Orlicz-Sobolev capacity of balls

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Abstract

Our aim in this note is to estimate the Orlicz-Sobolev capacity of balls.

1 Introduction and statement of results

For $0 < \alpha < n$ and a locally integrable function $f$ on $\mathbb{R}^n$, we define the Riesz potential $I_\alpha f$ of order $\alpha$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y) \, dy.$$ 

In the present note, we treat functions $f$ satisfying an Orlicz condition:

$$\int_{\mathbb{R}^n} \varphi_p(|f(y)|) \, dy < \infty. \quad (1.1)$$

Here $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on $(0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$c_1^{-1} \varphi(r) \leq \varphi(r^2) \leq c_1 \varphi(r) \quad \text{whenever } r > 0. \quad (\varphi 1)$$

We set

$$\varphi_p(0) = 0,$$

because we will see from $(\varphi 4)$ below that

$$\lim_{r \to 0^+} \varphi_p(r) = 0;$$

$$\lim_{r \to 0^-} \varphi_p(r) = 0;$$

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see [14, p205]. For an open set $G \subset \mathbb{R}^n$, we denote by $L^{\varphi_p}(G)$ the family of all locally integrable functions $g$ on $G$ such that
\[
\int_G \varphi_p(|g(x)|) \, dx < \infty,
\]
and define
\[
\|g\|_{\varphi_p, G} = \inf \left\{ \lambda > 0 : \int_G \varphi_p(|g(x)|/\lambda) \, dx \leq 1 \right\}.
\]
This is a quasi-norm in $L^{\varphi_p}(G)$. For $E \subset G$, the $(\alpha, \varphi_p)$-capacity is defined by
\[
C_{\alpha, \varphi_p}(E; G) = \inf \|f\|_{\varphi_p, G},
\]
where the infimum is taken over all functions $f$ such that $f = 0$ outside $G$ and $I_{\alpha} f(x) \geq 1$ for all $x \in E$ (cf. Adams and Hedberg [1], Meyers [10], Ziemer [17] and the second author [11, 12]).

Our aim in the present note is to give an estimate of $(\alpha, \varphi_p)$-capacity of balls. We denote by $B(x, r)$ the open ball centered at $x$ of radius $r$. For $R > 0$, consider
\[
\tilde{\varphi}_p(r) = \int_r^R [t^{n-\alpha p} \varphi(t^{-1})]^{-1/(p-1)} \, dt / t.
\]
As an extension of Adams and Hurri-Syrjänen [3, Theorem 2.11] and Joensuu [9, Corollary 6.3], we state our theorem in the following.

**Theorem A.** Suppose $p > 1$ and
\[
\tilde{\varphi}_p(0) = \infty.
\]
For $R > 0$, there exists a constant $A > 0$ such that
\[
A^{-1} \tilde{\varphi}_p(r)^{(p-1)/p} \leq C_{\alpha, \varphi_p}(B(x, r); B(x, R)) \leq A \tilde{\varphi}_p(r)^{(p-1)/p}
\]
whenever $0 < r < R/2$.

Recently Joensuu [9, Corollary 6.3] treated the case when $\varphi$ is nondecreasing. His main idea was to use the rearrangement equivalent norm for $\|f\|_{\varphi_p, G}$ ([5, 7, 8]), as an extension of Adams and Hurri-Syrjänen [3, Theorem 2.11] in the case when $\varphi(t) = (\log(e + t))^\beta$ with $p = n/\alpha > 1$ and $0 \leq \beta \leq p - 1$. Our proof will be done straightforward from the definition of capacity, and several technical assumptions posed in [9] are removed.

Throughout this note, let $A$ denote various constants independent of the variables in question and $A(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$. 
Remark 1.1. If \( \tilde{\varphi}_p(0) < \infty \), then \( C_{\alpha,\varphi_p}(\{0\}; B(0, R)) > 0 \). In this case \( I_\alpha f \) is continuous when \( f \in L^{p^*}(\mathbb{R}^n) \) vanishes outside a compact set; for this fact, we refer the reader to the paper [14, 16].

Remark 1.2. We here introduce another capacity. For a set \( E \subset \mathbb{R}^n \) and an open set \( G \subset \mathbb{R}^n \), we define

\[
B_{\alpha,\varphi_p}(E; G) = \inf \int_G \varphi_p(f(y)) \, dy,
\]

where the infimum is taken over all nonnegative measurable functions \( f \) on \( \mathbb{R}^n \) such that \( f = 0 \) outside \( G \) and \( I_\alpha f(x) \geq 1 \) for all \( x \in E \). With the aid of Adams and Hurri-Syrjänen [3], Joensuu [7, 8, 9] and Mizuta [12, Section 8.3, Lemma 3.1], [11], one can find a constant \( A > 1 \) such that

\[
A^{-1} \tilde{\varphi}_p(r)^{-(p-1)} \leq B_{\alpha,\varphi_p}(B(x, r); B(x, R)) \leq A \tilde{\varphi}_p(r)^{-(p-1)}
\]

for \( 0 < r < R/2 \) and \( x \in \mathbb{R}^n \). Hence, in view of Theorem A, there is a constant \( A > 1 \) such that

\[
A^{-1} B_{\alpha,\varphi_p}(B(x, r); B(x, R))^{1/p} \leq C_{\alpha,\varphi_p}(B(x, r); B(x, R)) \leq A B_{\alpha,\varphi_p}(B(x, r); B(x, R))^{1/p}
\]

for \( 0 < r < R/2 \) and \( x \in \mathbb{R}^n \).

We write \( f \sim g \) if there exists a constant \( A \) so that \( A^{-1} g \leq f \leq A g \).

Example 1.3. For \( n = \alpha p \), consider the function

\[
\varphi(t) = (\log(e + t))^\beta.
\]

If \( \beta < p - 1 \), then

\[
\tilde{\varphi}_p(r) \sim (\log(e + 1/r))^{-\beta/(p-1)+1}
\]

for \( 0 < r < 1 \). In this case

\[
C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \sim (\log(e + 1/r))^{(\beta-p+1)/p}
\]

whenever \( 0 < r < R/2 \) and \( x_0 \in \mathbb{R}^n \).

If \( \beta = p - 1 \), then

\[
\tilde{\varphi}_p(r) \sim \log(e + (\log(e + 1/r)))
\]

for \( 0 < r < 1 \). In this case

\[
C_{\alpha,\varphi_p}(B(x_0, r); B(x_0, R)) \sim (\log(e + (\log(e + 1/r))))^{-(p-1)/p}
\]

whenever \( 0 < r < R/2 \) and \( x_0 \in \mathbb{R}^n \).

For further related results, see Aissaoui and A. Benkirane [4], Adams and Hurri-Syrjänen [2], Edmunds and Evans [6] and Mizuta and Shimomura [14, 15, 16].
2 Proof of Theorem A

First we collect properties which follow from condition (\(\varphi 1\)) (see [12], [14, Lemma 2.3], [13, Section 7]).

(\(\varphi 2\)) \(\varphi\) satisfies the doubling condition, that is, there exists \(c_2 > 1\) such that
\[
c^{-1}_2 \varphi(r) \leq \varphi(2r) \leq c_2 \varphi(r) \quad \text{whenever } r > 0.
\]

(\(\varphi 3\)) For each \(\gamma > 0\), there exists \(c_3 = c_3(\gamma) \geq 1\) such that
\[
c^{-1}_3 \varphi(r) \leq \varphi(r^\gamma) \leq c_3 \varphi(r) \quad \text{whenever } r > 0.
\]

(\(\varphi 4\)) For each \(\gamma > 0\), there exists \(c_4 = c_4(\gamma) \geq 1\) such that
\[
s^\gamma \varphi(s) \leq c_4 t^\gamma \varphi(t) \quad \text{whenever } 0 < s < t.
\]

(\(\varphi 5\)) For each \(\gamma > 0\), there exists \(c_5 = c_5(\gamma) \geq 1\) such that
\[
t^{-\gamma} \varphi(t) \leq c_5 s^{-\gamma} \varphi(s) \quad \text{whenever } 0 < s < t.
\]

(\(\varphi 6\)) If \(\varphi\) and \(\varphi_1\) are positive monotone functions on \([0, \infty)\) satisfying (\(\varphi 1\)), then for each \(\gamma > 0\) then there exists a constant \(c_6 = c_6(\gamma) \geq 1\) such that
\[
c^{-1}_6 \varphi(r) \leq \varphi(r^\gamma \varphi_1(r)) \leq c_6 \varphi(r) \quad \text{whenever } r > 0.
\]

Remark 2.1. For each \(A_1 > 0\) there exists \(A_2 > 0\) such that
\[
A_1 \varphi_p(r) \geq \varphi_p(A_2 r) \quad \text{whenever } r > 0. \tag{2.1}
\]

Remark 2.2. If \(\alpha p < n\), then we see from (\(\varphi 2\)) and (\(\varphi 5\)) that
\[
\tilde{\varphi}_p(r) \sim [r^{n-\alpha p} \varphi(r^{-1})]^{-1/(p-1)} \tag{2.2}
\]
whenever \(0 < r < R/2\).

Remark 2.3. If \(n = \alpha p\) and \(0 < R \leq 1\), then \(\tilde{\varphi}_p\) is of logarithmic type on \([0, R^2]\), that is, there exists \(c > 0\) such that
\[
c^{-1} \tilde{\varphi}_p(r) \leq \tilde{\varphi}_p(r^2) \leq c \tilde{\varphi}_p(r) \quad \text{whenever } 0 \leq r \leq R^2.
\]
In fact, we see from (φ1) that

\[
\tilde{\varphi}_p(r^2) = \int_{r^2}^R [\varphi(t^{-1})]^{-1/(p-1)} \, dt/t
\]

\[
= \int_{r^2}^R [\varphi(t^{-1})]^{-1/(p-1)} \, dt/t + \int_{r^2}^R [\varphi(t^{-1})]^{-1/(p-1)} \, dt/t
\]

\[
= 2 \int_{r}^R [\varphi(t^{-2})]^{-1/(p-1)} \, dt/t + \int_{R^2}^R [\varphi(t^{-1})]^{-1/(p-1)} \, dt/t
\]

\[
\leq 2c_1^{1/(p-1)} \int_{r}^R [\varphi(t^{-1})]^{-1/(p-1)} \, dt/t + \int_{R^2}^R [\varphi(t^{-1})]^{-1/(p-1)} \, dt/t
\]

\[
\leq (2c_1^{1/(p-1)} + 1)\tilde{\varphi}_p(r)
\]

whenever \(0 < r \leq R^2\). Since \(\tilde{\varphi}_p(r) \leq \tilde{\varphi}_p(r^2)\), we see that \(\tilde{\varphi}_p\) is of logarithmic type on \([0, R^2]\).

If \(R^2 < r < R\), then one sees that \(\tilde{\varphi}_p(r) \sim \varphi(R^{-1})^{-1/(p-1)} \log(R/r)\).

Here let us give an upper estimate of \((\alpha, \varphi_p)\)-capacity of balls.

**Lemma 2.4.** There exists a constant \(A > 0\) such that

\[
C_{\alpha, \varphi_p}(B(x_0, r); B(x_0, 2r)) \leq A|\varphi|^{n-\alpha}p\varphi(p^{-1})^{1/p}
\]

whenever \(r > 0\) and \(x_0 \in \mathbb{R}^n\).

**Proof.** Without loss of generality we may assume that \(x_0 = 0\). For simplicity, set

\[
\psi(r) = |r^{n-\alpha}\varphi(r^{-1})|^{1/p}
\]

For \(r > 0\), consider the function

\[
f_r(y) = |y|^{-\alpha}
\]

for \(r < |y| < 2r\) and \(f_r = 0\) elsewhere. If \(x \in B(0, r)\) and \(y \in B(0, 2r) \setminus B(0, r)\), then \(|x - y| < 3r\), so that

\[
I_{\alpha}f_r(x) \geq (3r)^{n-\alpha} \int_{B(0, 2r) \setminus B(0, r)} |y|^{-\alpha} \, dy = A_1
\]

with a constant \(A_1 = A_1(\alpha, n) > 0\). It follows from the definition of capacity that

\[
C_{\alpha, \varphi_p}(B(0, r); B(0, 2r)) \leq \|f_r/A_1\|_{\varphi_p, B(0, 2r)}.
\]

Here, in view of (φ6) with \(\varphi_1(r) = \varphi(r^{-1})^{-1/p}\), we see that

\[
\int_{B(0, 2r)} \varphi_p(f_r(y)/\psi(r)) \, dy \leq A_2 \int_{B(0, 2r) \setminus B(0, r)} r^{-\alpha p}\psi(r)^{-p}\varphi(r^{-1}) \, dy
\]

\[
= A_3
\]
with constants $A_2 = A_2(c_6) > 0$ and $A_3 = A_3(c_6, n) > 0$. Hence, in view of (2.1), we can find $A_4 > 0$ such that
\[ \|f_r\|_{\mathcal{P}, B(0, 2r)} \leq A_4 \psi(r). \]
Now we establish
\[ C_{\alpha, \varphi_p}(B(0, r); B(0, 2r)) \leq A_1^{-1} \|f_r\|_{\mathcal{P}, B(0, 2r)} \leq A_1^{-1} A_4 \psi(r), \]
which proves the lemma. □

For $0 < R \leq 1$, we take $r_0 = r_0(R) > 0$ such that $r < r_0 \tilde{\varphi}_p(r)^{1/n} \leq \sqrt{r}$ for $0 < r < r_0$ and
\[ \int_{r_0}^R \left[ \varphi(t^{-1}) \right]^{-1/(p-1)} dt/t \geq 2 \int_{R^2} \left[ \varphi(t^{-1}) \right]^{-1/(p-1)} dt/t. \quad (2.3) \]

By Lemma 2.4 and Remark 2.2, we obtain the following result.

Corollary 2.5. Suppose $\alpha p < n$. Then there exists a constant $A > 0$ independent of $R$ such that
\[ C_{\alpha, \varphi_p}(B(x_0, r); B(x_0, R)) \leq A \tilde{\varphi}_p(r)^{-(p-1)/p} \]
whenever $0 < r < R/2$ and $x_0 \in \mathbb{R}^n$.

Next we prove the following result.

Lemma 2.6. Let $\alpha p = n$ and $0 < R \leq 1$. Then there exists a constant $A > 0$ independent of $R$ such that
\[ C_{\alpha, \varphi_p}(B(x_0, r); B(x_0, R)) \leq A \tilde{\varphi}_p(r)^{-(p-1)/p} \]
whenever $0 < r < r_0$ and $x_0 \in \mathbb{R}^n$.

Proof. Suppose $\alpha p = n$, $0 < R \leq 1$ and $x_0 = 0$. For $0 < r < r_0$ and $0 < K < 1$, consider the function
\[ f_{r, K}(y) = |y|^{-\alpha} \left[ \varphi(K|y|^{-1}) \right]^{-1/(p-1)} \]
for $r < |y| < KR$ and $f_{r, K} = 0$ elsewhere. If $x \in B(0, r)$ and $y \in B(0, R) \setminus B(0, r)$, then $|x - y| < 2|y|$, so that
\[ I_{\alpha f_{r, K}}(x) \geq 2^{\alpha-n} \int_{B(0, KR) \setminus B(0, r)} |y|^{\alpha-n} f_{r, K}(y) \, dy \]
\[ \geq 2^{\alpha-n} \omega_{n-1} \int_r^{KR} \left[ \varphi(K/t) \right]^{-1/(p-1)} dt/t \]
\[ = 2^{\alpha-n} \omega_{n-1} \tilde{\varphi}_p(r/K), \]
where $\omega_{n-1}$ is the surface measure of the boundary of the unit ball in $\mathbb{R}^n$. If $K = \tilde{\varphi}_p(r)^{-1/n}(< 1)$, then we see from (φ1) and (2.3) that

$$
\tilde{\varphi}_p(r/K) = \int_{r/K}^R [\varphi(1/t)]^{-1/(p-1)} \, dt/t
$$

$$
\geq \int_{\sqrt{K}}^R [\varphi(1/t)]^{-1/(p-1)} \, dt/t
$$

$$
\geq 2c_1^{-1/(p-1)} \int_{r}^{R^2} [\varphi(1/t)]^{-1/(p-1)} \, dt/t
$$

$$
\geq 2c_1^{-1/(p-1)} \left( \int_{r}^{R} [\varphi(1/t)]^{-1/(p-1)} \, dt/t \right) - 2^{-1} \int_{r_0}^{R} [\varphi(1/t)]^{-1/(p-1)} \, dt/t
$$

$$
\geq c_1^{-1/(p-1)} \tilde{\varphi}_p(r).
$$

Thus it follows that

$$
I_{\alpha} f_{r,K}(x) \geq \frac{2^{\alpha-n-n} \omega_{n-1}}{c_1^{1/(p-1)}} \tilde{\varphi}_p(r) = A_1 \tilde{\varphi}_p(r)
$$

with a constant $A_1 = 2^{\alpha-n-n} \omega_{n-1} c_1^{-1/(p-1)}$, which implies

$$
C_{\alpha,\varphi_p}(B(0,r); B(0,R)) \leq \| f_{r,K}/\{A_1 \tilde{\varphi}_p(r)\}\|_{\varphi_p,B(0,R)} = \{A_1 \tilde{\varphi}_p(r)\}^{-1} \| f_{r,K}\|_{\varphi_p,B(0,R)}.
$$

Here note from (φ6) with $\varphi_1(r) = \varphi(r)^{-1/p}$ that

$$
\int_{B(0,KR)} \varphi_p(K \alpha f_{r,K}(y)) \, dy
$$

$$
\leq c_6 \left( K^{-1/\alpha} \varphi(K \alpha \| f_{r,K}\|_{\varphi_p,B(0,R)} K^{-1}) \right) \beta \{ \varphi(1/t) \}^{1/\alpha} \varphi(K \alpha \| f_{r,K}\|_{\varphi_p,B(0,R)}) \, dy
$$

$$
= A_2 K\alpha \int_{r}^{KR} [\varphi(1/t)]^{-1/(p-1)} \, dt/t \leq A_2
$$

with $K = \tilde{\varphi}_p(r)^{-1/\alpha}$ and $A_2 = c_6 \omega_{n-1}$. This implies by (2.1) that there exists a constant $A_3 > 0$ such that

$$
\| f_{r,K}\|_{\varphi_p,B(0,R)} \leq A_3 K^{-\alpha} = A_3 \tilde{\varphi}_p(r)^{-1/p}.
$$

Now it follows that

$$
C_{\alpha,\varphi_p}(B(0,r); B(0,R)) \leq A_1^{-1} \tilde{\varphi}_p(r)^{-1} \| f_{r,K}\|_{\varphi_p,B(0,R)}
$$

$$
\leq A_1^{-1} A_3 \tilde{\varphi}_p(r)^{-1+1/p}.
$$

Thus the lemma is proved.
By Corollary 2.5 and Lemma 2.6, we find the following result.

**Theorem 2.7.** Suppose \( p > 1 \) and \( 0 < R \leq 1 \). Then there exist constants \( A > 0 \) independent of \( R \) and \( r_0 = r_0(R) > 0 \) such that

\[
C_{\alpha, \varphi_p}(B(x_0, r); B(x_0, R)) \leq A \tilde{\varphi}_p(r)^{-(p-1)/p}
\]

whenever \( 0 < r < r_0 \) and \( x_0 \in \mathbb{R}^n \).

**Remark 2.8.** Suppose \( p > 1 \). Then for each \( R > 0 \) one can find a constant \( A(R) > 0 \) such that

\[
C_{\alpha, \varphi_p}(B(x_0, r); B(x_0, R)) \leq A(R) \tilde{\varphi}_p(r)^{-(p-1)/p}
\]

whenever \( 0 < r < R/2 \) and \( x_0 \in \mathbb{R}^n \).

In fact, if \( 0 < R \leq 1 \) and \( 0 < r < r_0 \), then this is a consequence of Theorem 2.7. If \( 0 < R \leq 1 \) and \( r_0 \leq r < R/2 \), then

\[
C_{\alpha, \varphi_p}(B(x_0, R/2); B(x_0, R)) = C_{\alpha, \varphi_p}(B(x_0, R/2); B(x_0, R))
\]

and hence one can take \( A(R) > 0 \) such that

\[
C_{\alpha, \varphi_p}(B(x_0, R/2); B(x_0, R)) \leq A(R) \tilde{\varphi}_p(r_0)^{-(p-1)/p}.
\]

The case \( R \geq 1 \) is similarly treated.

Next we give a lower estimate of \((\alpha, \varphi_p)\)-capacity of balls.

**Theorem 2.9.** For \( R > 0 \), there exists a constant \( A = A(R) > 0 \) such that

\[
\tilde{\varphi}_p(r)^{-(p-1)/p} \leq AC_{\alpha, \varphi_p}(B(x_0, r); B(x_0, R))
\]

whenever \( 0 < r < R/2 < \infty \) and \( x_0 \in \mathbb{R}^n \).

**Proof.** As above we assume that \( x_0 = 0 \). For \( 0 < r < R/2 \), take a nonnegative measurable function \( f \) on \( B(0, R) \) such that

\[
I_\alpha f(x) \geq 1 \quad \text{for } x \in B(0, r).
\]

Then we have by Fubini’s theorem

\[
\int_{B(0, r)} dx \leq \int_{B(0, r)} I_\alpha f(x) dx
\]

\[
\leq \int_{B(0, R)} \left( \int_{B(0, r)} |x-y|^{\alpha-n} dx \right) f(y) dy
\]

\[
\leq A_1 r^n \int_{B(0, R)} (r+|y|)^{\alpha-n} f(y) dy,
\]
so that

\[ 1 \leq A_1 \int_{B(0,R)} (r + |y|)^{\alpha - n} f(y) \, dy. \tag{2.4} \]

We show that

\[ \int_{B(0,R)} (r + |y|)^{\alpha - n} f(y) \, dy \leq A_2 \tilde{\varphi}_p(r)^{-1/p+1}\|f\|_{\varphi_p,B(0,R)}. \tag{2.5} \]

For this purpose, suppose \( \|f\|_{\varphi_p,B(0,R)} \leq 1 \). Then, considering

\[ k(y) = \tilde{\varphi}_p(r + |y|)^{-1/p}(r + |y|)^{-\alpha}[(r + |y|)^{n-\alpha p}\varphi((r + |y|)^{-1})]^{-1/(p-1)}, \]

we find by (\( \varphi_4 \)), (\( \varphi_6 \)) and Remark 2.2

\[
\begin{align*}
&\int_{B(0,R/2)} (r + |y|)^{\alpha - n} f(y) \, dy \\
&\leq \int_{B(0,R/2)} (r + |y|)^{\alpha - n} k(y) \, dy \\
&\quad + A_3 \int_{B(0,R/2)} (r + |y|)^{n - n} f(y) \left( \frac{f(y)}{k(y)} \right)^{p-1} \frac{\varphi(f(y))}{\varphi(k(y))} \, dy \\
&\leq A_4 \left\{ \int_{B(0,R/2)} \tilde{\varphi}_p(t)^{-1/p} [t^{n-\alpha p}\varphi(t^{-1})]^{-1/(p-1)} \, dt/t \\
&\quad + \int_{B(0,R)} \tilde{\varphi}_p(r + |y|)^{(p-1)/p}\varphi(f(y)) \, dy \right\} \\
&\leq A_5 \left\{ \tilde{\varphi}_p(r)^{1-1/p} + \tilde{\varphi}_p(r)^{1/p-1/p} \int_{B(0,R)} \varphi(f(y)) \, dy \right\} \\
&\leq 2A_5 \tilde{\varphi}_p(r)^{1-1/p}.
\end{align*}
\]

Next, considering

\[ k = \tilde{\varphi}_p(R/2)^{-1/p}(R/2)^{-\alpha}[(R/2)^{n-\alpha p}\varphi((R/2)^{-1})]^{-1/(p-1)} \]

\[ \sim \tilde{\varphi}_p(R/2)^{1-1/p}(R/2)^{-\alpha}, \]
we find by ($\phi_4$), ($\phi_6$) and Remark 2.2

$$\int_{B(0,R) \setminus B(0,R/2)} (r + |y|)^{\alpha \alpha} f(y) \, dy$$

$$\leq (R/2)^{\alpha \alpha} \int_{B(0,R) \setminus B(0,R/2)} f(y) \, dy$$

$$\leq (R/2)^{\alpha \alpha} \int_{B(0,R) \setminus B(0,R/2)} k \, dy$$

$$+ A_6(R/2)^{\alpha \alpha} \int_{B(0,R) \setminus B(0,R/2)} f(y) \left( \frac{f(y)}{k} \right)^{p-1} \frac{\varphi(f(y))}{\varphi(k)} \, dy$$

$$\leq A_7 \tilde{\varphi}_p(R/2)^{1-1/p} \left( 1 + \int_{B(0,R)} \varphi_p(f(y)) \, dy \right)$$

$$\leq 2A_7 \tilde{\varphi}_p(r)^{1-1/p}$$

Thus

$$\int_{B(0,R)} (r + |y|)^{\alpha \alpha} f(y) \, dy \leq A_7 \tilde{\varphi}_p(r)^{1-1/p}$$

denotes arbitrary constant $A > 0$ such that

whenever $\|f\|_{\varphi_p, B(0,R)} \leq 1$, which implies (2.5).

In view of (2.4), (2.5) and the definition of capacity, we find

$$1 \leq A_7 \tilde{\varphi}_p(r)^{1-1/p} C_{\alpha,\varphi_1,\varphi_p}(B(0,r); B(0,R)),$$

which gives the conclusion.

Proof of Theorem A. Theorem A follows from Theorems 2.7 and 2.9 together with Remark 2.8.

3 \hspace{1cm} C_{\alpha,\varphi_1,-capacity}

In this section, we deal with the case $p = 1$. For this purpose, set

$$\varphi_1(r) = r \varphi(r)$$

and

$$\tilde{\varphi}_1(r) = r^{n-\alpha} \varphi(r^{-1}).$$

Here suppose further that $\varphi(r)$ is nondecreasing on $(0, \infty)$.

Theorem B. For $R > 0$, there exists a constant $A > 0$ such that

$$A^{-1} \tilde{\varphi}_1(r) \leq C_{\alpha,\varphi_1}(B(x,r); B(x,R)) \leq A \tilde{\varphi}_1(r)$$
whenever $0 < r < R/2$.

The proof is quite similar to that of Theorem A, and thus we omit it.

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