Sobolev embeddings for Riesz potential spaces of variable exponents near 1 and Sobolev's exponent

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Abstract

Our aim in this paper is to deal with Sobolev's embeddings for Riesz potentials of order α for functions f satisfying the Orlicz type condition

$$\int |f(y)|^{p(y)}(\log(c+|f(y)|))^{q(y)}dy < \infty,$$

where $p(\cdot)$ and $q(\cdot)$ are variable exponents satisfying the log-Hölder conditions.

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1 Introduction

The Sobolev space is a useful tool of the study for the existence and regularity of solutions of partial differential equations. The famous Sobolev inequality says that the Riesz potential $I_\alpha f$ of order α with $f \in L^p(\mathbf{R}^n)$ belongs to $L^{p^*}(\mathbf{R}^n)$ when $1 < p < \infty$ and $1/p^* = 1/p - \alpha/n > 0$. If f satisfies the Orlicz condition

$$\int |f(y)|^p(\log(e+|f(y)|))^q dy < \infty,$$

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then it is known (see e.g. Cianchi [1]) that

$$\int \left(|I_\alpha f(x)|(\log(e+|f(y)|))^{q/p}\right)^{p^*} dx < \infty.$$

When $p = 1$, the situation is a little different (see O'Neil [13]).

In the present paper, we aim to establish Sobolev's inequality for Riesz potentials of functions in the Orlicz spaces of variable exponent. In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see for example Orlicz [14], Kováčik-Rákosník [9], Edmunds-Rákosník [4] and Růžička [15]. In the limiting case we are also concerned with exponential integrabilities of Trudinger type and continuity for Riesz potentials.

For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^n$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y) dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^n} (1+|y|)^{\alpha-n}|f(y)|dy < \infty \tag{1.1}$$

(see [10, Theorem 1.1, Chapter 2]).

In this paper, following Cruz-Uribe and Fiorenza [2], we consider variable exponents $p(\cdot)$ and $q(\cdot)$ such that

$$|p(x)-p(y)| \le \frac{a}{\log(e+1/|x-y|)} \qquad \text{for all } x, y \in \mathbf{R}^n, \tag{1.2}$$

$$|q(x)-q(y)| \le \frac{b}{\log(e+\log(e+1/|x-y|))} \qquad \text{for all } x, y \in \mathbf{R}^n, \tag{1.3}$$

$$1 \le p^- \equiv \inf_{x\in\mathbf{R}^n} p(x) \le \sup_{x\in\mathbf{R}^n} p(x) \equiv p^+ < \infty \tag{1.4}$$

and

$$-\infty < q^- \equiv \inf_{x\in\mathbf{R}^n} q(x) \le \sup_{x\in\mathbf{R}^n} q(x) \equiv q^+ < \infty \tag{1.5}$$

for $a, b > 0$. Moreover, suppose there exist $\varepsilon_0 > 0$ and $C > 0$ such that

$$s^{p(x)-1}(\log(e+s))^{q(x)-\varepsilon_0} \le Ct^{p(x)-1}(\log(e+t))^{q(x)-\varepsilon_0} \tag{1.6}$$

whenever $0 < s < t$ and $x \in \mathbf{R}^n$. This is true if

$$K(p(x)-1)+q(x) \ge \varepsilon_0$$

for some positive constant K.

Let G be a bounded Borel set in $\mathbf{R}^n$. Let $\Phi(x,t)$ be a nonnegative function on $G \times \mathbf{R}$ such that

(1) $\Phi(\cdot, t)$ is measurable for each t.

(2) $\Phi(x, \cdot)$ is continuous and convex for each fixed $x \in G$.

Define the norm by

$$\|f\|_{\Phi(\cdot,\cdot)(G)} = \inf \left\{ \lambda > 0 : \int_G \Phi\left(x, |f(x)/\lambda|\right) dx \le 1 \right\}$$

and denote by $\Phi(\cdot, \cdot)(G)$ the space of all measurable functions f on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} < \infty$.

If $p(x) < n/\alpha$, then we set

$$1/p^*(x) = 1/p(x) - \alpha/n.$$

Define

$$\Phi(x,t) = t^{p(x)}(\log(c+t))^{q(x)},$$

$$\Psi(x,t) = \left\{ t(\log(c+t))^{q(x)/p(x)} \right\}^{p^*(x)}$$

and

$$\widetilde{\Psi}(x,t) = \left\{ t(\log(c+t))^{q(x)/p(x) - p(x)/p^*(x)} \right\}^{p^*(x)},$$

where $c \ge e$ is chosen so that $\Phi(x, \cdot)$, $\Psi(x, \cdot)$ $\widetilde{\Psi}(x, \cdot)$ are all convex on $[0, \infty)$.

For $0 < \delta < n/\alpha$, divide G into four sets:

$$\begin{aligned}
G_1 &= \{x \in G : 1 \le p(x) < 1 + \delta\}, \\
G_2 &= \{x \in G : 1 + \delta \le p(x) < n/\alpha\}, \\
G_3 &= \{x \in G : p(x) \ge n/\alpha \text{ and } q(x) < p(x) - 1\}, \\
G_4 &= \{x \in G : p(x) \ge n/\alpha \text{ and } q(x) \ge p(x) - 1\}.
\end{aligned}$$

Denote by χ_E the characteristic function of a measurable set E.

Our main result is the following, which is an extension of Futamura-Mizuta [5], Futamura-Mizuta-Shimomura [6] and Harjulehto-Hästö [7].

THEOREM 1.1. *Let $p(\cdot)$ and $q(\cdot)$ be as above. Then there exist constants $c_1, c_2, c_3, c_4 > 0$ such that*

$$\begin{aligned}
\int_G \Bigg\{ & \widetilde{\Psi}(x, I_\alpha f_1(x))\chi_{G_1}(x) + \Psi(x, c_1^{-1}\gamma_1(x)^{-1} I_\alpha f_2(x))\chi_{G_2}(x) \\
&+ \exp\left(\frac{I_\alpha f_3(x)^{p(x)/(p(x)-q(x)-1)}}{(c_2\gamma_3(x))^{p(x)/(p(x)-q(x)-1)}} \right) \chi_{G_3}(x) \\
&+ \exp\left(\exp\left(\frac{I_\alpha f_4(x)^{p(x)/(p(x)-1)}}{c_3^{p(x)/(p(x)-1)}} \right) \right) \chi_{G_4}(x) \Bigg\} dx \le c_4
\end{aligned}$$

for all nonnegative measurable functions f *on* G *with* $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$*, where* $f_j = f\chi_{G_j} (j = 1, 2, 3, 4)$*,*

$$\gamma_1(x) = p^*(x)^{(q(x)+p(x)-1)/p(x)}(\log p^*(x))^{q(x)/p(x)}$$

and

$$\gamma_3(x) = \gamma_2(x)^{-(p(x)-1)/p(x)}(\log(1/\gamma_2(x)))^{q(x)/p(x)}$$

with $\gamma_2(x) = \min\{p(x) - q(x) - 1, 1/2\}$*.*

The proof is given by discussing the cases (i) $1 \le p(x) \le p^+ < n/\alpha$ (Section 2), (ii) $1 < p^- \le p(x) < n/\alpha$ (Section 3), (iii) $p^- \ge n/\alpha$, $q(x) < p(x) - 1$ (Section 4) and (iv) $p^- \ge n/\alpha$, $q(x) \ge p(x) - 1$ (Section 5), separately.

In their paper[7], Harjulehto-Hästö gave an integrability result of Sobolev functions by diving the domain of integration into countably many measurable sets.

With the aid of O'Neil [13], one sees that if $f \in L^1(G)$, then

$$\int_G |I_\alpha f(x)|^{p^*}(\log(e + |I_\alpha f(x)|))^{-\beta}dx < \infty$$

when $\beta > 1$; this is not true when $\beta = 1$. Theorem 1.1 extends his result to the valuable exponent case. In case $p > 1$, the maximal function is a crucial tool by Hedberg's trick (see Hedberg [8]). In case $p^- = 1$, our strategy is to give an estimate of $I_\alpha f$ by use of a logarithmic type potential

$$\int_{\{y\in G:|x-y|^{-\varepsilon}<f(y)\}} (\log(c + |x-y|^{-1}))^{\varepsilon-1}|x-y|^{-n}(\log(c + f(y)))^{-\varepsilon}g(y)\ dy,$$

which plays a role of maximal functions, where $g(y) = \Phi(y, f(y))$. Thus our proof is quite different from that of O'Neil [13].

In section 7, we are concerned with continuity of Riesz potentials when $p(x) \ge n/\alpha$ and $q(x) > p(x) - 1$ for $x \in \mathbf{R}^n$.

We define the logarithmic potential for a locally integrable function f on $\mathbf{R}^n$ by

$$I_n f(x) = \int_{\mathbf{R}^n} \left(\log^+(1/|x-y|)\right) f(y)dy,$$

where $\log^+ r = \max\{0, \log r\}$. Here it is natural to assume that

$$\int_{\mathbf{R}^n} (\log(e + |y|))|f(y)|dy < \infty \tag{1.7}$$

(see [10, Theorem 1.1, Chapter 2]).

Finally, in Section 8, we show the exponential integrability of logarithmic potentials $I_n f$.

2 Variable exponents near 1

Throughout this paper, let C denote various positive constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

Let us begin with Sobolev's inequality for Riesz potentials of functions in $\Phi(\cdot,\cdot)(G)$ when $1 \le p(x) \le p^+ < n/\alpha$, which gives an extension of O'Neil [13].

THEOREM 2.1. *Let $p(\cdot)$ and $q(\cdot)$ be as in the Introduction. Suppose*

$$1 \le p(x) \le p^+ < n/\alpha$$

for $x \in \mathbf{R}^n$. Then there exists a constant $c_1 > 0$ such that

$$\|I_\alpha f\|_{\widetilde{\Psi}(\cdot,\cdot)(G)} \le c_1 \|f\|_{\Phi(\cdot,\cdot)(G)}$$

for all $f \in \Phi(\cdot,\cdot)(G)$.

When $p(x) = p = 1$ and $q(x) = \varepsilon_0 > 0$ for $x \in \mathbf{R}^n$, Theorem 2.1 says that

$$\left(\int_G |I_\alpha f(x)|^{p^*} (\log(e + |I_\alpha f(x)|))^{\varepsilon_0 p^* - 1} dx \right)^{1/p^*} \le C$$

for all measurable functions f satisfying

$$\int_G |f(y)| (\log(e + |f(y)|))^{\varepsilon_0} dy \le 1,$$

which is a consequence of O'Neil [13].

The case $p^- > 1$ is treated in the next section, by use of maximal functions. However the maximal operator fails to be bounded in $L^{p(\cdot)}(G)$ when $p^- = 1$. To show Theorem 2.1, we introduce the logarithmic type potential

$$J \equiv \int_{\{y \in G : |x-y|^{-\varepsilon} < f(y)\}} \rho_{\varepsilon-1}(|x-y|)(\log(c + f(y)))^{-\varepsilon} g(y)\, dy,$$

which plays a role of maximal functions; here $g(y) = \Phi(y, |f(y)|)$ and $\rho_{\varepsilon-1}(r) = r^{-n}(\log(c + 1/r))^{\varepsilon-1}$ for $0 < \varepsilon < \varepsilon_0/2$ with ε_0 in (1.6).

We use the notation $B(x, r)$ to denote the open ball centered at $x \in \mathbf{R}^n$ of radius $r > 0$.

LEMMA 2.2. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 2.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. If*

$$\delta = J^{-1/n}(\log(c + J))^{-p(x)/n},$$

then

$$\begin{aligned} I &\equiv \int_{\{y \in G \cap B(x,\delta) : |x-y|^{-\varepsilon} < f(y)\}} |x-y|^{\alpha-n}(\log(c + 1/|x-y|))^{-\varepsilon}(\log(c + f(y)))^{\varepsilon} f(y)\, dy \\ &\le C\{J^{1/p^*(x)}(\log(c + J))^{p(x)/p^*(x) - q(x)/p(x)} + 1\}. \end{aligned}$$

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. First consider the case when $J \ge 1$. We have by (1.6) for $k > 0$

$$
\begin{aligned}
I &\le k(\log(c+k))^{\varepsilon} \int_{\{y \in G \cap B(x,\delta): |x-y|^{-\varepsilon} < f(y)\}} |x-y|^{\alpha-n} (\log(c+1/|x-y|))^{-\varepsilon} dy \\
&\quad + \int_{\{y \in B(x,\delta): |x-y|^{-\varepsilon} < f(y)\}} |x-y|^{\alpha-n} (\log(c+1/|x-y|))^{-\varepsilon} (\log(c+f(y)))^{\varepsilon} f(y) \\
&\quad \times C \left(\frac{f(y)}{k} \right)^{p(y)-1} \left(\frac{\log(c+f(y))}{\log(c+k)} \right)^{q(y)-2\varepsilon} dy \\
&\le C \Big\{ k(\log(c+k))^{\varepsilon} \delta^{\alpha} (\log(c+1/\delta))^{-\varepsilon} \\
&\quad + \delta^{\alpha} (\log(c+1/\delta))^{1-2\varepsilon} \int_{\{y \in B(x,\delta): |x-y|^{-\varepsilon} < f(y)\}} \rho_{\varepsilon-1}(|x-y|) (\log(c+f(y)))^{-\varepsilon} g(y) \\
&\qquad \times \left(\frac{1}{k} \right)^{p(y)-1} \left(\frac{1}{\log(c+k)} \right)^{q(y)-2\varepsilon} dy \Big\}.
\end{aligned}
$$

Set

$$
k = J^{1/p(x)} (\log(c+J))^{-q(x)/p(x)}.
$$

Since

$$
\frac{\log J}{\log(1/\delta)} = \frac{\log J}{\log(J^{1/n} (\log(c+J))^{p(x)/n})} \le C,
$$

we see that if $y \in B(x,\delta)$, then

$$
J^{-p(y)} = J^{-p(y)+p(x)} J^{-p(x)} \le J^{a/\log(1/\delta)} J^{-p(x)} \le C J^{-p(x)}
$$

and similarly

$$
(\log(c+J))^{p(y)} \le (\log(c+J))^{a/\log(1/\delta)} (\log(c+J))^{p(x)} \le C (\log(c+J))^{p(x)},
$$

so that

$$
k^{-p(y)} \le C J^{-1} (\log(c+J))^{q(x)}
$$

and

$$
(\log(c+k))^{-q(y)} \le C (\log(c+J))^{-q(x)}.
$$

Consequently it follows that

$$
\begin{aligned}
I &\le C \big\{ J^{1/p(x)} (\log(c+J))^{-q(x)/p(x)} (\log(c+J))^{\varepsilon} \delta^{\alpha} (\log(c+1/\delta))^{-\varepsilon} \\
&\quad + \delta^{\alpha} (\log(c+1/\delta))^{1-2\varepsilon} J^{1/p(x)} (\log(c+J))^{-q(x)/p(x)+2\varepsilon} \big\} \\
&\le C J^{1/p(x)-\alpha/n} (\log(c+J))^{-q(x)/p(x)-\alpha p(x)/n+1}.
\end{aligned}
$$

In the case $J \le 1$, we set $k = 1$. The above considerations gives $I \le C$. Now the result follows. □

Lemma 2.3. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 2.1. Suppose $p^+ < n/\alpha$. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{\alpha-n/p(x)}(\log(c+1/\delta))^{-q(x)/p(x)}$$

for all $x \in G$ and $\delta > 0$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. For $y \in G \setminus B(x,\delta)$, set

$$N(x,y) = |x-y|^{-n/p(x)}(\log(c+|x-y|^{-1}))^{-q(x)/p(x)}.$$

By conditions (1.2), (1.3) and (1.6), we see that

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} N(x,y)dy \\
&+ C\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\left(\frac{f(y)}{N(x,y)}\right)^{p(y)-1}\left(\frac{\log(c+f(y))}{\log(c+N(x,y))}\right)^{q(y)} dy \\
\le\ & C\Big\{\delta^{\alpha-n/p(x)}(\log(c+1/\delta))^{-q(x)/p(x)} \\
&+ \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n/p(x)}(\log(c+1/|x-y|))^{-q(x)/p(x)} g(y)dy\Big\} \\
\le\ & C\delta^{\alpha-n/p(x)}(\log(c+1/\delta))^{-q(x)/p(x)}\left(1+\int_{G\setminus B(x,\delta)} g(y)dy\right) \\
\le\ & C\delta^{\alpha-n/p(x)}(\log(c+1/\delta))^{-q(x)/p(x)},
\end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(c+f(y)))^{q(y)}$, as required. □

We denote by $|E|$ the volume of E.

Proof of Theorem 2.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. For

$$\delta = J^{-1/n}(\log(c+J))^{-p(x)/n},$$

write

$$\begin{aligned}
I_\alpha f(x) &= \int_{G\cap B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
&= I_1 + I_2.
\end{aligned}$$

For $0 < \varepsilon < \min\{\varepsilon_0/2, \alpha\}$, set

$$J = \int_{\{y\in G: |x-y|^{-\varepsilon} < f(y)\}} \rho_{\varepsilon-1}(|x-y|)(\log(c+f(y)))^{-\varepsilon} g(y)\ dy.$$

In view of Lemma 2.2, we find

$$\begin{aligned} I_1 &\le \int_{G\cap B(x,\delta)} |x-y|^{\alpha-n-\varepsilon}dy \\ &\quad + \int_{\{y\in B(x,\delta):|x-y|^{-\varepsilon}<f(y)\}} |x-y|^{\alpha-n}\frac{(\log(c+f(y)))^{\varepsilon}}{(\log(c+|x-y|^{-\varepsilon}))^{\varepsilon}} f(y)dy \\ &\le C\big\{1+J^{1/p^*(x)}(\log(c+J))^{p(x)/p^*(x)-q(x)/p(x)}\big\}. \end{aligned}$$

Moreover, Lemma 2.3 yields

$$I_2 \le C\delta^{\alpha-n/p(x)}(\log(c+1/\delta))^{-q(x)/p(x)},$$

so that

$$\begin{aligned} I_\alpha f(x) &\le C\big\{1+J^{1/p^*(x)}(\log(c+J))^{p(x)/p^*(x)-q(x)/p(x)} \\ &\qquad\qquad + \delta^{\alpha-n/p(x)}(\log(c+1/\delta))^{-q(x)/p(x)}\big\} \\ &\le C\big\{1+J^{1/p^*(x)}(\log(c+J))^{p(x)/p^*(x)-q(x)/p(x)}\big\}. \end{aligned}$$

Hence we have

$$\int_G \widetilde{\Psi}(x, I_\alpha f(x))\ dx \le C\int_G (1+J)dx.$$

By using Fubini's theorem, we obtain

$$\begin{aligned} &\int_G \widetilde{\Psi}(x, I_\alpha f(x))\ dx \\ &\le C\left\{\int_G \left(\int_{\{y\in G:|x-y|^{-\varepsilon}<f(y)\}} \rho_{\varepsilon-1}(|x-y|)dx\right)(\log(c+f(y)))^{-\varepsilon}g(y)dy + |G|\right\} \\ &\le C\left\{\int_G g(y)dy+|G|\right\} \le C, \end{aligned}$$

which completes the proof. □

REMARK 2.4. Let $p(x) = p = 1$ and

$$q(x) = \frac{1}{\log(\log(e+|x|^{-1}))}$$

for $x \in \mathbf{B} = B(0,1)$. Here, note that $q(\cdot)$ satisfies the condition (1.3) since

$$\begin{aligned} &\frac{1}{\log(\log(1/t))} - \frac{1}{\log(\log(1/s))} \\ &= \int_0^{t-s} (\log(\log(1/(r+s))))^{-2}(\log(1/(r+s)))^{-1}\frac{dr}{r+s} \\ &\le C\int_0^{t-s} (\log(\log(1/r)))^{-2}(\log(1/r))^{-1}\frac{dr}{r} \\ &= \frac{C}{\log(\log(1/(t-s)))} \end{aligned}$$

whenever $0 < s < t < 1/e$. Set $p^* = n/(n-\alpha)$. Then one can find $f \in \Phi(\cdot,\cdot)(\mathbf{B})$ such that

$$\int_{\mathbf{B}} |I_\alpha f(x)|^{p^*} (\log(c + |I_\alpha f(x)|))^{p^* q(x)-1}\, dx = \infty.$$

To show this, for $0 < \gamma < 1/p^*$, let f be a nonnegative function on $\mathbf{B}$ such that

$$f(y) = |y|^{-n}(\log(c + 1/|y|))^{-1}(\log(c + \log(c + 1/|y|)))^{-\gamma-1}.$$

Then we have

$$\int_{\mathbf{B}} f(y)(\log(c + f(y)))^{q(y)}\, dy < \infty$$

and for $x \in \mathbf{B}$

$$\begin{aligned}
I_\alpha f(x) &\geq \int_{B(0,|x|/2)} |x-y|^{\alpha-n} f(y)\, dy \\
&\geq C|x|^{\alpha-n} \int_{B(0,|x|/2)} |y|^{-n}(\log(c+1/|y|))^{-1} \\
&\qquad\qquad \times (\log(c + \log(c+1/|y|)))^{-\gamma-1}\, dy \\
&\geq C|x|^{\alpha-n}(\log(c + \log(c + 1/|x|)))^{-\gamma}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\int_{\mathbf{B}} |I_\alpha f(x)|^{p^*}(\log(c + |I_\alpha f(x)|))^{p^* q(x)-1}\, dx \\
\geq\ & C\int_{\mathbf{B}} |x|^{-n}(\log(c+1/|x|))^{-1}(\log(c+\log(c+1/|x|)))^{-\gamma p^*}\, dx = \infty
\end{aligned}$$

when $\gamma < 1/p^*$.

3 Variable exponents near Sobolev's exponent

Set

$$\Psi(x,t) = \left\{t(\log(c+t))^{q(x)/p(x)}\right\}^{p^*(x)}$$

and denote by $\Psi(\cdot,\cdot)(G)$ the family of all measurable functions f on G such that

$$\|f\|_{\Psi(\cdot,\cdot)(G)} = \inf\left\{\lambda > 0 : \int_G \Psi\left(x, |f(x)/\lambda|\right) dx \leq 1\right\} < \infty.$$

THEOREM 3.1. *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.2)–(1.5) *such that*

$$1 < p^- \leq p(x) < n/\alpha$$

for $x \in G$. Then there exists a constant $c_1 > 0$ such that

$$\|\gamma_1(\cdot)^{-1} I_\alpha f\|_{\Psi(\cdot,\cdot)(G)} \leq c_1 \|f\|_{\Phi(\cdot,\cdot)(G)}$$

for all $f \in \Phi(\cdot,\cdot)(G)$, where

$$\gamma_1(x) = p^*(x)^{(q(x)+p(x)-1)/p(x)}(\log p^*(x))^{q(x)/p(x)}.$$

The case $\sup_{x\in G} p(x) < n/\alpha$ was shown in the authors [11, Theorem 2.8].

Remark 3.2. For $0 < \delta < 1$, we can find $f \in \Phi(\cdot,\cdot)(G)$ such that

$$\int_G \Psi(x, \gamma_1(\cdot)^{-\delta} I_\alpha f(x))dx = \infty$$

with $q(x) = 0$, so that the weight $\gamma_1(\cdot)^{-1}$ in Theorem 3.1 is needed.

For this, consider

$$p(x) = \frac{n}{\alpha} - \frac{1}{\log(1/|x|)}$$

for $x \in B_0 = B(0, 1/e)$ and

$$f(y) = |y|^{-\alpha}(\log(1/|y|))^{-\beta}$$

for $y \in B_0$. If $\alpha/n < \beta < 1$, then

$$\int_{B_0} f(y)^{p(y)} dy < \infty.$$

If $x \in B_0$, then

$$\begin{aligned} I_\alpha f(x) &\geq \int_{\{y\in B_0:|y|\geq |x|/2\}} |x-y|^{\alpha-n} f(y)dy \\ &\geq C\int_{\{y\in B_0:|y|\geq |x|/2\}} |y|^{-n}(\log(1/|y|))^{-\beta}dy \geq C(\log(1/|x|))^{-\beta+1} \end{aligned}$$

when $\beta < 1$. Now take β such that $\alpha/n < \beta < 1$ and

$$-\delta(1-\alpha/n) - \beta + 1 > 0.$$

Since $p^*(x) = (n/\alpha)^2 \log(1/|x|) - n/\alpha$, there exists a constant $c_0 > 0$ such that $\gamma_1(x) \leq C(\log(1/|x|))^{1-\alpha/n}$ and $p^*(x) > c_0 \log(1/|x|)$ for $x \in B_0$. Hence we find

$$\begin{aligned} \int_{B_0} \Psi(x, \gamma_1(\cdot)^{-\delta} I_\alpha f(x))dx &\geq C\int_{B_0} (\log(1/|x|))^{p^*(x)(-\delta(1-\alpha/n)-\beta+1)}dx \\ &\geq C\int_{B_0} (\log(1/|x|))^{c_0(-\delta(1-\alpha/n)-\beta+1)\log(1/|x|)}dx = \infty. \end{aligned}$$

For a proof of Theorem 3.1, we prepare several results.

Lemma 3.3. *Suppose $0 < a \leq R_0$ and $0 \leq b \leq R_0$. Then there exists a constant $C(R_0) > 0$ such that*

$$\int_\delta^{1/2} t^{-a}(\log(1/t))^{-b}\frac{dt}{t} \leq C(R_0)a^{-b-1}\delta^{-a}(\log(1/\delta))^{-b}$$

for all $0 < \delta < 1/2$.

Proof. Note that $u_a(s) = s^{-a}(\log(1/s))^{-b}$ attains a minimum value of $e^b b^{-b} a^b$ at $s = e^{-b/a}$ for $0 < s < 1$. If $1/2 \le e^{-b/a}$, then u_a is decreasing on $(0, 1/2]$. Hence

$$u_a(t) \le u_a(\delta) \quad \text{for} \quad 0 < \delta \le t < 1/2.$$

If $e^{-b/a} < 1/2$, then u_a is decreasing on $(0, e^{-b/a}]$ and increasing on $[e^{-b/a}, 1/2]$. Hence, in the case $e^{-b/a} \le \delta$ we have

$$u_a(t) \le \frac{u_a(1/2)}{u_a(e^{-b/a})} u_a(\delta) = \frac{2^a(\log 2)^{-b}}{e^b b^{-b} a^b} u_a(\delta) \quad \text{for} \quad 0 < \delta \le t < 1/2,$$

and, in the case $0 < \delta < e^{-b/a}$ we have

$$\begin{aligned} u_a(t) &\le u_a(\delta) \quad \text{for} \quad 0 < \delta \le t < e^{-b/a}, \\ u_a(t) &\le \frac{2^a(\log 2)^{-b}}{e^b b^{-b} a^b} u_a(\delta) \quad \text{for} \quad e^{-b/a} \le t < 1/2. \end{aligned}$$

Therefore, we obtain

$$u_a(t) \le C(R_0) a^{-b} u_a(\delta) \quad \text{for} \quad 0 < \delta \le t < 1/2, \tag{3.1}$$

so that

$$\begin{aligned} \int_\delta^{1/2} t^{-a}(\log(1/t))^{-b} \frac{dt}{t} &\le C(R_0)(a/2)^{-b} u_{a/2}(\delta) \int_\delta^{1/2} t^{-a/2} \frac{dt}{t} \\ &\le C(R_0) 2^{b+1} a^{-b-1} \delta^{-a} (\log(1/\delta))^{-b} \end{aligned}$$

for all $0 < \delta < 1/2$, as required. □

LEMMA 3.4. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 3.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$\int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C\gamma_1(x) \delta^{\alpha - n/p(x)} (\log(1/\delta))^{-q(x)/p(x)}$$

for all $x \in G$ and $0 < \delta < 1/2$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

First note that

$$\begin{aligned} \int_{G \setminus B(x, p^*(x)^{-1/n})} |x-y|^{\alpha-n} f(y) dy &\le p^*(x)^{1-\alpha/n} \int_G f(y) dy \\ &\le p^*(x)^{1-\alpha/n} \int_G \{1 + g(y)\} dy \\ &\le Cp^*(x)^{1-\alpha/n} \\ &\le Cp^*(x)^{1-1/p(x)} \\ &\le C\gamma_1(x) \end{aligned}$$

since $p^*(x)^{1/p^*(x)} \le C$, where $g(y) = f(y)^{p(y)}(\log(c+f(y)))^{q(y)}$ as before.

Setting $\eta(x) = p^*(x)^{-1/p(x)}(\log p^*(x))^{q(x)/p(x)}$ and $N(x,y) = |x-y|^{-n/p(x)}(\log(1/|x-y|))^{-q(x)/p(x)}$, we have

$$\begin{aligned}
&\int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & \int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n}\{\eta(x)N(x,y)\}dy \\
&+ \int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) \\
&\qquad \times C\left(\frac{f(y)}{\eta(x)N(x,y)}\right)^{p(y)-1}\left(\frac{\log(c+f(y))}{\log(c+\eta(x)N(x,y))}\right)^{q(y)} dy.
\end{aligned}$$

Here note that for $y \in B(x,p^*(x)^{-1/n})$

$$\{\eta(x)N(x,y)\}^{-p(y)} \le C\eta(x)^{-p(x)}|x-y|^n(\log(1/|x-y|))^{q(x)}$$

since

$$p^*(x)^{p(y)} \le p^*(x)^{a/(\log p^*(x)^{1/n})}p^*(x)^{p(x)} \le Cp^*(x)^{p(x)}$$

and

$$\eta(x)^{-p(y)} \le C\eta(x)^{-p(x)}.$$

Further, noting that

$$\log(c+st) \le C\log(c+s)\log(c+t) \qquad \text{when } s,t>0, \tag{3.2}$$

we obtain

$$\begin{aligned}
\{\log(c+\eta(x)N(x,y))\}^{-q(y)} &\le C(\log p^*(x))^{q(y)}(\log(1/|x-y|))^{-q(y)} \\
&\le C(\log p^*(x))^{q(x)}(\log(1/|x-y|))^{-q(x)}
\end{aligned}$$

for $y \in B(x,p^*(x)^{-1/n})$. It follows from Lemma 3.3 and (3.1) that

$$\begin{aligned}
&\int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & C\Big\{\eta(x)(n/p(x)-\alpha)^{-q(x)/p(x)-1}\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)} \\
&+\eta(x)^{1-p(x)}(\log p^*(x))^{q(x)} \\
&\qquad \times \int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n/p(x)}(\log(1/|x-y|))^{-q(x)/p(x)}g(y)dy\Big\} \\
\le\ & C\Big\{\gamma_1(x)\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)} + \eta(x)^{1-p(x)}(\log p^*(x))^{q(x)} \\
&\times (n/p(x)-\alpha)^{-q(x)/p(x)}\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} g(y)dy\Big\} \\
\le\ & C\gamma_1(x)\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\left(1+\int_{B(x,p^*(x)^{-1/n})\setminus B(x,\delta)} g(y)dy\right) \\
\le\ & C\gamma_1(x)\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)},
\end{aligned}$$

which proves the lemma. $\square$

For a locally integrable function f on G, we set

$$f_B = \frac{1}{|B|} \int_{B \cap G} f(y)\ dy.$$

We consider the maximal function Mf defined by

$$Mf(x) = \sup_B |f|_B,$$

where the supremum is taken over all balls $B = B(x, r)$. Diening [3] was the first who proved the local boundedness of maximal functions in the Lebesgue spaces of variable exponents satisfying the log-Hölder condition. In our case, we need the following result (see also D. Cruz-Uribe and A. Fiorenza [2]) :

PROPOSITION 3.5 ([11, Theorem 2.7]). *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.2) – (1.5) *such that*

$$p^- > 1.$$

Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then

$$\int_G \Phi(x, Mf(x)) dx \le C.$$

PROOF OF THEOREM 3.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Write

$$f = f\chi_{\{y \in G : f(y) < 1\}} + f\chi_{\{y \in G : f(y) \ge 1\}} = f_1 + f_2.$$

Then $I_\alpha f_1(x) \le C$.

We have by Lemma 3.4

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy + \int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \\ &\le C\big\{\delta^\alpha Mf(x) + \gamma_1(x) \delta^{\alpha - n/p(x)} (\log(1/\delta))^{-q(x)/p(x)}\big\} \end{aligned}$$

for $0 < \delta < 1/2$. Here, considering

$$\delta = C(\gamma_1(x)^{-1} Mf(x))^{-p(x)/n} (\log(\gamma_1(x)^{-1} Mf(x)))^{-q(x)/n}$$

when $\gamma_1(x)^{-1} Mf(x) \ge 1$, we find

$$I_\alpha f(x) \le C\gamma_1(x)^{\alpha p(x)/n} Mf(x)^{1-\alpha p(x)/n} (\log(c + \gamma_1(x)^{-1} Mf(x)))^{-\alpha q(x)/n}.$$

If $\gamma_1(x)^{-1} Mf(x) < 1$, then

$$I_\alpha f(x) \le C\big\{Mf(x) + \gamma_1(x)\big\} \le C\gamma_1(x).$$

Hence it follows that

$$\begin{aligned}\gamma_1(x)^{-1}I_\alpha f(x) &\le C\big\{(\gamma_1(x)^{-1}Mf(x))^{1-\alpha p(x)/n}(\log(c+\gamma_1(x)^{-1}Mf(x)))^{-\alpha q(x)/n}+1\big\}\\ &\le C\big\{Mf(x)^{p(x)/p^*(x)}(\log(c+Mf(x)))^{-\alpha q(x)/n}+1\big\}\end{aligned}$$

since $\gamma_1(x)^{-1/p^*(x)} \le C$, so that

$$\big\{c_1^{-1}\gamma_1(x)^{-1}I_\alpha f(x)(\log(c+\gamma_1(x)^{-1}I_\alpha f(x)))^{q(x)/p(x)}\big\}^{p^*(x)} \le C\left\{\Phi(x,Mf(x))+1\right\}.$$

By Proposition 3.5, we have

$$\int_G \Psi(x, c_1^{-1}\gamma_1(x)^{-1}I_\alpha f(x))dx \le C\int_G \left\{\Phi(x,Mf(x))+1\right\}dx \le C,$$

which completes the proof. □

4 Trudinger's exponential integrability

This section is concerned with the exponential integrability of Trudinger's type.

THEOREM 4.1. *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.2)–(1.5) *such that*

$$p(x) \ge n/\alpha \qquad \text{and} \qquad q(x) < p(x)-1$$

for $x \in G$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_G \exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_3(x))^{p(x)/(p(x)-q(x)-1)}}\right)dx \le c_2$$

for all nonnegative measurable functions f on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$, where

$$\gamma_3(x) = \gamma_2(x)^{-(p(x)-1)/p(x)}(\log(1/\gamma_2(x)))^{q(x)/p(x)}$$

with $\gamma_2(x) = \min\{p(x)-q(x)-1, 1/2\}$.

COROLLARY 4.2. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 4.1. Then there exists a constant $c_3 > 0$ such that*

$$\int_G \left\{\exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_3\gamma_3(x))^{p(x)/(p(x)-q(x)-1)}}\right)-1\right\}dx \le 1$$

for all nonnegative measurable functions f on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

REMARK 4.3. For $0 < \delta < 1$, we can find $f \in \Phi(\cdot,\cdot)(G)$ such that

$$\int_{B_0} \exp((\gamma_3(x)^{-\delta}I_\alpha f(x))^{p(x)/(p(x)-q(x)-1)})dx = \infty.$$

For this, consider

$$p(x) = \frac{n}{\alpha} + \frac{1}{\log(1/|x|)}$$

for $x \in B_0 = B(0, 1/4)$ and

$$f(y) = |y|^{-\alpha}(\log(1/|y|))^{-1}(\log\log(1/|y|))^{-\beta}$$

for $y \in B_0$. If $q(x) = p(x) - 1 - 1/\log\log(1/|x|)$ and $\beta > \alpha/n$, then

$$\int_{B_0} f(y)^{p(y)}(\log f(y))^{q(y)} dy < \infty.$$

If $x \in B_0$, then, as in Remark 3.2, we find

$$I_\alpha f(x) \ \geq \ C(\log\log(1/|x|))^{-\beta+1}$$

when $\beta < 1$. Now take β and $\varepsilon > 0$ such that $\alpha/n < \beta < 1$ and

$$-\delta(1 - \alpha/n + \varepsilon) - \beta + 1 > 0.$$

Since

$$\gamma_3(x) \leq C(\log\log(1/|x|))^{1-\alpha/n}(\log\log\log(1/|x|))^{(n-\alpha)/n} \leq C(\log\log(1/|x|))^{1-\alpha/n+\varepsilon}$$

and

$$p(x)/(p(x) - q(x) - 1) > (n/\alpha)\log\log(1/|x|),$$

we have

$$\begin{aligned} &\int_{B_0} \exp((\gamma_3(x)^{-\delta} I_\alpha f(x))^{p(x)/(p(x)-q(x)-1)})dx \\ \geq \ &\int_{B_0} \exp((C\log\log(1/|x|))^{(-\delta(1-\alpha/n+\varepsilon)-\beta+1)(n/\alpha)\log\log(1/|x|)})dx = \infty. \end{aligned}$$

Before proving Theorem 4.1, we prepare the following result.

Lemma 4.4. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 4.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \leq 1$. Then*

$$\int_{G \setminus B(x,\delta)} |x - y|^{\alpha-n} f(y)dy \leq C\gamma_3(x)\left(\log(1/\delta)\right)^{(p(x)-q(x)-1)/p(x)}$$

for all $x \in G$ and $0 < \delta < 1/2$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \leq 1$.
First note that

$$\int_{G \setminus B(x,\gamma_2(x)^{1/n})} |x - y|^{\alpha-n} f(y)dy \leq C\gamma_2(x)^{(\alpha-n)/n} \leq C\gamma_2(x)^{-1+1/p(x)}.$$

Setting $\eta(x) = \gamma_2(x)^{1/p(x)}(\log(1/\gamma_2))^{q(x)/p(x)}$ and $N(x,y) = |x-y|^{-n/p(x)}(\log(1/|x-y|))^{-(q(x)+1)/p(x)}$, we have

$$\begin{aligned}
&\int_{B(x,\gamma_2(x)^{1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\leq\ & \int_{B(x,\gamma_2(x)^{1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} \{\eta(x)N(x,y)\}dy \\
&+ \int_{B(x,\gamma_2(x)^{1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) \\
&\qquad \times C\left(\frac{f(y)}{\eta(x)N(x,y)}\right)^{p(y)-1}\left(\frac{\log(c+f(y))}{\log(c+\eta(x)N(x,y))}\right)^{q(y)} dy.
\end{aligned}$$

If $y \in B(x,\gamma_2(x)^{1/n})$, then $\gamma_2(x)^{-p(y)} \leq C\gamma_2(x)^{-p(x)}$, so that

$$\eta(x)^{-p(y)} \leq C\eta(x)^{-p(x)}.$$

Hence

$$\{\eta(x)N(x,y)\}^{-p(y)} \leq C\eta(x)^{-p(x)}|x-y|^n(\log(1/|x-y|))^{q(x)+1}$$

and by inequality (3.2)

$$\{\log(c+\eta(x)N(x,y))\}^{-q(y)} \leq C(\log(1/\gamma_2(x)))^{q(x)}(\log(1/|x-y|))^{-q(x)}$$

for $y \in B(x,\gamma_2(x)^{1/n})$. Consequently it follows from Lemma 3.3 that

$$\begin{aligned}
&\int_{B(x,\gamma_2(x)^{1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\leq\ & C\Big\{\eta(x)\gamma_2(x)^{-1}(\log(1/\delta))^{(p(x)-q(x)-1)/p(x)} + \eta(x)^{-p(x)+1}(\log(1/\gamma_2(x)))^{q(x)} \\
&\times \int_{B(x,\gamma_2(x)^{1/n})\setminus B(x,\delta)} |x-y|^{\alpha-n/p(x)}(\log(1/|x-y|))^{(p(x)-q(x)-1)/p(x)} g(y)dy\Big\} \\
\leq\ & C\gamma_3(x)(\log(1/\delta))^{(p(x)-q(x)-1)/p(x)}\left(1+\int_{B(x,\gamma_2(x)^{1/n})\setminus B(x,\delta)} g(y)dy\right) \\
\leq\ & C\gamma_3(x)(\log(1/\delta))^{(p(x)-q(x)-1)/p(x)},
\end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(c+f(y)))^{q(y)}$ as before.

Thus we have proved that

$$\begin{aligned}
\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy &\leq C\{\gamma_2(x)^{-1+1/p(x)} + \gamma_3(x)(\log(1/\delta))^{(p(x)-q(x)-1)/p(x)}\} \\
&\leq C\gamma_3(x)(\log(1/\delta))^{(p(x)-q(x)-1)/p(x)}
\end{aligned}$$

for $0 < \delta < 1/2$, which gives the lemma. □

Proof of Theorem 4.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then Lemma 4.4 gives

$$I_\alpha f(x) \le C\big\{\delta^\alpha Mf(x) + \gamma_3(x)(\log(1/\delta))^{(p(x)-q(x)-1)/p(x)}\big\}$$

for $0 < \delta < 1/2$. Here, considering

$$\delta = C(\gamma_3(x)^{-1}Mf(x))^{-1/\alpha}(\log(\gamma_3(x)^{-1}Mf(x)))^{(p(x)-q(x)-1)/(\alpha p(x))}$$

when $\gamma_3(x)^{-1}Mf(x) \ge 1$, we find

$$\begin{aligned} I_\alpha f(x) &\le C\big\{\gamma_3(x)(\log(c+\gamma_3(x)^{-1}Mf(x)))^{(p(x)-q(x)-1)/p(x)} + \gamma_3(x)\big\} \\ &\le C\gamma_3(x)(\log(c+Mf(x)))^{(p(x)-q(x)-1)/p(x)}. \end{aligned}$$

Hence it follows that

$$c_1^{-1}\gamma_3(x)^{-1}I_\alpha f(x) \le (\log(c+Mf(x)))^{(p(x)-q(x)-1)/p(x)},$$

so that

$$\exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_3(x))^{p(x)/(p(x)-q(x)-1)}}\right) \le c + Mf(x) \le C\{\Phi(x, Mf(x)) + 1\}.$$

By Proposition 3.5, we have

$$\int_G \exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_3(x))^{p(x)/(p(x)-q(x)-1)}}\right) dx \le C\left(\int_G \Phi(x, Mf(x))dx + 1\right) \le C,$$

as required. □

5 Trudinger's double exponential integrability

This section is devoted to the study of the double exponential integrability.

Theorem 5.1. *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.2)–(1.5) *such that*

$$p(x) \ge n/\alpha \qquad \text{and} \qquad q(x) \ge p(x) - 1$$

for $x \in \mathbf{R}^n$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_G \exp\left(\exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-1)}}{c_1^{p(x)/(p(x)-1)}}\right)\right) dx \le c_2$$

for all nonnegative measurable functions f on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

Lemma 5.2. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 5.1 and let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\left(\log(\log(1/\delta))\right)^{(p(x)-1)/p(x)}$$

for all $x \in G$ and $0 < \delta < 1/2$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

First note that

$$\int_{G\setminus B(x,1/4)} |x-y|^{\alpha-n} f(y)dy \le C.$$

Next, setting $N(x,y) = |x-y|^{-n/p(x)}(\log(1/|x-y|))^{-1}(\log(\log(1/|x-y|)))^{-1/p(x)}$, we have

$$\begin{aligned}
&\int_{B(x,1/4)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & \int_{B(x,1/4)\setminus B(x,\delta)} |x-y|^{\alpha-n} N(x,y)dy \\
&+ \int_{B(x,1/4)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\left(\frac{f(y)}{N(x,y)}\right)^{p(y)-1}\left(\frac{\log(c+f(y))}{\log(c+N(x,y))}\right)^{q(y)} dy \\
\le\ & C\Bigg\{(\log(\log(1/\delta)))^{(p(x)-1)/p(x)} + \int_{B(x,1/4)\setminus B(x,\delta)} |x-y|^{\alpha-n/p(x)} \\
&\times (\log(1/|x-y|))^{p(x)-q(x)-1}(\log(\log(1/|x-y|)))^{(p(x)-1)/p(x)} g(y)dy\Bigg\} \\
\le\ & C(\log(\log(1/\delta)))^{(p(x)-1)/p(x)},
\end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(c+f(y)))^{q(y)}$, as required. $\square$

Proof of Theorem 5.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then Lemma 5.2 gives

$$I_\alpha f(x) \le C\delta^\alpha Mf(x) + C(\log(\log(1/\delta)))^{(p(x)-1)/p(x)}$$

for $0 < \delta < 1/4$. Here, considering

$$\delta = CMf(x)^{-1/\alpha}(\log(\log(Mf(x))))^{(p(x)-1)/(\alpha p(x))}$$

when $Mf(x) \ge e^2$, we find

$$I_\alpha f(x) \le C(\log(c+\log(c+Mf(x))))^{(p(x)-1)/p(x)}.$$

Hence it follows that

$$c_1^{-1} I_\alpha f(x) \le (\log(\log(c+Mf(x))))^{(p(x)-1)/p(x)},$$

so that

$$\exp\left(\exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-1)}}{c_1^{p(x)/(p(x)-1)}}\right)\right) \le C\{\Phi(x, Mf(x)) + 1\}.$$

Now Proposition 3.5 yields

$$\int_G \exp\left(\exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-1)}}{c_1^{p(x)/(p(x)-1)}}\right)\right) dx \le C\left(\int_G \Phi(x, Mf(x))dx + 1\right) \le c_2,$$

as required. □

6 Proof of Theorem 1.1

Let f be a nonnegative measurable functions on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Let $p_1(x) = \min\{p(x), 1+\delta\}$ for $0 < \delta < n/\alpha$. Then Theorem 2.1 yields

$$\int_{G_1} \widetilde{\Psi}(x, I_\alpha f_1(x))dx \le C.$$

Letting $p_2(x) = \max\{p(x), 1+\delta\}$, we see by Theorem 3.1 that

$$\int_{G_2} \Psi(x, c_1^{-1}\gamma_1(x)^{-1} I_\alpha f_2(x))dx \le C,$$

where $c_1 > 0$ is in Theorem 3.1. Hence, in view of Theorems 4.1 and 5.1, Theorem 1.1 is proved. □

7 Continuity of Riesz potentials

In this section we are concerned with continuity properties of Riesz potentials.

THEOREM 7.1. *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.2)–(1.5) *such that*

$$p(x) \ge n/\alpha \qquad \text{and} \qquad q(x) > p(x) - 1$$

for $x \in \mathbf{R}^n$. If f is a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$, then $I_\alpha f(x)$ is continuous and

$$|I_\alpha f(z) - I_\alpha f(x)| \le C\gamma_5(x)(\log(1/|z-x|))^{-(q(x)-p(x)+1)/p(x)}$$

as $z \to x$ for each $x \in G$, where

$$\gamma_5(x) = \gamma_4(x)^{-(p(x)-1)/p(x)}(\log(1/\gamma_4(x)))^{q(x)/p(x)}$$

with $\gamma_4(x) = \min\{q(x) - p(x) + 1, 1/2\}$.

Lemma 7.2. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 7.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\gamma_5(x,\delta)\,(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)}$$

for all $x \in G$ and $0 < \delta < 1/4$, where

$$\gamma_5(x,t) = \gamma_4(x)^{-(p(x)-1)/p(x)-a/(p(x)\log(1/t))} \times (\log(1/\gamma_4(x)))^{q(x)/p(x)-aq(x)/(p(x)\log(1/t))+b/\log(\log(1/t))}.$$

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

Setting $\eta(x) = \gamma_4(x)^{1/p(x)}(\log(1/\gamma_4(x)))^{q(x)/p(x)}$ and $N(x,y) = |x-y|^{-n/p(x)}(\log(1/|x-y|))^{-(q(x)+1)/p(x)}$, we have

$$\begin{aligned} &\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ \le\ &\int_{B(x,\delta)} |x-y|^{\alpha-n}\eta(x)N(x,y)dy \\ &+ \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) \left(\frac{f(y)}{\eta(x)N(x,y)}\right)^{p(y)-1} \left(\frac{\log(c+f(y))}{\log(c+\eta(x)N(x,y))}\right)^{q(y)} dy. \end{aligned}$$

Note that

$$\{\eta(x)N(x,y)\}^{-p(y)} \le \eta(x)^{-p(x)-a/\log(1/\delta)}|x-y|^n(\log(1/|x-y|))^{q(x)+1}$$

and

$$\{\log(c+\eta(x)N(x,y))\}^{-q(y)} \le C(\log(1/\gamma_4(x)))^{q(x)+b/\log(\log(1/\delta))}(\log(1/|x-y|))^{-q(x)}$$

for $y \in B(x,\delta)$. Consequently it follows that

$$\begin{aligned} &\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ \le\ & C\Big\{\eta(x)\gamma_4(x)^{-1}(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)} \\ &+ \eta(x)^{-p(x)+1-a/\log(1/\delta)}(\log(1/\gamma_4(x)))^{q(x)+b/\log(\log(1/\delta))} \\ &\qquad \times \int_{B(x,\delta)} |x-y|^{\alpha-n/p(x)}(\log(1/|x-y|))^{-(q(x)-p(x)+1)/p(x)} g(y)dy\Big\} \\ \le\ & C\gamma_5(x,\delta)(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)}\left(1+\int_{B(x,\delta)} g(y)dy\right) \\ \le\ & C\gamma_5(x,\delta)(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)}, \end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(c+f(y)))^{q(y)}$, as required. □

LEMMA 7.3. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 7.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y)dy \le C\delta^{-1} (\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)}$$

for all $x \in G$ and $0 < \delta < 1/2$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

First note that

$$\int_{G\setminus B(x,1/2)} |x-y|^{\alpha-n-1} f(y)dy \le C.$$

Setting $N(x,y) = |x-y|^{-n/p(x)}(\log(1/|x-y|))^{-(q(x)+1)/p(x)}$, we have

$$\begin{aligned}
&\int_{B(x,1/2)\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y)dy \\
\le\ &\int_{B(x,1/2)\setminus B(x,\delta)} |x-y|^{\alpha-n-1} N(x,y)dy \\
&+ \int_{B(x,1/2)\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y) \left(\frac{f(y)}{N(x,y)}\right)^{p(y)-1} \left(\frac{\log(c+f(y))}{\log(c+N(x,y))}\right)^{q(y)} dy.
\end{aligned}$$

Since

$$\{N(x,y)\}^{-p(y)} \le C|x-y|^n (\log(1/|x-y|))^{q(x)+1}$$

and

$$\{\log(c+N(x,y))\}^{-q(y)} \le C(\log(1/|x-y|))^{-q(x)}$$

for $y \in B(x,1/2)$, it follows from Lemma 3.3 that

$$\begin{aligned}
&\int_{B(x,1/2)\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y)dy \\
\le\ & C\Big\{\delta^{-1}(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)} \\
&+ \int_{B(x,1/2)\setminus B(x,\delta)} |x-y|^{\alpha-n/p(x)-1}(\log(1/|x-y|))^{-(q(x)-p(x)+1)/p(x)} g(y)dy\Big\} \\
\le\ & C\delta^{-1}(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)} \left(1 + \int_{B(x,1/2)\setminus B(x,\delta)} g(y)dy\right) \\
\le\ & C\delta^{-1}(\log(1/\delta))^{-(q(x)-p(x)+1)/p(x)},
\end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(c+f(y)))^{q(y)}$, as required. □

PROOF OF THEOREM 7.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

Write

$$\begin{aligned} I_\alpha f(x) - I_\alpha f(z) &= \int_{B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)dy - \int_{B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{G \setminus B(x,2|x-z|)} (|x-y|^{\alpha-n} - |z-y|^{\alpha-n}) f(y)dy. \end{aligned}$$

By Lemma 7.2, we have

$$\int_{B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)dy \le C\gamma_5(x, 3|x-z|)(\log(1/|z-x|))^{-(q(x)-p(x)+1)/p(x)}$$

and

$$\begin{aligned} \int_{B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)dy &\le \int_{B(z,3|x-z|)} |z-y|^{\alpha-n} f(y)dy \\ &\le C\gamma_5(z, 3|x-z|)(\log(1/|z-x|))^{-(q(z)-p(z)+1)/p(z)} \end{aligned}$$

for $0 < |x-z| < 1/2$. On the other hand, by the mean value theorem for analysis, we have by Lemma 7.3

$$\begin{aligned} &\int_{G \setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y)dy \\ &\le C|x-z| \int_{G \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)dy \\ &\le C(\log(1/|z-x|))^{-(q(x)-p(x)+1)/p(x)}. \end{aligned}$$

Now we establish

$$\begin{aligned} |I_\alpha f(x) - I_\alpha f(z)| &\le C\{\gamma_5(x, 3|x-z|)(\log(1/|z-x|))^{-(q(x)-p(x)+1)/p(x)} \\ &\quad + \gamma_5(z, 3|x-z|)(\log(1/|z-x|))^{-(q(z)-p(z)+1)/p(z)}\} \end{aligned}$$

for $0 < |x-z| < 1/4$, which implies

$$|I_\alpha f(z) - I_\alpha f(x)| \le C\gamma_5(x)(\log(1/|z-x|))^{-(q(x)-p(x)+1)/p(x)}$$

as $z \to x$ for all $x \in G$. □

8 Logarithmic potentials

In this section we discuss Sobolev's theorem for logarithmic potentials.

THEOREM 8.1. *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.3) *such that $p(x) = 1$ and*

$$0 \le q(x) < 1$$

for $x \in \mathbf{R}^n$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_G \left\{ \exp\left((c_1 I_n f(x))^{1/(1-q(x))} \right) - 1 \right\} dx \le c_2$$

for all nonnegative measurable functions f on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$.

To show this, we estimate $I_n f$ by the logarithmic potential

$$J = \int_G \rho_{-\beta}(|x-y|) g(y)\, dy,$$

where $\rho_{-\beta}(r) = r^{-n}(\log(2+1/r))^{-\beta}$ with $\beta > 1$ and $g(y) = f(y)(\log(e+f(y)))^{q(y)}$.

LEMMA 8.2. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 8.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$F \equiv \int_{B(x,\delta)} \rho_{-\beta}(|x-y|) f(y)\, dy \le CJ \left\{ (\log(e+J))^{-q(x)} + (\log(e+1/\delta))^{-q(x)} \right\}$$

for all $x \in G$ and $\delta > 0$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. We have for $k > 0$

$$F \le k \int_G \rho_{-\beta}(|x-y|) dy + \int_{B(x,\delta)} \rho_{-\beta}(|x-y|) f(y) \left(\frac{\log(e+f(y))}{\log(e+k)} \right)^{q(y)} dy.$$

If $J \le \delta^{-n}$, then we set

$$k = J(\log(e+J))^{-q(x)}.$$

Since $\delta \le J^{-1/n}$, we see that

$$(\log(e+k))^{-q(y)} \le C(\log(e+J))^{-q(x)}$$

for $y \in B(x,\delta)$. Consequently it follows that

$$F \le CJ(\log(e+J))^{-q(x)}.$$

If $J > \delta^{-n}$, then we set

$$k = \delta^{-n}(\log(e+1/\delta))^{-q(x)}$$

and obtain

$$\begin{aligned} F &\le C\left\{ \delta^{-n}(\log(e+1/\delta))^{-q(x)} + (\log(e+1/\delta))^{-q(x)} J \right\} \\ &\le C(\log(e+1/\delta))^{-q(x)} J. \end{aligned}$$

Now the result follows. □

LEMMA 8.3. *Let $p(\cdot)$ and $q(\cdot)$ be as in Theorem 8.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Then*

$$\int_{G \setminus B(x,\delta)} \log^+(1/|x-y|) f(y) dy \le C(\log(e+1/\delta))^{-q(x)+1}$$

for all $x \in G$ and $\delta > 0$.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. Let $0 < \gamma < n$. For $y \in G \setminus B(x,\delta)$ and $\delta > 0$, set

$$N(x,y) = |x-y|^{-\gamma}.$$

By condition (1.3), we see that

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \log^+(1/|x-y|) f(y)dy \le \int_G \log^+(1/|x-y|) N(x,y)dy \\
&+ \int_{G\setminus B(x,\delta)} \log^+(1/|x-y|) f(y) \left(\frac{\log(e+f(y))}{\log(e+N(x,y))} \right)^{q(y)} dy \\
\le\ & C\left\{ 1 + \int_{G\setminus B(x,\delta)} (\log(e+1/|x-y|))^{-q(y)+1} g(y)dy \right\} \\
\le\ & C\left\{ 1 + (\log(e+1/\delta))^{-q(x)+1} \int_{G\setminus B(x,\delta)} g(y)dy \right\} \\
\le\ & C(\log(e+1/\delta))^{-q(x)+1},
\end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(e+f(y)))^{q(y)}$, as required. □

Proof of Theorem 8.1. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} \le 1$. For $x \in G$ and $\delta > 0$, write

$$\begin{aligned}
I_n f(x) &= \int_{B(x,\delta)} \log^+(1/|x-y|) f(y)dy + \int_{G\setminus B(x,\delta)} \log^+(1/|x-y|) f(y)dy \\
&= I_1 + I_2.
\end{aligned}$$

For $\beta > 1$, we infer from Lemma 8.2 that

$$\begin{aligned}
I_1 &\le C\delta^n (\log(e+1/\delta))^{1+\beta} \int_{B(x,\delta)} \rho_{-\beta}(|x-y|) f(y)dy \\
&\le C\delta^n (\log(e+1/\delta))^{1+\beta} J \left\{ (\log(e+1/\delta))^{-q(x)} + (\log(e+J))^{-q(x)} \right\}.
\end{aligned}$$

Hence, in view of Lemma 8.3, we find

$$\begin{aligned}
I_n f(x) \le\ & C\big\{ \delta^n (\log(e+1/\delta))^{1+\beta} J \left\{ (\log(e+1/\delta))^{-q(x)} + (\log(e+J))^{-q(x)} \right\} \\
& + (\log(e+1/\delta))^{-q(x)+1} \big\}.
\end{aligned}$$

Now, considering $\delta = J^{-1/n}(\log(e+J))^{-\beta/n}$, we find

$$I_n f(x) \le C(\log(e+J))^{-q(x)+1},$$

so that

$$\exp((c_1 I_n f(x))^{1/(1-q(x))}) \le e + J.$$

Integrating both sides over G gives

$$\int_G \exp((c_1 I_n f(x))^{1/(1-q(x))})\ dx \le c_2,$$

which proves the required result. □

THEOREM 8.4. *Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on $\mathbf{R}^n$ satisfying* (1.2) – (1.5) *such that*

$$p(x) > 1 \qquad \text{or} \qquad q(x) \geq 1$$

for all $x \in \overline{G}$. If f is a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} < \infty$, then $I_n f$ is continuous on G.

Proof. Let f be a nonnegative measurable function on G with $\|f\|_{\Phi(\cdot,\cdot)(G)} < \infty$. Then note that

$$\int_G f(y)(\log(e + f(y)))dy < \infty.$$

Hence, it follows from [10, Theorem 9.1, Section 5.9] that $I_n f$ is continuous on G. □

In the same manner as Lemmas 8.2 and 8.3 we can show that

$$I_n f(z) - I_n f(x) = o(|z - x|^{n(1-1/p(x))}(\log(1/|z - x|))^{\gamma})$$

as $z \to x, x \in G$ when $p(x) < n'$ and $\gamma > 2 - (q(x) + 1)/p(x)$.

For further results, we refer the reader to the paper by Ohno ([12]).

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