $\beta < \gamma$, $\beta \leq u(\beta) \in M$,

(b) for each $\beta < \gamma$, $\beta \in M$ iff $u(\beta) = \beta$,

(c) if $\beta' < \beta < \gamma$, then $u(\beta') \leq u(\beta)$.

Moreover set
$$Z = \{u(\beta) : \beta < \gamma\}.$$
Then $Z \subseteq [0, \gamma] \cap M$ and u can be considered as a function on γ onto Z, i.e., $u : \gamma \to Z$.

Claim 1. *It follows from* $\mathrm{cof}(\gamma) > \omega$ *that* $Z = [0, \gamma] \cap M$, $\gamma \in Z$ *and* $[0, \gamma) \cap M$ *is bounded in* $[0, \gamma)$.

Give Z the order topology. Since Z is countable, by Claim 1, it is homeomorphic to a successor ordinal $< \omega_1$. Therefore Z is compact, so is $Y = \mathcal{K}(Z) = \mathcal{CL}(Z)$. Now for every $\alpha \in Z$, let
$$d(\alpha) = \sup\{\delta + 1 : \delta \in [0, \alpha) \cap Z\}.$$
Then $d(\alpha) = \sup\{\delta + 1 : \delta \in [0, \alpha) \cap M\}$ holds and d can be considered as a function on Z into γ, that is, $d : Z \to \gamma$. Obviously we have:

(d) for each $\alpha \in Z$, $d(\alpha) \leq \alpha$,

(e) if α', $\alpha \in Z$ with $\alpha' < \alpha$, then $d(\alpha') \leq d(\alpha)$.

Claim 2. $u : \gamma \to Z$ *and* $d : Z \to \gamma$ *are both continuous.*

Claim 3. *The functions* u *and* d *have the following properties:*

(1) *For every* $\beta < \gamma$, $d(u(\beta)) = \sup\{\delta + 1 : \delta \in \beta \cap M\} \leq \beta$.

(2) *For every* $\alpha \in Z$, $u(d(\alpha)) = \alpha$ *holds, i.e.,* $u \circ d$ *is the identity map on* Z.

(3) *For every* $\beta < \gamma$ *and* $\alpha \in Z$, *if* $\beta < d(\alpha)$, *then* $u(\beta) < d(\alpha) \leq \alpha$.

(4) *If* $\beta' < \beta < \gamma$, $\alpha \in Z$ *and* $d(\alpha) \in (\beta', \beta]$, *then* $\alpha \in (u(\beta'), u(\beta)]$.

Consider the continuous functions $\tilde{u} : X \to Y$ and $\tilde{d} : Y \to X$ defined by
$$\tilde{u}(F) = u[F], \quad \tilde{d}(H) = d[H] \quad \text{for } F \in X \text{ and } H \in Y.$$
The following claim is Claim 6 in [8].

Claim 4. *For every* $F \in X$, $f(F) = f(\tilde{d}(\tilde{u}(F)))$.

To finish the proof, we need further two claims. First note $\gamma \in u[T]$, because T is cofinal in γ.

Claim 5. $u[T]$ *is closed in* Z, *therefore* $u[T] \in Y$.

Proof of Claim 5. Let $\alpha \in Z \setminus u[T]$. It follows from $\alpha \in Z \subseteq M$ that $u(\alpha) = \alpha \notin u[T]$, therefore $\alpha \notin T$. Moreover it follows from (2) in Claim 3 that $u(d(\alpha)) = \alpha \notin u[T]$, thus we have $d(\alpha) \notin T$. Since T is closed and $d(\alpha) = \sup\{\delta+1 : \delta \in [0, \alpha) \cap M\}$, there is $\delta \in \alpha \cap M$ such that $(\delta, d(\alpha)] \cap T = \emptyset$.

First we show the fact $(\delta, \alpha] \cap T = \emptyset$. To see this, assume on the contrary that there is $\xi \in (\delta, \alpha] \cap T$. Since $(\delta, d(\alpha)] \cap T = \emptyset$ and $\alpha \notin T$, we have $\xi \in (d(\alpha), \alpha) \cap T$. Note by $\xi \leq \alpha \in M$ that $\xi \leq u(\xi) \leq \alpha$ holds. Now it follows from $\alpha \notin u[T]$ and $\xi \in T$ that $u(\xi) < \alpha$. Therefore by $u(\xi) \in M$, we have $u(\xi) + 1 \leq d(\alpha)$. Then $d(\alpha) < \xi \leq u(\xi) < d(\alpha)$ holds, a contradiction.

Next we show $(\delta, \alpha] \cap u[T] = \emptyset$. To see this, let $\xi \in T$. By the fact above, we have $\xi \leq \delta$ or $\alpha < \xi$. When $\xi \leq \delta$, we have by $\delta \in M$, $u(\xi) \leq \delta$. When $\alpha < \xi$, we have $\alpha < \xi \leq u(\xi)$. Thus in either cases, we have $u(\xi) \notin (\delta, \alpha]$, this shows $(\delta, \alpha] \cap u[T] = \emptyset$.

Finally, it follows from $\alpha, \delta \in Z$ with $\delta < \alpha$ that $(\delta, \alpha] \cap Z$ is a neighborhood of α disjoint from $u[T]$. We see that $u[T]$ is closed in Z.

Now define $\bar{u} : X_T \to Y$ by $\bar{u}(F) = u[F]$ for every $F \in X_T$. Obviously $\bar{u}$ extends the function $\tilde{u}$.

Claim 6. *$\bar{u}$ is continuous.*

Proof of Claim 6. It suffices to see that $\bar{u}$ is continuous at T in X_T. To see this, let $\mathcal{U}$ be a neighborhood of $\bar{u}(T) = u[T]$ in Y. By Lemma 2 in [8], we may assume that

$$\mathcal{U} = \langle(\beta_{n-1}, \alpha_{n-1}] \cap Z, (\beta_{n-2}, \alpha_{n-2}] \cap Z, \ldots, (\beta_0, \alpha_0] \cap Z\rangle_Y,$$

where $\beta_i, \alpha_i \in Z (i < n)$, such that

(1) $\alpha_0 = \max u[T] = \gamma$ and $\{\alpha_i : i < n\} \subseteq u[T]$,

(2) $\alpha_{i+1} \leq \beta_i < \alpha_i (i < n)$, where $\alpha_n = -1$.

Fact 1. $T \in \langle(\beta_{n-1}, \alpha_{n-1}], (\beta_{n-2}, \alpha_{n-2}], \ldots, (\beta_0, \alpha_0)\rangle_{\mathcal{CL}([0,\gamma))}$.

First let $i < n$. By $u[T] \in \mathcal{U}$, we have $u[T] \cap ((\beta_i, \alpha_i] \cap Z) \neq \emptyset$. Pick $\xi \in T$ with $\beta_i < u(\xi) \leq \alpha_i$. Then we have $\beta_i < \xi \leq u(\xi) \leq \alpha_i$ because of $\alpha_i, \beta_i \in Z \subseteq M$. Thus $T \cap (\beta_i, \alpha_i] \neq \emptyset$ for $1 \leq i < n$ and $T \cap (\beta_0, \alpha_0) \neq \emptyset$.

Next let $\xi \in T$. It follows from $u[T] \in \mathcal{U}$ that there is $i < n$ such that $u(\xi) \in (\beta_i, \alpha_i] \cap Z$. Then, as above, we have $\beta_i < \xi \leq u(\xi) \leq \alpha_i$. Thus we have $\xi \in (\beta_0, \alpha_0)$ in the case $i = 0$, and $\xi \in (\beta_i, \alpha_i]$ in the case $1 \leq i < n$. These arguments show Fact 1.

By Fact 1, the set

$$\mathcal{W} = \langle(\beta_{n-1}, \alpha_{n-1}], (\beta_{n-2}, \alpha_{n-2}], \ldots, (\beta_0, \alpha_0)\rangle_{\mathcal{CL}([0,\gamma))} \cap X_T$$

is a neighborhood of T in X_T. The next fact completes the proof of Claim 6.

Fact 2. $\bar{u}[\mathcal{W}] \subseteq \mathcal{U}$.

Let $F \in \mathcal{W}$. Since $\bar{u}(T) = u[T] \in \mathcal{U}$, we may assume $F \in X$ and we will show $\bar{u}(F) \in \mathcal{U}$.

First let $i < n$. When $i = 0$, by $F \in \mathcal{W}$, we can take $\xi \in F \cap (\beta_0, \alpha_0)$. Then $\beta_0 < \xi \leq u(\xi) \leq \alpha_0 = \gamma$. When $1 \leq i < n$, we can take $\xi \in F \cap (\beta_i, \alpha_i]$. Then $\beta_i < \xi \leq u(\xi) \leq \alpha_i$. In either cases, $u[F] \cap ((\beta_i, \alpha_i] \cap Z) \neq \emptyset$.

Next let $\xi \in F$. By $F \in \mathcal{W}$, we have either $\xi \in (\beta_0, \alpha_0)$ or for some $1 \leq i < n$ $\xi \in (\beta_i, \alpha_i]$. Then, as above, in either cases we have $u(\xi) \in (\beta_i, \alpha_i] \cap Z$.

These arguments show $\bar{u}(F) = u[F] \in \mathcal{U}$.

In order to end the proof let $f_T : X_T \to \mathbb{I}$ be the function defined by the composition of the continuous functions $\bar{u}$, $\tilde{d}$ and f, i.e., $f_T = f \circ \tilde{d} \circ \bar{u}$. By Claim 4, for every $F \in X$

$$f_T(F) = f(\tilde{d}(\bar{u}(F))) = f(\tilde{d}(\tilde{u}(F))) = f(F).$$

Therefore f_T is a continuous extension of f over X_T. $\square$

Now we establish our principal theorem.

Theorem 2.3. *If γ is an ordinal with $\mathrm{cof}(\gamma) > \omega$, then $\mathcal{K}([0,\gamma))$ is C-embedded in $\mathcal{CL}([0,\gamma))$.*

Proof. Let $f : \mathcal{K}([0,\gamma)) \longrightarrow \mathbb{R}$ be a continuous function. Since $([0,\gamma))$ is pseudocompact, we can assume that $f : \mathcal{K}([0,\gamma)) \to \mathbb{I}$. By Lemma 2.2, for every $T \in \mathcal{CL}([0,\gamma)) \setminus \mathcal{K}([0,\gamma))$, there is a function $f_T : \mathcal{K}([0,\gamma)) \cup \{T\} \longrightarrow \mathbb{I}$ extending f such that f_T is continuous at T in $\mathcal{K}([0,\gamma)) \cup \{T\}$. Since $f_T|_{\mathcal{K}([0,\gamma))} = f$ and $\mathcal{K}([0,\gamma))$ is open in $\mathcal{K}([0,\gamma)) \cup \{T\}$, then f_T is continuous for each $T \in \mathcal{CL}([0,\gamma)) \setminus \mathcal{K}([0,\gamma))$. Define $F = \bigcup_{T \in \mathcal{CL}([0,\gamma)) \setminus \mathcal{K}([0,\gamma))} f_T$. Of course $F : \mathcal{CL}([0,\gamma)) \longrightarrow \mathbb{I}$ is a well defined function. By Lemma 2.1, F is a continuous function. Therefore $\mathcal{K}([0,\gamma))$ is C-embedded in $\mathcal{CL}([0,\gamma))$. $\square$

Corollary 2.4. *Let γ be an ordinal number. Then the following are equivalent.*

(1) $\mathcal{K}([0,\gamma))$ *is C-embedded in* $\mathcal{CL}([0,\gamma))$.
(2) $\mathcal{K}([0,\gamma))$ *is C^*-embedded in* $\mathcal{CL}([0,\gamma))$.
(3) $\mathrm{cof}(\gamma) \neq \omega$

Proof. "(1) $\to$ (2)" is obvious and "(2) $\to$ (3)" is Corollary 3.16 in [1].

(3) $\to$ (1) : Assume that $\mathrm{cof}\gamma \neq \omega$. If $\mathrm{cof}(\gamma) < \omega$ then $[0,\gamma)$ is compact and the assertion is trivial. Suppose that $\mathrm{cof}(\gamma) > \omega$, then apply the theorem above. $\square$

3. On Questions

Remember that a space X is strongly 0-dimensional iff the Stone-Čech compactification $\beta(X)$ is 0-dimensional. Therefore if X is a dense C^*-embedded subspace of Y, then X is strongly 0-dimensional iff Y is strongly 0-dimensional. This fact, (c) in Introduction and Corollary 2.4 immediately give an affirmative answer to Question 1.4 (3):

Corollary 3.1. *If γ is an ordinal with $\mathrm{cof}(\gamma) \neq \omega$, then $\mathcal{CL}([0,\gamma))$ is strongly 0-dimensional.*

However the following still remains open:

Question 3.2. *Is $\mathcal{CL}([0,\gamma))$ strongly 0-dimensional if $\gamma > \omega$ and $\mathrm{cof}(\gamma) = \omega$?*

Remark 3.3. (1) Ohta informed to the first author that the following two results also show the corollary above:

(i) [9, Proposition 4.13.1] A space X is 0-dimensional if and only if $\mathcal{K}(X)$ is 0-dimensional.
(ii) [2] If X is normal and ω-bounded, then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta(X))$.

Because, if $\mathrm{cof}(\gamma) \neq \omega$, then $[0,\gamma)$ is normal and ω-bounded. Therefore by (ii) $\beta(\mathcal{CL}([0,\gamma))) = \mathcal{CL}(\beta([0,\gamma))) = \mathcal{CL}([0,\gamma]) = \mathcal{K}([0,\gamma])$. Since $[0,\gamma]$ is 0-dimensional, by (i) so is $\mathcal{K}([0,\gamma])$, therefore $\mathcal{CL}([0,\gamma))$ is strongly 0-dimensional.

(2) Remark that Ohta's approach does not use elementary submodel techniques. As remarked above, there is also another proof of Theorem 2.3 (thus Corollary 2.4) without using elementary submodel techniques. Therefore we may be considered as getting a proof, which does not use elementary submodel techniques, of the strong 0-dimensionality of $\mathcal{K}([0,\gamma))$ in the case $\mathrm{cof}(\gamma) \neq \omega$, see Question 1.4 (4).

The proof of Theorem 3.6 appeared in [7], which uses elementary submodel techniques, can be directly translated to a proof without using such techniques as follows. Also note that there is a roundabout way to translate it, see [5]. In order to simplify the proof we introduce the following notation.

Notation 3.4. *For subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{CL}(X)$, let*

$$\mathcal{A} \biguplus \mathcal{B} = \begin{cases} \mathcal{A} & \text{if } \mathcal{B} = \emptyset, \\ \mathcal{B} & \text{if } \mathcal{A} = \emptyset, \\ \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\} & \text{otherwise.} \end{cases}$$

Obviously the operation $\biguplus$ on $\mathcal{CL}(X)$ is associative and commutative. Also we have:

Lemma 3.5. *Let X be a normal space.*

(1) *Let $\langle U_1, \ldots, U_n \rangle$ and $\langle V_1, \ldots, V_m \rangle$ be basic open sets. Then*
$$\langle U_1, \ldots, U_n \rangle \biguplus \langle V_1, \ldots, V_m \rangle = \langle U_1, \ldots, U_n, V_1, \ldots, V_m \rangle$$
holds.

(2) *For every pair of open sets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{CL}(X)$, $\mathcal{U} \biguplus \mathcal{V}$ is an open set of $\mathcal{CL}(X)$.*

Proof. (1) It is evident that
$$\langle U_1, \ldots, U_n \rangle \biguplus \langle V_1, \ldots, V_m \rangle \subseteq \langle U_1, \ldots, U_n, V_1, \ldots, V_m \rangle .$$
We will prove the other containment using induction on the number of open sets of X comprising the right basic open set. Let $n \in \mathbb{N}$ and let $U_1, \ldots, U_n$, V be non-empty open sets of X. Take $T \in \langle U_1, \ldots, U_n, V \rangle$. If $T \subseteq \bigcup_{1 \le i \le n} U_i$, then fix $t \in T \cap V$. So $T = T \cup \{t\} \in \langle U_1, \ldots, U_n \rangle \biguplus \langle V \rangle$. Suppose $T \cap (X \setminus \bigcup_{1 \le i \le n} U_i) \neq \emptyset$. Let $F = T \setminus \bigcup_{1 \le i \le n} U_i$. Since X is normal there is an open set W of X such that $F \subseteq W \subseteq \mathrm{Cl}(W) \subseteq V$. Let $T_0 = T \cap \mathrm{Cl}(W)$. Then $T_0 \in \langle V \rangle$. For each $i \le n$, fix $x_n \in T \cap U_n$. Then $T_1 = (T \setminus W) \cup \{x_1, \ldots, x_n\} \in \langle U_1, \ldots, U_n \rangle$. So $T = T_1 \cup T_0 \in \langle U_1, \ldots, U_n \rangle \biguplus \langle V \rangle$. This is the end of our first step. Suppose that
$$\langle U_1, \ldots, U_n \rangle \biguplus \langle V_1, \ldots, V_m \rangle = \langle U_1, \ldots, U_n, V_1, \ldots, V_m \rangle$$
is true. Then for every non-empty open set V_{m+1} of X,
$$\langle U_1, \ldots, U_n \rangle \biguplus \langle V_1, \ldots, V_{m+1} \rangle = \langle U_1, \ldots, U_n \rangle \biguplus (\langle V_1, \ldots, V_m \rangle \biguplus \langle V_{m+1} \rangle) =$$
$$(\langle U_1, \ldots, U_n \rangle \biguplus \langle V_1, \ldots, V_m \rangle) \biguplus \langle V_{m+1} \rangle = \langle U_1, \ldots, U_n, V_1, \ldots, V_m \rangle \biguplus \langle V_{m+1} \rangle$$
$$= \langle U_1, \ldots, U_n, V_1, \ldots, V_{m+1} \rangle .$$
(2) is a consequence of (1). □

Theorem 3.6. ([6, 7]) *Let γ be a regular uncountable ordinal number. Then $\mathcal{K}([0,\gamma))$ is normal.*

Proof. Suppose that $\mathcal{K}([0,\gamma))$ is not normal. Let $\mathcal{A}$ and $\mathcal{B}$ be a couple of non-empty disjoint closed sets of $\mathcal{K}([0,\gamma))$ such that both can not be separated by disjoint open sets of $\mathcal{K}([0,\gamma))$. Since $\mathcal{K}([0,\gamma])$ is normal, there is $T \in \mathcal{K}([0,\gamma]) \setminus \mathcal{K}([0,\gamma))$ such that $T \in \mathrm{Cl}_{\mathcal{K}([0,\gamma])}\mathcal{A} \cap \mathrm{Cl}_{\mathcal{K}([0,\gamma])}\mathcal{B}$. Obviously $\gamma \in T$.

For every $\alpha < \gamma$, fix a neighborhood base $\mathcal{V}_\alpha$ at $T \cap [0, \alpha]$ in $\mathcal{K}([0, \alpha])$ with $|\mathcal{V}_\alpha| < \gamma$ whenever $T \cap [0, \alpha] \neq \emptyset$, where remark that the weight of the space $\mathcal{K}([0, \alpha])$ is less than or equal to $|\alpha|$ when α is infinite.

If $T \cap [0, \alpha] = \emptyset$, then set $\mathcal{V}_\alpha = \{\emptyset\}$.

Now fix $\alpha < \gamma$. Since by the lemma above, for every $\mathcal{U} \in \mathcal{V}_\alpha$, $\mathcal{U} \uplus (\alpha, \gamma]^+$ is a neighborhood of T in $\mathcal{K}([0, \gamma])$, we can fix

$$A(\mathcal{U}, \alpha) \in \mathcal{A} \cap (\mathcal{U} \biguplus (\alpha, \gamma]^+), \quad B(\mathcal{U}, \alpha) \in \mathcal{B} \cap (\mathcal{U} \biguplus (\alpha, \gamma]^+).$$

Remark that $A(\mathcal{U}, \beta)$'s and $B(\mathcal{U}, \beta)$'s are bounded in γ, moreover $\mathcal{V}_\beta$'s are of size $< \gamma$, therefore

$$C = \{\alpha < \gamma : \forall \beta < \alpha \ (\sup(\bigcup_{\mathcal{U} \in \mathcal{V}_\beta} A(\mathcal{U}, \beta) \cup \bigcup_{\mathcal{U} \in \mathcal{V}_\beta} B(\mathcal{U}, \beta)) < \alpha)\}$$

is a closed unbounded set in $[0, \gamma)$. Pick $\alpha \in C$ and set $R = (T \cap [0, \alpha]) \cup \{\alpha\}$. Then obviously $R \in \mathcal{K}([0, \gamma))$. The following claim completes the proof.

Claim: $R \in \mathcal{A} \cap \mathcal{B}$.

It suffices to see that

$$R \in \mathrm{Cl}_{\mathcal{K}([0,\gamma])} \mathcal{A} \cap \mathrm{Cl}_{\mathcal{K}([0,\gamma])} \mathcal{B}.$$

Let $\mathcal{W}$ be a neighborhood of R in $\mathcal{K}([0, \gamma])$. By [8, Lemma 2] and that α is the largest element of R, we may assume $\mathcal{W} = \mathcal{U} \uplus (\beta, \alpha]^+$ for some $\beta < \alpha$ and $\mathcal{U} \in \mathcal{V}_\beta$. Since $A(\mathcal{U}, \beta) \in \mathcal{U} \uplus (\beta, \gamma]^+$ and $\beta < \alpha \in C$, we have $\sup A(\mathcal{U}, \beta) < \alpha$, therefore

$$A(\mathcal{U}, \beta) \in (\mathcal{U} \biguplus (\beta, \alpha]^+) \cap \mathcal{A} = \mathcal{W} \cap \mathcal{A}.$$

This proves $R \in \mathrm{Cl}_{\mathcal{K}([0,\gamma])} \mathcal{A}$. Similarly we have $R \in \mathrm{Cl}_{\mathcal{K}([0,\gamma])} \mathcal{B}$. $\square$

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References

[1] J. Angoa, Y. F. Ortiz-Castillo, A. Tamariz-Mascarua *Compact like properties on hyperspaces*, to appear.

[2] J. Ginsburg, *On the Stone-Čech compactification of the space of closed sets*, Trans. Amer. Math. Soc. 215 (1976), 301-311.

[3] J. Ginsburg, *Some results on the countable compactness and pseudocompactness of hyperspaces*, Canad. J. Math. 27(1975), no. 6, 1392-1399.

[4] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[5] Y. Hirata, N. Kemoto, *Orthocompactness versus normality in hyperspaces*, Topology Appl. **159**, (2012) 1169-1178.

[6] N. Kemoto, *Normality and countable paracompactness of hyperspaces of ordinals*, Topology Appl. **154**, (2007) 358-363.

[7] N. Kemoto, *Erratum to "Normality and countable paracompactness of hyperspaces of ordinals"*, Topology Appl. **157**, Issue 15 (2010) 2446-2447.

[8] N. Kemoto, J. Terasawa, *Strong zero-dimensionality of hyperspaces*, Topology Appl. **157**, (2010) 2376-2382.

[9] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., **71** (1951), 152-182.

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