Orlicz capacity of balls

Yoshihiro Mizuta and Takao Ohno

Abstract

The notion of classical Newton capacity has been generalized to various forms. Among others, Meyers introduced a general notion of L^p-capacity, which is defined by general potentials of functions in the Lebesgue space L^p and such notion of capacity has been proved to provide rich results in the nonlinear potential theory as well as in the study of various function spaces and partial differential equations; see e.g., Adams-Hedberg [1]. The most useful L^p-capacity is Riesz capacity. The aim in this note is to estimate the Riesz capacity of balls $B(x,r)$ centered at x of radius r in the Orlicz setting.

1 Introduction and statement of results

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $I_\alpha f$ of order α by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

In the present note, we treat functions f satisfying an Orlicz condition :

$$\int_{\mathbf{R}^n} |f(y)|^p \varphi(|f(y)|)\, dy < \infty, \tag{1.1}$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on the interval $(0,\infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

(φ1) $\qquad c_1^{-1}\varphi(r) \le \varphi(r^2) \le c_1\varphi(r) \qquad$ whenever $r > 0$.

We set $\Phi_{p,\varphi}(r) = r^p\varphi(r)$ and

$$\Phi_{p,\varphi}(0) = 0,$$

because we will see from (φ4) below that

$$\lim_{r\to 0+} \Phi_{p,\varphi}(r) = 0;$$

2000 Mathematics Subject Classification : Primary 46E30, 31B15
Key words and phrases : Orlicz space, Morrey space, capacity, Riesz potentials

see [16, p205].

We denote by $L^{\Phi_{p,\varphi}}(\mathbf{R}^n)$ the family of all locally integrable functions g on $\mathbf{R}^n$ such that

$$\Phi_{p,\varphi}(g) \equiv \int_{\mathbf{R}^n} \Phi_{p,\varphi}(|g(x)|)\, dx < \infty.$$

Let G be a bounded open set in $\mathbf{R}^n$. For $E \subset G$, the relative $(\alpha, \Phi_{p,\varphi})$-capacity is defined by

$$C_{\alpha,\Phi_{p,\varphi}}(E; G) = \inf_f \Phi_{p,\varphi}(f),$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \geq 1 \qquad \text{for all } x \in E$$

(cf. Adams-Hedberg [1], Meyers [12], Ziemer [21] and the first author [13, 14]).

Our first aim in the present note is to give an estimate of $(\alpha, \Phi_{p,\varphi})$-capacity of balls. We denote by $B(x, r)$ the open ball centered at x of radius r. For fixed α, set

$$\varphi_p(r) = \int_r^1 [t^{n-\alpha p}\varphi(t^{-1})]^{-1/(p-1)}\, dt/t$$

when $0 \leq r < 1$.

As an extension of Adams-Hurri-Syrjänen [3], Joensuu [11] and Mizuta [14, Section 8.3, Lemma 3.1], we state our theorem in the following.

Theorem A. *There exists a constant* $A_1 > 0$ *such that*

$$A_1^{-1}\varphi_p(r)^{-(p-1)} \leq C_{\alpha,\Phi_{p,\varphi}}(B(0,r); B(0,1)) \leq A_1\varphi_p(r)^{-(p-1)}$$

whenever $0 < r < 1/2$.

Theorem A was proved by Adams-Hurri-Syrjänen [3, Theorem 2.11] in the case when $\varphi(t) = (\log(e+t))^\beta$ with $p = n/\alpha > 1$ and $0 \leq \beta \leq p-1$. Recently Joensuu [9, 10, 11] treated the case when φ is nondecreasing. Their main idea was to use the rearrangement equivalent norm for $\|f\|_{L^{\Phi_{p,\varphi}}(\mathbf{R}^n)}$ ([6, 9, 10]). Our proof will be done straightforward from the definition of capacity, and several technical assumptions posed in [11] are removed. If $\varphi_p(0) < \infty$, then $C_{\alpha,\Phi_{p,\varphi}}(\{0\}; B(0,1)) > 0$, and $I_\alpha f$ is continuous when $f \in L^{\Phi_{p,\varphi}}(\mathbf{R}^n)$ has compact support, in view of Mizuta-Shimomura [18] .

If $\alpha p < n$, then we see from (φ4) below that

$$\varphi_p(r) \sim [r^{n-\alpha p}\varphi(r^{-1})]^{-1/(p-1)}$$

when $0 < r < 1/2$, where we write $f \sim g$ if there exists a constant $A > 0$ so that $A^{-1}g \leq f \leq Ag$.

Corollary 1.1. *If* $\alpha p < n$, *then*

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r);B(0,1)) \sim r^{n-\alpha p}\varphi(r^{-1})$$

whenever $0 < r < 1/2$.

Example 1.2. For $n = \alpha p$, consider the function

$$\varphi(t) = (\log(e+t))^{\beta}.$$

If $\beta < p-1$, then

$$\varphi_p(r) \sim (\log(e+1/r))^{-\beta/(p-1)+1}$$

for $0 < r < 1$. In this case

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r);B(0,1)) \sim (\log(e+1/r))^{\beta-p+1}$$

whenever $0 < r < 1/2$.

If $\beta = p-1$, then

$$\varphi_p(r) \sim \log(e+(\log(e+1/r)))$$

for $0 < r < 1$. In this case

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r);B(0,1)) \sim (\log(e+(\log(e+1/r))))^{-p+1}$$

whenever $0 < r < 1/2$.

Remark 1.3. We here introduce another capacity. Define

$$\|g\|_{L^{\Phi_{p,\varphi}}(\mathbf{R}^n)} = \inf\{\lambda > 0 : \Phi_{p,\varphi}(g/\lambda) \le 1\}.$$

This is a quasi-norm in $L^{\Phi_{p,\varphi}}(\mathbf{R}^n)$. For a set $E \subset G$, we define

$$N_{\alpha,\Phi_{p,\varphi}}(E;G) = \inf_f \|f\|^p_{L^{\Phi_{p,\varphi}}(\mathbf{R}^n)},$$

where the infimum is taken over all nonnegative measurable functions f on $\mathbf{R}^n$ such that $f = 0$ outside G and $I_{\alpha(x)}f(x) \ge 1$ for all $x \in E$. With the aid of [8], one can find a constant $A > 1$ such that

$$A^{-1}\varphi_p(r)^{-(p-1)} \le N_{\alpha,\Phi_{p,\varphi}}(B(0,r);B(0,1)) \le A\varphi_p(r)^{-(p-1)}$$

for $0 < r < 1/2$. This implies that two quantities $C_{\alpha,\Phi_{p,\varphi}}(B(0,r);B(0,1))$ and $N_{\alpha,\Phi_{p,\varphi}}(B(0,r);B(0,1))$ are comparable. We do not know whether two capacites $C_{\alpha,\Phi_{p,\varphi}}(\cdot;B(0,1))$ and $N_{\alpha,\Phi_{p,\varphi}}(\cdot;B(0,1))$ are comparable or not.

Next consider the Morrey-Orlicz space $\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)$ of all functions g such that

$$\sup_{x\in\mathbf{R}^n, r>0} r^\nu \fint_{B(x,r)} \Phi_{p,\varphi}(|g(y)|)\,dy < \infty$$

with the quasi-norm

$$\|g\|_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)} = \inf\left\{\lambda > 0 : \sup_{x\in\mathbf{R}^n, r>0} r^\nu \fint_{B(x,r)} \Phi_{p,\varphi}(|g(y)|/\lambda)\,dy \le 1\right\}.$$

For fundamental properties of Morrey-Orlicz space, we refer the reader to the papers by Adams-Xiao [4], Mizuta-Nakai-Ohno-Shimomura [15] and Mizuta-Shimomura [19, 20] .

Our second aim in the present note is to give an estimate of Morrey-Orlicz capacity of balls defined by

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(E;G) = \inf_f \|f\|^p_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)},$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \ge 1 \qquad \text{for all } x \in E.$$

For fixed α, ν, consider

$$\tilde{\varphi}_p(r) = \int_r^1 [t^{\nu-\alpha p}\varphi(t^{-1})]^{-1/p}\,dt/t$$

when $0 < r < 1$.

THEOREM B. *Suppose $\alpha p \le \nu < n$. There exists a constant $A_2 > 0$ such that*

$$A_2^{-1}\tilde{\varphi}_p(r)^{-p} \le N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(B(0,r);B(0,1)) \le A_2\tilde{\varphi}_p(r)^{-p}$$

whenever $0 < r < 1/2$.

This is an extension of Adams-Xiao [4, Theorem 5.3].

The case $\nu = n$ was treated by Theorem A. If $\tilde{\varphi}_p(0) < \infty$, then $N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(\{0\};B(0,1)) > 0$, and $I_\alpha f$ is continuous on $\mathbf{R}^n$ when $f \in \mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)$ has compact support, in view of Mizuta-Shimomura [19, 20] .

COROLLARY 1.4. *If $\alpha p < \nu$, then*

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(B(0,r);B(0,1)) \sim r^{\nu-\alpha p}\varphi(r^{-1})$$

whenever $0 < r < 1/2$.

EXAMPLE 1.5. Let $\varphi(t) = (\log(e + t))^\beta$. If $\alpha p < \nu$, then

$$\tilde{\varphi}_p(r) \sim [r^{\nu-\alpha p}(\log(e + 1/r))^\beta]^{-1/p}$$

for $0 < r < 1/2$, so that

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_p,\varphi}}(B(0,r); B(0,1)) \sim r^{\nu-\alpha p}(\log(e + 1/r))^\beta$$

whenever $0 < r < 1/2$.

If $\alpha p = \nu$ and $\beta < p$, then

$$\tilde{\varphi}_p(r) \sim (\log(e + 1/r))^{1-\beta/p}$$

for $0 < r < 1/2$, so that

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_p,\varphi}}(B(0,r); B(0,1)) \sim (\log(e + 1/r))^{\beta-p}$$

whenever $0 < r < 1/2$.

For further related results, see Aissaoui-Benkirane [5], Adams-Hurri-Syrjänen [2], Edmunds-Evans [7] and Mizuta-Shimomura [16, 17, 18].

Throughout this note, let A denote various constants independent of the variables in question and $A(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

2 Proof of Theorem A

First we collect properties which follow from condition ($\varphi1$) (see [14], [16, Lemma 2.3], [15, Section 7]).

($\varphi2$) φ satisfies the doubling condition, that is, there exists $c_2 > 1$ such that

$$c_2^{-1}\varphi(r) \le \varphi(2r) \le c_2\varphi(r) \qquad \text{whenever } r > 0.$$

($\varphi3$) For each $\gamma > 0$, there exists $c_3 = c_3(\gamma) \ge 1$ such that

$$c_3^{-1}\varphi(r) \le \varphi(r^\gamma) \le c_3\varphi(r) \qquad \text{whenever } r > 0.$$

($\varphi4$) For each $\gamma > 0$, there exists $c_4 = c_4(\gamma) \ge 1$ such that

$$s^\gamma\varphi(s) \le c_4 t^\gamma\varphi(t) \qquad \text{whenever } 0 < s < t.$$

($\varphi5$) For each $\gamma > 0$, there exists $c_5 = c_5(\gamma) \ge 1$ such that

$$t^{-\gamma}\varphi(t) \le c_5 s^{-\gamma}\varphi(s) \qquad \text{whenever } 0 < s < t.$$

(φ6) If φ and φ_1 are positive monotone functions on $[0,\infty)$ satisfying (φ1), then for each $\gamma > 0$ there exists a constant $c_6 = c_6(\gamma) \geq 1$ such that

$$c_6{}^{-1}\varphi(r) \leq \varphi(r^\gamma \varphi_1(r)) \leq c_6 \varphi(r) \qquad \text{whenever } r > 0.$$

Consider the function

$$\bar{\Phi}_{p,\varphi}(r) = \int_0^r \sup_{0<s<t} s^{-1}\Phi_{p,\varphi}(s)\, dt.$$

Then $\bar{\Phi}_{p,\varphi}$ is nondecreasing and

$$\bar{\Phi}_{p,\varphi}(r) \sim \Phi_{p,\varphi}(r)$$

by (φ4). Thus $\|\cdot\|_{L^{\bar{\Phi}_{p,\varphi}}(\mathbf{R}^n)}$ defines an equivalent norm to $\|\cdot\|_{L^{\Phi_{p,\varphi}}(\mathbf{R}^n)}$, and

$$C_{\alpha,\Phi_{p,\varphi}}(E, B(0,1)) \sim C_{\alpha,\bar{\Phi}_{p,\varphi}}(E, B(0,1))$$

for $E \subset B(0,1/2)$.

Lemma 2.1. *There exists a constant $A > 0$ such that*

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r), B(0,1)) \leq A\varphi_p(r)^{-(p-1)}$$

for $0 < r < 1/2$.

Proof. We prove this lemma only when $\alpha p = n$. Set $K = \varphi_p(r)$. If $r > 0$ is small, say $0 < r < r_0$ $\big(< 1/2\big)$, then we see from (φ4) that

$$rK^{1/\alpha} < \sqrt{r}.$$

For $0 < r < r_0$, consider the function

$$f_r(y) = K^{-1}|y|^{-\alpha}[\varphi(K^{-1/\alpha}|y|^{-1})]^{-1/(p-1)}$$

for $r < |y| < K^{-1/\alpha}$ and $f_r = 0$ elsewhere. If $y \in B(0,1) \setminus B(0,r)$ and $x \in B(0,r)$, then $|x - y| < 2|y|$. Hence we have

$$\begin{aligned} I_\alpha f_r(x) &\geq 2^{\alpha-n} \int_{B(0,1)\setminus B(0,r)} |y|^{\alpha-n} f_r(y)\, dy \\ &\geq 2^{\alpha-n}\omega_{n-1}K^{-1} \int_{rK^{1/\alpha}}^1 [\varphi(t^{-1})]^{-1/(p-1)}\, dt/t \\ &\geq AK^{-1} \int_{\sqrt{r}}^1 [\varphi(t^{-1})]^{-1/(p-1)}\, dt/t \\ &\geq A, \end{aligned}$$

since $\varphi_p(r)$ is of log-type, where ω_{n-1} denotes the area of the unit sphere. Hence we obtain

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r), B(0,1)) \le A\Phi_{p,\varphi}(f_r).$$

Since $\varphi(f_r) \le A\varphi(K^{-1/\alpha}|y|^{-1})$ by (φ6), we obtain

$$\begin{aligned}\int_{B(0,1)} \Phi_{p,\varphi}(f_r(y))\, dy &\le AK^{-p} \int_{B(0,K^{-1/\alpha})\setminus B(0,r)} |y|^{-\alpha p}\varphi(K^{-1/\alpha}|y|^{-1})^{-p/(p-1)+1}\, dy \\ &\le AK^{-p+1}.\end{aligned}$$

Thus it follows that

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r), B(0,1)) \le AK^{-p+1}$$

for $0 < r < r_0$.

If $r_0 \le r < 1/2$, then

$$\begin{aligned}C_{\alpha,\Phi_{p,\varphi}}(B(0,r), B(0,1)) &\le C_{\alpha,\Phi_{p,\varphi}}(B(0,1/2), B(0,1)) \\ &= A \\ &\le AK^{-p+1},\end{aligned}$$

which proves the lemma. □

LEMMA 2.2. *There exists a constant $A > 0$ such that*

$$C_{\alpha,\Phi_{p,\varphi}}(B(0,r), B(0,1)) \ge A\varphi_p(r)^{-(p-1)}.$$

for $0 < r < 1/2$.

Proof. We prove this only when $\alpha p = n$, since the case $\alpha p < n$ is proved similarly. For $0 < r < 1/2$, take a nonnegative measurable function f on $B(0,1)$ such that

$$I_\alpha f(x) \ge 1 \qquad \text{for } x \in B(0,r).$$

Then we have by Fubini's theorem

$$\begin{aligned}\int_{B(0,r)} dx &\le \int_{B(0,r)} I_\alpha f(x)\, dx \\ &\le \int_{B(0,1)} \left(\int_{B(0,r)} |x-y|^{\alpha-n}\, dx \right) f(y)\, dy \\ &\le Ar^n \int_{B(0,1)} (r+|y|)^{\alpha-n} f(y)\, dy,\end{aligned}$$

so that

$$1 \le A \int_{B(0,1)} (r+|y|)^{\alpha-n} f(y)\, dy.$$

We show that

$$1 \leq A[\varphi_p(r)]^{p-1}\Phi_{p,\varphi}(f). \tag{2.1}$$

Let $0 < \varepsilon < 1$ and $K = \varphi_p(r)$. Considering

$$k(y) = \varepsilon K^{-1}(r+|y|)^{-\alpha}[\varphi(K^{-1/\alpha}(r+|y|)^{-1})]^{-1/(p-1)},$$

we obtain

$$\begin{aligned}
&\int_{B(0,1)} (r+|y|)^{\alpha-n} f(y)\, dy \leq \int_{B(0,1)} (r+|y|)^{\alpha-n} k(y)\, dy \\
&\quad + A\int_{B(0,1)} (r+|y|)^{\alpha-n} f(y) \left(\frac{f(y)}{k(y)}\right)^{p-1} \frac{\varphi(f(y))}{\varphi(k(y))}\, dy \\
\leq\ & A\varepsilon K^{-1} \int_0^1 [\varphi(K^{-1/\alpha}(r+t)^{-1})]^{-1/(p-1)}(r+t)^{-1}\, dt \\
&\quad + A(\varepsilon)K^{p-1}\int_{B(0,1)} f(y)^p \varphi(f(y))\, dy \\
\leq\ & A\varepsilon + A(\varepsilon)K^{p-1}\Phi_{p,\varphi}(f),
\end{aligned}$$

since

$$\begin{aligned}
\varphi(k(y)) &\geq A(\varepsilon)\varphi((K^{-1/\alpha}(r+|y|)^{-1})^{\alpha}[\varphi(K^{-1/\alpha}(r+|y|)^{-1})]^{-1/(p-1)}) \\
&\geq A(\varepsilon)\varphi(K^{-1/\alpha}(r+|y|)^{-1})
\end{aligned}$$

by ($\varphi 2$) and ($\varphi 6$), and

$$\begin{aligned}
\int_r^1 [\varphi(K^{-1/\alpha}t^{-1})]^{-1/(p-1)}\, dt/t &\leq \int_{rK^{1/\alpha}}^{K^{1/\alpha}} [\varphi(s^{-1})]^{-1/(p-1)}\, ds/s \\
&\leq AK + \int_1^{K^{1/\alpha}} [\varphi(s^{-1})]^{-1/(p-1)}\, ds/s \\
&\leq AK + A\max\{1, [\varphi(K^{1/\alpha})]^{-1/(p-1)}\}\log K \\
&\leq AK
\end{aligned}$$

by ($\varphi 6$). Now it follows that

$$1 \leq A\varepsilon + A(\varepsilon)K^{p-1}\Phi_{p,\varphi}(f).$$

By taking $A\varepsilon = 1/2$, we obtain (2.1), and hence

$$1 \leq A\varphi_p(r)^{p-1}C_{\alpha,\Phi_{p,\varphi}}(B(0,r); B(0,1)),$$

which proves the conclusion. □

Now we establish Theorem A by use of Lemmas 2.1 and 2.2.

3 Proof of Theorem B

To prove Theorem B, we prepare some lemmas.

LEMMA 3.1. *If $\alpha p \le \nu < n$, then there exists a constant $A > 0$ such that*

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(B(0,r);B(0,1)) \le A\tilde{\varphi}_p(r)^{-p}$$

whenever $0 < r \le 1/2$.

Proof. We show this lemma only when $\alpha p = \nu$. For $0 < r < 1/2$ and $K = \tilde{\varphi}_p(r)$, consider the function

$$f_r(y) = K^{-1}|y|^{-\alpha}[\varphi(|y|^{-1})]^{-1/p}$$

for $r < |y| < 1$ and $f_r = 0$ elsewhere. If $x \in B(0,r)$, then, as in the proof of Lemma 2.1, we have

$$I_\alpha f_r(x) \ \ge \ A,$$

so that

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(B(0,r);B(0,1)) \le A\|f_r\|^p_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)}.$$

Setting

$$g_r(y) = |y|^{-\alpha}[\varphi(|y|^{-1})]^{-1/p},$$

we have

$$\int_{B(x,t)} \Phi_{p,\varphi}(g_r(y))\, dy \le A\int_{B(0,t)} |y|^{-\alpha p}\, dy \le At^{N-\alpha p},$$

which implies that $\|g_r\|_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)} \le A$. Hence it follows that

$$\|f_r\|_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)} \le AK^{-1}\|g_r\|_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)} \le AK^{-1},$$

which proves the lemma. □

LEMMA 3.2. *If $\alpha p \le \nu < n$, then there exists a constant $A > 0$ such that*

$$N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(B(0,r);B(0,1)) \ge A\tilde{\varphi}_p(r)^{-p}$$

whenever $0 < r \le 1/2$.

Proof. As before, we show this lemma only when $\alpha p = \nu$. For $0 < r < 1/2$, take a nonnegative measurable function f on $B(0,1)$ such that

$$I_\alpha f(x) \ge 1 \qquad \text{for } x \in B(0,r).$$

Then, as in the proof of Lemma 2.2, we have

$$1 \le A \int_{B(0,1)} (r+|y|)^{\alpha-n} f(y)\, dy.$$

We show that

$$1 \le A \tilde{\varphi}_p(r)^p \|f\|^p_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)}. \tag{3.1}$$

To show this, we may assume that $\|f\|_{\mathcal{M}^{\nu,\Phi_{p,\varphi}}(\mathbf{R}^n)} \le 1$. Considering

$$k(y) = |y|^{-\alpha}[\varphi(|y|^{-1})]^{-1/p},$$

we find

$$\begin{aligned}
⨍_{B(0,t)} f(y)\, dy &\le ⨍_{B(0,t)} k(y)\, dy + A ⨍_{B(0,t)} f(y) \left(\frac{f(y)}{k(y)}\right)^{p-1} \frac{\varphi(f(y))}{\varphi(k(y))}\, dy \\
&\le At^{-\alpha}[\varphi(t^{-1})]^{-1/p} + At^{\alpha(p-1)}[\varphi(t^{-1})]^{-1/p} ⨍_{B(0,t)} f(y)^p \varphi(f(y))\, dy \\
&\le At^{-\alpha}[\varphi(t^{-1})]^{-1/p},
\end{aligned}$$

so that

$$\begin{aligned}
\int_{B(0,1)} (r+|y|)^{\alpha-n} f(y)\, dy &\le A \int_0^1 \left(\int_{B(0,t)} f(y)\, dy \right) (r+t)^{\alpha-n-1} dt \\
&\le A \int_0^1 (r+t)^{-1}[\varphi((r+t)^{-1})]^{-1/p}\, dt \\
&\le A\tilde{\varphi}(r),
\end{aligned}$$

which gives (3.1). Thus we obtain

$$1 \le A\tilde{\varphi}_p(r)^p N_{\alpha,\mathcal{M}^{\nu,\Phi_{p,\varphi}}}(B(0,r); B(0,1)),$$

which proves the conclusion. □

Now we establish Theorem B by use of Lemmas 3.1 and 3.2.

References

[1] D. R. Adams and L. I. Hedberg, Function spaces and potential theory, Springer-Verlag, Berlin Heidelberg New York, 1996.

[2] D. R. Adams and R. Hurri-Syrjänen, Vanishing exponential integrability for functions whose gradients belong to $L^n(\log(e+L))^\alpha$, J. Funct. Anal. **197** (2003), 162-178.

[3] D. R. Adams and R. Hurri-Syrjänen, Capacity estimates, Proc. Amer. Math. Soc. **131** (2003), 1159-1167.

[4] D. R. Adams and J. Xiao, Nonlinear potential analysis on Morrey spaces and their capacities, Indiana Univ. Math. J. **53** (2004), no. 6, 1629-1663.

[5] N. Aissaoui and A. Benkirane, Capacités dans les espaces d'Orlicz, Ann. Sci. Math. Québec **18** (1994), 1-23.

[6] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, Inc., New York, 1988.

[7] D. E. Edmunds and W. D. Evans, Hardy operators, function spaces and embeddings, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

[8] T. Futamura, Y. Mizuta, T. Ohno and T. Shimomura, Orlicz-Sobolev capacity of balls, to appear in Illinois J. Math.

[9] J. Joensuu, On null sets of Sobolev-Orlicz capacities, to appear in Illinois J. Math.

[10] J. Joensuu, Orlicz-Sobolev capacities and their null sets, to appear in Rev. Mat. Complut.

[11] J. Joensuu, Estimates for certain Orlicz-Sobolev capacities of an Euclidean ball, preprint.

[12] N. G. Meyers, A theory of capacities for potentials in Lebesgue classes, Math. Scand. **8** (1970), 255-292.

[13] Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions, Hiroshima Math. J. **23**, 79-153 (1993).

[14] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosyo, Tokyo, 1996.

[15] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, Boundedness of fractional integral operators on Morrey spaces and Sobolev embeddings for generalized Riesz potentials, Math. Soc. Japan **62** (2010), 707–744.

[16] Y. Mizuta and T. Shimomura, Differentiability and Hölder continuity of Riesz potentials of functions in Orlicz classes, Analysis **20** (2000), 201-223.

[17] Y. Mizuta and T. Shimomura, Vanishing exponential integrability for Riesz potentials of functions in Orlicz classes, Illinois J. Math. **51** (2007), 1039-1060.

[18] Y. Mizuta and T. Shimomura, Continuity properties of Riesz potentials of Orlicz functions, Tohoku Math. J. **61** (2009), 225-240.

[19] Y. Mizuta and T. Shimomura, Continuity properties for Riesz potentials of functions in Morrey spaces of variable exponent, Math. Inequal. Appl. **13** (2010), 99-122.

[20] Yoshihiro Mizuta and Tetsu Shimomura, Sobolev's inequality for Riesz potentials of functions in Morrey spaces of integral form, Math. Nachr. **283** (2010), 1336–1352.

[21] W. P. Ziemer, Weakly differentiable functions, Springer-Verlag, New York, 1989.

Department of Mechanical Systems Engineering
Hiroshima Institute of Technology
2-1-1 Miyake,Saeki-ku,Hiroshima, 731-5193 Japan
e-mail : yoshihiromizuta3@gmail.com
and
Faculty of Education and Welfare Science
Oita University
Dannoharu Oita-city 870-1192, Japan
e-mail : t-ohno@oita-u.ac.jp