Orlicz-Sobolev capacity of balls

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Abstract

Our aim in this note is to estimate the Orlicz-Sobolev capacity of balls.

1 Introduction and statement of results

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $I_\alpha f$ of order α by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

In the present note, we treat functions f satisfying an Orlicz condition :

$$\int_{\mathbf{R}^n} \varphi_p(|f(y)|)\, dy < \infty. \tag{1.1}$$

Here $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on $(0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$(\varphi 1)$ $\qquad c_1^{-1}\varphi(r) \le \varphi(r^2) \le c_1 \varphi(r) \qquad$ whenever $r > 0$.

We set

$$\varphi_p(0) = 0,$$

because we will see from $(\varphi 4)$ below that

$$\lim_{r \to 0+} \varphi_p(r) = 0;$$

see [14, p205]. For an open set $G \subset \mathbf{R}^n$, we denote by $L^{\varphi_p}(G)$ the family of all locally integrable functions g on G such that

$$\int_G \varphi_p(|g(x)|)\, dx < \infty,$$

and define

$$\|g\|_{\varphi_p,G} = \inf\left\{\lambda > 0 : \int_G \varphi_p(|g(x)|/\lambda)\, dx \le 1\right\}.$$

This is a quasi-norm in $L^{\varphi_p}(G)$. For $E \subset G$, the (α, φ_p)-capacity is defined by

$$C_{\alpha,\varphi_p}(E; G) = \inf \|f\|_{\varphi_p,G},$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \ge 1 \qquad \text{for all } x \in E$$

(cf. Adams and Hedberg [1], Meyers [10], Ziemer [17] and the second author [11, 12]).

Our aim in the present note is to give an estimate of (α, φ_p)-capacity of balls. We denote by $B(x, r)$ the open ball centered at x of radius r. For $R > 0$, consider

$$\tilde{\varphi}_p(r) = \int_r^R [t^{n-\alpha p}\varphi(t^{-1})]^{-1/(p-1)}\, dt/t.$$

As an extension of Adams and Hurri-Syrjänen [3, Theorem 2.11] and Joensuu [9, Corollary 6.3], we state our theorem in the following.

THEOREM A. *Suppose* $p > 1$ *and*

$$\tilde{\varphi}_p(0) = \infty.$$

For $R > 0$, *there exists a constant* $A > 0$ *such that*

$$A^{-1}\tilde{\varphi}_p(r)^{-(p-1)/p} \le C_{\alpha,\varphi_p}(B(x, r); B(x, R)) \le A\tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < R/2$.

Recently Joensuu [9, Corollary 6.3] treated the case when φ is nondecreasing. His main idea was to use the rearrangement equivalent norm for $\|f\|_{\varphi_p,G}$ ([5, 7, 8]), as an extension of Adams and Hurri-Syrjänen [3, Theorem 2.11] in the case when $\varphi(t) = (\log(e + t))^\beta$ with $p = n/\alpha > 1$ and $0 \le \beta \le p - 1$. Our proof will be done straightforward from the definition of capacity, and several technical assumptions posed in [9] are removed.

Throughout this note, let A denote various constants independent of the variables in question and $A(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

Remark 1.1. If $\tilde{\varphi}_p(0) < \infty$, then $C_{\alpha,\varphi_p}(\{0\}; B(0,R)) > 0$. In this case $I_\alpha f$ is continuous when $f \in L^{\varphi_p}(\mathbf{R}^n)$ vanishes outside a compact set; for this fact, we refer the reader to the paper [14, 16].

Remark 1.2. We here introduce another capacity. For a set $E \subset \mathbf{R}^n$ and an open set $G \subset \mathbf{R}^n$, we define

$$B_{\alpha,\varphi_p}(E; G) = \inf \int_G \varphi_p(f(y))\, dy,$$

where the infimum is taken over all nonnegative measurable functions f on $\mathbf{R}^n$ such that $f = 0$ outside G and $I_\alpha f(x) \geq 1$ for all $x \in E$. With the aid of Adams and Hurri-Syrjänen [3], Joensuu [7, 8, 9] and Mizuta [12, Section 8.3, Lemma 3.1], [11], one can find a constant $A > 1$ such that

$$A^{-1}\tilde{\varphi}_p(r)^{-(p-1)} \leq B_{\alpha,\varphi_p}(B(x,r); B(x,R)) \leq A\tilde{\varphi}_p(r)^{-(p-1)}$$

for $0 < r < R/2$ and $x \in \mathbf{R}^n$. Hence, in view of Theorem A, there is a constant $A > 1$ such that

$$A^{-1}B_{\alpha,\varphi_p}(B(x,r); B(x,R))^{1/p} \leq C_{\alpha,\varphi_p}(B(x,r); B(x,R)) \leq AB_{\alpha,\varphi_p}(B(x,r); B(x,R))^{1/p}$$

for $0 < r < R/2$ and $x \in \mathbf{R}^n$.

We write $f \sim g$ if there exists a constant A so that $A^{-1}g \leq f \leq Ag$.

Example 1.3. For $n = \alpha p$, consider the function

$$\varphi(t) = (\log(e + t))^\beta.$$

If $\beta < p - 1$, then

$$\tilde{\varphi}_p(r) \sim (\log(e + 1/r))^{-\beta/(p-1)+1}$$

for $0 < r < 1$. In this case

$$C_{\alpha,\varphi_p}(B(x_0,r); B(x_0,R)) \sim (\log(e + 1/r))^{(\beta-p+1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

If $\beta = p - 1$, then

$$\tilde{\varphi}_p(r) \sim \log(e + (\log(e + 1/r)))$$

for $0 < r < 1$. In this case

$$C_{\alpha,\varphi_p}(B(x_0,r); B(x_0,R)) \sim (\log(e + (\log(e + 1/r))))^{-(p-1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

For further related results, see Aissaoui and A. Benkirane [4], Adams and Hurri-Syrjänen [2], Edmunds and Evans [6] and Mizuta and Shimomura [14, 15, 16].

2 Proof of Theorem A

First we collect properties which follow from condition ($\varphi 1$) (see [12], [14, Lemma 2.3], [13, Section 7]).

($\varphi 2$) φ satisfies the doubling condition, that is, there exists $c_2 > 1$ such that

$$c_2^{-1}\varphi(r) \le \varphi(2r) \le c_2\varphi(r) \qquad \text{whenever } r > 0.$$

($\varphi 3$) For each $\gamma > 0$, there exists $c_3 = c_3(\gamma) \ge 1$ such that

$$c_3^{-1}\varphi(r) \le \varphi(r^\gamma) \le c_3\varphi(r) \qquad \text{whenever } r > 0.$$

($\varphi 4$) For each $\gamma > 0$, there exists $c_4 = c_4(\gamma) \ge 1$ such that

$$s^\gamma\varphi(s) \le c_4 t^\gamma \varphi(t) \qquad \text{whenever } 0 < s < t.$$

($\varphi 5$) For each $\gamma > 0$, there exists $c_5 = c_5(\gamma) \ge 1$ such that

$$t^{-\gamma}\varphi(t) \le c_5 s^{-\gamma}\varphi(s) \qquad \text{whenever } 0 < s < t.$$

($\varphi 6$) If φ and φ_1 are positive monotone functions on $[0, \infty)$ satisfying ($\varphi 1$), then for each $\gamma > 0$ then there exists a constant $c_6 = c_6(\gamma) \ge 1$ such that

$$c_6{}^{-1}\varphi(r) \le \varphi(r^\gamma\varphi_1(r)) \le c_6\varphi(r) \qquad \text{whenever } r > 0.$$

REMARK 2.1. For each $A_1 > 0$ there exists $A_2 > 0$ such that

$$A_1\varphi_p(r) \ge \varphi_p(A_2 r) \qquad \text{whenever } r > 0. \tag{2.1}$$

REMARK 2.2. If $\alpha p < n$, then we see from ($\varphi 2$) and ($\varphi 5$) that

$$\tilde{\varphi}_p(r) \sim [r^{n-\alpha p}\varphi(r^{-1})]^{-1/(p-1)} \tag{2.2}$$

whenever $0 < r < R/2$.

REMARK 2.3. If $n = \alpha p$ and $0 < R \le 1$, then $\tilde{\varphi}_p$ is of logarithmic type on $[0, R^2]$, that is, there exists $c > 0$ such that

$$c^{-1}\tilde{\varphi}_p(r) \le \tilde{\varphi}_p(r^2) \le c\tilde{\varphi}_p(r) \qquad \text{whenever } 0 \le r \le R^2.$$

In fact, we see from ($\varphi 1$) that

$$\begin{aligned}
\tilde{\varphi}_p(r^2) &= \int_{r^2}^{R} [\varphi(t^{-1})]^{-1/(p-1)}\ dt/t \\
&= \int_{r^2}^{R^2} [\varphi(t^{-1})]^{-1/(p-1)}\ dt/t + \int_{R^2}^{R} [\varphi(t^{-1})]^{-1/(p-1)}\ dt/t \\
&= 2\int_{r}^{R} [\varphi(t^{-2})]^{-1/(p-1)}\ dt/t + \int_{R^2}^{R} [\varphi(t^{-1})]^{-1/(p-1)}\ dt/t \\
&\leq 2c_1^{1/(p-1)} \int_{r}^{R} [\varphi(t^{-1})]^{-1/(p-1)}\ dt/t + \int_{R^2}^{R} [\varphi(t^{-1})]^{-1/(p-1)}\ dt/t \\
&\leq (2c_1^{1/(p-1)} + 1)\tilde{\varphi}_p(r)
\end{aligned}$$

whenever $0 < r \leq R^2$. Since $\tilde{\varphi}_p(r) \leq \tilde{\varphi}_p(r^2)$, we see that $\tilde{\varphi}_p$ is of logarithmic type on $[0, R^2]$.

If $R^2 < r < R$, then one sees that $\tilde{\varphi}_p(r) \sim \varphi(R^{-1})^{-1/(p-1)} \log(R/r)$.

Here let us give an upper estimate of (α, φ_p)-capacity of balls.

LEMMA 2.4. *There exists a constant $A > 0$ such that*

$$C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, 2r)) \leq A[r^{n-\alpha p}\varphi(r^{-1})]^{1/p}$$

whenever $r > 0$ and $x_0 \in \mathbf{R}^n$.

Proof. Without loss of generality we may assume that $x_0 = 0$. For simplicity, set

$$\psi(r) = [r^{n-\alpha p}\varphi(r^{-1})]^{1/p}.$$

For $r > 0$, consider the function

$$f_r(y) = |y|^{-\alpha}$$

for $r < |y| < 2r$ and $f_r = 0$ elsewhere. If $x \in B(0, r)$ and $y \in B(0, 2r)\setminus B(0, r)$, then $|x - y| < 3r$, so that

$$I_\alpha f_r(x) \geq (3r)^{\alpha-n} \int_{B(0,2r)\setminus B(0,r)} |y|^{-\alpha}\ dy = A_1$$

with a constant $A_1 = A_1(\alpha, n) > 0$. It follows from the definition of capacity that

$$C_{\alpha,\varphi_p}(B(0, r); B(0, 2r)) \leq \|f_r/A_1\|_{\varphi_p,B(0,2r)}.$$

Here, in view of ($\varphi 6$) with $\varphi_1(r) = \varphi(r^{-1})^{-1/p}$, we see that

$$\begin{aligned}
\int_{B(0,2r)} \varphi_p(f_r(y)/\psi(r))\ dy &\leq A_2 \int_{B(0,2r)\setminus B(0,r)} r^{-\alpha p}\psi(r)^{-p}\varphi(r^{-1})\ dy \\
&= A_3
\end{aligned}$$

with constants $A_2 = A_2(c_6) > 0$ and $A_3 = A_3(c_6, n) > 0$. Hence, in view of (2.1), we can find $A_4 > 0$ such that

$$\|f_r\|_{\varphi_p, B(0,2r)} \le A_4 \psi(r).$$

Now we establish

$$\begin{aligned} C_{\alpha,\varphi_p}(B(0,r); B(0,2r)) &\le A_1^{-1} \|f_r\|_{\varphi_p, B(0,2r)} \\ &\le A_1^{-1} A_4 \psi(r), \end{aligned}$$

which proves the lemma. □

For $0 < R \le 1$, we take $r_0 = r_0(R) > 0$ such that $r < r\tilde{\varphi}_p(r)^{1/n} \le \sqrt{r}$ for $0 < r < r_0$ and

$$\int_{r_0}^{R} [\varphi(t^{-1})]^{-1/(p-1)} dt/t \ge 2 \int_{R^2}^{R} [\varphi(t^{-1})]^{-1/(p-1)} dt/t. \qquad (2.3)$$

By Lemma 2.4 and Remark 2.2, we obtain the following result.

Corollary 2.5. *Suppose $\alpha p < n$. Then there exists a constant $A > 0$ independent of R such that*

$$C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \le A \tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

Next we prove the following result.

Lemma 2.6. *Let $\alpha p = n$ and $0 < R \le 1$. Then there exists a constant $A > 0$ independent of R such that*

$$C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \le A \tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < r_0$ and $x_0 \in \mathbf{R}^n$.

Proof. Suppose $\alpha p = n$, $0 < R \le 1$ and $x_0 = 0$. For $0 < r < r_0$ and $0 < K < 1$, consider the function

$$f_{r,K}(y) = |y|^{-\alpha} [\varphi(K|y|^{-1})]^{-1/(p-1)}$$

for $r < |y| < KR$ and $f_{r,K} = 0$ elsewhere. If $x \in B(0, r)$ and $y \in B(0, R) \setminus B(0, r)$, then $|x - y| < 2|y|$, so that

$$\begin{aligned} I_\alpha f_{r,K}(x) &\ge 2^{\alpha-n} \int_{B(0,KR)\setminus B(0,r)} |y|^{\alpha-n} f_{r,K}(y)\, dy \\ &\ge 2^{\alpha-n} \omega_{n-1} \int_{r}^{KR} [\varphi(K/t)]^{-1/(p-1)}\, dt/t \\ &= 2^{\alpha-n} \omega_{n-1} \tilde{\varphi}_p(r/K), \end{aligned}$$

where ω_{n-1} is the surface measure of the boundary of the unit ball in $\mathbf{R}^n$. If $K = \tilde{\varphi}_p(r)^{-1/n}(<1)$, then we see from ($\varphi$1) and (2.3) that

$$\begin{aligned}
\tilde{\varphi}_p(r/K) &= \int_{r/K}^{R} [\varphi(1/t)]^{-1/(p-1)}\, dt/t \\
&\geq \int_{\sqrt{r}}^{R} [\varphi(1/t)]^{-1/(p-1)}\, dt/t \\
&\geq 2c_1^{-1/(p-1)} \int_{r}^{R^2} [\varphi(1/t)]^{-1/(p-1)}\, dt/t \\
&\geq 2c_1^{-1/(p-1)} \left(\int_{r}^{R} [\varphi(1/t)]^{-1/(p-1)}\, dt/t - 2^{-1} \int_{r_0}^{R} [\varphi(1/t)]^{-1/(p-1)}\, dt/t \right) \\
&\geq c_1^{-1/(p-1)} \tilde{\varphi}_p(r).
\end{aligned}$$

Thus it follows that

$$I_\alpha f_{r,K}(x) \geq 2^{\alpha-n} \omega_{n-1} c_1^{-1/(p-1)} \tilde{\varphi}_p(r) = A_1 \tilde{\varphi}_p(r)$$

with a constant $A_1 = 2^{\alpha-n}\omega_{n-1}c_1^{-1/(p-1)}$, which implies

$$C_{\alpha,\varphi_p}(B(0,r); B(0,R)) \leq \|f_{r,K}/\{A_1\tilde{\varphi}_p(r)\}\|_{\varphi_p, B(0,R)} = \{A_1\tilde{\varphi}_p(r)\}^{-1}\|f_{r,K}\|_{\varphi_p, B(0,R)}.$$

Here note from (φ6) with $\varphi_1(r) = \varphi(r)^{-1/p}$ that

$$\begin{aligned}
&\int_{B(0,KR)} \varphi_p(K^\alpha f_{r,K}(y))\, dy \\
\leq\ & c_6 \int_{B(0,KR)\setminus B(0,r)} (K/|y|)^{\alpha p} [\varphi(K|y|^{-1})]^{-p/(p-1)} \varphi(K|y|^{-1})\, dy \\
=\ & A_2 K^{\alpha p} \int_{r}^{KR} [\varphi(K/t)]^{-1/(p-1)}\, dt/t \leq A_2
\end{aligned}$$

with $K = \tilde{\varphi}_p(r)^{-1/n}$ and $A_2 = c_6\omega_{n-1}$. This implies by (2.1) that there exists a constant $A_3 > 0$ such that

$$\|f_{r,K}\|_{\varphi_p, B(0,R)} \leq A_3 K^{-\alpha} = A_3 \tilde{\varphi}_p(r)^{1/p}.$$

Now it follows that

$$\begin{aligned}
C_{\alpha,\varphi_p}(B(0,r); B(0,R)) &\leq A_1^{-1}\tilde{\varphi}_p(r)^{-1}\|f_{r,K}\|_{\varphi_p, B(0,R)} \\
&\leq A_1^{-1} A_3 \tilde{\varphi}_p(r)^{-1+1/p}.
\end{aligned}$$

Thus the lemma is proved. □

By Corollary 2.5 and Lemma 2.6, we find the following result.

THEOREM 2.7. *Suppose $p > 1$ and $0 < R \le 1$. Then there exist constants $A > 0$ independent of R and $r_0 = r_0(R) > 0$ such that*

$$C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \le A\tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < r_0$ and $x_0 \in \mathbf{R}^n$.

REMARK 2.8. Suppose $p > 1$. Then for each $R > 0$ one can find a constant $A(R) > 0$ such that

$$C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \le A(R)\tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

In fact, if $0 < R \le 1$ and $0 < r < r_0$, then this is a consequence of Theorem 2.7. If $0 < R \le 1$ and $r_0 \le r < R/2$, then

$$C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \le C_{\alpha,\varphi_p}(B(x_0, R/2); B(x_0, R))$$

and hence one can take $A(R) > 0$ such that

$$C_{\alpha,\varphi_p}(B(x_0, R/2); B(x_0, R)) \le A(R)\tilde{\varphi}_p(r_0)^{-(p-1)/p}.$$

The case $R \ge 1$ is similarly treated.

Next we give a lower estimate of (α, φ_p)-capacity of balls.

THEOREM 2.9. *For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$\tilde{\varphi}_p(r)^{-(p-1)/p} \le AC_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R))$$

whenever $0 < r < R/2 < \infty$ and $x_0 \in \mathbf{R}^n$.

Proof. As above we assume that $x_0 = 0$. For $0 < r < R/2$, take a nonnegative measurable function f on $B(0, R)$ such that

$$I_\alpha f(x) \ge 1 \qquad \text{for } x \in B(0, r).$$

Then we have by Fubini's theorem

$$\begin{aligned}
\int_{B(0,r)} dx &\le \int_{B(0,r)} I_\alpha f(x)\, dx \\
&\le \int_{B(0,R)} \left(\int_{B(0,r)} |x - y|^{\alpha-n}\, dx \right) f(y)\, dy \\
&\le A_1 r^n \int_{B(0,R)} (r + |y|)^{\alpha-n} f(y)\, dy,
\end{aligned}$$

so that

$$1 \le A_1 \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\ dy. \tag{2.4}$$

We show that

$$\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\ dy \le A_2 \tilde{\varphi}_p(r)^{-1/p+1} \|f\|_{\varphi_p, B(0,R)}. \tag{2.5}$$

For this purpose, suppose $\|f\|_{\varphi_p, B(0,R)} \le 1$. Then, considering

$$k(y) = \tilde{\varphi}_p(r+|y|)^{-1/p}(r+|y|)^{-\alpha}[(r+|y|)^{n-\alpha p}\varphi((r+|y|)^{-1})]^{-1/(p-1)},$$

we find by (φ4), (φ6) and Remark 2.2

$$\begin{aligned}
& \int_{B(0,R/2)} (r+|y|)^{\alpha-n} f(y)\ dy \\
\le\ & \int_{B(0,R/2)} (r+|y|)^{\alpha-n} k(y)\ dy \\
& + A_3 \int_{B(0,R/2)} (r+|y|)^{\alpha-n} f(y) \left(\frac{f(y)}{k(y)}\right)^{p-1} \frac{\varphi(f(y))}{\varphi(k(y))}\ dy \\
\le\ & A_4 \left\{ \int_r^R \tilde{\varphi}_p(t)^{-1/p}[t^{n-\alpha p}\varphi(t^{-1})]^{-1/(p-1)}\ dt/t \right. \\
& \left. + \int_{B(0,R)} \tilde{\varphi}_p(r+|y|)^{(p-1)/p} \varphi_p(f(y))\ dy \right\} \\
\le\ & A_5 \left\{ \tilde{\varphi}_p(r)^{1-1/p} + \tilde{\varphi}_p(r)^{(p-1)/p} \int_{B(0,R)} \varphi_p(f(y))\ dy \right\} \\
\le\ & 2A_5 \tilde{\varphi}_p(r)^{1-1/p}.
\end{aligned}$$

Next, considering

$$\begin{aligned}
k &= \tilde{\varphi}_p(R/2)^{-1/p}(R/2)^{-\alpha}[(R/2)^{n-\alpha p}\varphi((R/2)^{-1})]^{-1/(p-1)} \\
&\sim \tilde{\varphi}_p(R/2)^{1-1/p}(R/2)^{-\alpha},
\end{aligned}$$

we find by (φ4), (φ6) and Remark 2.2

$$\begin{aligned}
&\int_{B(0,R)\setminus B(0,R/2)} (r+|y|)^{\alpha-n} f(y)\,dy \\
\leq\ & (R/2)^{\alpha-n} \int_{B(0,R)\setminus B(0,R/2)} f(y)\,dy \\
\leq\ & (R/2)^{\alpha-n} \int_{B(0,R)\setminus B(0,R/2)} k\,dy \\
& + A_6 (R/2)^{\alpha-n} \int_{B(0,R)\setminus B(0,R/2)} f(y) \left(\frac{f(y)}{k}\right)^{p-1} \frac{\varphi(f(y))}{\varphi(k)}\,dy \\
\leq\ & A_7 \tilde{\varphi}_p (R/2)^{1-1/p} \left(1 + \int_{B(0,R)} \varphi_p(f(y))\,dy\right) \\
\leq\ & 2A_7 \tilde{\varphi}_p (R/2)^{1-1/p} \\
\leq\ & 2A_7 \tilde{\varphi}_p (r)^{1-1/p}.
\end{aligned}$$

Thus

$$\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y)\,dy \leq A_8 \tilde{\varphi}_p(r)^{1-1/p}$$

whenever $\|f\|_{\varphi_p, B(0,R)} \leq 1$, which implies (2.5).

In view of (2.4), (2.5) and the definition of capacity, we find

$$1 \leq A_9 \tilde{\varphi}_p(r)^{1-1/p} C_{\alpha,\varphi_p}(B(0,r); B(0,R)),$$

which gives the conclusion. □

Proof of Theorem A. Theorem A follows from Theorems 2.7 and 2.9 together with Remark 2.8.

□

3 C_{α,φ_1}-capacity

In this section, we deal with the case $p = 1$. For this purpose, set

$$\varphi_1(r) = r\varphi(r)$$

and

$$\tilde{\varphi}_1(r) = r^{n-\alpha}\varphi(r^{-1}).$$

Here suppose further that $\varphi(r)$ is nondecreasing on $(0,\infty)$.

THEOREM B. *For $R > 0$, there exists a constant $A > 0$ such that*

$$A^{-1}\tilde{\varphi}_1(r) \leq C_{\alpha,\varphi_1}(B(x,r); B(x,R)) \leq A\tilde{\varphi}_1(r)$$

whenever $0 < r < R/2$.

The proof is quite similar to that of Theorem A, and thus we omit it.

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