Maximal functions, Riesz potentials and Sobolev embeddings on Musielak-Orlicz-Morrey spaces of variable exponent in $\mathbf{R}^n$

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Abstract

Our aim in this paper is to deal with Sobolev embeddings for Riesz potentials of variable order with functions in variable exponent Musielak-Orlicz-Morrey spaces.

1 Introduction

The space introduced by Morrey [36] in 1938 has become a useful tool of the study for the existence and regularity of partial differential equations. Variable exponent spaces have been studied in many articles over the past decades, for a survey see [15, 20, 45]. These investigations have dealt with the spaces themselves, e.g. [10, 16, 19, 24], with related differential equations [2, 5, 12], and with applications [1, 7, 44]. In the present paper, we aim to establish Sobolev embeddings for Riesz potentials of variable order with functions in variable exponent Musielak-Orlicz-Morrey spaces.

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r and denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbf{R}^n$. In our discussions, the boundedness of the Hardy-Littlewood maximal operator is a crucial tool as in Hedberg [23]. It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^n)$ if $p > 1$ (see [47]). Chiarenza-Frasca [8] generalized the boundedness of the maximal operator by replacing Lebesgue spaces by Morrey spaces $L^{p,\nu}(\mathbf{R}^n)$, where Morrey space $L^{p,\nu}(\mathbf{R}^n)$ is a family of $f \in L^1_{loc}(\mathbf{R}^n)$ satisfying the Morrey condition

$$\sup_{x\in\mathbf{R}^n, r>0} \frac{r^\nu}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy < \infty$$

2000 Mathematics Subject Classification : Primary 31B15, 46E30

Key words and phrases : Maximal functions, Musielak-Orlicz-Morrey space of variable exponent, Riesz potential, Sobolev embeddings, Sobolev's inequality

for $\nu \geq 0$ (see also Nakai [38]). Further, the boundedness of the maximal operator was also studied on Orlicz-Morrey spaces (see [39, 40, 41]). In [13], Diening showed that the maximal operator was bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ if the variable exponent p, which is a constant outside a ball, satisfies the locally log-Hölder condition and $\inf p > 1$ (see condition (P2) in Section 2). Cruz-Uribe-Fiorenza-Neugebauer [11] proved the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbf{R}^n)$ when p satisfies the log-Hölder condition on $\mathbf{R}^n$ and $\inf p > 1$ (see condition (P3) in Section 2). In the case of bounded open sets, Almeida-Hasanov-Samko [4] and Mizuta-Shimomura [33] studied the boundedness of the maximal operator for the variable exponent Morrey spaces. Let G be a bounded open set in $\mathbf{R}^n$ and $d_G = \text{diam } G$. Further, for a nonnegative measurable function ω on $G \times (0, d_G)$ satisfying some conditions, Guliyev-Hasanov-Samko [17, 18] showed that the maximal operator is bounded on the generalized variable exponent Morrey space $L^{p(\cdot),\omega}(G)$ (see Remark 3.2 below).

On the other hand, the maximal operator is bounded from Orlicz spaces $L \log L(G)$ to $L^1(G)$, while the maximal operator is not bounded on $L^1(G)$. In the variable exponent case, a variable exponent p with $\inf p = 1$ may approach or attain 1 in value, so that the boundedness of the maximal operator on the variable exponent Orlicz space $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ has been investigated, e.g. [9, 29, 31]. Also, in the case $\inf p > 1$, the boundedness of the maximal operator was studied on the variable exponent Orlicz space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$ (see [21, 25, 32]). For the variable exponent Orlicz-Morrey spaces, we refer the reader to [28].

Our first task in this paper is to establish the boundedness of maximal operators in the variable exponent Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(\mathbf{R}^n)$ (see Theorem 3.1), where p is a variable exponent satisfying the log-Hölder condition and $\inf p > 1$, and Φ and κ are of the form $\Phi(x,t) = t^{p(x)}\varphi(x,t)$ and $\kappa(x,t) = t^{\nu(x)}\psi(x,t)$ (see Section 2 for the definition of Φ and κ). In the previous paper [28], the authors gave the bounded sets version when $\Phi(x,r) = r^{p(x)}(\log(e+r))^{q(x)}$ and $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$. For the case $\inf p = 1$, see Section 5.

For a measurable function $\alpha : \mathbf{R}^n \to (0,n)$, we consider the Riesz potential of variable order α for a locally integrable function f on $\mathbf{R}^n$ defined by

$$I_{\alpha(x)}f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha(x)-n} f(y)dy.$$

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha f\|_{L^{p^*}(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}$$

for $f \in L^p(\mathbf{R}^n)$, $0 < \alpha < n$ and $1 < p < n/\alpha$. Sobolev's inequality has been studied in many articles and settings. If $f \in L^{p,\nu}(\mathbf{R}^n)$, then it is shown (see Adams [3] and Peetre [43]) that $I_\alpha f$ satisfies Sobolev's inequality whenever $\nu > \alpha p$, where $1 < p < \infty$. Mizuta-Shimomura [34] dealt with Sobolev's embeddings for Riesz potentials of functions in Orlicz spaces $L^\Phi(G)$, where $\Phi(x,t) = t^{p(x)}\varphi(x,t)$ and φ is a monotone log-type function. The version for Orlicz-Morrey spaces was also studied by Nakai [39]. Diening [14] has established embedding results for Riesz potentials with functions in $L^{p(\cdot)}(\mathbf{R}^n)$. See also [6]. In the case of bounded open sets, Almeida-Hasanov-Samko [4] and Mizuta-Shimomura [33] studied Sobolev's embeddings for Riesz potentials of functions in the variable exponent Morrey spaces. Further, the version for the generalized variable exponent Morrey space $L^{p(\cdot),\omega}(G)$ was discussed by Guliyev-Hasanov-Samko [17, 18].

When $p = 1$, the situation is a little different. O'Neil [42] showed that I_α is bounded operator from $L^1(\log L)^{1-\alpha/n}(G)$ to $L^{n/(n-\alpha)}(G)$ if $1-\alpha/n > 0$. Recently, the authors [26] gave a result on Sobolev embeddings for Riesz potentials of functions in $L^{1,\kappa}(G)$ with $\kappa(r) = r^\nu(\log(2+1/r))^\beta$ (see also [27, 46]). In [29], the authors showed that $I_\alpha f$ satisfies Sobolev's inequality with functions in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$. For the variable exponent Orlicz-Morrey spaces, we refer to [28].

As an application of the boundedness of maximal functions, we shall give a Morrey version of Sobolev inequality for $I_{\alpha(x)}f$ with functions f in $L^{\Phi,\kappa}(\mathbf{R}^n)$ if $\inf p > 1$ (see Theorem 4.1), as an extension of Adams [3], Almeida-Hasanov-Samko [4], the authors [28], Mizuta-Shimomura [33] and O'Neil [42]. Further, we deal with the case $\inf p = 1$ in Section 5.

The structure of rest of this paper is as follows. The next section is for notation and conventions used throughout the paper. In Section 3, we prove Theorem 3.1. In Section 4, we prove Theorem 4.1 by Theorem 3.1 and Hedberg's trick [23]. In Section 5, we are concerned with Sobolev embeddings for $I_{\alpha(x)}f$ with functions f in $L^{\Phi,\kappa}(\mathbf{R}^n)$ when $\inf p = 1$, which extend the results by the authors [28].

2 Notation and conventions

Throughout this paper, let C denote various constants independent of the variables in question. For non-negative functions f and g, we write $f \sim g$ if there exists a constant $C > 0$ so that $C^{-1}g \le f \le Cg$. For an integrable function u on a measurable set $E \subset \mathbf{R}^n$ of positive measure, we define the integral mean over E by

$$\fint_E u(x)\,dx = \frac{1}{|E|}\int_E u(x)\,dx.$$

We will introduce Musielak-Orlicz-Morrey spaces. We consider a function $\Phi(x,t) : \mathbf{R}^n \times [0,\infty) \to [0,\infty)$ satisfying the following conditions: $\Phi(x,t)$ is measurable on $\mathbf{R}^n$ for each $t \ge 0$ and convex on $[0,\infty)$ for each $x \in \mathbf{R}^n$, and $\Phi(x,0) = 0$. Further we consider a function $\kappa(x,r) : \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ satisfying the following condition:

(κ1) $\kappa(x,t) \le Ct^n$ for all $x \in \mathbf{R}^n$ and $t \ge 1$.

We introduce Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(\mathbf{R}^n)$ by the family of all measurable functions f with finite norm

$$\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} = \inf\left\{\lambda > 0 : \sup_{x\in\mathbf{R}^n, r>0} \kappa(x,r) \fint_{B(x,r)} \Phi(y, |f(y)|/\lambda)\, dy \le 1\right\}.$$

Note that $L^{\Phi,\kappa}(\mathbf{R}^n)$ is a Banach space with respect to the norm $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)}$. Here we may consider the fractional maximal operator defined by

$$M_{\Phi,\kappa} f(x) = \sup_{r>0} \kappa(x,r) \fint_{B(x,r)} \Phi(y, |f(y)|)\ dy.$$

Then it is worth to see by the doubling condition of $\Phi(y,\cdot)$ that

$$\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \sim \|M_{\Phi,\kappa} f\|_{L^\infty(\mathbf{R}^n)}.$$

In this paper, we treat the following special Φ and κ.

Let p be a continuous function on $\mathbf{R}^n$ satisfying

(P1) $1 \le p_- \equiv \inf_{x\in\mathbf{R}^n} p(x) \le p_+ \equiv \sup_{x\in\mathbf{R}^n} p(x) < \infty$;

(P2) $|p(x) - p(y)| \le \dfrac{C}{\log(e + 1/|x-y|)}$ for all $x, y \in \mathbf{R}^n$;

(P3) $|p(x) - p(y)| \le \dfrac{C}{\log(e + |x|)}$ for all $x, y \in \mathbf{R}^n$ with $|y| \ge \dfrac{|x|}{2}$.

If p satisfies (P2) and (P3), then p is said to satisfy the log-Hölder condition on $\mathbf{R}^n$. By (P3), p has a finite limit $p(\infty)$ at ∞ and

(P3′) $|p(x) - p(\infty)| \le \dfrac{C}{\log(e + |x|)}$ for all $x \in \mathbf{R}^n$.

We say that a function $\varphi : \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ is of log-type if it satisfies

(φ0) $\varphi(\cdot, r)$ is measurable for all $r > 0$ and $\varphi(x,\cdot)$ is continuous for a.e. $x \in \mathbf{R}^n$,

(φ1) $0 < \inf_{x\in\mathbf{R}^n} \varphi(x,1) \le \sup_{x\in\mathbf{R}^n} \varphi(x,1) < \infty$,

(φ2) $c_1^{-1} \le \dfrac{\varphi(x,r)}{\varphi(x,s)} \le c_1$ for all $x \in \mathbf{R}^n$ and $2^{-1}s \le r \le 2s$,

(φ3) $c_2^{-1} \le \dfrac{\varphi(x,r)}{\varphi(x,s)} \le c_2$ for all $x \in \mathbf{R}^n$ and $\min\{s, s^2\} \le r \le \max\{s, s^2\}$.

If $\varphi(x,r)$ is of log-type, then $\varphi(x, 1/r)$ is also of log-type. Further, if $\varepsilon > 0$, then $r^\varepsilon \varphi(x,r)$ is uniformly quasi-increasing on $(0,\infty)$, that is,

(φ) $r^\varepsilon \varphi(x,r) \le c_3 s^\varepsilon \varphi(x,s)$ for all $x \in \mathbf{R}^n$ and $0 < r < s < \infty$

(see e.g. [27]).

Further we say that φ satisfies the log-Hölder condition if it satisfies

(φ4) $\dfrac{1}{c_4} \le \dfrac{\varphi(x,r^{-1})}{\varphi(y,r^{-1})} \le c_4$ for all $x, y \in \mathbf{R}^n$ with $|x-y| < r$ and $r \le 1$,

(φ5) $\dfrac{1}{c_5} \le \dfrac{\varphi(x,t)}{\varphi(y,t)} \le c_5$ for all $x, y \in \mathbf{R}^n$ and $\max\{(1+|x|)^{-1}, (1+|y|)^{-1}\} \le t < 1$.

The constants c_1–c_5 are independent of $x \in \mathbf{R}^n$ and $r, s \in (0,\infty)$.

For instance

$$\varphi_1(x,t) := \big(\log(e+t)\big)^{q(x)}$$

satisfies those conditions if $q \in L^\infty(\mathbf{R}^n)$ is non-negative and log log-Hölder continuous, i.e.,

$$|q(x) - q(y)| \le \frac{C}{\log\log(e^2 + 1/|x-y|)} \quad \text{for all } x, y \in \mathbf{R}^n$$

(cf. [9]). Of course we may also add further logarithms, e.g.

$$\varphi_2(x,t) := \big(\log(e+t)\big)^{q(x)} \big(\log\log(e^2+t)\big)^{r(x)},$$

etc. We also give another example which satisfies conditions above:

$$\varphi_3(x,t) := \big(\log(e+t)\big)^{q_1(x)} \big(\log(e+1/t)\big)^{q_2(x)},$$

where q_1 and q_2 are in $L^\infty(\mathbf{R}^n)$ and satisfy

$$|q_1(x) - q_1(y)| \le \frac{C}{\log\log(e^2 + 1/|x-y|)} \quad \text{for all } x, y \in \mathbf{R}^n,$$

and

$$|q_2(x) - q_2(y)| \le \frac{C}{\log\log(e^2 + |x|)} \quad \text{for all } x, y \in \mathbf{R}^n \text{ with } |y| \ge \frac{|x|}{2}.$$

The function φ_3 also satisfies the condition ($\varphi_\infty 1$) in Section 4.

For functions φ satisfying all the conditions (φ0)–(φ5), set

$$\Phi(x,t) = \begin{cases} t^{p(x)}\varphi(x,t), & t > 0, \\ 0, & t = 0. \end{cases}$$

Suppose

($\Phi 1$) $t^{-1}\Phi(x,t)$ is uniformly quasi-increasing on $(0,\infty)$ for fixed $x \in \mathbf{R}^n$.

Here note that if $\Phi(x,t)$ is convex for each $x \in \mathbf{R}^n$, then ($\Phi 1$) holds, that is, $t^{-1}\Phi(x,t)$ in non-decreasing for each $x \in \mathbf{R}^n$. Further, note that if $p_- > 1$, then ($\Phi 1$) is satisfied by (φ). It is useful to note that

$$\bar{\Phi}(x,t) = \int_0^t \left\{ \sup_{0<r\le s} r^{-1}\Phi(x,r) \right\} ds \tag{2.1}$$

is convex and $\bar{\Phi}(x,t/c) \le \Phi(x,t) \le \bar{\Phi}(x,2t)$ for some constant $c > 0$ by ($\Phi 1$). This means that Φ is quasi-convex.

Given Φ as above, let $L^{\Phi}(\mathbf{R}^n)$ denote the set of all measurable functions f such that $\|f\|_{L^{\Phi}(\mathbf{R}^n)} < \infty$, where

$$\|f\|_{L^{\Phi}(\mathbf{R}^n)} = \inf\left\{ \lambda > 0 : \int_{\mathbf{R}^n} \bar{\Phi}(y, |f(y)|/\lambda)\, dy \le 1 \right\}.$$

Then the variable exponent Musielak-Orlicz space $L^{\Phi}(\mathbf{R}^n)$ is complete with respect to the norm $\|f\|_{L^{\Phi}(\mathbf{R}^n)}$ (cf [37]).

Next we define the variable exponent Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(\mathbf{R}^n)$. For a measurable function ν on $\mathbf{R}^n$ satisfying

($\nu 1$) $0 < \nu_- \equiv \inf_{x\in\mathbf{R}^n} \nu(x) \le \nu_+ \equiv \sup_{x\in\mathbf{R}^n} \nu(x) \le n$

and a log-type function $\psi(x,r)$ on $\mathbf{R}^n \times (0,\infty)$, that is, satisfying ($\varphi 0$)–($\varphi 3$), set

$$\kappa(x,r) = r^{\nu(x)}\psi(x,r).$$

Now, given Φ and κ as above, we denote by $L^{\Phi,\kappa}(\mathbf{R}^n)$ the family of all measurable functions f with finite norm

$$\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} = \inf\left\{ \lambda > 0 : \sup_{x\in\mathbf{R}^n, r>0} \kappa(x,r) \fint_{B(x,r)} \bar{\Phi}(y, |f(y)|/\lambda)\, dy \le 1 \right\}.$$

Remark 2.1. The conditions ($\kappa 1$) and ($\nu 1$) are natural. In fact, if $\kappa(x,r)/r^n \to \infty$ for some $x \in \mathbf{R}^n$ as $r \to \infty$ or $1/\kappa(x,r) \to 0$ for all $x \in \mathbf{R}^n$ as $r \to 0$, then $L^{\Phi,\kappa}(\mathbf{R}^n) = \{0\}$.

Remark 2.2. The logarithms contained in the modulus of continuity of the exponent p are natural, because they represent a quite standard assumption on the exponent in order to get the boundedness of the maximal operator (see [13]). Also, the $\log\log$-Hölder condition is natural, e.g. see [30].

3 Maximal functions

For a locally integrable function f on $\mathbf{R}^n$, we consider the maximal function Mf defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x,r)$.

First we prove the boundedness of maximal operator in $L^{\Phi,\kappa}(\mathbf{R}^n)$, which gives an extension of [4, 28, 41].

THEOREM 3.1. *Suppose $p_- > 1$. Then there exists a constant $C > 0$ such that*

$$\|Mf\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le C\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)}.$$

In [28], we studied a bounded sets version of Theorem 3.1 when $\Phi(x,r) = r^{p(x)}(\log(e+r))^{q(x)}$ and $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$.

The conclusion of Theorem 3.1 is equivalent to

$$\|M_{\Phi,\kappa}(Mf)\|_{L^\infty(\mathbf{R}^n)} \le C\|M_{\Phi,\kappa}f\|_{L^\infty(\mathbf{R}^n)}$$

for some constant $C > 0$.

REMARK 3.2. Guliyev-Hasanov-Samko [17, 18] treated the boundedness of the maximal functions on a bounded domain G in the case when $\Phi(x,r) = r^{p(x)}$. In fact, in [17], setting $\omega(x,r) = \kappa(x,r)^{-1/p(x)}$, they assume the condition

$$\int_r^{d_G} \omega(x,t)\frac{dt}{t} \le C\omega(x,r) \qquad \text{for all } x \in G \text{ and } 0 < r < d_G,$$

instead of our log-type conditions posed on ν and ψ.

When $\kappa(x,r) = r^n$, we can prove the following result with a small change, which is an extension of [11, 14, 25, 32].

COROLLARY 3.3. *Suppose $p_- > 1$. Then the operator M is bounded from $L^\Phi(\mathbf{R}^n)$ to itself, that is, there exists a constant $C > 0$ such that*

$$\|Mf\|_{L^\Phi(\mathbf{R}^n)} \le C\|f\|_{L^\Phi(\mathbf{R}^n)}$$

for all $f \in L^\Phi(\mathbf{R}^n)$.

For a proof of Theorem 3.1, we prepare some lemmas.

For a measurable function f on $\mathbf{R}^n$ and a ball $B(x,r)$, let

$$I = I(x,r) = \fint_{B(x,r)} |f(y)|\, dy \quad \text{and} \quad J = J(x,r) = \fint_{B(x,r)} |g(y)|\, dy,$$

where $g(y) = \Phi(y, |f(y)|)$.

Let us begin with the following result.

LEMMA 3.4 (cf. [32, Lemma 2.6]). *There exists a constant $C > 0$ such that*

$$I \le CJ^{1/p(x)}\varphi(x,J)^{-1/p(x)} \tag{3.1}$$

whenever f is a measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$ such that $f \ge 1$ or $f = 0$ on $\mathbf{R}^n$.

Proof. Let f be a nonnegative measurable function f on $\mathbf{R}^n$ with $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$ such that $f \ge 1$ or $f = 0$ on $\mathbf{R}^n$. Then

$$\kappa(x,r)J \le C.$$

First suppose $J \ge 1$; then $\kappa(x,r) \le CJ^{-1} \le C$, so that r is (uniformly) bounded by (φ) and ($\nu1$). Set $k = cJ^{1/p(x)}\varphi(x,J)^{-1/p(x)}$, where c is chosen such that $k \ge 1$. Then we see from (φ) and (P1) that

$$k \le C\kappa(x,r)^{-1/p(x)}\varphi(x,1/\kappa(x,r))^{-1/p(x)} \le Cr^{-a}$$

for some $a > \nu_+/p_-$. Moreover, if $y \in B(x,r)$, then note from (P2) that

$$k^{|p(x)-p(y)|} \le k^{C/\log(e+1/|x-y|)} \le (Cr^{-a})^{C/\log(e+1/r)} \le C,$$

so that $k^{p(x)} \le Ck^{p(y)}$; and by ($\varphi3$) and ($\varphi4$)

$$\frac{\varphi(x,k)}{\varphi(y,k)} \le C\frac{\varphi(x,k^{1/a})}{\varphi(y,k^{1/a})} \le C,$$

since $|x-y| < r \le Ck^{-1/a}$. Hence $\Phi(x,k) \le C\Phi(y,k)$ for $y \in B(x,r)$, so that it follows from ($\Phi1$) that

$$\begin{aligned} I &\le k + C\fint_{B(x,r)} f(y)\left(\frac{f(y)^{-1}\Phi(y,f(y))}{k^{-1}\Phi(y,k)}\right)dy \\ &\le k + Ck\Phi(x,k)^{-1}\fint_{B(x,r)} g(y)dy \\ &= k + Ck\Phi(x,k)^{-1}J \\ &\le Ck \end{aligned}$$

since $\Phi(x,k)^{-1} \le J^{-1}$ by (φ) and (P1).

If $J \le 1$, then, since $f \ge 1$ or $f = 0$, we have by ($\Phi1$) and ($\varphi1$)

$$\begin{aligned} I &\le C\fint_{B(x,r)} g(y)dy \\ &= CJ \\ &\le CJ^{1/p(x)}\varphi(x,J)^{-1/p(x)}, \end{aligned}$$

as required. □

Lemma 3.5. *There exists a constant $C > 0$ such that*

$$I \le C\left\{J^{1/p(x)}\varphi(x,J)^{-1/p(x)} + (1+|x|)^{-n}\right\} \tag{3.2}$$

for all measurable functions f on $\mathbf{R}^n$ such that $0 \le f < 1$ on $\mathbf{R}^n$.

Proof. Let f be a measurable function on $\mathbf{R}^n$ such that $0 \le f < 1$ on $\mathbf{R}^n$. Write

$$\begin{aligned} f &= f\chi_{\{y\in B(x,r)\cap B(0,|x|/2):(1+|y|)^{-n-1}\le f(y)<1\}} + f\chi_{\{y\in B(x,r)\cap B(0,|x|/2):0\le f(y)<(1+|y|)^{-n-1}\}} \\ &\quad + f\chi_{\{y\in B(x,r)\setminus B(0,|x|/2):(1+|x|/2)^{-n-1}\le f(y)<1\}} + f\chi_{\{y\in B(x,r)\setminus B(0,|x|/2):0\le f(y)<(1+|x|/2)^{-n-1}\}} \\ &= f_1+f_2+f_3+f_4, \end{aligned}$$

where χ_E denotes the characteristic function of E. Write

$$I = I_1 + I_2 + I_3 + I_4,$$

where $I_j = \fint_{B(x,r)} f_j(y)dy$ for $j = 1,2,3,4$.

It is easy to see that $I_4 \le (1+|x|)^{-n}$.

To estimate I_1–I_3, set $k = J^{1/p(x)}\varphi(x,J)^{-1/p(x)}$.

Case 1 : $(1+|y|)^{-n-1} \le f(y) < 1$ or $f(y) = 0$ for $y \in B(0,|x|/2)$.

In this case, as in the proof of Lemma 3.4, note from (P3), (φ3) and (φ5) that

$$f(y)^{-|p(x)-p(y)|} \le f(y)^{-C/\log(e+|y|)} \le (1+|y|)^{C/\log(e+|y|)} \le C$$

and

$$\frac{\varphi(x,f(y))}{\varphi(y,f(y))} \le C\frac{\varphi(x,f(y)^{1/(n+1)})}{\varphi(y,f(y)^{1/(n+1)})} \le C$$

for $y \in B(0,|x|/2)$, so that $\Phi(x,f(y)) \le C\Phi(y,f(y))$ for $y \in B(0,|x|/2)$. Hence we have by (φ), (Φ1) and (P1)

$$\begin{aligned} I_1 &\le Cr^{-n}\int_{B(x,r)\cap B(0,|x|/2)} f(y)dy \\ &\le C\left\{k + r^{-n}\int_{B(x,r)\cap B(0,|x|/2)} f(y)\left(\frac{f(y)^{-1}\Phi(x,f(y))}{k^{-1}\Phi(x,k)}\right)dy\right\} \\ &\le C\left\{k + k\Phi(x,k)^{-1}\fint_{B(x,r)} g(y)dy\right\} \\ &= C\left\{k + k\Phi(x,k)^{-1}J\right\} \\ &\le CJ^{1/p(x)}\varphi(x,J)^{-1/p(x)}. \end{aligned}$$

Case 2 : $(1+|x|)^{-n-1} \le f(y) < 1$ or $f(y) = 0$ for $y \in B(x,r)\setminus B(0,|x|/2)$.

This case is treated in the same manner as Case 1.

Case 3 : $0 \le f(y) < (1+|y|)^{-n-1}$ for $y \in B(0,|x|/2)$.

If $0 < r \le |x|/2$, then

$$I_2 \le C(1+|x|)^{-n-1},$$

since $|x| \sim |y|$. If $r > |x|/2$, then

$$I_2 \le Cr^{-n} \int_{B(0,3r)} f(y)dy \le C(1+r)^{-n} \le C(1+|x|)^{-n},$$

so that

$$I_2 \le C(1+|x|)^{-n}.$$

Now we obtain

$$I \le C\left\{J^{1/p(x)}\varphi(x,J)^{-1/p(x)} + (1+|x|)^{-n}\right\}$$

and thus the proof is completed. □

By Lemmas 3.4 and 3.5, we have

COROLLARY 3.6. *There exists a constant $C > 0$ such that*

$$I \le C\left\{J^{1/p(x)}\varphi(x,J)^{-1/p(x)} + (1+|x|)^{-n}\right\},$$

whenever f is a measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

LEMMA 3.7. *Suppose $p_- > 1$, and take p_0 such that $1 < p_0 < p_-$. Then there exists a constant $C > 0$ such that*

$$\Phi(x, Mf(x)) \le C\left[\{Mg_0(x)\}^{p_0} + (1+|x|)^{-A}\right],$$

whenever f is a measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$, where $g_0(y) = \Phi_0(y,|f(y)|) = \Phi(y,|f(y)|)^{1/p_0}$ and $A = np_0$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$. Write

$$f = f\chi_{\{y: f(y)\ge 1\}} + f\chi_{\{y: 0\le f(y)<1\}} = f_1 + f_2.$$

Let $1 < p_0 < p_-$. Since we see from (φ) and (φ1)

$$\fint_{B(x,r)} \Phi(y, f_1(y))^{1/p_0} dy \le C \fint_{B(x,r)} \Phi(y, f(y))dy \le C\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^n$ and $r > 0$, applying Lemma 3.4 with $p(x)$ and $\varphi(x,r)$ replaced by $p(x)/p_0$ and $\varphi(x,r)^{1/p_0}$, respectively, we obtain

$$\Phi(x, Mf_1(x)) \le C\{Mg_1(x)\}^{p_0},$$

where $g_1(y) = \Phi(y, f_1(y))^{1/p_0}$. Moreover, in view of Lemma 3.5, we have

$$\Phi(x, Mf_2(x)) \le C\left[\{Mg_2(x)\}^{p_0} + (1+|x|)^{-A}\right]$$

for $A = np_0$, where $g_2(y) = \Phi(y, f_2(y))^{1/p_0}$. Thus it follows that

$$\Phi(x, Mf(x)) \le C\left[\{Mg_0(x)\}^{p_0} + (1+|x|)^{-A}\right],$$

where $g_0(y) = \Phi(y, f(y))^{1/p_0}$, as required. □

We need the following result in the constant case (see [38, Theorem 1]); for this, note from (ν1) and (φ) that

$$\int_r^\infty \kappa(x,t)^{-1}t^{-1}dt \le C\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^n$ and $r > 0$.

LEMMA 3.8. *Suppose $p_0 > 1$. Let f be a measurable function on $\mathbf{R}^n$ satisfying*

$$⨍_{B(x,r)} |f(y)|^{p_0}\, dy \le \kappa(x,r)^{-1} \tag{3.3}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Then there exists a constant $C > 0$ such that

$$⨍_{B(z,r)} \{Mf(x)\}^{p_0}\, dx \le C\kappa(z,r)^{-1}$$

for all $z \in \mathbf{R}^n$ and $r > 0$, where the constant C is independent of f satisfying (3.3).

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$. Since $\kappa(x,r)^{-1} \to \infty$ uniformly as $r \to 0$ by (φ) and (ν1), for $z \in \mathbf{R}^n$ and $r > 0$, by Lemmas 3.7 and 3.8 and (κ1), we find

$$\begin{aligned} ⨍_{B(z,r)} \Phi(x, Mf(x))\, dx &\le C ⨍_{B(z,r)} \left[\{Mg_0(x)\}^{p_0} + (1+|x|)^{-A}\right] dx \\ &\le C\{\kappa(z,r)^{-1} + (1+r)^{-n}\} \\ &\le C\kappa(z,r)^{-1}, \end{aligned}$$

where $g_0(y) = \Phi(y, f(y))^{1/p_0}$ and $A = np_0$ as in Lemma 3.7. □

4 Sobolev's inequality

For a measurable function $\alpha : \mathbf{R}^n \to (0, n)$ satisfying

(α1) $0 < \alpha_- \equiv \operatorname{ess\,inf}_{x\in\mathbf{R}^n} \alpha(x) \le \operatorname{ess\,sup}_{x\in\mathbf{R}^n} \alpha(x) \equiv \alpha_+ < n$,

we define the Riesz potential of variable order α for a locally integrable function f on $\mathbf{R}^n$ by

$$I_{\alpha(x)} f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha(x)-n} f(y)dy.$$

As an application of the boundedness of maximal functions in $L^{\Phi,\kappa}(\mathbf{R}^n)$ and Hedberg's trick [23], we are going to establish the Morrey version of Sobolev's type inequality for Riesz potentials $I_{\alpha(x)} f$ of functions $f \in L^{\Phi,\kappa}(\mathbf{R}^n)$. For this purpose, we need an auxiliary function $\varphi_\infty : (0,\infty) \to (0,\infty)$ of log-type for which there exists a constant $c_\infty > 1$ such that

(φ_∞1) $\dfrac{1}{c_\infty} \le \dfrac{\varphi(x,(1+|x|)^{-1})}{\varphi_\infty((1+|x|)^{-1})} \le c_\infty$ for all $x \in \mathbf{R}^n$;

one may take $\varphi_\infty(t) = \limsup_{|y|\to\infty} \varphi(y,t)$ by (φ5).

For $\kappa(x,r) = r^{\nu(x)}\psi(x,r)$, assume further that

(ν2) ν is log-Hölder continuous at ∞, that is, there exists $\nu(\infty)$ such that

$$|\nu(x) - \nu(\infty)| \le \frac{C}{\log(e+|x|)} \quad \text{for all } x \in \mathbf{R}^n.$$

Further we need

($\nu\alpha$1) $\displaystyle\operatorname*{ess\,inf}_{x\in\mathbf{R}^n} (\nu(x)/p(x) - \alpha(x)) > 0$;

($\nu\alpha$2) $\displaystyle\operatorname*{ess\,inf}_{x\in\mathbf{R}^n\setminus B(0,1)} (\nu(x)/p(\infty) - \alpha(x)) > 0$.

We consider the Sobolev exponent

$$1/p^*(x) = 1/p(x) - \alpha(x)/\nu(x) \tag{4.1}$$

and the modular

$$\Psi(x,r) = \{r\varphi(x,r)^{1/p(x)}\psi(x,1/r)^{\alpha(x)/\nu(x)}\}^{p^*(x)}. \tag{4.2}$$

THEOREM 4.1. *Suppose $p_- > 1$. Then there exists a constant $C > 0$ such that*

$$⨍_{B(z,r)} \Psi(x, |I_{\alpha(x)} f(x)|)\, dx \le C\kappa(z,r)^{-1}$$

whenever $z \in \mathbf{R}^n$, $r > 0$ and $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

As in Remark 3.2, Theorem 4.1 was treated on bounded open sets by Guliyev-Hasanov-Samko [17, 18] in the case when $\Phi(x,r) = r^{p(x)}$. For the case when $\Phi(x,r) = r^{p(x)}(\log(e+r))^{q(x)}$ and $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$, see [28]. For related results, see also [3, 4, 8, 33, 35].

Corollary 4.2. *Let $\Phi(r) = r^p(\log(e+r))^q$ and $\kappa(r) = r^\nu(\log(e+1/r))^\beta$, where $1 < p < \infty, -\infty < q < \infty, 0 < \nu \le n$ (if $\nu = n$, then $\beta = 0$). Set $1/p^* = 1/p - \alpha/\nu$. Then, for $0 < \alpha < \nu/p$, there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(|I_\alpha f(x)|)\, dx \le C\kappa(r)^{-1}$$

whenever $z \in \mathbf{R}^n$, $r > 0$ and $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$, where

$$\Psi(r) = \{r(\log(e+r))^{q/p+\beta\alpha/\nu}\}^{p^*}.$$

Remark 4.3. Condition $(\nu\alpha 2)$ is needed, as was pointed out by Hästö [22].

For the proof of Theorem 4.1, let us begin with the following technical lemma.

Lemma 4.4 (cf. [28, Lemma 2.7]). *Let τ be a real number and let $\lambda : (0,\infty) \to (0,\infty)$ satisfy the doubling condition. Suppose h is a nonnegative measurable function on $\mathbf{R}^n$ such that*

$$\int_{B(0,r)} h(y)dy \le \lambda(r)$$

for all $r > 0$. Then there exist a constant $C > 0$ such that

$$\int_{B(0,r_2)\setminus B(0,r_1)} |y|^{-\tau} h(y)dy \le C \int_{r_1}^{2r_2} t^{-\tau}\lambda(t)\frac{dt}{t}$$

whenever $0 < r_1 \le r_2 < \infty$.

Lemma 4.5. *Let $0 < \sigma < np(\infty)/\nu(\infty)$. Then there exists a constant $C > 0$ such that*

$$\fint_{B(x,r)} f(y)dy \le C\kappa(x,r)^{-1/p(x)}\varphi(x,1/r)^{-1/p(x)}$$

for all $x \in \mathbf{R}^n$, $0 < r < 2(1+|x|)^\sigma$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$. Then we have

$$J = \fint_{B(x,r)} \Phi(y, f(y))dy \le C\kappa(x,r)^{-1}$$

for all $r > 0$. Noting from (P3') and $(\nu 2)$ that

$$(1+r)^{p(x)/\nu(x)} \sim (1+r)^{p(\infty)/\nu(\infty)}$$

for $0 < r < 2(1+|x|)^\sigma$, we see from Corollary 3.6, (φ), (P1) and (ν1) that

$$\begin{aligned} \fint_{B(x,r)} f(y)dy &\le C\left\{J^{1/p(x)}\varphi(x,J)^{-1/p(x)} + (1+|x|)^{-n}\right\} \\ &\le C\left\{\kappa(x,r)^{-1/p(x)}\varphi(x,1/\kappa(x,r))^{-1/p(x)} + (1+r)^{-n/\sigma}\right\} \\ &\le C\kappa(x,r)^{-1/p(x)}\varphi(x,1/r)^{-1/p(x)}, \end{aligned}$$

as required. □

LEMMA 4.6. *For σ as in Lemma 4.5, there exists a constant $C > 0$ such that*

$$\int_{B(x,(1+|x|)^\sigma)\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)dy \le C\delta^{\alpha(x)}\kappa(x,\delta)^{-1/p(x)}\varphi(x,\delta^{-1})^{-1/p(x)}$$

for all $x \in \mathbf{R}^n$, $\delta > 0$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

Proof. Let

$$\eta = \operatorname*{ess\,inf}_{x\in\mathbf{R}^n}(\nu(x)/p(x) - \alpha(x)).$$

Then $\eta > 0$ by ($\nu\alpha$1). By Lemmas 4.4 and 4.5, (P1) and (φ), we have for all $x \in \mathbf{R}^n$ and $\delta > 0$

$$\begin{aligned} &\int_{B(x,(1+|x|)^\sigma)\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\,dy \\ \le\ & C\int_\delta^{2(1+|x|)^\sigma} t^{\alpha(x)}\kappa(x,t)^{-1/p(x)}\varphi(x,1/t)^{-1/p(x)}\,\frac{dt}{t} \\ =\ & C\int_\delta^{2(1+|x|)^\sigma} t^{\alpha(x)-\nu(x)/p(x)+\eta/2}\psi(x,t)^{-1/p(x)}\varphi(x,1/t)^{-1/p(x)}t^{-\eta/2}\,\frac{dt}{t} \\ \le\ & C\delta^{\alpha(x)}\kappa(x,\delta)^{-1/p(x)}\varphi(x,1/\delta)^{-1/p(x)}, \end{aligned}$$

which completes the proof. □

LEMMA 4.7. *For $\sigma \ge 1$, there exists a constant $C > 0$ such that*

$$\fint_{B(x,r)} f(y)dy \le C\kappa(x,r)^{-1/p(\infty)}\varphi_\infty(1/r)^{-1/p(\infty)}$$

for all $x \in \mathbf{R}^n$, $r \ge (1+|x|)^\sigma$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$. Letting

$$k(y) = \kappa(x,1+|y|)^{-1/p(\infty)}\varphi_\infty(1/(1+|y|))^{-1/p(\infty)},$$

we have by (Φ1)

$$\int_{B(x,r)} f(y)dy \le \int_{B(0,2r)} k(y)dy + C\int_{B(x,r)} f(y)\left(\frac{f(y)^{-1}\Phi(y,f(y))}{k(y)^{-1}\Phi(y,k(y))}\right)dy$$

for $r \ge (1+|x|)^\sigma$. Since $k(y)$ is bounded by (φ), (P1) and $(\nu 1)$, we obtain by (φ), (P1), (P3$'$) and $(\nu 1)$

$$k(y)^{-|p(\infty)-p(y)|} \le Ck(y)^{-C/\log(e+|y|)} \le C$$

and

$$\frac{\varphi_\infty(k(y))}{\varphi(y,k(y))} \le C\frac{\varphi_\infty((1+|y|)^{-1})}{\varphi(y,(1+|y|)^{-1})} \le C,$$

so that

$$\Phi_\infty(k(y)) \le C\Phi(y,k(y))$$

for $y \in B(x,r)$, where $\Phi_\infty(t) = t^{p(\infty)}\varphi_\infty(t)$. Hence we find by (P1) and (φ)

$$\int_{B(0,2r)} k(y)dy \le Cr^n\kappa(x,r)^{-1/p(\infty)}\varphi_\infty(1/r)^{-1/p(\infty)}$$

since $\nu(x)/p(\infty) \le \nu^+/p(\infty) < n$, and

$$\begin{aligned}\int_{B(x,r)} f(y)dy &\le C\Big\{r^n\kappa(x,r)^{-1/p(\infty)}\varphi_\infty(1/r)^{-1/p(\infty)} \\ &\quad + \kappa(x,r)^{(p(\infty)-1)/p(\infty)}\varphi_\infty(1/r)^{-1/p(\infty)}\int_{B(x,r)}\Phi(y,f(y))\,dy\Big\} \\ &\le Cr^n\kappa(x,r)^{-1/p(\infty)}\varphi_\infty(1/r)^{-1/p(\infty)}\end{aligned}$$

by (φ), (P1), $(\nu 1)$ and $1+|y| \le 3r$, as required. □

LEMMA 4.8. *For $\sigma \ge 1$, there exists a constant $C>0$ such that*

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus B(x,(1+|x|)^\sigma)} |x-y|^{\alpha(x)-n}f(y)dy \\ &\le C(1+|x|)^{\sigma\alpha(x)}\kappa(x,(1+|x|)^\sigma)^{-1/p(\infty)}\varphi(x,(1+|x|)^{-1})^{-1/p(\infty)}\end{aligned}$$

for all $x \in \mathbf{R}^n$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

Proof. Since

$$(1+|x|)^{1/p(x)} \sim (1+|x|)^{1/p(\infty)}$$

by (P3), it follows from Lemmas 4.4 and 4.7 (P1), (φ) and $(\nu\alpha 2)$ that

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus B(x,(1+|x|)^\sigma)} |x-y|^{\alpha(x)-n}f(y)dy \\ &\le C\int_{(1+|x|)^\sigma}^\infty t^{\alpha(x)}\kappa(x,t)^{-1/p(\infty)}\varphi_\infty(1/t)^{-1/p(\infty)}\frac{dt}{t} \\ &\le C(1+|x|)^{\sigma\alpha(x)}\kappa(x,(1+|x|)^\sigma)^{-1/p(\infty)}\varphi_\infty((1+|x|)^{-\sigma})^{-1/p(\infty)}.\end{aligned}$$

□

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We may assume that $f \geq 0$. Take σ, ς and ς' such that $n/\nu(\infty) < \varsigma' < \varsigma < \sigma < np(\infty)/\nu(\infty)$. For $\delta > 0$, write

$$\begin{aligned} I_{\alpha(x)} f(x) &\leq \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)dy + \int_{B(x,(1+|x|)^\sigma)\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)dy \\ &\quad + \int_{\mathbf{R}^n \setminus B(x,(1+|x|)^\sigma)} |x-y|^{\alpha(x)-n} f(y)dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Note that

$$I_1 \leq \sum_{j=0}^{\infty} (2^{-j-1}\delta)^{\alpha(x)-n} \int_{B(x,2^{-j}\delta)\setminus B(x,2^{-j-1}\delta)} f(y)dy \leq C\delta^{\alpha(x)} Mf(x).$$

Moreover, Lemmas 4.6 and 4.8 yield

$$I_2 \leq C\delta^{\alpha(x)} \kappa(x,\delta)^{-1/p(x)} \varphi(x,1/\delta)^{-1/p(x)}$$

and

$$I_3 \leq C(1+|x|)^{-\varsigma\nu(x)/p^*(x)}$$

by (φ), (P1) and $(\nu1)$.

Hence we find

$$I_{\alpha(x)} f(x) \leq C \left\{ \delta^{\alpha(x)} Mf(x) + \delta^{\alpha(x)} \kappa(x,\delta)^{-1/p(x)} \varphi(x,1/\delta)^{-1/p(x)} + (1+|x|)^{-\varsigma\nu(x)/p^*(x)} \right\}.$$

Now, letting

$$\delta = \{Mf(x)\}^{-p(x)/\nu(x)} \psi(x, Mf(x))^{-1/\nu(x)} \varphi(x, 1/Mf(x))^{-1/\nu(x)},$$

we obtain from (φ), (P1) and $(\nu1)$

$$\begin{aligned} I_{\alpha(x)} f(x) &\leq C\Big[\{Mf(x)\}^{p(x)/p^*(x)} \psi(x, 1/Mf(x))^{-\alpha(x)/\nu(x)} \varphi(x, Mf(x))^{-\alpha(x)/\nu(x)} \\ &\quad + (1+|x|)^{-\varsigma\nu(x)/p^*(x)} \Big], \end{aligned}$$

so that we have by (φ), (P1), $(\nu1)$ and $(\nu\alpha1)$

$$\Psi(x, I_{\alpha(x)} f(x)) \leq C \left\{ \Phi(x, Mf(x)) + (1+|x|)^{-\varsigma'\nu(x)} \right\}.$$

Here note from (P3$'$) that

$$\begin{aligned} ⨍_{B(z,r)} (1+|x|)^{-\varsigma'\nu(x)}\, dx &\leq C ⨍_{B(z,r)} (1+|x|)^{-\varsigma'\nu(\infty)}\, dx \\ &\leq C\kappa(z,r)^{-1}, \end{aligned}$$

since $\varsigma'\nu(\infty) > n$ and (κ1). Hence it follows from Theorem 3.1 that

$$\begin{aligned} ⨍_{B(z,r)} \Psi(x, I_{\alpha(x)}f(x))\,dx &\le C\left\{ ⨍_{B(z,r)} \Phi(x, Mf(x))\,dx + ⨍_{B(z,r)} (1+|x|)^{-\varsigma'\nu(x)}\,dx \right\} \\ &\le C\kappa(z,r)^{-1} \end{aligned}$$

for $z \in \mathbf{R}^n$ and $r > 0$, which completes the proof of Theorem 4.1. □

5 Sobolev's inequality in the case $p_- = 1$

This section is concerned with Sobolev's inequality when $p_- = 1$. For this purpose, we further need the following technical conditions:

($\kappa\varphi$1) $\kappa(x,r)\varphi(x,1/r) \le Cr^n$ for $x \in \mathbf{R}^n$ and $r > 1$;

(φ_∞2) $r^{-1}\Phi_\infty(r)$ is quasi-increasing on $(0,\infty)$, where $\Phi_\infty(r) = r^{p(\infty)}\varphi_\infty(r)$;

($\kappa\gamma$1) there is $\gamma > 1$ such that

$$⨍_{B(z,r)} \kappa(x,(1+|x|))^{-1}(\log(e+|x|))^{-\gamma}\,dx \le C\kappa(z,r)^{-1}$$

for $z \in \mathbf{R}^n$ and $r > 0$.

THEOREM 5.1. *For γ as in ($\kappa\gamma$1), there exists a constant $C > 0$ such that*

$$⨍_{B(z,r)} \Psi(x, |I_{\alpha(x)}f(x)|)(\log(e + |I_{\alpha(x)}f(x)| + |I_{\alpha(x)}f(x)|^{-1}))^{-\gamma}\,dx \le C\kappa(z,r)^{-1}$$

whenever $z \in \mathbf{R}^n$, $r > 0$ and $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

For a proof of Theorem 5.1, we need the following result.

LEMMA 5.2. *Let $\gamma > 1$. Then there exists a constant $C > 0$ such that*

$$⨍_{B(z,r)} Mg(x)(\log(e + Mg(x) + Mg(x)^{-1}))^{-\gamma}\,dx \le C\kappa(z,r)^{-1}$$

for all $z \in \mathbf{R}^n$, $r > 0$ and $g \ge 0$ satisfying

$$⨍_{B(z,r)} g(y)\,dy \le \kappa(z,r)^{-1}$$

for all $z \in \mathbf{R}^n$ and $r > 0$.

Proof. For $1 < \gamma \le 2$, we see that $t(\log(\gamma + t + 1/t))^{-\gamma}$ is increasing on $(0, \infty)$ and

$$t(\log(e + t + 1/t))^{-\gamma} \le C(\gamma) t(\log(\gamma + t + t^{-1}))^{-\gamma}.$$

Let $z \in \mathbf{R}^n$ and $r > 0$. We set

$$g = g_0 + g_1, \quad g_0 = g\chi_{B(z,2r)}.$$

Let

$$I_j = \int_{B(z,r)} Mg_j(x)(\log(\gamma + Mg_j(x) + Mg_j(x)^{-1}))^{-\gamma}\, dx, \quad j = 0, 1.$$

Then

$$\int_{B(z,r)} Mg(x)(\log(\gamma + Mg(x) + Mg(x)^{-1}))^{-\gamma}\, dx \le C(I_0 + I_1).$$

We have

$$I_0 = \int_0^\infty \lambda(t)\, d(t(\log(\gamma + t + t^{-1}))^{-\gamma}),$$

where $\lambda(t) = |\{x \in B(z, 2r) : Mg_0(x) > t\}|$. Here we note from [47, Theorem 1, Chapter 1] that

$$\lambda(t) \le Ct^{-1} \int_{\{x \in B(z,2r) : g_0(x) > t/2\}} g_0(x)\, dx$$

for $t > 0$. Using Fubini's theorem, we obtain

$$\begin{aligned} I_0 &= \int_0^\infty \lambda(t)\, d(t(\log(\gamma + t + t^{-1}))^{-\gamma}) \\ &\le C \int_{B(z,2r)} g_0(x) \left\{ \int_0^{2g_0(x)} t^{-1} d(t(\log(\gamma + t + t^{-1}))^{-\gamma}) \right\} dx \\ &\le C \int_{B(z,2r)} g_0(x)\, dx = C \int_{B(z,2r)} g(x)\, dx \\ &\le Cr^n \kappa(z, r)^{-1}. \end{aligned}$$

Next we see from (φ) that for $x \in B(z, r)$

$$Mg_1(x) \le C \sup_{t \ge r} t^{-n} \int_{B(z,2t)} g(y)\, dy \le C \sup_{t \ge r} \kappa(z, t)^{-1} \le C\kappa(z, r)^{-1},$$

so that

$$I_1 \le \int_{B(z,r)} Mg_1(x)\, dx \le Cr^n \kappa(z, r)^{-1}.$$

Thus we obtain

$$⨍_{B(z,r)} Mg(x)(\log(e + Mg(x) + Mg(x)^{-1}))^{-\gamma}\, dx \le C\kappa(z, r)^{-1},$$

which proves the lemma. □

REMARK 5.3. Theorem 5.1 and Lemma 5.2 don't hold in case $\gamma = 1$ (see [29, Remark 2]).

In fact, consider $\Phi(x,r) = r^{p(x)}, \kappa(r) = r^n, 0 < \alpha < n$ and p is an exponent of the form

$$p(x) = 1 + c/\log(e + |x|)$$

with $c > 0$. If $f = 1$ on $B(0,1)$ and $f = 0$ elsewhere, then

$$\int_{\mathbf{R}^n \setminus B(0,2)} Mf(x)^{p(x)} (\log(e + Mf(x)^{-1}))^{-1} dx = \infty$$

and

$$\int_{\mathbf{R}^n \setminus B(0,2)} I_\alpha f(x)^{p^*(x)} (\log(e + I_\alpha f(x)^{-1}))^{-1} dx = \infty,$$

where $1/p^*(x) = 1/p(x) - \alpha/n$.

To treat the case $p_- = 1$, we modify some lemmas in Section 4.

LEMMA 5.4. *There exists a constant $C > 0$ such that*

$$\fint_{B(x,r)} f(y) dy \le C\kappa(x,r)^{-1/p(x)} \varphi(x, 1/r)^{-1/p(x)}$$

for all $x \in \mathbf{R}^n$, $0 < r < 2(1 + |x|)$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

This is proved in the same manner as Lemma 4.5, by using condition ($\kappa\varphi 1$).

Lemma 5.4 yields the following result in the same manner as Lemma 4.6.

LEMMA 5.5. *There exists a constant $C > 0$ such that*

$$\int_{B(x,1+|x|)\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) dy \quad \le \quad C\delta^{\alpha(x)} \kappa(x,\delta)^{-1/p(x)} \varphi(x, \delta^{-1})^{-1/p(x)}$$

for all $x \in \mathbf{R}^n$, $\delta > 0$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

We next establish the following result.

LEMMA 5.6. *There exists a constant $C > 0$ such that*

$$\fint_{B(x,r)} f(y) dy \le C\kappa(x,r)^{-1/p(\infty)} \varphi_\infty(1/r)^{-1/p(\infty)}$$

for all $x \in \mathbf{R}^n$, $r \ge 1 + |x|$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)} \le 1$.

To show this, in the proof of Lemma 4.7, we replace k by $k(y) = (1 + |y|)^{-\lambda}$ with $\lambda > n$, and then use ($\kappa\varphi 1$) and ($\varphi_\infty 2$).

Finally we prepare the following lemma.

LEMMA 5.7. *There exists a constant $C>0$ such that*

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus B(x,1+|x|)}|x-y|^{\alpha(x)-n}f(y)dy\\ \le\ & C(1+|x|)^{\alpha(x)}\kappa(x,1+|x|)^{-1/p(\infty)}\varphi((1+|x|)^{-1})^{-1/p(\infty)}\end{aligned}$$

for all $x\in\mathbf{R}^n$ and $f\ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(\mathbf{R}^n)}\le 1$.

In fact, if $p(\infty)=1$, then we see from the proof of Lemma 4.8 that

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus B(x,1+|x|)}|x-y|^{\alpha(x)-n}f(y)dy\\ \le\ & C\int_{(1+|x|)}^{\infty}t^{\alpha(x)}\kappa(x,t)^{-1}\varphi_\infty(1/t)^{-1}\frac{dt}{t}\\ \le\ & C(1+|x|)^{\alpha(x)}\kappa(x,(1+|x|))^{-1}\varphi_\infty((1+|x|)^{-1})^{-1}\end{aligned}$$

by Lemma 5.6.

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. As in the proof of Theorem 4.1, we have by Lemmas 5.5 and 5.7

$$\begin{aligned}I_{\alpha(x)}f(x)\ \le\ & C\big[\{Mf(x)\}^{p(x)/p^*(x)}\varphi(x,Mf(x))^{-\alpha(x)/\nu(x)}\psi(x,1/Mf(x))^{-\alpha(x)/\nu(x)}\\ &+(1+|x|)^{\alpha(x)}\kappa(x,1+|x|)^{-1/p(x)}\varphi(x,(1+|x|)^{-1})^{-1/p(x)}\big],\end{aligned}$$

so that

$$\begin{aligned}&\Psi(x,I_{\alpha(x)}f(x))(\log(e+I_{\alpha(x)}f(x)+I_{\alpha(x)}f(x)^{-1}))^{-\gamma}\\ \le\ & C\left\{\Phi(x,Mf(x))(\log(e+Mf(x)+Mf(x)^{-1}))^{-\gamma}+\kappa(x,1+|x|)^{-1}(\log(e+|x|))^{-\gamma}\right\}\\ \le\ & C\left\{Mg(x)(\log(e+Mg(x)+Mg(x)^{-1}))^{-\gamma}+\kappa(x,1+|x|)^{-1}(\log(e+|x|))^{-\gamma}\right\}\end{aligned}$$

by Corollary 3.6 and ($\kappa\varphi 1$), where $g(y)=\Phi(y,f(y))$. Hence, we obtain by Lemma 5.2 and ($\kappa\gamma 1$)

$$\begin{aligned}&\fint_{B(z,r)}\Psi(x,I_{\alpha(x)}f(x))(\log(e+I_{\alpha(x)}f(x)+I_{\alpha(x)}f(x)^{-1}))^{-\gamma}\,dx\\ \le\ & C\left\{\fint_{B(z,r)}Mg(x)(\log(e+Mg(x)+Mg(x)^{-1}))^{-\gamma}\,dx\right.\\ &\left.+\fint_{B(z,r)}\kappa(x,1+|x|)^{-1}(\log(e+|x|))^{-\gamma}\,dx\right\}\\ \le\ & C\kappa(z,r)^{-1}\end{aligned}$$

for $z\in\mathbf{R}^n$ and $r>0$, which completes the proof of Theorem 5.1. □

Acknowledgement We would like to express our thanks to the referees for their kind comments and suggestions.

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