Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponent

Yoshihiro Mizuta, Eiichi Nakai, Takao Ohno and Tetsu Shimomura

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Abstract

Let $\alpha, \nu, \beta, p$ and $q$ be all variable exponents. Our aim in this paper is to deal with Sobolev embeddings for Riesz potentials of order $\alpha$ with functions $f$ in Morrey spaces $L^{\Phi,\nu,\beta}(G)$ with $\Phi(t) = t^p(\log(e + t))^q$ over a bounded open set $G \subset \mathbb{R}^n$. Here $p$ and $q$ satisfy the log-Hölder and the loglog-Hölder conditions, respectively. Also the case when $p$ attains the value 1 in some parts of the domain is included in our results.

1 Introduction

Let $G$ be a bounded open set in $\mathbb{R}^n$. We denote by $d_G$ the diameter of $G$.

For a measurable function $\alpha : \mathbb{R}^n \to (0, n)$, we define the Riesz potential of order $\alpha$ for an integrable function $f$ on $G$ by

$$I_{\alpha(x)}f(x) = \int_{\mathbb{R}^n} |x - y|^\alpha f(y) \, dy.$$ 

Here and in what follows we assume that $f = 0$ outside $G$. We also assume that $\alpha_- \equiv \text{ess inf}_{x \in \mathbb{R}^n} \alpha(x) > 0$.

We denote by $B(x, r)$ the ball $\{ y \in \mathbb{R}^n : |y - x| < r \}$ with center $x$ and of radius $r > 0$, and by $|B(x, r)|$ its Lebesgue measure, i.e. $|B(x, r)| = \sigma_n r^n$, where $\sigma_n$ is the volume of the unit ball in $\mathbb{R}^n$. We define the integral mean of $f$ over $B(x, r)$ by

$$\int_{B(x,r)} f(y) \, dy = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy.$$ 

Following Cruz-Uribe and Fiorenza [5], we consider continuous functions $p : \mathbb{R}^n \to [1, \infty)$ and $q : \mathbb{R}^n \to \mathbb{R}$, which are called variable exponents. In this paper, we consider variable exponents $p$ and $q$ on $\mathbb{R}^n$ such that

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be the set of all measurable functions whenever $0 < q < \infty$ for some constant $c > 0$. We set $q_+ = \inf_{x \in \mathbb{R}^n} q(x) \leq \sup_{x \in \mathbb{R}^n} q(x) = q_+ < \infty$.

For bounded measurable functions $p : \mathbb{R}^n \to (0, n]$, let $L^{p, \nu, \beta}(G)$ be the set of all measurable functions $f$ on $G$ such that $\|f\|_{L^{p, \nu, \beta}(G)} < \infty$, where

$$\|f\|_{L^{p, \nu, \beta}(G)} = \inf \left\{ \lambda > 0 : \sup_{x \in G, d < r < d_G} \Phi(y, |f(y)|/\lambda) dy \leq 1 \right\};$$

we set $f = 0$ outside $G$. For the constant Morrey spaces, we refer to [19], [27] and [15, 22, 24, 25]. For simplicity, in the case $\nu \equiv n$ and $\beta \equiv 0$, $L^{p, \nu, \beta}(G)$ is denoted by $L^{p, \nu}(\log L)^{d} (G)$.

Throughout this paper, we assume that (P1), (P2), (Q1) and (Q2) hold and that there exists a constant $K > 0$ such that

$$K(p(x) - 1) + q(x) > 0$$

for all $x \in \mathbb{R}^n$. In this case we can find $c_0 > e$ such that, for each fixed $x \in \mathbb{R}^n$, $\Phi(x, r) \equiv r^{p(x)}(\log(c_0 + r))^{q(x)}$ is convex on $[0, \infty)$, $\lim_{r \to 0} \Phi(x, r) = \Phi(x, 0) = 0$ and $\lim_{r \to \infty} \Phi(x, r) = \infty$ (see [9, Theorem 5.1]). Then $\| \cdot \|_{L^{p, \nu, \beta}(G)}$ is a quasi norm, since

$$\Phi(x, c^{-1}r) \leq \Phi(x, r) \leq \Phi(x, cr),$$

for some constant $c > 0$ independent of $x \in \mathbb{R}^n$ and $r \geq 0$. Furthermore, $t^{-1} \Phi(x, t)$ is uniformly almost increasing on $(0, \infty)$, that is, there exists a constant $C > 0$ such that

$$s^{-1} \Phi(x, s) \leq C t^{-1} \Phi(x, t),$$

whenever $0 < s < t$ and $x \in \mathbb{R}^n$.

Our aim in this paper is to discuss the boundedness of the operator $I_\alpha : f \mapsto I_\alpha(x)f(x)$ from the Morrey space $L^{p, \nu, \beta}(G)$ to another Morrey space $L^{p, \nu, \beta}(G)$ with suitable $\Psi(x, r)$. When $p_- = \inf_{x \in \mathbb{R}^n} p(x) > 1$, the maximal functions are a crucial tool as in Hedberg [8], where an easy proof of Sobolev’s inequality for Riesz potentials is given. Since we are mainly concerned with the case $p_- = 1$, our strategy is to find an estimate of Riesz potentials by use of another Riesz-type potentials of 0 order, which plays a role of the maximal functions (see Sections 2
and 3). Our result contains the known result, as a special case, that $I_\alpha$ is bounded from $L^1(\log L)^q(G)$ to $L^{p'}(\log L)^{p'_q-1}(G)$ for $p^* = n/(n - \alpha)$ and $q > 0$ (O’Neil [26, Theorem 5.2]); see Remark 2.3.

In Section 4, we investigate the case $p_- > 1$. For this purpose, we first show the boundedness of the Hardy-Littlewood maximal operator $M$. Our result contains the known result, as a special case, that $I_\alpha$ is bounded from $L^p(\log L)^q(G)$ to $L^{p'}(\log L)^{p'_q}(G)$ for $p^* = np/(n - \alpha p)$ and $q \in \mathbb{R}$ (O’Neil [26, Theorem 4.7]); see Remark 4.7. For related results, see [1, 3, 4, 16, 17, 18].

In Section 5, we are concerned with Morrey version of Trudinger’s type exponential integrability for $I_{\alpha(x)}f(x)$ in the case $p_- \geq 1$. Our result contains the result of Trudinger [30] and [12, Corollaries 4.6 and 4.8] as special cases (Remark 5.4). The result is also an improvement of [15, Theorems 4.4 and 4.5]. For related results, see [2, 4, 10, 11, 28, 31].

In the last section we discuss the continuity of $I_{\alpha(x)}f(x)$. For a function $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, let $\Lambda_\phi(G)$ be the set of all functions $f$ on $G$ such that $\|f\|_{\Lambda_\phi(G)} < \infty$, where

$$||f||_{\Lambda_\phi} = \sup_{x,y \in G, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, |x - y|) + \phi(y, |x - y|)}.$$ 

See [23] for the function space $\Lambda_\phi$. If $\phi(x, r) = r^{\gamma(x)}$, then we denote $\Lambda_\phi(G)$ by $\text{Lip}_{\gamma(x)}(G)$. In the last section we show the boundedness of the operator $I_{\alpha(x)}$ from $L^{\Phi_{\alpha(x)}}(G)$ to $\Lambda_\phi(G)$ under some conditions. It is known that $I_\alpha$ is bounded from $L^p(G)$ to $\text{Lip}_{\gamma}(G)$ for $0 < \gamma = \alpha - n/p < 1$. We extend this fact to the boundedness of $I_{\alpha(x)}$ from $L^p(G)$ to $\text{Lip}_{\gamma(x)}(G)$ as a corollary (Corollary 6.2).

Throughout this paper, let $C$ denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

## 2 Sobolev’s inequality in the case $p_- = 1$

Recall that $\alpha : \mathbb{R}^n \to (0, n)$, $\nu : \mathbb{R}^n \to (0, n]$ and $\beta : \mathbb{R}^n \to \mathbb{R}$ are bounded measurable functions and $\alpha_- > 0$. Throughout this section, we assume that

$$\inf_{x \in \mathbb{R}^n} (1/p(x) - \alpha(x)/\nu(x)) > 0. \quad (2.1)$$

In this case we have $\nu_\alpha > 0$.

Our first aim is to give the following Morrey version of Sobolev’s type inequality for Riesz potentials of functions satisfying Morrey conditions. We consider the Sobolev exponent

$$1/p(x) = 1/p(x) - \alpha(x)/\nu(x) \quad (2.2)$$

and the new modular function

$$\Psi(x, t) = t^{p(x)}(\log e + t)^{p(x)q(x)/p(x) + \alpha(x)\beta(x)/\nu(x)} \quad (2.3)$$
Theorem 2.1. Let \( p_- = 1 \). Suppose that (2.1) holds. Then, for each \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that

\[
\int_{B(z,r)} \Psi(x, |I_{\alpha(x)}f(x)|)(\log(e + |I_{\alpha(x)}f(x)|))^{(1+\varepsilon)} \, dx \leq Cr^{-\nu(z)(\log(e + 1/r))^{-\beta(z)-\varepsilon}}
\]

whenever \( z \in G \), \( 0 < r < d_G \) and \( \|f\|_{L^{\Phi,\nu,\beta(G)}} \leq 1 \).

Remark 2.2. For \( \eta \in \mathbb{R} \), set

\[
\tilde{\Psi}(x, t) = \Psi(x, t)(\log(e + t))^{-\eta} = t^{p^*(x)}(\log(e + t))^{p^*(x)(q(x)/p(x) + \alpha(x)\beta(x)/\nu(x)) - \eta}
\]

Then \( \tilde{\Psi}(x, t) \) satisfies the condition (1.1) with \( p(x) \) and \( q(x) \) replaced by \( p^*(x) \) and \( p^*(x)(q(x)/p(x) + \alpha(x)\beta(x)/\nu(x)) - \eta \), respectively, and thus \( \|\cdot\|_{L^{\Phi,\nu,\beta(G)}} \) is a quasi norm.

Remark 2.3. In this theorem, we can not take \( \varepsilon = 0 \) (see [11, Remark 3.3] and O’Neil [26, Theorem 5.2]).

This theorem gives the following norm version.

Corollary 2.4. Let \( p_- = 1 \). Suppose that (2.1) holds. Then, for \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that

\[
\|I_{\alpha(x)}f\|_{L^{\Phi,\nu,\beta(G)}} \leq C\|f\|_{L^{\Phi,\nu,\beta(G)}}.
\]

For \( \varepsilon > 0 \), setting

\[
\rho_\varepsilon(r) = r^{-n}(\log(e + 1/r))^{-\varepsilon-1},
\]

we consider the logarithmic potential

\[
J_\varepsilon f(x) = \int_{G} \rho_\varepsilon(|x - y|)g(y) \, dy,
\]

where \( g(y) = \Phi(y, |f(y)|) = |f(y)|^{p(y)}(\log(e + |f(y)|))^{q(y)} \). Write

\[
I_{\alpha(x)}f(x) = \int_{B(x,\delta)} |x - y|^\alpha f(y) \, dy + \int_{G\setminus B(x,\delta)} |x - y|^\alpha f(y) \, dy = I_1(\delta) + I_2(\delta).
\]

Following the Hedberg trick [8], we give an estimate of \( I_1(\delta) \) by \( J_\varepsilon f(x) \), instead of maximal functions. After this, we give an estimate of \( I_2(\delta) \) by use of Young’s inequality. Finally, taking \( \delta \) suitably, we obtain an estimate of \( I_{\alpha(x)}f(x) \) by \( J_\varepsilon f(x) \). For this purpose, we prepare some lemmas.

Let us begin with an estimate of \( I_1(\delta) \) by \( J_\varepsilon f(x) \).

Lemma 2.5. For \( 0 < \delta \leq d_G \), \( x \in G \) and a nonnegative integrable function \( f \) on \( G \), set

\[
I_1(\delta) = \int_{B(x,\delta)} |x - y|^\alpha f(y) \, dy.
\]
Let $\varepsilon > 0$ be fixed and set $J = J_\varepsilon f(x)$ for simplicity. Then there exists a constant $C > 0$ such that

$$I_1(\delta) \leq C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}$$

$$+ \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)J}\}.$$

**Proof.** For $k > 0$, we have by (1.2)

$$I_1(\delta) \leq k \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} \, dy$$

$$+ C \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) \left(\frac{f(y)}{k}\right)^{p(y)-1} \left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} \, dy$$

$$\leq C\left\{k\delta^{\alpha(x)} + \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} g(y) \left(\frac{1}{k}\right)^{p(y)-1} \left(\frac{1}{\log(e+k)}\right)^{q(y)} \, dy\right\}$$

$$\leq C\left\{k\delta^{\alpha(x)} + \delta^{\alpha(x)}(\log(e+1/\delta))^{1+\varepsilon}$$

$$\times \int_{B(x,\delta)} \rho(x) |x-y| g(y) \left(\frac{1}{k}\right)^{p(y)-1} \left(\frac{1}{\log(e+k)}\right)^{q(y)} \, dy\right\}.$$

We set

$$k = \delta^{-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}.$$

For $y \in B(x,\delta)$, note from (P2) that

$$|(p(x) - p(y)) \log k| \leq C$$

so that

$$k^{-p(y)} \leq Ck^{-p(x)}.$$  \hspace{1cm} (2.4)

Similarly, by (Q2) we have

$$(\log(e+k))^{-q(y)} \leq C(\log(e+k))^{-q(x)}.$$  \hspace{1cm} (2.5)

Consequently it follows from (2.4) and (2.5) that

$$I_1(\delta) \leq C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}$$

$$+ \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)J}\}.$$

Now the result follows. \hfill \Box

Next we give an estimate for

$$I_2(\delta) = \int_{G \setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) \, dy.$$
Lemma 2.6. There exists a constant $C > 0$ such that
\[ \int_{B(x,r)} f(y)dy \leq C r^{-\nu(x)/p(x)} (\log(e + 1/r))^{-\beta(x)/p(x)} \]
for all $x \in G$, $0 < r < d_G$ and $f \geq 0$ satisfying $\|f\|_{L^{\nu,\beta}(G)} \leq 1$.

Proof. For $k > 0$, we have by (1.2)
\[
\int_{B(x,r)} f(y)dy \leq k + C \int_{B(x,r)} f(y) \left( \frac{f(y)}{k} \right)^{p(y)-1} \frac{\log(e + f(y))}{\log(e + k)} dy
\]
\[
= k + C \int_{B(x,r)} g(y)k^{-p(y)+1} (\log(e + k))^{-\gamma(y)} dy,
\]
where $g(y) = f(y)^{p(y)} (\log(e + f(y)))^{\gamma(y)}$ as before. Setting
\[ k = r^{-\nu(x)/p(x)} (\log(e + 1/r))^{-(\beta(x)/p(x))}, \]
we find by (P2) and (Q2)
\[
\int_{B(x,r)} f(y)dy \leq k + C kr^{\nu(x)/(\log(e + 1/r))^{\beta(x)}} \int_{B(x,r)} g(y) dy
\]
\[
\leq Ck
\]
\[
= C r^{-\nu(x)/p(x)} (\log(e + 1/r))^{-(\beta(x)/p(x))},
\]
as required. \hfill \Box

Lemma 2.7. Let $\lambda, \mu, \nu, \tau$ and $\gamma$ are real numbers. Suppose $h$ is a nonnegative measurable function on $\mathbb{R}^n$ such that
\[ \int_{B(0,r)} h(y)dy \leq r^{-\lambda} (\log(e + 1/r))^{-\mu} \]
for all $r > 0$. Then there exist a constant $C > 0$ such that
\[ \int_{B(0,r_2) \setminus B(0,r_1)} |y|^{-\tau} \log(1/|y|)^{-\gamma} h(y)dy \leq C \int_{r_1}^{2r_2} t^{-\tau-\lambda} (\log(e + 1/t))^{-\mu-\gamma} \frac{dt}{t} \]
whenever $0 < r_1 \leq r_2 < \infty$.

Proof. By the integration by parts we have
\[
\int_{B(0,r_2) \setminus B(0,r_1)} |y|^{-\tau} \log(1/|y|)^{-\gamma} h(y)dy
\]
\[
\leq \int_{r_1}^{r_2} \left( \int_{B(0,t)} f(y)dy \right) d(-t^{-\tau} (\log(1/t))^{-\gamma}) + r_2^{-\tau} (\log(1/r_2))^{-\gamma} \int_{B(0,r_2)} f(y)dy.
\]
Hence it suffices to note that
\[ r_2^{-\tau} (\log(1/r_2))^{-\gamma} \int_{B(0,r_2)} f(y)dy \leq r_2^{-\tau-\lambda} (\log(e + 1/r_2))^{-\mu-\gamma}
\]
\[ \leq C \int_{r_2}^{2r_2} t^{-\tau-\lambda} (\log(e + 1/t))^{-\mu-\gamma} \frac{dt}{t}. \]
\hfill \Box
Lemma 2.8. There exists a constant $C > 0$ such that

$$I_2(\delta) \leq C \delta^{\alpha(x) - \nu(x)/p(x)} (\log(e + 1/\delta))^{-q(x)/p(x) - \beta(x)/p(x)}$$

for all $x \in G$, $0 < \delta < d_G$ and $f \geq 0$ satisfying $\|f\|_{L^{q,p,\beta}(G)} \leq 1$.

Proof. Let

$$\eta = \text{ess inf}_{x \in G} (\nu(x)/p(x) - \alpha(x)).$$

Then $\eta > 0$ by (2.1). By Lemmas 2.6 and 2.7 we have for all $x \in G$ and $0 < \delta < d_G$

$$I_2(\delta) \leq C \int_{\delta}^{2d_G} t^{\alpha(x) - \nu(x)/p(x)} (\log(e + 1/t))^{-q(x)/p(x) - \beta(x)/p(x)} \frac{dt}{t} \leq C \delta^{\alpha(x) - \nu(x)/p(x)} (\log(e + 1/\delta))^{-q(x)/p(x) - \beta(x)/p(x)} \int_{\delta}^{2d_G} t^{-\eta/2} \frac{dt}{t},$$

which completes the proof.

What remains for the proof of Theorem 2.1 is to give a Morrey property for $J_\varepsilon f(x)$.

Lemma 2.9. There exists a constant $C > 0$ such that

$$\int_{B(z,r)} J_\varepsilon f(x) \, dx \leq C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z) - \varepsilon}$$

for all $z \in G$, $0 < r < d_G$ and $f \geq 0$ satisfying $\|f\|_{L^{q,p,\beta}(G)} \leq 1$.

Proof. For $z \in G$ and $0 < r < d_G$, write

$$J_\varepsilon f(x) = \int_{B(z,2r)} \rho_\varepsilon(|x - y|) g(y) \, dy + \int_{G \setminus B(z,2r)} \rho_\varepsilon(|x - y|) g(y) \, dy = J_1(x) + J_2(x).$$

Then we have

$$\int_{B(z,r)} J_1(x) \, dx \leq \int_{B(z,2r)} \left( \int_{B(z,r)} \rho_\varepsilon(|x - y|) dx \right) g(y) \, dy \leq C r^{-n} (\log(e + 1/r))^{-\varepsilon} \int_{B(z,2r)} g(y) \, dy \leq C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z) - \varepsilon}$$

and

$$\int_{B(z,r)} J_2(x) \, dx \leq C \int_{G \setminus B(z,2r)} \rho_\varepsilon(|z - y|) g(y) \, dy \leq C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z) - \varepsilon},$$

where we use Lemma 2.7 for the last inequality.
Now we are ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* We may assume that $f \geq 0$. For $\delta > 0$, write

$$I_{\alpha}(x) f(x) = I_1(\delta) + I_2(\delta).$$

In view of Lemma 2.5, we find

$$I_1(\delta) \leq C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e + 1/\delta))^{-(q(x)+\beta(x))/p(x)}$$

$$+ \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e + 1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)J}\}.$$

Moreover, Lemma 2.8 yields

$$I_2(\delta) \leq C\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e + 1/\delta))^{-(q(x)+\beta(x))/p(x)},$$

so that

$$I_{\alpha}(x) f(x) \leq C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e + 1/\delta))^{-(q(x)+\beta(x))/p(x)}$$

$$+ \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e + 1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)J}\}.$$

Now, letting $\delta = \min\{d_G, J^{-1/\nu(x)}(\log(e + J))^{-(\beta(x)+(1+\varepsilon))/\nu(x)}\}$, we obtain

$$I_{\alpha}(x) f(x) \leq C\{1 + J^{1/\nu(x)}(\log(e + J))^{-\alpha(x)\beta(x)/\nu(x)-q(x)/p(x)+(1+\varepsilon)/p^*(x)}\}.$$

By Lemma 2.9, we obtain

$$\int_{B(z,r)} \Psi(x, I_{\alpha}(x) f(x)) (\log(e + |I_{\alpha}(x) f(x)|))^{-(1+\varepsilon)} dx$$

$$\leq C \int_{B(z,r)} (1 + J) dx$$

$$\leq C r^{-\nu(x)} (\log(e + 1/r))^{-\beta(x)-\varepsilon}$$

for $z \in G$ and $0 < r < d_G$, which completes the proof of Theorem 2.1. \qed

**Example 2.10.** Let

$$\omega(t) = \begin{cases} 
0 & \text{when } t = 0, \\
1/\log(1/|t|) & \text{when } 0 < |t| < r_0, \\
1/\log(1/r_0) & \text{when } |t| \geq r_0
\end{cases}$$

and

$$\eta(t) = \begin{cases} 
0 & \text{when } t = 0, \\
1/\log \log(1/|t|) & \text{when } 0 < |t| < r_0, \\
1/\log \log(1/r_0) & \text{when } |t| \geq r_0
\end{cases}$$

for $0 < r_0 < 1/4$. Consider

$$p(x) = p(x_1, x_2) = 1 + a \omega(x_2),$$

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and 
\[ q(x) = q(x_1, x_2) = b \eta(x_2), \]
where \( a > 0 \) and \( b > 0 \). Then, note that \( p(\cdot) \) satisfies the conditions (P1) and (P2) and \( q(\cdot) \) satisfies the conditions (Q1) and (Q2). Let \( \gamma > 1 \). If 
\[ f(y) = |y_2|^{-1}(\log(e + 1/|y_2|))^{-\gamma}, \]
then note that 
\[ f(y) \]

\[
\int_{B(z,r)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)}dy \leq Cr^{-1} \int_0 |y_2|^{-1}(\log(e + 1/|y_2|))^{-\gamma}dy_2 \leq Cr^{-1}(\log(e + 1/r))^{-\beta}
\]
for all \( z \in B = B(0, 1) \) and \( r > 0 \), when \( \beta = \gamma - 1 > 0 \). Here we may assume that \( x_2 \neq 0 \). Setting \( Q(x) = \{ y = (y_1, y_2) \in B : |x_1 - y_1| < |x_2|, |y_2| < |x_2| \} \), we note that
\[
I_\alpha f(x) \geq \int_{Q(x)} |x - y|^{a-2} f(y)dy \geq C|x_2|^{a-2} \int_{Q(x)} f(y)dy \geq C|x_2|^{a-1} \int_0^{2|x_2|} |y_2|^{-1}(\log(2 + |y_2|^{-1}))^{-\beta-1}dy_2 \geq C|x_2|^{a-1}(\log(2 + |x_2|^{-1}))^{-\beta},
\]
Since 
\[ 1/p^*(x) - 1/p^*(y) = 1/p(x) - 1/p(y), \]
we see that 
\[
\int_{B(0,r)} I_\alpha f(x)^{p^*(x)}(\log(e + I_\alpha f(x)))^{(q(x)/p(x)+\alpha\beta)p^*(x)-(1+\varepsilon)}dx \geq C \int_{B(0,r)} |x_2|^{-1}(\log(e + 1/|x_2|))^{-\beta-\varepsilon-1}dx \geq C^{-1}(\log(e + 1/r))^{-\beta-\varepsilon}
\]
for all \( 0 < r < 1 \).

This implies that Theorem 2.1 is best possible as to the exponents appearing in the Morrey condition.

### 3 Sobolev’s inequality in the case \( p_- = 1 \) and \( q_- > 0 \)

Let \( p_- = 1 \). In this section we assume that there exists a constant \( q_0 > 0 \) such that 
\[ s^{p(x)-1}(\log(e + s))^{q(x)-q_0} \leq t^{p(x)-1}(\log(e + t))^{q(x)-q_0}, \]
whenever \( 0 < s < t \) and \( x \in \mathbb{R}^n \). Let \( p^* \) and \( \Psi \) be as in (2.2) and (2.3), respectively. Under this assumption, Theorem 2.1 is shown to be valid for \( \varepsilon = 0 \).
**Theorem 3.1.** Let $p_\ast = 1$. Suppose that $(2.1)$ and $(3.1)$ hold. Then there exists a constant $C > 0$ such that

$$\int_{B(x,r)} \Psi(x, |I_\alpha(x)f(x)|)(\log(e + |I_\alpha(x)f(x)|))^{-1} \, dx \leq C r^{-\nu(x)}(\log(e + 1/r))^{-\beta(x)}$$

for all $z \in G$, $0 < r < d_G$ and $f$ satisfying $\|f\|_{L^{q_1,\nu,\beta}(G)} \leq 1$.

**Corollary 3.2.** Let $p_\ast = 1$. Suppose that $(2.1)$ and $(3.1)$ hold. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{q_1,\nu,\beta}(G)} \leq C \|f\|_{L^{q_1,\nu,\beta}(G)},$$

where $\tilde{\Psi}_1(x,t) = \Psi(x,t)(\log(e + t))^{-1}$.

**Remark 3.3.** If $p(x) = 1$, $q(x) = q > 0$, $\nu(x) = n$ and $\beta(x) = 0$, then $p_\ast = n/(n - \alpha)$ and the Riesz operator $I_\alpha$ is bounded from $L^1(\log L)^q(G)$ to $L^{p_\ast}(\log L)^{p_\ast q - 1}(G)$, which is a consequence of O’Neil [26, Theorem 5.2].

For $\varepsilon > 0$, let

$$\rho_\varepsilon(r) = r^{-n}(\log(e + 1/r))^\varepsilon - 1.$$ For a nonnegative measurable function $f$ on $G$, we define the logarithmic potential

$$L_\varepsilon f(x) = \int_{\{y \in G: |x - y|^{-\varepsilon} < f(y)\}} \rho_\varepsilon(|x - y|)(\log(e + f(y)))^{-\varepsilon} g(y) \, dy,$$

where $g(y) = f(y)^{p(y)}(\log(e + f(y)))^{q(y)}$.

For the proof of Theorem 3.1, we need to modify Lemmas 2.5 and 2.9 in the following manner.

**Lemma 3.4.** Let $0 < \varepsilon \leq q_0/2$ and

$$F(\delta) = \int_{\{y \in B(x,\delta): |x - y|^{-\varepsilon} < f(y)\}} |x - y|^{\alpha(x) - n} \left( \frac{\log(e + f(y))}{\log(e + 1/|x - y|)} \right)^\varepsilon f(y) \, dy$$

for $0 < \delta < d_G$ and a nonnegative measurable function $f$ on $G$. Then there exists a constant $C > 0$ such that

$$F(\delta) \leq C \{\delta^{\alpha(x) - \nu(x)/p(x)}(\log(e + 1/\delta))^{-\nu(x)/p(x)} \delta^{\alpha(x) + (p(x) - 1)\nu(x)/p(x)}(\log(e + 1/\delta))^{\beta(x) - \nu(x)/p(x)} |x - f(y)|^{\beta(x) - \nu(x)/p(x)} L_\varepsilon f(x) \}.$$

**Proof.** Let $E = \{y \in B(x,\delta): |x - y|^{-\varepsilon} < f(y)\}$. For $k > 0$, let

$$E_k^1 = \{y \in B(x,\delta): |x - y|^{-\varepsilon} < f(y) \leq k\}, \quad E_k^2 = E \setminus E_k^1.$$
Then we have
\[
\int_{E_k^x} |x - y|^{\alpha(x) - n} \left( \frac{\log(e + f(y))}{\log(e + 1/|x - y|)} \right)^\epsilon f(y) \, dy \\
\leq k (\log(e + k))^\epsilon \int_{B(x, \delta)} |x - y|^{\alpha(x) - n} (\log(e + 1/|x - y|))^{-\epsilon} \, dy \\
= Ck (\log(e + k))^\epsilon \int_0^\delta t^{\alpha(x) - 1} (\log(e + 1/t))^{-\epsilon} \, dt \\
\leq Ck (\log(e + k))^\epsilon \delta^{\alpha(x) - \alpha - \alpha/2} (\log(e + 1/\delta))^{-\epsilon} \int_0^\delta t^{\alpha - 1/2 - 1} \, dt \\
= Ck (\log(e + k))^\epsilon \delta^{\alpha(x)} (\log(e + 1/\delta))^{-\epsilon},
\]
and, using (3.1),
\[
\int_{E_k^x} |x - y|^{\alpha(x) - n} \left( \frac{\log(e + f(y))}{\log(e + 1/|x - y|)} \right)^\epsilon f(y) \, dy \\
\leq \int_{E_k^x} |x - y|^{\alpha(x) - n} \left( \frac{\log(e + f(y))}{\log(e + 1/|x - y|)} \right)^\epsilon f(y) \\
\quad \times C \left( \frac{f(y)}{k} \right)^{p(y) - 1} \left( \frac{1}{\log(e + k)} \right)^{q(y) - 2\epsilon} \, dy \\
= C \int_{E_k^x} |x - y|^{\alpha(x) - n} (\log(e + 1/|x - y|))^{-\epsilon} (\log(e + f(y)))^{-\epsilon} g(y) \\
\quad \times \left( \frac{1}{k} \right)^{p(y) - 1} \left( \frac{1}{\log(e + k)} \right)^{q(y) - 2\epsilon} \, dy \\
\leq C \delta^{\alpha(x)} (\log(e + 1/\delta))^{1 - 2\epsilon} \int_{E} \rho_{-\epsilon}(|x - y|)(\log(e + f(y)))^{-\epsilon} g(y) \\
\quad \times \left( \frac{1}{k} \right)^{p(y) - 1} \left( \frac{1}{\log(e + k)} \right)^{q(y) - 2\epsilon} \, dy.
\]
Hence
\[
F(\delta) \leq C \left\{ k (\log(e + k))^\epsilon \delta^{\alpha(x)} (\log(e + 1/\delta))^{-\epsilon} \\
+ \delta^{\alpha(x)} (\log(e + 1/\delta))^{1 - 2\epsilon} \int_{E} \rho_{-\epsilon}(|x - y|)(\log(e + f(y)))^{-\epsilon} g(y) \\
\quad \times \left( \frac{1}{k} \right)^{p(y) - 1} \left( \frac{1}{\log(e + k)} \right)^{q(y) - 2\epsilon} \, dy \right\}.
\]
We set
\[
k = \delta^{-\nu(x)/p(x)} (\log(e + 1/\delta))^{-q(x) + \beta(x)/p(x)}.
\]
Then we have for \( y \in B(x, \delta) \),
\[
k^{-p(y)} \leq C k^{-p(x)}
\]
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and
\[(\log(e + k))^{-\eta(y)} \leq C(\log(e + k))^{-\eta(x)}\]

by (2.4) and (2.5). Consequently it follows that
\[F(\delta) \leq C\{\delta^{\alpha(x) - \eta(x)/p(x)}(\log(e + 1/\delta))^{-\eta(x) + \beta(x)/p(x)}
\]
\[+ \delta^{\alpha(x) + (p(x)-1)\eta(x)/p(x)}(\log(e + 1/\delta))^{\beta(x) - \eta(x) + \beta(x)/p(x)} + L_\varepsilon f(x)\}.
\]

Now the result follows.

**Lemma 3.5.** There exists a constant \(C > 0\) such that
\[\int_{B(z,r)} L_\varepsilon f(x) \, dx \leq Cr^{-\eta(z)}(\log(e + 1/r))^{-\beta(z)}\]
for all \(z \in G\), \(0 < r < d_G\) and \(f \geq 0\) satisfying \(\|f\|_{L^{r,\eta,\beta}(G)} \leq 1\).

**Proof.** Let \(f\) be a nonnegative measurable function on \(G\) satisfying \(\|f\|_{L^{r,\eta,\beta}(G)} \leq 1\).
Write
\[L_\varepsilon f(x) = \int_{\{y \in B(z,2r) : |x-y|^{-\epsilon} < f(y)\}} \rho_{-\varepsilon}(|x-y|)(\log(e + f(y)))^{-\epsilon} g(y) \, dy \]
\[+ \int_{\{y \in G \setminus B(z,2r) : |x-y|^{-\epsilon} < f(y)\}} \rho_{-\varepsilon}(|x-y|)(\log(e + f(y)))^{-\epsilon} g(y) \, dy \]
\[= L_1(x) + L_2(x),\]
where \(g(y) = f(y)^{\eta(y)}(\log(e + f(y)))^{\eta(y)}\). By Fubini’s theorem, we have
\[\int_{B(z,r)} L_1(x) \, dx \]
\[\leq C \int_{B(z,2r)} \left( \int_{\{y \in G : |x-y|^{-\epsilon} < f(y)\}} \rho_{-\varepsilon}(|x-y|) \, dx \right) (\log(e + f(y)))^{-\epsilon} g(y) \, dy \]
\[\leq C \int_{B(z,2r)} g(y) \, dy \leq Cr^{-\eta(z)}(\log(e + 1/r))^{-\beta(z)}.
\]

For \(L_2\), note that
\[L_2(x) \leq C \int_{G \setminus B(z,2r)} |x-y|^{-n}(\log(e + 1/|x-y|))^{-1} g(y) \, dy \]
\[\leq C \int_{G \setminus B(z,2r)} |z-y|^{-n}(\log(e + 1/|z-y|))^{-1} g(y) \, dy \]
for \(x \in B(z,r)\). Hence, as in the proof of Lemma 2.7, we see that
\[\int_{B(z,r)} L_2(x) \, dx \leq Cr^n \int_{G \setminus B(z,2r)} |z-y|^{-n}(\log(e + 1/|z-y|))^{-1} g(y) \, dy \]
\[\leq Cr^n \int_{2r}^{2d_G} t^{-\eta(z)}(\log(e + 1/t))^{-\beta(z)-1} \frac{dt}{t} \]
\[\leq Cr^{-\eta(z)}(\log(e + 1/r))^{-\beta(z)-1}.
\]
Thus this lemma is proved.
Proof of Theorem 3.1. We may assume that $f \geq 0$. For $\varepsilon = \min\{\alpha_-/2, q_0/2\}$ and $x \in \mathbb{R}^n$, set $L = L_\varepsilon f(x)$.

For $\delta > 0$, write

$$I_\alpha(x)f(x) = \int_{B(x,\delta)} |x - y|^\alpha f(y) \, dy + \int_{G \setminus B(x,\delta)} |x - y|^\alpha f(y) \, dy = I_1(\delta) + I_2(\delta).$$

In view of Lemma 3.4, we find

$$I_1(\delta) \leq \int_{B(x,\delta)} |x - y|^\alpha f(y) \, dy + \int_{\{y \in B(x,\delta) : |x - y| < \varepsilon \}} |x - y|^\alpha f(y) \, dy \leq C \left\{ \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{-\nu(x)} + \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{\nu(x)} \right\} \leq C \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{-\nu(x)}/p(x),$$

with $L = L_\varepsilon f(x)$. Moreover, Lemma 2.8 yields

$$I_2(\delta) \leq C \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{-\nu(x)/p(x)},$$

so that

$$I_\alpha(x)f(x) \leq C \left\{ \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{-\nu(x)}/p(x) \right\} \left\{ \left( \log(e + 1/\delta) \right)^{-\nu(x)/p(x)} + \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{\nu(x)/p(x)} \right\} \leq C \left\{ 1 + \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{-\nu(x)/p(x)} \right\} \leq C \left\{ 1 + \delta^\alpha \nu(x)/p(x) \left( \log(e + 1/\delta) \right)^{-\nu(x)/p(x)} \right\}.$$

In view of Lemma 3.5, we find

$$\frac{\Psi(x, I_\alpha(x)f(x))}{(\log(e + I_\alpha(x)f(x)))^{-1}} \leq C \int_{B(x,r)} (1 + L) \, dx \leq C r^{-\nu(x)} \left( \log(e + 1/r) \right)^{-\beta(x)},$$

which completes the proof of Theorem 3.1. \hfill \Box

Example 3.6. Let

$$\omega(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\log(1/|t|) & \text{when } 0 < |t| < r_0, \\ 1/\log(1/r_0) & \text{when } |t| \geq r_0 \end{cases}$$

and

$$\eta(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\log(1/|t|) & \text{when } 0 < |t| < r_0, \\ 1/\log(1/r_0) & \text{when } |t| \geq r_0 \end{cases}$$
for $0 < r_0 < 1/4$. Consider
\[ p(x) = p(x_1, x_2) = 1 + a \omega(x_2), \]
and
\[ q(x) = q(x_1, x_2) = q + b \eta(x_2), \]
where $a > 0$, $q > 0$ and $b > 0$. Let $\gamma \in \mathbb{R}$. If
\[ f(y) = |y_2|^{-1}(\log(e + 1/|y_2|))^{-\gamma}, \]
then note that
\[
\int_{B(z,r)} f(y)^p(y)(\log(e + f(y)))^q(y) dy \leq Cr^{-1}(\log(e + 1/r))^{-\beta}
\]
for all $z \in B = B(0, 1)$ and $r > 0$, when $\beta = \gamma - 1 - q > 0$. Further, for $0 < \alpha < 1$, we have
\[ I_\alpha f(x) \geq C|x_2|^{\alpha-1}(\log(e + 1/|x_2|))^{-\gamma+1} \]
for $x \in B(0, 1)$. Take $\gamma$ such that $\gamma < \delta + 1 + q$ for $\delta > 0$. Then we see that
\[
\int_{B(0,r)} I_\alpha f(x)^p(x)(\log(e + I_\alpha f(x)))^{(q(x)/p(x)+\alpha\beta)p^*(x)-1+\delta} dx \\
\geq C \int_{B(0,r)} |x_2|^{-1}(\log(e + 1/|x_2|))^{-\beta-1+\delta} dx = \infty
\]
for all $0 < r < 1$ and $\delta > 0$. This implies that Theorem 3.1 is best possible as to the exponents appearing in the Morrey condition.

4 Sobolev’s inequality in the case $p_- > 1$

In this section, we are concerned with the case $p_- > 1$. In this case, (1.1) holds for $K \geq -q_-/(p_- - 1)$.

We first show the boundedness of the Hardy-Littlewood maximal operator:
\[ Mf(x) = \sup_B \int_B |f(y)| dy, \]
where the supremum is taken over all balls $B$ containing $x$.

**Theorem 4.1.** Suppose $p_- > 1$ and $\nu_- > 0$ Then there exists a constant $C > 0$ such that
\[
\int_{B(z,r)} Mf(x)^p(x)(\log(e + Mf(x)))^{q(x)} dx \leq C r^{-\nu(x)}(\log(e + 1/r))^{-\beta(x)}
\]
for all $z \in G$, $0 < r < d_G$ and $f$ with $\|f\|_{L^{p_,\nu_-,\beta}(G)} \leq 1$.

**Remark 4.2.** For the constant case, we refer the reader to [25].
To prove Theorem 4.1, we prepare several lemmas. Let us begin with the following result, which is a consequence of [20, Theorem 1].

**Lemma 4.3 ([20, Theorem 1]).** Suppose \( p_0 > 1 \) and \( \nu_- > 0 \). Let \( f \) be a measurable function on \( G \) satisfying
\[
\int_{B(x,r)} |f(y)|^{p_0} \, dy \leq r^{-\nu(x)} (\log(e + 1/r))^{-\beta(x)}
\]
for all \( x \in G \) and \( 0 < r < d_G \). Then there exists a constant \( C > 0 \) such that
\[
\int_{B(z,r)} Mf(x)^{p_0} \, dx \leq Cr^{-\nu(x)} (\log(e + 1/r))^{-\beta(x)}
\]
for all \( z \in G \) and \( 0 < r < d_G \), where the constant \( C \) is independent of \( f \) satisfying (4.1).

**Lemma 4.4.** Suppose \( \nu_- > 0 \). Let \( f \) be a nonnegative measurable function on \( G \) satisfying \( \|f\|_{L^{\Phi,\nu,\beta}(G)} \leq 1 \) such that
\[
f(x) \geq 1 \quad \text{or} \quad f(x) = 0 \quad \text{for each} \quad x \in G.
\]
Then there exists a constant \( C > 0 \) such that
\[
Mf(x)^{p(x)} (\log(e + Mf(x)))^{q(x)} \leq CMg(x)
\]
for all \( x \in G \), where \( g(y) = f(y)^{p(y)} (\log(e + f(y)))^{q(y)} \). In the above, the constant \( C \) is independent of \( f \).

**Proof.** Let
\[
H = H_{x,r} = \int_{B(x,r)} g(y) \, dy.
\]
We shall show
\[
\int_{B(x,r)} f(y) \, dy \leq CH^{1/p(x)} (\log(e + H))^{-q(x)/p(x)}
\]
for all \( x \in G \) and \( 0 < r < d_G \). Then
\[
Mf(x) \leq CMg(x)^{1/p(x)} (\log(e + Mg(x)))^{-q(x)/p(x)}.
\]
This implies the desired conclusion.

To show (4.3), first consider the case when \( H \geq 1 \). Set
\[
k = H^{1/p(x)} (\log(e + H))^{-q(x)/p(x)}.
\]
Then we have
\[
\int_{B(x,r)} f(y) \, dy \leq k + C \int_{B(x,r)} f(y) \left( \frac{f(y)}{k} \right)^{p(y)-1} \left( \frac{\log(e + f(y))}{\log(e + k)} \right)^{q(y)} \, dy
\]
\[
= k + C \int_{B(x,r)} g(y) k^{-p(y)+1} (\log(e + k))^{-q(y)} \, dy.
\]
Since
\[ H \leq r^{-\nu(x)}(\log(e + 1/r))^{-\beta(x)} \]
for all \( x \in G \) and \( 0 < r < d_G \), we obtain for \( y \in B(x, r) \), as in the proof of Lemma 2.5,
\[ k^{-p(y)} \leq Ck^{-p(x)} = CH^{-1}(\log(e + H))^{q(x)} \]
and
\[ (\log(e + k))^{-q(y)} \leq C(\log(e + k))^{-q(x)} \leq C(\log(e + H))^{-q(x)}. \]
Consequently (4.3) follows.
In the case \( H \leq 1 \), we find
\[ H \leq CH^{1/p(x)}(\log(e + H))^{-q(x)/p(x)}. \]
Since \( f(y) \geq 1 \) or \( f(y) = 0 \) for each \( y \in G \), we have
\[ g(y) = f(y) \cdot f(y)^{p(y)-1}(\log(e + f(y)))^{q(y)} \geq Cf(y) \]
for some \( C > 0 \) and hence
\[ \int_{B(x,r)} f(y) \, dy \leq CH. \]
This shows (4.3). \( \square \)

**Proof of Theorem 4.1.** We may assume that \( f \geq 0 \). Write
\[ f = f \chi_{\{y:f(y)\geq 1\}} + f \chi_{\{y:f(y)<1\}} = f_1 + f_2, \]
where \( \chi_E \) denotes the characteristic function of \( E \). Take \( p_0 \) such that \( 1 < p_0 < p_- \). Since
\[
\int_{B(x,r)} f_1(y)^{p(y)/p_0}(\log(e + f_1(y)))^{q(y)/p_0} \, dy \\
\leq C \int_{B(x,r)} f_1(y)^{p(y)}(\log(e + f_1(y)))^{q(y)} \, dy \leq Cr^{-\nu(x)}(\log(e + 1/r))^{-\beta(x)}
\]
for all \( x \in G \) and \( 0 < r < d_G \), applying Lemma 4.4 with \( p(x) \) and \( q(x) \) replaced by \( p(x)/p_0 \) and \( q(x)/p_0 \), respectively, we obtain
\[ Mf_1(x)^{p(x)/p_0}(\log(e + Mf_1(x)))^{q(x)/p_0} \leq CMg_1(x), \]
where \( g_1(y) = f_1(y)^{p(y)/p_0}(\log(e + f_1(y)))^{q(y)/p_0} \). Note that \( g_1 \) satisfies (4.1). Since \( Mf_2 \leq 1 \), it follows that
\[ Mf(x)^{p(x)}(\log(e + Mf(x)))^{q(x)} \leq C(1 + Mg_1(x)^{p_0}). \]
Hence, by Lemma 4.3, we see that
\[
\int_{B(z,r)} Mf(x)^{p(x)}(\log(e + Mf(x)))^{q(x)} \, dx \leq C \int_{B(z,r)} (1 + Mg_1(x)^{p_0}) \, dx \\
\leq Cr^{-\nu(x)}(\log(e + 1/r))^{-\beta(z)}
\]
for all \( z \in G \) and \( 0 < r < d_G \), as required. \( \square \)
Now we give a Morrey version of Sobolev’s inequality for Riesz potentials. Let $p^*$ and $\Psi$ be as in (2.2) and (2.3), respectively.

**Theorem 4.5.** Suppose that $p_- > 1$ and (2.1) holds. Then there exists a constant $C > 0$ such that

$$\int_{B(z,r)} \Psi(x, |I_\alpha f(x)|) |dx| \leq C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$, $0 < r < d_G$ and $f$ satisfying $\|f\|_{L^{p^*,\nu,\beta}(G)} \leq 1$.

This theorem gives the following norm version, which is simpler than Corollaries 2.4 and 3.2.

**Corollary 4.6 (cf. [16, Theorem 4.3]).** Suppose that $p_- > 1$ and (2.1) holds. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{p^*,\nu,\beta}(G)} \leq C \|f\|_{L^{p,\nu,\beta}(G)}.$$

**Remark 4.7.** If $p(x) = p > 1$, $q(x) = q \in \mathbb{R}$, $\nu(x) = n$ and $\beta(x) = 0$, then $p^* = np/(n - \alpha p)$ and the operator $I_\alpha$ is bounded from $L^p(\log L)^q(G)$ to $L^{p^*}(\log L)^{q/p}(G)$, which is shown by O’Neil [26, Theorem 4.7].

For further related results, we refer the reader to the papers [3, 16, 17].

**Remark 4.8.** Theorem 4.5 is best possible as to the exponents appearing in the Morrey condition.

**Proof of Theorem 4.5.** We may assume that $f \geq 0$, as before. By Lemma 2.8, we find

$$I_\alpha f(x) = \int_{B(x,\delta)} |x - y|^{-\alpha(x) - n} f(y) dy + \int_{G \setminus B(x,\delta)} |x - y|^{-\alpha(x) - n} f(y) dy$$

$$\leq C \left\{ \delta^{\alpha(x)} M f(x) + \delta^{\alpha(x) - \nu(x)/p(x)} (\log(e + 1/\delta))^{-(q(x) + \beta(x))/\nu(x)} \right\}.$$

Considering

$$\delta = \min \left\{ d_G, M f(x)^{-p(x)/\nu(x)} (\log(e + M f(x)))^{-(q(x) + \beta(x))/\nu(x)} \right\},$$

we have

$$I_\alpha f(x) \leq C \left\{ 1 + M f(x)^{-p(x)/\nu(x)} (\log(e + M f(x)))^{-\alpha(x)(q(x) + \beta(x))/\nu(x)} \right\}$$

$$= C \left\{ 1 + M f(x)^{p(x)/p(x)} (\log(e + M f(x)))^{-\alpha(x)(q(x) + \beta(x))/\nu(x)} \right\}.$$ 

Then we find

$$\Psi(x, I_\alpha f(x)) \leq C \left\{ 1 + M f(x)^{p(x)} (\log(e + M f(x)))^{q(x)} \right\}$$

for all $x \in G$. It follows from Theorem 4.1 that

$$\int_{B(z,r)} \Psi(x, I_\alpha f(x)) |dx| \leq C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$, as required. □
5 Trudinger’s inequality

This section is concerned with Morrey version of Trudinger’s type exponential integrability for Riesz potentials, in case
\[
\text{ess inf}_{x \in \mathbb{R}^n} (\alpha(x) - \nu(x)/p(x)) \geq 0,
\]
which is equivalent to
\[
\text{ess sup}_{x \in \mathbb{R}^n} (1/p(x) - \alpha(x)/\nu(x)) \leq 0.
\]

Set
\[
\Gamma(x, r) = c_0 \int_1^r (\log(e + t))^{-(q(x) + \beta(x))/p(x)} \frac{dt}{t}
\]
for \(x \in \mathbb{R}^n\) and \(r \geq 2\), where we choose \(c_0\) such that \(\inf_{x \in \mathbb{R}^n} \Gamma(x, 2) = 2\). For convenience, set \(\Gamma(x, r) = (\Gamma(x, 2)/2)r\) when \(r < 2\). Note that there exists a constant \(C > 0\) such that
\[
C^{-1} \leq \frac{\Gamma(x, r^2)}{\Gamma(x, r)} \leq C \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } r \geq 2,
\]
since \(-(q(x) + \beta(x))/p(x)\) is bounded. Let
\[
s_x = \sup_{r \geq 2} \Gamma(x, r) = c_0 \int_1^\infty (\log(1 + t))^{-(q(x) + \beta(x))/p(x)} \frac{dt}{t}.
\]
Then \(2 < s_x \leq \infty\) and \(\Gamma(x, \cdot)\) is bijective from \([0, \infty)\) to \([0, s_x)\). We denote by \(\Gamma^{-1}(x, \cdot)\) the inverse function of \(\Gamma(x, \cdot)\). If \(s_x < \infty\), we set \(\Gamma^{-1}(x, r) = \infty\) for \(r \geq s_x\).

**Theorem 5.1.** Suppose \(\nu_- > 0\) and (5.1) holds. Let \(\varepsilon\) be a measurable function on \(\mathbb{R}^n\) such that
\[
\text{ess inf}_{x \in \mathbb{R}^n} (\nu(x)/p(x) - \varepsilon(x)) > 0 \text{ and } 0 < \varepsilon_- \leq \varepsilon_+ < \alpha_-. \tag{5.2}
\]
Then there exist constants \(c_1, c_2 > 0\) such that
\[
\int_{B(z, r)} \Gamma^{-1} \left( x, \frac{|I_{a(x)}f(x)|}{c_1} \right) dx \leq c_2 r^{s(x) - \varepsilon(x)/p(z)}
\]
for all \(z \in G\), \(0 < r < d_G\) and \(f\) satisfying \(\|f\|_{L^{\infty, \nu, \beta}(G)} \leq 1\). In the above \(|I_{a(x)}f(x)|/c_1 < s_x\) for a.e. \(x \in B(z, r)\).

**Remark 5.2.** Let \(\alpha, p, q, \nu, \beta, \varepsilon\) be all constants and \(0 < \varepsilon < \alpha\).

1. If \(q + \beta < p\), then, for \(r \geq 2\),
\[
C^{-1} \Gamma(r) \leq (\log(e + r))^{1-(q+\beta)/p} \leq C \Gamma(r)
\]
and
\[
\Gamma^{-1}(C^{-1}r) \leq \exp(r^{p/(p-q-\beta)}) \leq \Gamma^{-1}(Cr).
\]
(2) If \( q + \beta = p \), then, for \( r \geq 2 \),
\[
C^{-1}\Gamma(r) \leq \log(\log(e + r)) \leq C\Gamma(r)
\]
and
\[
\Gamma^{-1}(C^{-1}r) \leq \exp(\exp(r)) \leq \Gamma^{-1}(Cr).
\]

**Corollary 5.3.** Under the assumptions in Theorem 5.1, there exist constants \( c_1, c_2 > 0 \) such that

1. in case \( \operatorname{ess} \sup_{x \in \mathbb{R}^n} (q(x) + \beta(x))/p(x) < 1 \),
   \[
   \int_{B(z,r)} \exp \left( \frac{|I_\alpha f(x)|^{p(x)/(p(x) - q(x) - \beta)}}{c_1} \right) \, dx \leq c_2 r^\epsilon(z)^{-\nu/p(z)},
   \]
2. in case \( \operatorname{ess} \inf_{x \in \mathbb{R}^n} (q(x) + \beta(x))/p(x) \geq 1 \),
   \[
   \int_{B(z,r)} \exp \left( \exp \left( \frac{|I_\alpha f(x)|}{c_1} \right) \right) \, dx \leq c_2 r^\epsilon(z)^{-\nu/p(z)}
   \]
for all \( z \in G, 0 < r < d_G \) and \( f \) satisfying \( \|f\|_{L^{q,\nu,\beta}(G)} \leq 1 \).

**Remark 5.4.** When \( p, q, \beta, \alpha, \nu \) are all constants such that \( p = 1, q = 0, \beta < 1 \) and \( \alpha = \nu \), this is due to Corollaries 4.6 and 4.8 in [12]. In particular, the case \( p = 1, q = \beta = 0, \alpha = \nu = 1 \) and \( r = d_G \) coincides with the result by Trudinger [30]. A weaker result is shown by Mizuta and Shimomura [15, Theorem 4.4].

To prove the theorem, we use the following lemmas. The first lemma can be proved with minor changes of the proof of Lemma 2.8.

**Lemma 5.5.** Suppose that \( \nu_- > 0 \) and (5.1) holds. Then there exists a constant \( C > 0 \) such that
\[
\int_{G \setminus B(x,\delta)} |x - y|^{|\alpha(x) - n|} f(y) \, dy \leq C\Gamma(x, 1/\delta)
\]
for all \( x \in G, 0 < \delta < d_G \) and \( f \geq 0 \) satisfying \( \|f\|_{L^{q,\nu,\beta}(G)} \leq 1 \).

**Lemma 5.6.** Let \( \epsilon \) be a measurable function on \( G \) satisfying (5.2). Setting \( \rho(z, r) = r^{\epsilon(z)}(\log(e + 1/r))^{(q(z) + \beta(z))/p(z)} \), define
\[
I_{\rho(z)} f(x) = \int_{G} \frac{\rho(z, |x - y|)}{|x - y|^n} f(y) \, dy.
\]
Then there exists a constant \( C > 0 \) such that
\[
\int_{B(z, r)} I_{\rho(z)} f(x) \, dx \leq C r^\epsilon(z)^{-\nu(z)/p(z)}
\]
for all \( z \in G, 0 < r < d_G \) and \( f \geq 0 \) satisfying \( \|f\|_{L^{q,\nu,\beta}(G)} \leq 1 \).
Proof. Write

\[
I_\rho(z)f(x) = \int_{B(z,2r)} \frac{\rho(z,|x-y|)}{|x-y|^n} f(y) \, dy + \int_{G\setminus B(z,2r)} \frac{\rho(z,|x-y|)}{|x-y|^n} f(y) \, dy
\]

\[
= I_1(x) + I_2(x).
\]

By Fubini’s theorem and Lemma 2.6, we have

\[
\int_{B(z,r)} I_1(x) \, dx = \int_{B(z,2r)} \left( \int_{B(z,r)} \frac{\rho(z,|x-y|)}{|x-y|^n} \, dx \right) f(y) \, dy
\]

\[
\leq \int_{B(z,2r)} \left( \int_{B(y,r)} \frac{\rho(z,|x-y|)}{|x-y|^n} \, dx \right) f(y) \, dy
\]

\[
= n\sigma_n \int_{B(z,2r)} \left( \int_0^{3r} \frac{\rho(z,t)}{t} \, dt \right) f(y) \, dy
\]

\[
\leq C\rho(z,3r) \int_{B(z,2r)} f(y) \, dy
\]

\[
\leq C\rho(z,3r)(2r)^{n-\nu(z)/p(z)}(\log(e + 1/(2r)))^{-(q(z)+\beta(z))/p(z)}
\]

\[
\leq C\rho(z,3r)(2r)^{n+\varepsilon(z)-\nu(z)/p(z)}.
\]

For \(I_2\), note that

\[
I_2(x) \leq C \int_{G\setminus B(z,2r)} \frac{\rho(z,|z-y|)}{|z-y|^n} f(y) \, dy \quad \text{for} \quad x \in B(z, r),
\]

since there exists a constant \(C > 0\) such that

\[
C^{-1} \leq \frac{\rho(z,r)}{\rho(z,s)} \leq C \quad \text{for} \quad z \in G, \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.
\]

Hence we have by Lemmas 2.6 and 2.7

\[
I_2(x) \leq C \int_{2r}^{2dr} \frac{\rho(z,t)}{t^n} t^{\nu(z)/p(z)}(\log(e + 1/t))^{-(q(z)+\beta(z))/p(z)} \frac{dt}{t}
\]

\[
\leq C \int_{2r}^{2dr} t^{\varepsilon(z)-\nu(z)/p(z)} \frac{dt}{t}
\]

\[
\leq C(r^{\varepsilon(z)-\nu(z)/p(z)}).
\]

Thus this lemma is proved. \(\square\)

Proof of Theorem 5.1. We have only to treat nonnegative \(f\) with \(\|f\|_{L^{\nu(p),\beta}(G)} \leq 1\).

By Lemma 5.5 we find

\[
I_\alpha(x)f(x) = \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) \, dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) \, dy
\]

\[
= \int_{B(x,\delta)} |x-y|^{\alpha(x)-\varepsilon(z)}(\log(e + 1/|x-y|))^{-(q(z)+\beta(z))/p(z)} \frac{\rho(z,|x-y|)}{|x-y|^n} f(y) \, dy
\]

\[+ C\Gamma(x,1/\delta)
\]

\[
\leq C \{ \delta^{\alpha(x)-\varepsilon(z)}(\log(e + 1/\delta))^{-(q(z)+\beta(z))/p(z)} I_\rho(z)f(x) + \Gamma(x,1/\delta) \}
\]

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for \( \delta > 0 \). Considering
\[
\delta = \min \left\{ d_G, \left( \frac{\Gamma(x, I_{\rho(z)}f(x))(\log(e + I_{\rho(z)}f(x)))^{(q(z) + \beta(z))/p(z)}}{I_{\rho(z)}f(x)} \right)^{1/(\alpha(x) - \gamma(z))} \right\},
\]
we have the inequality
\[
I_{\alpha(x)}f(x) \leq c_1 \max \left\{ 1, \Gamma(x, I_{\rho(z)}f(x)) \right\},
\]
for some constant \( c_1 > 0 \). Since \( 1 \leq \Gamma(x, 1) = \Gamma(x, 2)/2, \Gamma^{-1}(x, 1) \leq 1 \). Then
\[
\int_{B(z, r)} \Gamma^{-1}(x, I_{\alpha(z)}f(x) dx \leq \int_{B(z, r)} \left\{ 1 + I_{\rho(z)}f(x) \right\} dx
\]
for all \( z \in G \) and \( 0 < r < d_G \). Hence Lemma 5.6 gives the conclusion.

6 Continuity

In this section we are concerned with continuity for Riesz potentials when (5.1) and the following condition hold:
\[
\mathcal{H}(x, r) \equiv \int_0^r t^\alpha(x, -\nu(x)/p(x))(\log(e + 1/t))^{-(q(x) + \beta(x))/p(x)} dt \leq \infty.
\]
In this case \( \mathcal{H}(x, r) \to 0 \) as \( r \to 0 \) and \( \mathcal{H}(x, r) \leq \mathcal{H}(x, 2r) \leq C \mathcal{H}(x, r) \) for some constant \( C > 0 \) independent of \( x \in \mathbb{R}^n \) and \( 0 < r < \infty \).

**Theorem 6.1.** Let \( 0 < \theta \leq 1 \) and \( \gamma(x) = \alpha(x) - \nu(x)/p(x) \). Suppose that \( \alpha \in \text{Lip}_p(G) \), \( \nu_+ > 0 \) and \( 0 \leq \gamma_+ \leq \gamma_- < \theta \). If \( f \) is a measurable function on \( G \) satisfying \( \|f\|_{L^{0, \nu, \beta}(G)} \leq 1 \), then \( I_{\alpha(z)}f \) is continuous on \( G \). Moreover, there exists a constant \( C > 0 \) such that
\[
|I_{\alpha(z)}f(x) - I_{\alpha(z)}f(z)| \leq C\{\mathcal{H}(x, |x - z|) + \mathcal{H}(z, |x - z|)\}
\]
for all \( x, z \in G \), where the constant \( C \) is independent of \( f \) satisfying \( \|f\|_{L^{0, \nu, \beta}(G)} \leq 1 \). That is, the operator \( I_{\alpha(z)} \) is bounded from \( L^{0, \nu, \beta}(G) \) to \( \Lambda_H(G) \).

**Corollary 6.2.** Let \( 0 < \theta \leq 1 \) and \( \gamma(x) = \alpha(x) - \nu(x)/p(x) \). Suppose \( \alpha \in \text{Lip}_p(G) \) and \( 0 < \gamma_- \leq \gamma_+ < \theta \). Then the operator \( I_{\alpha(z)} \) is bounded from \( L^{0, \nu, \beta}(G) \) to \( \text{Lip}_{\beta}^1(G) \).

**Remark 6.3.** The case when \( \alpha, p \) are constants and \( n = 1 \) is the result of Hardy-Littlewood [6, Theorem 12].

**Corollary 6.4.** Let \( \alpha, \nu \) and \( \beta \) be constants. Suppose
\[
0 \leq \alpha - \nu/p_- \leq \alpha - \nu/p_+ < 1, \quad \beta > \text{ess sup}_{x \in \mathbb{R}^n}(\rho(x) - q(x)).
\]
Then there exists a constant $C > 0$ such that
\[
|I_{a}f(x) - I_{a}f(z)| \\ \leq C\{ |x - z|^{|\alpha - \nu/p(x)}(\log(e + 1/|x - z|))^{-\nu/p(x)} + |x - z|^{|\alpha - \nu/p(z)}(\log(e + 1/|x - z|))^{-\nu/p(z)} \},
\]
for all $x, z \in G$ and for all $f$ satisfying $\|f\|_{L^{\phi,\nu,\beta}(G)} \leq 1$.

**Remark 6.5.** If $p(x) = 1$ and $q(x) = 0$, the corollary above is a special case of [21, Theorem 3.3]. If $p(x) = 1, q(x) = 0, \alpha = \nu$ and $\beta > 1$, the corollary above is [11, Theorem 1.1 (3)], where $\alpha, \nu$ and $\beta$ are constants. See also [21, 29].

To prove the theorems, we need the following lemmas.

**Lemma 6.6.** Let $0 < \theta \leq 1$. Suppose $\alpha \in \text{Lip}_{b}(G)$. Then there exists a constant $C > 0$ such that
\[
||x - y|^{\alpha(x) - n} - |z - y|^{\alpha(z) - n}| \leq C(||x - z||x - y|^{\alpha(x) - n - 1} + |x - z|^{\theta}|x - y|^{\alpha(x) - n - \theta}),
\]
for all $x, y, z \in G$ satisfying $|x - y| \geq 2|x - z|$.

**Proof.** Let $r = |x - y|$ and $s = |z - y|$. Then $1/2 \leq r/s \leq 2$ and
\[
|r^{\alpha(x) - n} - s^{\alpha(z) - n}| \leq |r^{\alpha(x) - n} - s^{\alpha(x) - n}| + |s^{\alpha(x) - n} - s^{\alpha(z) - n}|
\]
\[= |r - s||\alpha(x) - n| \tilde{r}^{\alpha(x) - n - 1} + |\alpha(x) - \alpha(z)||\log s| s^{\tilde{\alpha} - n}
\]
\[\leq C(||x - z||r^{\alpha(x) - n - 1} + |x - z|^{\theta}s^{\alpha(x) - n - \theta}s^{\tilde{\alpha} - \alpha(x)})
\]
\[\leq C(||x - z||r^{\alpha(x) - n - 1} + |x - z|^{\theta}\tilde{r}^{\alpha(x) - n - \theta}),
\]
where $\tilde{r} = (1 - t)r + ts$ and $\tilde{\alpha} = (1 - u)\alpha(x) + u\alpha(z)$ for some $0 < t, u < 1$. \hfill \Box

The following two lemmas can be proved in the same manner as Lemma 2.8.

**Lemma 6.7.** Suppose $\nu_{-} > 0$. Then there exists a constant $C > 0$ such that
\[
\int_{B(x,r) \setminus B(x,s)} |x - y|^{\alpha(x) - n}f(y)dy \leq C \int_{s}^{r} t^{\alpha(x) - \nu(x)/p(x)} (\log(e + 1/t))^{-(q(x) + \beta(x))/p(x)} dt/t
\]
for all $x \in G, 0 < 2s < r < \infty$ and for all $f \geq 0$ satisfying $\|f\|_{L^{\phi,\nu,\beta}(G)} \leq 1$.

**Lemma 6.8.** Let $\theta > 0$. Suppose $\nu_{-} > 0$ and
\[
\text{ess sup}_{x \in G} (\alpha(x) - \nu(x)/p(x)) < \theta.
\]

Then there exists a constant $C > 0$ such that
\[
\int_{\mathbb{R}^{n} \setminus B(x,r)} |x - y|^{\alpha(x) - \theta}f(y) dy \leq C_{\nu}r^{\alpha(x) - \nu(x)/p(x) - \theta} (\log(e + 1/r))^{-(q(x) + \beta(x))/p(x)},
\]
for all $x \in G, r > 0$ and for all $f \geq 0$ satisfying $\|f\|_{L^{\phi,\nu,\beta}(G)} \leq 1$. 22
Proof of Theorem 6.1. We may assume that $f \geq 0$. Write

\[
I_\alpha(x)f(x) - I_\alpha(z)f(z) = \int_{B(x,2|x-z|)} |x - y|^{\alpha(x) - n} f(y) \, dy - \int_{B(x,2|x-z|)} |z - y|^{\alpha(z) - n} f(y) \, dy
\]

\[
+ \int_{G \setminus B(x,2|x-z|)} (|x - y|^{\alpha(x) - n} - |z - y|^{\alpha(z) - n}) f(y) \, dy
\]

for $x, z \in G$. Using Lemma 6.7, we have

\[
\int_{B(x,2|x-z|)} |x - y|^{\alpha(x) - n} f(y) \, dy \leq C \int_0^{2|x-z|} t^{\alpha(x) - \nu(x)/p(x)} \left( \log(e + 1/t) \right)^{-(q(x) + \beta(x))/p(x)} \frac{dt}{t}
\]

\[
\leq C \mathcal{H}(x, |x - z|),
\]

and

\[
\int_{B(x,2|x-z|)} |x - y|^{\alpha(z) - n} f(y) \, dy \leq \int_{B(z,3|x-z|)} |x - y|^{\alpha(z) - n} f(y) \, dy
\]

\[
\leq C \mathcal{H}(z, |x - z|).
\]

On the other hand, by Lemmas 6.6 and 6.8, we have

\[
\int_{G \setminus B(x,2|x-z|)} \left( |x - y|^{\alpha(x) - n} - |z - y|^{\alpha(z) - n} \right) f(y) \, dy
\]

\[
\leq C \left\{ |x - z| \int_{G \setminus B(x,2|x-z|)} |x - y|^{\alpha(x) - n - 1} f(y) \, dy
\]

\[
+ |x - z|^{\theta} \int_{G \setminus B(x,2|x-z|)} |x - y|^{\alpha(x) - n - \theta} f(y) \, dy \right\}
\]

\[
\leq C |x - z|^{\alpha(x) - \nu(x)/p(x)} \left( \log(e + 1/|x - z|) \right)^{-(q(x) + \beta(x))/p(x)}
\]

\[
\leq C \mathcal{H}(x, |x - z|).
\]

Then we have the conclusion. \qed

References


The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail: yomizuta@hiroshima-u.ac.jp

and

Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
E-mail: enakai@cc.osaka-kyoiku.ac.jp

and

General Arts,
Hiroshima National College of Maritime Technology,
Higashino Oosakikamijima Toyotagun 725-0231, Japan
E-mail: ohno@hiroshima-cmt.ac.jp

and

Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail: tshimo@hiroshima-u.ac.jp