The present study conceptualizes explorative proving, and applies this idea to four domain-specific frameworks for the curriculum development of Algebra, Geometry, Function, and Data Handling. We highlight the significance and the way of mapping these frameworks to instructional units of “Course of Study” in Japan, and argue that the former have a better curriculum reform potential.

NECESSITY FOR CURRICULUM FRAMEWORKS OF EXPLORATIVE PROVING

The teaching and learning of proof is recognized internationally as a key component of mathematics curricula (Hanna & de Villiers, 2012). However, it remains the case that students at the junior high school level (and beyond) experience difficulties in learning proofs in mathematics (e.g. Harel & Sowder, 2007; Martinez and Pedemonte, 2014). To overcome their difficulties by not only reflecting on the nature of mathematics, but also cultivating generic competencies of authentic explorative thinking (Miyazaki & Fujita, 2015), we put forward a series of frameworks for the development of curriculum in explorative proving in lower secondary education (G7-9). We argue that they fare better than Japanese current “Course of Study” equivalents. In the first part of this paper, we provide a conceptualization of Explorative Proving in school mathematics. In the second part, we lay out the details about how we can implement this idea for curriculum frameworks in domains of junior high school mathematics such as Algebra, Geometry, Function, and Data Handling. In the last part, we compare our proposals with the existing “Course of Study” units, and argue that our proposals could offer a more effective way of teaching explorative proving.

EXPLORATIVE PROVING IN SCHOOL MATHEMATICS

Based on Fawcett (1938), Waerden (1967), and Lakatos (1976), we argue that proving activities in mathematics involve producing statements inductively, deductively, and analogically, planning and constructing proofs, looking back over proving processes and overcoming global/local counter-examples or errors, and utilizing already-proved statements in the context of working on further proofs (see Fig. 1) to reflect the nature of proving as an activity in mathematics (Freudenthal, 1971).

By considering insights from the above, we define explorative proving as having the following three components: producing propositions, producing proofs
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(planning and construction), and looking back (examining, improving, and advancing) (see Fig. 1) (Miyazaki & Fujita, 2015).

In developing domain-specific frameworks, producing proofs is focused on, and its elements (planning and construction; axes) and their interactions are represented a two-axes model in which each axis is divide into two or three parts according to the characteristics of domain, and intersections on the model means the level of learning of constructing and planning proofs. These levels should be subsequently transited, and learning how to construct proofs precedes learning how to plan them. For each stage, the component “Looking back (examining, improving, and advancing)” can be also expected and encouraged as explorative proving.

DOMAIN-SPECIFIC FRAMEWORK FOR CURRICULUM DEVELOPMENT

Offering a detailed proposal, this section shows how to theoretically implement the idea of explorative proving in curriculum frameworks in accordance with the characteristics of the domains of junior high school mathematics. We focus on producing proofs, namely planning and construction.

**Geometry**

Formal proofs in geometry in junior high school have a logical structure that connects premises and conclusions via deductive reasoning by adopting singular and universal propositions. By focusing on this characteristic, we propose the following learning levels planning a proof:

GP1: Clarifying what -and how- can be used to connect premises and conclusion.

GP2: Considering how to think backward from a conclusion, think forward from premises, and how to connect them.

In geometry, “Planning a proof” refers to the activity of seeking ways to connect premises and conclusions by deductive reasoning (Tsujiyama and Yui, 2018). This activity needs to expand the network of propositions that can be deduced from premises, and the other network of propositions that can be deduced from conclusions, and to seek the common propositions within the two networks. The first learning level (GP1) refers to differentiating the objects (what can be used to connect premises and conclusions) from the methods (how the objects can be used) that are necessary to plan a proof. GP1 also refers to the use of objects and methods in order to connect premises and conclusions. The advanced learning level (GP2) consists in thinking forward from premises to conclusions, thinking backward in the opposite direction, and making use of them in order to connect premises and conclusions.

When it comes to “Constructing a proof,” the following learning levels can be considered:

GC1: Forming and expressing the deductive connection between premises and conclusions in a singular statement.

GC2: Forming and expressing the deductive connection between premises and conclusions while differentiating universal instantiation and hypothetical syllogisms from deductive reasoning.

“Constructing a proof” consists in finding the common propositions in two relational networks, and expressing the deductive connection between premises and conclusions, which are suggested by planning. Especially in a geometrical proof, premises and conclusions can be connected mainly by hypothetical syllogisms based on singular propositions. Considering a proof in more detail, each
singular proposition would be deduced with a universal proposition (e.g., theorems). This deduction can be realized by universal instantiation. Finally, constructing a proof can be achieved by expressing the connection with language, diagram, etc. The first learning level (GC1) is dedicated to expressing the part of connection based on a hypothetical syllogism. The advanced learning level (GC2) is dedicated to differentiating universal instantiation and hypothetical syllogism from deductive reasoning, and to expressing singular propositions and universal propositions with a clear distinction.

The goal of our proposed curriculum is to help students achieve levels at GP2 and GC2 by the end of junior high school. However, it would be perhaps unrealistic to expect from students to construct proofs immediately at the highest stages. Therefore, it is necessary to set up several intermediate levels. By setting Stage O where there is no differentiation between planning a proof and constructing a proof, we can set five hypothetical stages if we combine two kinds of level related to planning and constructing. The transition from the lowest stage (Stage O) to the highest stage (GP2, GC2) needs to go through Stages GP1, GC1, in order to enhance planning and constructing a proof. Therefore, the transition can be divided into an earlier component \([0 \Rightarrow (GP1, GC1)]\) and a later component \([GP1, GC1) \Rightarrow (GP2, GC2)]\). Thus, we can establish the two transition processes as learning progressions (Empson, 2011)(see Fig. 2).

**Algebra**

Miwa (1996) has illustrated three processes in the use of symbolic expressions: *Express, Transform, and Read* (Fig. 3). While planning a proof, one should consider how the premises and conclusions of a proposition can be symbolically expressed (*Express and Read* process, respectively), but also how to transform symbolic expressions representing premises into symbolic expressions representing conclusions (*Transform* process).

Taking the above into account, we propose the following learning levels for planning:

**AP1:** Clarifying how the premises of a proposition can be symbolically expressed by dividing a proposition into premises and conclusions.

**AP2:** Clarifying how the premises and conclusions of a proposition can be symbolically expressed by considering what the letters stand for.

**AP3:** Clarifying how the premises and conclusions of a proposition can be symbolically expressed and transformed to connect with each other by considering what the letters stand for.
To construct proofs in algebra, we propose the following learning levels:

AC1: Forming and expressing the deductive connection between premises and conclusions by using symbolic expressions and by explaining the relationship between premises, conclusions of a proposition, and symbolic expressions.

AC2: After proving the above and showing what the letters stand for, explaining the overall relationship between symbolic expressions and a proposition.

“Constructing a proof” consists in revealing the connection between premises and conclusions with the use of symbolic expressions. According to Miwa (1996), for students to understand the generality of proof in algebra, it is necessary to understand that literal symbols express arbitrary numbers. Therefore, we propose two levels in the Express process (Fig. 3). In the first level (AC1), students explain the relationship between the premises of a proposition and symbolic expressions. In the second (AC2), by being conscious of the generality of a proposition, they reveal what the letters stand for.

With regard to the Read process (Fig. 3), Miwa (1996) suggested that we need to read or interpret the result of transforming a symbolic expression (“Symbolic Expression*” in Fig. 3) in the context of the original situation in order to gain insight or uncover a new interpretation. Therefore, in constructing a proof, we propose two levels: In the first level (AC1), students explain the relationship between the result of transformations and the conclusions of a proposition. In the second (AC2), students explain the overall relationship between symbolic expressions and a proposition.

The goal of our curriculum is to help students achieve AP3 and AC2 levels by the end of junior high school. Three transition processes are assumed (see Fig. 4): In Transition process I, students aim to clarify how the premises can be symbolically expressed by grasping the idea of a proof in algebra. In Transition process II, students aim to think backward from the conclusions by being conscious of the generality of a proposition. In Transition process III, students aim to consider how to transform symbolic expressions.

Transition process I reflects the transition from Stage O to (AP1, AC1) via AC1. This process passes through AC1, not AP1, because it is necessary to have a chance to construct a proof in order to learn how to “plan a proof” as with the domain of Geometry. Similarly, Transition process II reflects the transition from Stage (AP1, AC1) to (AP2, AC2) via (AP1, AC2). Finally, Transition process III is from (AP2, AC2) to (AP3, AC2) (see Fig. 4). Like geometry, the component “Looking back” plays important roles in each level. For each stage, the component “Looking back (Examining, Improving, and Advancing, EIA)” can be expected and encouraged as explorative proving.

**Function**

In lower secondary education, one of the central aims of learning functions is to foster students’ functional thinking, stressing the significance of its applicability. Hence, “producing propositions” in
this process includes not only conjecturing about the properties of the functions, but also interpreting phenomena or predicting unknown situations by considering the relations among variables in the real world. We focus on the latter activity. In this process, proving refers to justifying the prediction or interpretation produced by using functions.

In the case of proving a prediction or interpretation (as well as in mathematical proofs), it is essential to inspect whether the inference process is appropriate. Such a process is similar to mathematical modeling, which is often grasped by following three sub-processes: formulating ($f$), employing ($e$) and interpreting ($i$). Therefore, we propose the following stages of learning:

$f1$: Showing what functions to be used in problem solving

$f2$: Showing mathematical evidence for the judgment about what function is to be used in problem solving

$f3$: Showing how the original real problem situation has been idealized or simplified

$e1$: Describing the entire process from the given conditions to mathematical conclusions

$e2$: Describing the entire process that leads to mathematical conclusions by means of specifying the mathematical model

$i1$: Taking into account the results of interpreting mathematical conclusions in the context of the original real problem situation

$i2$: Taking into account the results of interpreting mathematical conclusions that refer to limitations, and specifying their cause

The goal of our curriculum in the domain of function is to be able to construct the proof at stages $f3$, $e2$ and $i2$ (hereafter, we give a brief account in the form “proof at ($f3; e2; i2$)”). We set the following levels for proof construction:

FC1: Understanding the necessity of three sub-processes (formulating, employing, and interpreting) as a frame for proof, and constructing proofs at ($f1; e1; i1$).

FC2: Upgrading the description of proofs in terms of mathematical evidence, and constructing proofs at ($f2; e2; i1$).

FC3: Upgrading the description of proofs in terms of idealization and/or simplification, and constructing proofs at ($f3; e2; i2$).

In addition to being able to construct proofs autonomously, learning how to plan them is also required. Thus, we set the following levels for planning (corresponding to the levels of proof construction):

FP1: Investigating functions, and their usages to connect a premise and a conclusion from

Fig. 5. Function-specific framework
the viewpoint of three sub-processes of mathematical modeling.

FP2: Investigating functions, and their usages to connect a premise and a conclusion from the viewpoint of criteria that justify three sub-processes of mathematical modeling.

FP3: Investigating functions, and their usages to connect a premise and a conclusion from the viewpoint of whole process of mathematical modeling.

By considering following points, we could postulate six transition processes of learning in the domain of Function (see Fig. 5).

**Data Handling**

While in the domains of algebra and geometry students are expected to prove the logical necessity of propositions, the domain of data handling deals with the plausibility of claims in real world contexts. Given this characteristic, we use the term *justification*, rather than *proof*. To characterize the meaning of justification, we employ Toulmin’s (2003) layout of arguments. His model focuses on the soundness of practical arguments in everyday life situations, and thus it is relevant to the data handling domain. We use the simplified version of Toulmin’s scheme where each inference contains three elements: the claim being argued, the datum used to justify the claim, and the warrant describing how the datum supports the claim. In our research, the datum means something obtained by statistically analyzing raw data (e.g., mean, median, and histogram), rather than the raw data themselves. *Planning a justification* refers to thinking about what datum is appropriate for supporting a claim, and *constructing a justification* refers to actually producing an argument that consists of the claim, datum, and warrant.

<table>
<thead>
<tr>
<th>Task. The number set below shows how many books each of 20 grade 7 students in a secondary school read for one month. Can a student who read three books be regarded as reading relatively more books in this group? Explain your answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 0 1 1 8 1 10</td>
</tr>
<tr>
<td>2 3 1 12 2 3 1</td>
</tr>
<tr>
<td>1 0 5 2 1 10 (books)</td>
</tr>
</tbody>
</table>

**Fig. 6. Example of a task (left panel) and justification (right panel)**

We briefly illustrate the above characterization with the task in Fig. 6 taken from a Japanese mathematics textbook for secondary school students. Before working on this task, students are expected to be familiar with several types of descriptive statistics such as mean, median, and mode. Thus, students could plan which average they should consider for answering the question posed in the task. Accordingly, students may construct a justification (as in Fig. 6, right panel).

Fig. 7 represents our framework for curriculum development in the domain of data handling. In this framework, we differentiate two levels of justification: DC1 involves constructing a justification based on a single datum, and DC2 involves constructing a justification based on multiple data (DP1 and DP2 involve planning the respective justifications). This is because, given that the data handling domain is related to uncertain empirical phenomena, a single datum sometimes may not be enough to justify a claim, and additional data may be necessary for strengthening the justification —for
instance, in the task shown in Fig. 6, students may reinforce their justification by further taking a histogram into account.

In this framework, the transition from Stage O to Stage (DP1, DC1) is considered in the same way as the frameworks in other domains (e.g., geometry). With respect to the subsequent transition, the necessity for employing multiple data would arise in a situation where a single datum is not sufficient to represent the empirical phenomenon. In this case, on the one hand, the transition of planning would precede that of constructing since it is likely that students first plan what data they would add in order to strengthen their justification. On the other hand, it would not be reasonable to set up a stage exclusively concentrating on this kind of planning, namely Stage (DP2, DC1), because this stage means that students plan to employ multiple data but actually construct a justification using a single datum alone. Thus, we consider the direct transition from Stage (DP1, DC1) to Stage (DP2, DC2) as shown in Fig. 7. For each stage, the component “Looking back (examining, improving, and advancing)” can be also expected and encouraged as explorative proving.

TOWARD DEVELOPING DOMAIN-SPECIFIC CURRICULUM BASED ON THESE FRAMEWORKS

In order to develop a domain-specific curriculum based on the previous frameworks, we examined the existing implemented curriculum “Course of Study” in Japan, and compared the implemented units with the transition processes that we proposed above. For example, in the case of geometry in Japanese junior high schools, our proposed framework can achieve to develop the desirable and realizable curriculum, echoing the idea of explorative proving, while “Course of Study” only requires realizing the idea, but does not propose the way to realize it as curriculum. Moreover, “Course of Study” requires that students learn various properties of plane and space figures mainly based on congruency and similarity, and also the meaning of proofs, and how to prove formally. Although “Course of Study” encourages the gradual introduction of formal proofs until the end of Grade 8, it does not offer a clear plan on how to gradually implement the learning processes of planning and constructing a proof. By combining local transitions of our frameworks with units in “Course of Study”, the developed curriculum can propose teachers with a realizable plan on how to gradually implement the learning processes, and evaluate students’ ability.

In order to show the advantages of our proposal, we show the table combining the implemented units in “Course of Study” with two transition processes in our theoretical framework (Table 1). Each unit has a plethora of contents. We have stipulated a correspondence between those and the local transitions. Taking this correspondence as a basis in an ongoing study, we have been implementing experimental lessons (Cobb et al., 2003), and investigate the feasibility of our provisional frameworks and curriculum by conducting a careful observation of students and their work (the so-called “method of lesson study”; Lewis, Perry and Murata, 2006).
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<table>
<thead>
<tr>
<th>Units in “Course of Study”</th>
<th>Local transitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Properties of parallel lines and angles</td>
<td>(GP1, GC1) → (GP1, GC2)</td>
</tr>
<tr>
<td>Properties of angles of polygons</td>
<td></td>
</tr>
<tr>
<td>Meaning of congruent and conditions of congruent triangles</td>
<td>(GP1, GC2) + EIA</td>
</tr>
<tr>
<td>Meaning of formal proofs and how to prove formally</td>
<td>(GP1, GC2) → (GP2, GC2)</td>
</tr>
<tr>
<td>Properties of triangles and quadrilaterals</td>
<td>(GP2, GC2) + EIA</td>
</tr>
</tbody>
</table>

Table 1: Correspondence of implemented units with local transitions in Grade 8 geometry

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