Weighted Orlicz-Riesz capacity of balls

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Abstract

Our aim in this note is to estimate the weighted Orlicz-Riesz capacity of balls.

1 Introduction and statement of results

Several versions of capacities for Orlicz-Riesz spaces have appeared in research papers, for example those by Aissaoui and Benkirane [5], Kuznetsov [14], Mizuta [17], Adams and Hurri-Syrjänen [3, 4], Joensuu [12]. The notion of capacity offers a standard way to characterize exceptional sets and is indispensable to an understanding of the local behavior of functions in Orlicz-Riesz spaces. Various capacity estimates also play an important role in the study of solutions to partial differential equations.

Recently the authors [8] gave an estimate of the Orlicz-Riesz capacity of balls, as an extension of Adams and Hurri-Syrjänen [4] and Joensuu [12]. An estimate of the weighted Sobolev capacity of balls can be found e.g. in Heinonen, Kilpeläinen and Martio [9].

A positive measurable function w on $\mathbf{R}^n$ is called an A_p weight (written as $w \in A_p$) if there exists a positive constant C_p such that

$$\left(\frac{w(B)}{|B|}\right)\left(\frac{w^{1/(1-p)}(B)}{|B|}\right)^{p-1} \leqq C_p \quad (< \infty)$$

for all balls B, where $1 < p < \infty$, $|\cdot|$ denotes the n-dimensional Lebesgue measure and

$$w(B) = \int_B w(y)\, dy.$$

As an example, we have that the function $w(x) = |x|^\delta$ is an A_p weight if and only if $-n < \delta < n(p-1)$. It is well known that, for an A_p weight w, the corresponding measure w is doubling, that is, $w(2B) \leq cw(B)$ for all balls $B = B(x,r)$; here the constant c depends only on n, p and C_p and $2B$ stands for the

2000 Mathematics Subject Classification: 46E35, 46E30, 31B15

Key words and phrases: weighted Orlicz-Riesz capacity, Riesz potential, weighted-Riesz spaces

enlarged ball $B(x, 2r)$. For these and other fundamental properties of A_p weights; see, for example, Heinonen, Kilpeläinen and Martio [9].

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $I_\alpha f$ of order α by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

In the present note, we treat functions f satisfying an Orlicz condition with an A_p weight ω:

$$\int_{\mathbf{R}^n} \varphi_p(|f(y)|)\omega(y)\, dy < \infty. \tag{1.1}$$

Here $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive quasi-increasing function on $(0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$(\varphi 1) \qquad c_1^{-1}\varphi(r) \le \varphi(r^2) \le c_1 \varphi(r) \qquad \text{whenever } r > 0.$$

We say that $\varphi(r)$ is quasi-increasing if there exists $c > 0$ such that

$$\varphi(s) \le c\varphi(t) \qquad \text{whenever } 0 < s < t.$$

We set

$$\varphi_p(0) = 0,$$

because we will see from $(\varphi 4)$ below that

$$\lim_{r \to 0+} \varphi_p(r) = 0;$$

see [19, p205].

For an open set $G \subset \mathbf{R}^n$, we denote by $L^{\varphi_p,\omega}(G)$ the family of all locally integrable functions f on G such that

$$\int_G \varphi_p(|f(y)|)\omega(y)\, dy < \infty,$$

and define

$$\|f\|_{\varphi_p,\omega,G} = \inf\left\{\lambda > 0 : \int_G \varphi_p(|f(y)|/\lambda)\omega(y)\, dy \le 1\right\}.$$

This is a quasi-norm in $L^{\varphi_p,\omega}(G)$. For $E \subset G$, the relative $(\alpha, \varphi_p, \omega)$-capacity is defined by

$$B_{\alpha,\varphi_p,\omega}(E; G) = \inf \int_G \varphi_p(|f(y)|)\omega(y)\, dy,$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \ge 1 \qquad \text{for all } x \in E$$

(cf. Adams and Hedberg [2], Meyers [15], Ziemer [22] and the first author [16, 17]).

Our first aim in the present note is to give an estimate of the modular capacity $B_{\alpha,\varphi_p,\omega}$ of open balls $B(x,r)$ centered at x of radius r, as an extension of Adams and Hurri-Syrjänen [4, Theorem 2.11], Joensuu [12], Heinonen, Kilpeläinen and Martio [9, Theorems 2.18 and 2.19] and the authors [8, Theorem A]. In fact, our first theorem is stated in the following.

THEOREM A (cf. [21, Lemma 7.3]). *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A > 0$ such that*

$$A^{-1}\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\,dt/t\right)^{1-p} \le B_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R))$$
$$\le A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\,dt/t\right)^{1-p}$$

whenever $B(x,r) \subset B(0,R/4)$.

We write $f \sim g$ if there exists a constant A so that $A^{-1}g \le f \le Ag$.

EXAMPLE 1.1. Let $\omega(x) = |x|^\delta$ and $\varphi(t) = (\log(e+t))^\beta$.

(1) If $\alpha p - n < \delta < n(p-1)$, then

$$\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t \sim r^{(-\alpha p+n+\delta)/(1-p)}(\log(e+1/r))^{\beta/(1-p)}$$

for $0 < r < R/2 < 1$. In this case

$$B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \sim r^{-\alpha p+n+\delta}(\log(e+1/r))^\beta.$$

(2) If $\alpha p - n = \delta$ and $\beta < p-1$, then

$$\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t \sim (\log(e+1/r))^{\beta/(1-p)+1}$$

for $0 < r < R/2 < 1$. In this case

$$B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \sim (\log(e+1/r))^{\beta+1-p}.$$

(3) If $\alpha p - n = \delta$ and $\beta = p-1$, then

$$\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t \sim \log(e+(\log(e+1/r)))$$

for $0 < r < R/2 < 1$. In this case

$$B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \sim (\log(e+(\log(e+1/r))))^{-(p-1)}.$$

(4) If $\alpha p - n > \delta > -n$ or $\alpha p - n = \delta$ and $\beta > p-1$, then

$$B_{\alpha,\varphi_p,\omega}(\{0\}; B(0,1)) > 0.$$

Next we are concerned with the norm capacity. For $E \subset G$, we define

$$C_{\alpha,\varphi_p,\omega}(E; G) = \inf \|f\|_{\varphi_p,\omega,G},$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \geq 1 \qquad \text{for all } x \in E.$$

Theorem B. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A > 0$ such that*

$$A^{-1} \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{(1-p)/p} \leq C_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R))$$
$$\leq A \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{(1-p)/p}$$

whenever $B(x,r) \subset B(0,R/4)$.

In view of Theorems A and B, we have the following result, which extends the results by Adams and Hurri-Syrjänen[4], Joensuu [12] and the authors [8].

Corollary 1.2. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there is a constant $A > 0$ such that*

$$A^{-1} B_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R))^{1/p}$$
$$\leq C_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R)) \leq A B_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R))^{1/p}$$

whenever $B(x,r) \subset B(0,R/4)$.

For further related results, we refer the reader to Adams [1], Adams and Hurri-Syrjänen [3], Edmunds and Evans [7], Kilpeläinen [13] and Mizuta and Shimomura [19, 20, 21].

Throughout this note, let A denote various constants independent of the variables in question and $A(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

2 Proof of Theorem A

First we collect properties which follow from condition ($\varphi 1$) (see [17], [19, Lemma 2.3], [18, Section 7]).

($\varphi 2$) φ satisfies the doubling condition, that is, there exists $c_2 > 1$ such that

$$c_2^{-1} \varphi(r) \leq \varphi(2r) \leq c_2 \varphi(r) \qquad \text{whenever } r > 0.$$

(φ3) For each $\gamma > 0$, there exists $c_3 = c_3(\gamma) \geq 1$ such that

$$c_3^{-1}\varphi(r) \leq \varphi(r^\gamma) \leq c_3\varphi(r) \qquad \text{whenever } r > 0.$$

(φ4) For each $\gamma > 0$, there exists $c_4 = c_4(\gamma) \geq 1$ such that

$$s^\gamma\varphi(s) \leq c_4 t^\gamma\varphi(t) \qquad \text{whenever } 0 < s < t.$$

(φ5) For each $\gamma > 0$, there exists $c_5 = c_5(\gamma) \geq 1$ such that

$$t^{-\gamma}\varphi(t) \leq c_5 s^{-\gamma}\varphi(s) \qquad \text{whenever } 0 < s < t.$$

(φ6) If φ and ψ are positive monotone functions on $[0,\infty)$ satisfying (φ1), then for each $\gamma > 0$ then there exists a constant $c_6 = c_6(\gamma) \geq 1$ such that

$$c_6{}^{-1}\varphi(r) \leq \varphi(r^\gamma\psi(r)) \leq c_6\varphi(r) \qquad \text{whenever } r > 0.$$

Let us begin with an upper estimate for modular $B_{\alpha,\varphi_p,\omega}$-capacity of balls.

Theorem 2.1. *Suppose $p > 1$ and ω is a positive locally integrable function on $\mathbf{R}^n$. Then there exists a constant $A > 0$ such that*

$$B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \leq A\left(\int_{2r}^{2R}\{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t\right)^{1-p}$$

whenever $0 < r < R/2 < \infty$.

Proof. For $r > 0$, consider the function

$$f_r(y) = |y|^{-\alpha}$$

for $r < |y| < 2r$ and $f_r = 0$ elsewhere. If $x \in B(0,r)$ and $y \in B(0,2r) \setminus B(0,r)$, then $|x-y| < 3r$, so that

$$I_\alpha f_r(x) \geq (3r)^{\alpha-n}\int_{B(0,2r)\setminus B(0,r)}|y|^{-\alpha}\,dy = A_1$$

with a constant $A_1 = A_1(\alpha,n) > 0$.

Now let $0 < r < R/2$, and take j_0 such that $2^{j_0+1}r \leq R < 2^{j_0+2}r$. For $\{a_j\}$ such that $a_j \geq 0$ and $\sum_{j=0}^{j_0}a_j = 1$, set

$$f = \sum_{j=0}^{j_0}a_j f_{2^jr}/A_1.$$

Then

$$I_\alpha f(x) = A_1^{-1}\sum_{j=0}^{j_0}a_j I_\alpha f_{2^jr}(x) \geq 1$$

for $x \in B(0,r)$. Therefore we have by ($\varphi 2$) and ($\varphi 3$) that

$$\begin{aligned} B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) &\le \int_{B(0,R)} \varphi_p(f(y))\omega(y)\,dy \\ &= \sum_{j=0}^{j_0} \int_{B(0,R)} [a_j f_{2^j r}(y)/A_1]^p \varphi(a_j f_{2^j r}(y)/A_1)\omega(y)\,dy \\ &\le A_2 \sum_{j=0}^{j_0} a_j^p (2^j r)^{-\alpha p} \varphi((2^j r)^{-1})\omega(B(0,2^{j+1}r)). \end{aligned}$$

Now, letting $K = \sum_{j=0}^{j_0} \{(2^j r)^{-\alpha p}\varphi((2^j r)^{-1})\omega(B(0,2^{j+1}r))\}^{1/(1-p)}$ and

$$a_j = \frac{\{(2^j r)^{-\alpha p}\varphi((2^j r)^{-1})\omega(B(0,2^{j+1}r))\}^{1/(1-p)}}{K},$$

we find

$$\begin{aligned} B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) &\le A_2 K^{-p} \sum_{j=0}^{j_0} \{(2^j r)^{-\alpha p}\varphi((2^j r)^{-1})\omega(B(0,2^{j+1}r))\}^{1/(1-p)} \\ &= A_2 K^{1-p}. \end{aligned}$$

Hence we have

$$\begin{aligned} K &\ge A_3 \sum_{j=0}^{j_0} \int_{2^{j+1}r}^{2^{j+3}r} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t \\ &\ge A_3 \int_{2r}^{2^{j_0+3}r} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t, \end{aligned}$$

so that

$$K \ge A_4 \int_{2r}^{2R} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t,$$

which proves the result. □

Corollary 2.2. If ω is doubling, then there exists a constant $A > 0$ such that

$$B_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \le A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\,dt/t\right)^{1-p}$$

whenever $B(x,r) \subset B(0,R/4)$.

In fact,

$$\begin{aligned} B_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) &\le B_{\alpha,\varphi_p,\omega}(B(x,r);B(x,R/2)) \\ &\le A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\,dt/t\right)^{1-p} \end{aligned}$$

whenever $B(x,r) \subset B(0,R/4)$.

Next we give a lower estimate for modular $B_{\alpha,\varphi_p,\omega}$-capacity of balls.

THEOREM 2.3. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}$$

whenever $0 < r < R/2 < \infty$.

Proof. For $0 < r < R/2$, take a nonnegative measurable function f on $B(0,R)$ such that

$$I_\alpha f(x) \geq 1 \qquad \text{for } x \in B(0,r).$$

Then we have by Fubini's theorem

$$\begin{aligned}
\int_{B(0,r)} dx &\leq \int_{B(0,r)} I_\alpha f(x) \, dx \\
&\leq \int_{B(0,R)} \left(\int_{B(0,r)} |x-y|^{\alpha-n} \, dx \right) f(y) \, dy \\
&\leq A_1 r^n \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \, dy.
\end{aligned}$$

For $\varepsilon > 0$ and $0 < \delta < \alpha$, we see from Hölder's inequality that

$$\begin{aligned}
& \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \, dy \\
= & \int_{\{y \in B(0,R): f(y) > \varepsilon (r+|y|)^{-\delta}\}} (r+|y|)^{\alpha-n} f(y) \, dy \\
& + \int_{\{y \in B(0,R): f(y) \leq \varepsilon (r+|y|)^{-\delta}\}} (r+|y|)^{\alpha-n} f(y) \, dy \\
\leq & \left(\int_{B(0,R)} [(r+|y|)^{\alpha-n} \{\varphi(\varepsilon (r+|y|)^{-\delta}) \omega(y)\}^{-1/p}]^{p'} dy \right)^{1/p'} \\
& \times \left(\int_{B(0,R)} \varphi_p(f(y)) \omega(y) \, dy \right)^{1/p} + \varepsilon \int_{B(0,R)} (r+|y|)^{\alpha-n-\delta} dy,
\end{aligned}$$

where $1/p + 1/p' = 1$. Since $\omega \in A_p$, we find by ($\varphi 2$) and ($\varphi 3$) that

$$\begin{aligned}
& \int_{B(0,R)} [(r+|y|)^{\alpha-n} \{\varphi(\varepsilon (r+|y|)^{-\delta}) \omega(y)\}^{-1/p}]^{p'} dy \\
\leq & \ A_2(\varepsilon) \int_0^R (r+t)^{(\alpha-n)p/(p-1)} \varphi((r+t)^{-1})^{1/(1-p)} \left(\int_{B(0,t)} \omega(y)^{1/(1-p)} \, dy \right) \frac{dt}{t} \\
\leq & \ A_3(\varepsilon) \int_0^R (r+t)^{\alpha p/(p-1)} \varphi((r+t)^{-1})^{1/(1-p)} \left(\int_{B(0,r+t)} \omega(y) \, dy \right)^{1/(1-p)} \frac{dt}{t} \\
\leq & \ A_3(\varepsilon) \int_r^{2R} \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt/t \\
\leq & \ A_4(\varepsilon) \int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt/t.
\end{aligned}$$

Thus we derive

$$\begin{aligned}
&\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\, dy \\
\le\ & A_4(\varepsilon)\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t\right)^{1/p'} \\
&\times \left(\int_{B(0,R)} \varphi_p(f(y))\omega(y)\, dy\right)^{1/p} + A_5\varepsilon,
\end{aligned}$$

so that

$$\begin{aligned}
1 \ \le\ & A(\varepsilon)\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t\right)^{1/p'} \\
&\times \left(\int_{B(0,R)} \varphi_p(f(y))\omega(y)\, dy\right)^{1/p} + A_7\varepsilon,
\end{aligned}$$

If $A_7\varepsilon = 1/2$, then we establish

$$B_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \ge A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t\right)^{1-p},$$

as required. □

Corollary 2.4. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$B_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \ge A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\, dt/t\right)^{1-p}$$

whenever $B(x,r) \subset B(0,R/4)$.

Now Theorem A follows from Corollaries 2.2 and 2.4.

3 Proof of Theorem B

Let us begin with an upper estimate for the norm $C_{\alpha,\varphi_p,\omega}$-capacity of balls.

Theorem 3.1. *Suppose that $p > 1$ and $\omega(B(0,r))$ satisfies the doubling condition. Then there exists a constant $A > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \le A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t\right)^{(1-p)/p}$$

whenever $0 < r < R/2 < \infty$.

Proof. Set

$$\varphi^*(r) = \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t$$

for $r > 0$. Consider the function

$$f(y) = |y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\varphi^*(r)^{-1/p}$$

for $r < |y| < R$ and $f = 0$ elsewhere. If $x \in B(0,r)$ and $y \in B(0,R)\setminus B(0,r)$, then $|x-y| < 2|y|$, so that

$$\begin{aligned} I_\alpha f(x) &\geq 2^{\alpha-n}\varphi^*(r)^{-1/p}\int_{B(0,R)\setminus B(0,r)} |y|^{-n}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\, dy \\ &= A_1\varphi^*(r)^{(p-1)/p} \end{aligned}$$

with a constant $A_1 = A_1(\alpha,n) > 0$. It follows from the definition of capacity that

$$C_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \leq A_1^{-1}\varphi^*(r)^{(1-p)/p}\|f\|_{\varphi_p,\omega,B(0,R)}.$$

Thus it suffices to show that

$$\|f\|_{\varphi_p,\omega,B(0,R)} \leq A_2.$$

For this propose, we first note that

$$\begin{aligned} &\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy \\ = &\int_{B(0,R)\setminus B(0,r)} f(y)^p\varphi(f(y))\omega(y)dy \\ = &\ \varphi^*(r)^{-1}\int_{B(0,R)\setminus B(0,r)} |y|^{-\alpha p}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{p/(1-p)} \\ &\qquad \times\varphi\left(|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\varphi^*(r)^{-1/p}\right)\omega(y)dy. \end{aligned}$$

Here we see from the doubling condition of $\omega(B(0,r))$ that

$$\begin{aligned} &|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\varphi^*(r)^{-1/p} \\ \leq &\ A_3|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\left(\int_r^{2R}\{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t\right)^{-1/p} \\ \leq &\ A_3|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\left(\int_{|y|}^{2|y|}\{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\, dt/t\right)^{-1/p} \\ \leq &\ A_4\varphi(|y|^{-1})^{-1/p}\omega(B(0,|y|))^{-1/p} \end{aligned}$$

for $y \in B(0,R)\setminus B(0,r)$. Further we can find constants $A_0 > 0$ and $\gamma > 0$ such that

$$\omega(B(0,t)) \geq A_0 t^\gamma$$

for all $t > 0$. Hence, as in Theorem 2.1, we obtain by (φ6)

$$
\begin{aligned}
& \int_{B(0,R)} \varphi_p(f(y))\omega(y)dy \\
\leq\ & A_5\varphi^*(r)^{-1} \int_{B(0,R)\setminus B(0,r)} |y|^{-\alpha p}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{p/(1-p)}\varphi(|y|^{-1})\omega(y)dy \\
\leq\ & A_6\varphi^*(r)^{-1} \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}dt/t \\
=\ & A_6,
\end{aligned}
$$

as required. □

COROLLARY 3.2. *Suppose that $p > 1$ and ω is doubling. Then there exists a constant $A > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \leq A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\ dt/t\right)^{(1-p)/p}$$

whenever $B(x,r) \subset B(0,R/4)$.

Next we give a lower estimate for the norm $C_{\alpha,\varphi_p,\omega}$-capacity of balls.

THEOREM 3.3. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \geq A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\ dt/t\right)^{(1-p)/p}$$

whenever $0 < r < R/2 < \infty$.

Proof. For $0 < r < R/2$ take a nonnegative measurable function f on $B(0,R)$ such that

$$I_\alpha f(x) \geq 1 \qquad \text{for } x \in B(0,r).$$

Then we have by Fubini's theorem

$$
\begin{aligned}
\int_{B(0,r)} dx &\leq \int_{B(0,r)} I_\alpha f(x)\ dx \\
&\leq \int_{B(0,R)} \left(\int_{B(0,r)} |x-y|^{\alpha-n}\ dx\right) f(y)\ dy \\
&\leq A_1 r^n \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\ dy,
\end{aligned}
$$

so that

$$1 \leq A_1 \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\ dy.$$

We show that

$$
\begin{aligned}
&\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\, dy \\
&\qquad \le A_2 \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)}\, dt/t \right)^{(p-1)/p} \|f\|_{\varphi_p,\omega,B(0,R)}.
\end{aligned}
$$

For this purpose, suppose that $\|f\|_{\varphi_p,\omega,B(0,R)} \le 1$. For $0 < \delta < \alpha$, we see from Hölder's inequality that

$$
\begin{aligned}
&\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\, dy \\
= \;& \int_{\{y\in B(0,R): f(y) > (r+|y|)^{-\delta}\}} (r+|y|)^{\alpha-n} f(y)\, dy \\
&+ \int_{\{y\in B(0,R): f(y) \le (r+|y|)^{-\delta}\}} (r+|y|)^{\alpha-n} f(y)\, dy \\
\le \;& \left(\int_{B(0,R)} [(r+|y|)^{\alpha-n} \{\varphi((r+|y|)^{-\delta}) \omega(y)\}^{-1/p}]^{p'} dy \right)^{1/p'} \\
&\times \left(\int_{B(0,R)} \varphi_p(f(y)) \omega(y)\, dy \right)^{1/p} + \int_{B(0,R)} (r+|y|)^{\alpha-n-\delta} dy.
\end{aligned}
$$

Here note from $\omega \in A_p$, $(\varphi 2)$ and $(\varphi 3)$ that

$$
\begin{aligned}
&\int_{B(0,R)} [(r+|y|)^{\alpha-n} \{\varphi(\varepsilon(r+|y|)^{-\delta}) \omega(y)\}^{-1/p}]^{p'} dy \\
\le \;& A_3 \int_0^R (r+t)^{(\alpha-n)p/(p-1)} \varphi((r+t)^{-1})^{1/(1-p)} \left(\int_{B(0,t)} \omega(y)^{1/(1-p)}\, dy \right) \frac{dt}{t} \\
\le \;& A_4 \int_0^R (r+t)^{\alpha p/(p-1)} \varphi((r+t)^{-1})^{1/(1-p)} \left(\int_{B(0,r+t)} \omega(y)\, dy \right)^{1/(1-p)} \frac{dt}{t} \\
\le \;& A_4 \int_r^{2R} \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)}\, dt/t \\
\le \;& A_5 \int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)}\, dt/t,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\, dy \\
\le \;& A_5 \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)}\, dt/t \right)^{1/p'} \\
&\times \left(\int_{B(0,R)} \varphi_p(f(y)) \omega(y)\, dy \right)^{1/p} + A_6 \\
\le \;& A_7 \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)}\, dt/t \right)^{1/p'}.
\end{aligned}
$$

Hence we establish

$$C_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \geq A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\,dt/t\right)^{(1-p)/p},$$

as required. □

Corollary 3.4. *Suppose $p>1$ and $\omega \in A_p$. For $R>0$, there exists a constant $A = A(R) > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \geq A\left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\,dt/t\right)^{(1-p)/p}$$

whenever $B(x,r) \subset B(0,R/4)$.

As in the proof of Theorem A, Theorem B follows readily from Corollaries 3.2 and 3.4.

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