Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponent

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November 14, 2009

Abstract

Let α, ν, β, p and q be all variable exponents. Our aim in this paper is to deal with Sobolev embeddings for Riesz potentials of order α with functions f in Morrey spaces $L^{\Phi,\nu,\beta}(G)$ with $\Phi(t) = t^p(\log(e+t))^q$ over a bounded open set $G \subset \mathbf{R}^n$. Here p and q satisfy the log-Hölder and the loglog-Hölder conditions, respectively. Also the case when p attains the value 1 in some parts of the domain is included in our results.

1 Introduction

Let G be a bounded open set in $\mathbf{R}^n$. We denote by d_G the diameter of G.

For a measurable function $\alpha : \mathbf{R}^n \to (0, n)$, we define the Riesz potential of order α for an integrable function f on G by

$$I_{\alpha(x)}f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha(x)-n} f(y)\, dy.$$

Here and in what follows we assume that $f = 0$ outside G. We also assume that $\alpha_- \equiv \operatorname{ess\,inf}_{x\in\mathbf{R}^n} \alpha(x) > 0$.

We denote by $B(x, r)$ the ball $\{y \in \mathbf{R}^n : |y - x| < r\}$ with center x and of radius $r > 0$, and by $|B(x, r)|$ its Lebesgue measure, i.e. $|B(x, r)| = \sigma_n r^n$, where σ_n is the volume of the unit ball in $\mathbf{R}^n$. We define the integral mean of f over $B(x, r)$ by

$$⨍_{B(x,r)} f(y)\, dy = \frac{1}{|B(x, r)|} \int_{B(x,r)} f(y)\, dy.$$

Following Cruz-Uribe and Fiorenza [5], we consider continuous functions $p : \mathbf{R}^n \to [1, \infty)$ and $q : \mathbf{R}^n \to \mathbf{R}$, which are called variable exponents. In this paper, we consider variable exponents p and q on $\mathbf{R}^n$ such that

2000 Mathematics Subject Classification : Primary 31B15, 46E30

Key words and phrases : Morrey spaces of variable exponent, Riesz potentials, Sobolev embeddings, Sobolev's inequality, Trudinger's exponential inequality, Lipschitz spaces of variable exponent

(P1) $1 \le p_- \equiv \inf_{x\in\mathbf{R}^n} p(x) \le \sup_{x\in\mathbf{R}^n} p(x) \equiv p_+ < \infty$;

(P2) $|p(x) - p(y)| \le C/\log(e + 1/|x-y|)$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$;

(Q1) $-\infty < q_- \equiv \inf_{x\in\mathbf{R}^n} q(x) \le \sup_{x\in\mathbf{R}^n} q(x) \equiv q_+ < \infty$;

(Q2) $|q(x) - q(y)| \le C/\log(e + (\log(e + 1/|x-y|)))$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

If p satisfies (P2) (resp. q satisfies (Q2)), then p (resp. q) is said to satisfy the log-Hölder (resp. loglog-Hölder) condition.

Set

$$\Phi(x,r) = \Phi_{p,q}(x,r) = r^{p(x)}(\log(e+r))^{q(x)}.$$

For bounded measurable functions $\nu : \mathbf{R}^n \to (0,n]$ and $\beta : \mathbf{R}^n \to \mathbf{R}$, let $L^{\Phi,\nu,\beta}(G)$ be the set of all measurable functions f on G such that $\|f\|_{L^{\Phi,\nu,\beta}(G)} < \infty$, where

$$\|f\|_{L^{\Phi,\nu,\beta}(G)} = \inf\left\{\lambda > 0 : \sup_{x\in G, 0<r<d_G} r^{\nu(x)}(\log(e+1/r))^{\beta(x)} \fint_{B(x,r)} \Phi(y, |f(y)|/\lambda)\, dy \le 1\right\};$$

we set $f = 0$ outside G. For the constant Morrey spaces, we refer to [19], [27] and [15, 22, 24, 25]. For simplicity, in the case $\nu \equiv n$ and $\beta \equiv 0$, $L^{\Phi,\nu,\beta}(G)$ is denoted by $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$.

Throughtout this paper, we assume that (P1), (P2), (Q1) and (Q2) hold and that there exists a constant $K > 0$ such that

$$K(p(x) - 1) + q(x) > 0 \tag{1.1}$$

for all $x \in \mathbf{R}^n$. In this case we can find $c_0 > e$ such that, for each fixed $x \in \mathbf{R}^n$, $\bar\Phi(x,r) \equiv r^{p(x)}(\log(c_0 + r))^{q(x)}$ is convex on $[0,\infty)$, $\lim_{r\to 0} \bar\Phi(x,r) = \bar\Phi(x,0) = 0$ and $\lim_{r\to\infty} \bar\Phi(x,r) = \infty$ (see [9, Theorem 5.1]). Then $\|\cdot\|_{L^{\Phi,\nu,\beta}(G)}$ is a quasi norm, since

$$\Phi(x, c^{-1}r) \le \bar\Phi(x,r) \le \Phi(x, cr),$$

for some constant $c > 0$ independent of $x \in \mathbf{R}^n$ and $r \ge 0$. Furthermore, $t^{-1}\Phi(x,t)$ is uniformly almost increasing on $(0,\infty)$, that is, there exists a constant $C > 0$ such that

$$s^{-1}\Phi(x,s) \le Ct^{-1}\Phi(x,t), \tag{1.2}$$

whenever $0 < s < t$ and $x \in \mathbf{R}^n$.

Our aim in this paper is to discuss the boundedness of the operator $I_\alpha : f \longrightarrow I_{\alpha(x)}f(x)$ from the Morrey space $L^{\Phi,\nu,\beta}(G)$ to another Morrey space $L^{\Psi,\nu,\beta}(G)$ with suitable $\Psi(x,r)$. When $p_- = \inf_{x\in\mathbf{R}^n} p(x) > 1$, the maximal functions are a crucial tool as in Hedberg [8], where an easy proof of Sobolev's inequality for Riesz potentials is given. Since we are mainly concerned with the case $p_- = 1$, our strategy is to find an estimate of Riesz potentials by use of another Riesz-type potentials of 0 order, which plays a role of the maximal functions (see Sections 2 and 3). Our result contains the known result, as a special case, that I_α is bounded from $L^1(\log L)^q(G)$ to $L^{p^*}(\log L)^{p^*q-1}(G)$ for $p^* = n/(n-\alpha)$ and $q > 0$ (O' Neil [26, Theorem 5.2]); see Remark 2.3.

In Section 4, we investigate the case $p_- > 1$. For this purpose, we first show the boundedness of the Hardy-Littlewood maximal operator M. Our result contains the known result, as a special case, that I_α is bounded from $L^p(\log L)^q(G)$ to $L^{p^*}(\log L)^{p^*q/p}(G)$ for $p^* = np/(n-\alpha p)$ and $q \in \mathbf{R}$ (O'Neil [26, Theorem 4.7]); see Remark 4.7. For related results, see [1, 3, 4, 16, 17, 18].

In Section 5, we are concerned with Morrey version of Trudinger's type exponential integrability for $I_{\alpha(x)}f(x)$ in the case $p_- \geq 1$. Our result contains the result of Trudinger [30] and [12, Corollaries 4.6 and 4.8] as special cases (Remark 5.4). The result is also an improvement of [15, Theorems 4.4 and 4.5]. For related results, see [2, 4, 10, 11, 28, 31].

In the last section we discuss the continuity of $I_{\alpha(x)}f(x)$. For a function $\phi : \mathbf{R}^n \times (0,\infty) \to (0,\infty)$, let $\Lambda_\phi(G)$ be the set of all functions f on G such that $\|f\|_{\Lambda_\phi(G)} < \infty$, where

$$\|f\|_{\Lambda_\phi} = \sup_{x,y\in G,\, x\neq y} \frac{2|f(x)-f(y)|}{\phi(x,|x-y|)+\phi(y,|x-y|)}.$$

See [23] for the function space Λ_ϕ. If $\phi(x,r) = r^{\gamma(x)}$, then we denote $\Lambda_\phi(G)$ by $\mathrm{Lip}_{\gamma(\cdot)}(G)$. In the last section we show the boundedness of the operator $I_{\alpha(\cdot)}$ from $L^{\Phi,\nu,\beta}(G)$ to $\Lambda_\phi(G)$ under some conditions. It is known that I_α is bounded from $L^p(G)$ to $\mathrm{Lip}_\gamma(G)$ for $0 < \gamma = \alpha - n/p < 1$. We extend this fact to the boundedness of $I_{\alpha(\cdot)}$ from $L^{p(\cdot)}$ to $\mathrm{Lip}_{\gamma(\cdot)}(G)$ as a corollary (Corollary 6.2).

Throughout this paper, let C denote various constants independent of the variables in question and $C(a,b,\cdots)$ be a constant that depends on $a,b,\cdots$.

2 Sobolev's inequality in the case $p_- = 1$

Recall that $\alpha : \mathbf{R}^n \to (0,n)$, $\nu : \mathbf{R}^n \to (0,n]$ and $\beta : \mathbf{R}^n \to \mathbf{R}$ are bounded measurable functions and $\alpha_- > 0$. Throughtout this section, we assume that

$$\operatorname*{ess\,inf}_{x\in\mathbf{R}^n} \,(1/p(x) - \alpha(x)/\nu(x)) > 0. \tag{2.1}$$

In this case we have $\nu_- \geq \alpha_- > 0$.

Our first aim is to give the following Morrey version of Sobolev's type inequality for Riesz potentials of functions satisfying Morrey conditions. We consider the Sobolev exponent

$$1/p^*(x) = 1/p(x) - \alpha(x)/\nu(x) \tag{2.2}$$

and the new modular function

$$\Psi(x,t) = t^{p^*(x)}(\log(e+t))^{p^*(x)(q(x)/p(x)+\alpha(x)\beta(x)/\nu(x))}. \tag{2.3}$$

THEOREM 2.1. *Let $p_- = 1$. Suppose that* (2.1) *holds. Then, for each $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(x, |I_{\alpha(x)}f(x)|)(\log(e+|I_{\alpha(x)}f(x)|))^{-(1+\varepsilon)}\, dx \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)-\varepsilon}$$

whenever $z \in G$, $0 < r < d_G$ and $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

REMARK 2.2. For $\eta \in \mathbf{R}$, set

$$\begin{aligned} \widetilde{\Psi}_\eta(x,t) &= \Psi(x,t)(\log(e+t))^{-\eta} \\ &= t^{p^*(x)}(\log(e+t))^{p^*(x)(q(x)/p(x)+\alpha(x)\beta(x)/\nu(x))-\eta} \end{aligned}$$

Then $\widetilde{\Psi}_\eta(x,t)$ satisfies the condition (1.1) with $p(x)$ and $q(x)$ replaced by $p^*(x)$ and $p^*(x)(q(x)/p(x)+\alpha(x)\beta(x)/\nu(x)) - \eta$, respectively, and thus $\|\cdot\|_{L^{\tilde{\Psi}_\eta,\nu,\beta}(G)}$ is a quasi norm.

REMARK 2.3. In this theorem, we can not take $\varepsilon = 0$ (see [11, Remark 3.3] and O'Neil [26, Theorem 5.2]).

This theorem gives the following norm version.

COROLLARY 2.4. *Let $p_- = 1$. Suppose that* (2.1) *holds. Then, for $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\|I_{\alpha(\cdot)}f\|_{L^{\tilde{\Psi}_\varepsilon,\nu,\beta}(G)} \le C\|f\|_{L^{\Phi,\nu,\beta}(G)}.$$

For $\varepsilon > 0$, setting

$$\rho_\varepsilon(r) = r^{-n}(\log(e+1/r))^{-\varepsilon-1},$$

we consider the logarithmic potential

$$J_\varepsilon f(x) = \int_G \rho_\varepsilon(|x-y|)g(y)\, dy,$$

where $g(y) = \Phi(y, |f(y)|) = |f(y)|^{p(y)}(\log(e+|f(y)|))^{q(y)}$. Write

$$\begin{aligned} I_{\alpha(x)}f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\, dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\, dy \\ &= I_1(\delta) + I_2(\delta). \end{aligned}$$

Following the Hedberg trick [8], we give an estimate of $I_1(\delta)$ by $J_\varepsilon f(x)$, instead of maximal functions. After this, we give an estimate of $I_2(\delta)$ by use of Young's inequality. Finally, taking δ suitably, we obtain an estimate of $I_{\alpha(x)}f(x)$ by $J_\varepsilon f(x)$. For this purpose, we prepare some lemmas.

Let us begin with an estimate of $I_1(\delta)$ by $J_\varepsilon f(x)$.

LEMMA 2.5. *For $0 < \delta \le d_G$, $x \in G$ and a nonnegative integrable function f on G, set*

$$I_1(\delta) = \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\, dy.$$

Let $\varepsilon > 0$ be fixed and set $J = J_\varepsilon f(x)$ for simplicity. Then there exists a constant $C > 0$ such that

$$\begin{aligned} I_1(\delta) &\le C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ &\quad + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)}J\}. \end{aligned}$$

Proof. For $k > 0$, we have by (1.2)

$$\begin{aligned} I_1(\delta) &\le k\int_{B(x,\delta)} |x-y|^{\alpha(x)-n}dy \\ &\quad + C\int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\left(\frac{f(y)}{k}\right)^{p(y)-1}\left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} dy \\ &\le C\left\{k\delta^{\alpha(x)} + \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} g(y)\left(\frac{1}{k}\right)^{p(y)-1}\left(\frac{1}{\log(e+k)}\right)^{q(y)} dy\right\} \\ &\le C\Bigg\{k\delta^{\alpha(x)} + \delta^{\alpha(x)}(\log(e+1/\delta))^{1+\varepsilon} \\ &\quad \times \int_{B(x,\delta)} \rho_\varepsilon(|x-y|)g(y)\left(\frac{1}{k}\right)^{p(y)-1}\left(\frac{1}{\log(e+k)}\right)^{q(y)} dy\Bigg\}. \end{aligned}$$

We set

$$k = \delta^{-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}.$$

For $y \in B(x,\delta)$, note from (P2) that

$$|(p(x)-p(y))\log k| \le C$$

so that

$$k^{-p(y)} \le Ck^{-p(x)}. \tag{2.4}$$

Similarly, by (Q2) we have

$$(\log(e+k))^{-q(y)} \le C(\log(e+k))^{-q(x)}. \tag{2.5}$$

Consequently it follows from (2.4) and (2.5) that

$$\begin{aligned} I_1(\delta) &\le C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ &\quad + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)}J\}. \end{aligned}$$

Now the result follows. □

Next we give an estimate for

$$I_2(\delta) = \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\,dy.$$

LEMMA 2.6. *There exists a constant $C > 0$ such that*

$$\fint_{B(x,r)} f(y)dy \le Cr^{-\nu(x)/p(x)}(\log(e+1/r))^{-(q(x)+\beta(x))/p(x)}$$

for all $x \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

Proof. For $k > 0$, we have by (1.2)

$$\begin{aligned}\fint_{B(x,r)} f(y)dy &\le k + C\fint_{B(x,r)} f(y)\left(\frac{f(y)}{k}\right)^{p(y)-1}\left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} dy \\ &= k + C\fint_{B(x,r)} g(y)k^{-p(y)+1}(\log(e+k))^{-q(y)}dy,\end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(e+f(y)))^{q(y)}$ as before. Setting

$$k = r^{-\nu(x)/p(x)}(\log(e+1/r))^{-(q(x)+\beta(x))/p(x)},$$

we find by (P2) and (Q2)

$$\begin{aligned}\fint_{B(x,r)} f(y)dy &\le k + Ckr^{\nu(x)}(\log(e+1/r))^{\beta(x)}\fint_{B(x,r)} g(y)\,dy \\ &\le Ck \\ &= Cr^{-\nu(x)/p(x)}(\log(e+1/r))^{-(q(x)+\beta(x))/p(x)},\end{aligned}$$

as required. □

LEMMA 2.7. *Let λ, μ, ν, τ and γ are real numbers. Suppose h is a nonnegative measurable function on $\mathbf{R}^n$ such that*

$$\int_{B(0,r)} h(y)dy \le r^{-\lambda}(\log(e+1/r))^{-\mu}$$

for all $r > 0$. Then there exist a constant $C > 0$ such that

$$\int_{B(0,r_2)\setminus B(0,r_1)} |y|^{-\tau}(\log(1/|y|))^{-\gamma}h(y)dy \le C\int_{r_1}^{2r_2} t^{-\tau-\lambda}(\log(e+1/t))^{-\mu-\gamma}\frac{dt}{t}$$

whenever $0 < r_1 \le r_2 < \infty$.

Proof. By the integration by parts we have

$$\begin{aligned}&\int_{B(0,r_2)\setminus B(0,r_1)} |y|^{-\tau}(\log(1/|y|))^{-\gamma}h(y)dy \\ &\le \int_{r_1}^{r_2}\left(\int_{B(0,t)} f(y)dy\right)d(-t^{-\tau}(\log(1/t))^{-\gamma}) + r_2^{-\tau}(\log(1/r_2))^{-\gamma}\int_{B(0,r_2)} f(y)dy.\end{aligned}$$

Hence it suffices to note that

$$\begin{aligned}r_2^{-\tau}(\log(1/r_2))^{-\gamma}\int_{B(0,r_2)} f(y)dy &\le r_2^{-\tau-\lambda}(\log(e+1/r_2))^{-\mu-\gamma} \\ &\le C\int_{r_2}^{2r_2} t^{-\tau-\lambda}(\log(e+1/t))^{-\mu-\gamma}\frac{dt}{t}.\end{aligned}$$

□

LEMMA 2.8. *There exists a constant $C > 0$ such that*

$$I_2(\delta) \le C\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}$$

for all $x \in G$, $0 < \delta < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

Proof. Let

$$\eta = \operatorname*{ess\,inf}_{x\in G}(\nu(x)/p(x) - \alpha(x)).$$

Then $\eta > 0$ by (2.1). By Lemmas 2.6 and 2.7 we have for all $x \in G$ and $0 < \delta < d_G$

$$\begin{aligned} I_2(\delta) &\le C\int_\delta^{2d_G} t^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/t))^{-q(x)/p(x)-\beta(x)/p(x)}\,\frac{dt}{t} \\ &\le C\delta^{\alpha(x)-\nu(x)/p(x)+\eta/2}(\log(e+1/\delta))^{-q(x)/p(x)-\beta(x)/p(x)}\int_\delta^{2d_G} t^{-\eta/2}\,\frac{dt}{t} \\ &\le C\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-q(x)/p(x)-\beta(x)/p(x)}, \end{aligned}$$

which completes the proof. □

What remains for the proof of Theorem 2.1 is to give a Morrey property for $J_\varepsilon f(x)$.

LEMMA 2.9. *There exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} J_\varepsilon f(x)\,dx \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)-\varepsilon}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

Proof. For $z \in G$ and $0 < r < d_G$, write

$$\begin{aligned} J_\varepsilon f(x) &= \int_{B(z,2r)} \rho_\varepsilon(|x-y|)g(y)\,dy + \int_{G\setminus B(z,2r)} \rho_\varepsilon(|x-y|)g(y)\,dy \\ &= J_1(x) + J_2(x). \end{aligned}$$

Then we have

$$\begin{aligned} \fint_{B(z,r)} J_1(x)\,dx &\le \int_{B(z,2r)} \left(\fint_{B(z,r)} \rho_\varepsilon(|x-y|)dx\right) g(y)\,dy \\ &\le Cr^{-n}(\log(e+1/r))^{-\varepsilon}\int_{B(z,2r)} g(y)\,dy \\ &\le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)-\varepsilon} \end{aligned}$$

and

$$\begin{aligned} \fint_{B(z,r)} J_2(x)\,dx &\le C\int_{G\setminus B(z,2r)} \rho_\varepsilon(|z-y|)g(y)\,dy \\ &\le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)-\varepsilon}, \end{aligned}$$

where we use Lemma 2.7 for the last inequality. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We may assume that $f \geq 0$. For $\delta > 0$, write

$$I_{\alpha(x)} f(x) \;=\; I_1(\delta) + I_2(\delta).$$

In view of Lemma 2.5, we find

$$\begin{aligned} I_1(\delta) \;\leq\; & C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ & + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)} J\}. \end{aligned}$$

Moreover, Lemma 2.8 yields

$$I_2(\delta) \leq C\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)},$$

so that

$$\begin{aligned} I_{\alpha(x)} f(x) \;\leq\; & C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ & + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+(1+\varepsilon)} J\}. \end{aligned}$$

Now, letting $\delta = \min\{d_G, J^{-1/\nu(x)}(\log(e+J))^{-(\beta(x)+(1+\varepsilon))/\nu(x)}\}$, we obtain

$$I_{\alpha(x)} f(x) \leq C\big\{1 + J^{1/p^*(x)}(\log(e+J))^{-\alpha(x)\beta(x)/\nu(x)-q(x)/p(x)+(1+\varepsilon)/p^*(x)}\big\}.$$

By Lemma 2.9, we obtain

$$\begin{aligned} & \fint_{B(z,r)} \Psi(x, I_{\alpha(x)} f(x))(\log(e+|I_{\alpha(x)} f(x)|))^{-(1+\varepsilon)}\, dx \\ \leq\; & C \fint_{B(z,r)} (1+J)\, dx \\ \leq\; & C r^{-\nu(z)}(\log(e+1/r))^{-\beta(z)-\varepsilon} \end{aligned}$$

for $z \in G$ and $0 < r < d_G$, which completes the proof of Theorem 2.1. □

EXAMPLE 2.10. Let

$$\omega(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\log(1/|t|) & \text{when } 0 < |t| < r_0, \\ 1/\log(1/r_0) & \text{when } |t| \geq r_0 \end{cases}$$

and

$$\eta(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\log\log(1/|t|) & \text{when } 0 < |t| < r_0, \\ 1/\log\log(1/r_0) & \text{when } |t| \geq r_0 \end{cases}$$

for $0 < r_0 < 1/4$. Consider

$$p(x) = p(x_1, x_2) = 1 + a\omega(x_2),$$

and

$$q(x) = q(x_1, x_2) = b\eta(x_2),$$

where $a > 0$ and $b > 0$. Then, note that $p(\cdot)$ satisfies the conditions (P1) and (P2) and $q(\cdot)$ satisfies the conditions (Q1) and (Q2). Let $\gamma > 1$. If

$$f(y) = |y_2|^{-1}(\log(e + 1/|y_2|))^{-\gamma},$$

then note that

$$\begin{aligned} \fint_{B(z,r)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy &\le Cr^{-1}\int_0^r |y_2|^{-1}(\log(e + 1/|y_2|))^{-\gamma} dy_2 \\ &\le Cr^{-1}(\log(e + 1/r))^{-\beta} \end{aligned}$$

for all $z \in \mathbf{B} = B(0,1)$ and $r > 0$, when $\beta = \gamma - 1 > 0$. Here we may assume that $x_2 \neq 0$. Setting $Q(x) = \{y = (y_1, y_2) \in \mathbf{B} : |x_1 - y_1| < |x_2|, |y_2| < |x_2|\}$, we note that

$$\begin{aligned} I_\alpha f(x) &\ge \int_{Q(x)} |x - y|^{\alpha-2} f(y) dy \\ &\ge C|x_2|^{\alpha-2}\int_{Q(x)} f(y) dy \\ &\ge C|x_2|^{\alpha-1}\int_0^{|x_2|} |y_2|^{-1}(\log(2 + |y_2|^{-1}))^{-\beta-1} dy_2 \\ &\ge C|x_2|^{\alpha-1}(\log(2 + |x_2|^{-1}))^{-\beta}, \end{aligned}$$

Since

$$1/p^*(x) - 1/p^*(y) = 1/p(x) - 1/p(y),$$

we see that

$$\begin{aligned} &\fint_{B(0,r)} I_\alpha f(x)^{p^*(x)}(\log(e + I_\alpha f(x)))^{(q(x)/p(x)+\alpha\beta)p^*(x)-(1+\varepsilon)} dx \\ \ge\ & C\fint_{B(0,r)} |x_2|^{-1}(\log(e + 1/|x_2|))^{-\beta-\varepsilon-1} dx \\ \ge\ & Cr^{-1}(\log(e + 1/r))^{-\beta-\varepsilon} \end{aligned}$$

for all $0 < r < 1$.

This implies that Theorem 2.1 is best possible as to the exponents appearing in the Morrey condition.

3 Sobolev's inequality in the case $p_- = 1$ and $q_- > 0$

Let $p_- = 1$. In this section we assume that there exists a constant $q_0 > 0$ such that

$$s^{p(x)-1}(\log(e + s))^{q(x)-q_0} \le t^{p(x)-1}(\log(e + t))^{q(x)-q_0}, \tag{3.1}$$

whenever $0 < s < t$ and $x \in \mathbf{R}^n$. Let p^* and Ψ be as in (2.2) and (2.3), respectively. Under this assumption, Theorem 2.1 is shown to be valid for $\varepsilon = 0$.

THEOREM 3.1. *Let $p_- = 1$. Suppose that* (2.1) *and* (3.1) *hold. Then there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(x, |I_{\alpha(x)}f(x)|)(\log(e + |I_{\alpha(x)}f(x)|))^{-1}\, dx \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)}$$

for all $z \in G$, $0 < r < d_G$ and f satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

COROLLARY 3.2. *Let $p_- = 1$. Suppose that* (2.1) *and* (3.1) *hold. Then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{\widetilde{\Psi}_1,\nu,\beta}(G)} \le C\|f\|_{L^{\Phi,\nu,\beta}(G)},$$

where $\widetilde{\Psi}_1(x,t) = \Psi(x,t)(\log(e+t))^{-1}$.

REMARK 3.3. If $p(x) = 1$, $q(x) = q > 0$, $\nu(x) = n$ and $\beta(x) = 0$, then $p^* = n/(n-\alpha)$ and the Riesz operator I_α is bounded from $L^1(\log L)^q(G)$ to $L^{p^*}(\log L)^{p^*q-1}(G)$, which is a consequence of O'Neil [26, Theorem 5.2].

For $\varepsilon > 0$, let

$$\rho_{-\varepsilon}(r) = r^{-n}(\log(e+1/r))^{\varepsilon-1}.$$

For a nonnegative measurable function f on G, we define the logarithmic potential

$$L_\varepsilon f(x) = \int_{\{y\in G: |x-y|^{-\varepsilon} < f(y)\}} \rho_{-\varepsilon}(|x-y|)(\log(e+f(y)))^{-\varepsilon} g(y)\, dy,$$

where $g(y) = f(y)^{p(y)}(\log(e+f(y)))^{q(y)}$.

For the proof of Theorem 3.1, we need to modify Lemmas 2.5 and 2.9 in the following manner.

LEMMA 3.4. *Let $0 < \varepsilon \le q_0/2$ and*

$$F(\delta) = \int_{\{y\in B(x,\delta): |x-y|^{-\varepsilon} < f(y)\}} |x-y|^{\alpha(x)-n} \left(\frac{\log(e+f(y))}{\log(e+1/|x-y|)}\right)^{\varepsilon} f(y)\, dy$$

for $0 < \delta < d_G$ and a nonnegative measurable function f on G. Then there exists a constant $C > 0$ such that

$$\begin{aligned} F(\delta) \;\le\; & C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ & + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+1} L_\varepsilon f(x)\}. \end{aligned}$$

Proof. Let $E = \{y \in B(x,\delta) : |x-y|^{-\varepsilon} < f(y)\}$. For $k > 0$, let

$$E_k^1 = \{y \in B(x,\delta) : |x-y|^{-\varepsilon} < f(y) \le k\}, \quad E_k^2 = E \setminus E_k^1.$$

Then we have

$$\begin{aligned}
&\int_{E_k^1} |x-y|^{\alpha(x)-n} \left(\frac{\log(e+f(y))}{\log(e+1/|x-y|)} \right)^{\varepsilon} f(y)\,dy \\
\leq\ & k(\log(e+k))^{\varepsilon} \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} (\log(e+1/|x-y|))^{-\varepsilon}\,dy \\
=\ & Ck(\log(e+k))^{\varepsilon} \int_0^{\delta} t^{\alpha(x)-1} (\log(e+1/t))^{-\varepsilon}\,dt \\
\leq\ & Ck(\log(e+k))^{\varepsilon} \delta^{\alpha(x)-\alpha_-/2} (\log(e+1/\delta))^{-\varepsilon} \int_0^{\delta} t^{\alpha_-/2-1}\,dt \\
=\ & Ck(\log(e+k))^{\varepsilon} \delta^{\alpha(x)} (\log(e+1/\delta))^{-\varepsilon},
\end{aligned}$$

and, using (3.1),

$$\begin{aligned}
&\int_{E_k^2} |x-y|^{\alpha(x)-n} \left(\frac{\log(e+f(y))}{\log(e+1/|x-y|)} \right)^{\varepsilon} f(y)\,dy \\
\leq\ & \int_{E_k^2} |x-y|^{\alpha(x)-n} \left(\frac{\log(e+f(y))}{\log(e+1/|x-y|)} \right)^{\varepsilon} f(y) \\
&\qquad \times C \left(\frac{f(y)}{k} \right)^{p(y)-1} \left(\frac{\log(e+f(y))}{\log(e+k)} \right)^{q(y)-2\varepsilon} dy \\
=\ & C \int_{E_k^2} |x-y|^{\alpha(x)-n} (\log(e+1/|x-y|))^{-\varepsilon} (\log(e+f(y)))^{-\varepsilon} g(y) \\
&\qquad \times \left(\frac{1}{k} \right)^{p(y)-1} \left(\frac{1}{\log(e+k)} \right)^{q(y)-2\varepsilon} dy \\
\leq\ & C\delta^{\alpha(x)} (\log(e+1/\delta))^{1-2\varepsilon} \int_E \rho_{-\varepsilon}(|x-y|) (\log(e+f(y)))^{-\varepsilon} g(y) \\
&\qquad \times \left(\frac{1}{k} \right)^{p(y)-1} \left(\frac{1}{\log(e+k)} \right)^{q(y)-2\varepsilon} dy.
\end{aligned}$$

Hence

$$\begin{aligned}
F(\delta) \leq C \Bigg\{ & k(\log(e+k))^{\varepsilon} \delta^{\alpha(x)} (\log(e+1/\delta))^{-\varepsilon} \\
& + \delta^{\alpha(x)} (\log(e+1/\delta))^{1-2\varepsilon} \int_E \rho_{-\varepsilon}(|x-y|) (\log(e+f(y)))^{-\varepsilon} g(y) \\
&\qquad \times \left(\frac{1}{k} \right)^{p(y)-1} \left(\frac{1}{\log(e+k)} \right)^{q(y)-2\varepsilon} dy \Bigg\}.
\end{aligned}$$

We set

$$k = \delta^{-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)}.$$

Then we have for $y \in B(x,\delta)$,

$$k^{-p(y)} \ \leq\ Ck^{-p(x)}$$

and

$$(\log(e+k))^{-q(y)} \quad \leq \quad C(\log(e+k))^{-q(x)}$$

by (2.4) and (2.5). Consequently it follows that

$$\begin{aligned} F(\delta) \quad &\leq \quad C\{\delta^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ &\quad + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)}(\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+1}L_\varepsilon f(x)\}. \end{aligned}$$

Now the result follows. □

LEMMA 3.5. *There exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} L_\varepsilon f(x)\, dx \leq Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)}$$

for all $z \in G$, $0 < r < d_G$ and $f \geq 0$ satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \leq 1$.

Proof. Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \leq 1$. Write

$$\begin{aligned} L_\varepsilon f(x) \quad &= \quad \int_{\{y\in B(z,2r):|x-y|^{-\varepsilon}<f(y)\}} \rho_{-\varepsilon}(|x-y|)(\log(e+f(y)))^{-\varepsilon}g(y)\, dy \\ &\quad + \int_{\{y\in G\setminus B(z,2r):|x-y|^{-\varepsilon}<f(y)\}} \rho_{-\varepsilon}(|x-y|)(\log(e+f(y)))^{-\varepsilon}g(y)\, dy \\ &= \quad L_1(x) + L_2(x), \end{aligned}$$

where $g(y) = f(y)^{p(y)}(\log(e+f(y)))^{q(y)}$. By Fubini's theorem, we have

$$\begin{aligned} &\int_{B(z,r)} L_1(x)\ dx \\ &\leq \quad C\int_{B(z,2r)} \left(\int_{\{y\in G:|x-y|^{-\varepsilon}<f(y)\}} \rho_{-\varepsilon}(|x-y|)dx\right)(\log(e+f(y)))^{-\varepsilon}g(y)dy \\ &\leq \quad C\int_{B(z,2r)} g(y)dy \leq Cr^{n-\nu(z)}(\log(e+1/r))^{-\beta(z)}. \end{aligned}$$

For L_2, note that

$$\begin{aligned} L_2(x) \quad &\leq \quad C\int_{G\setminus B(z,2r)} |x-y|^{-n}(\log(e+1/|x-y|))^{-1}g(y)\ dy \\ &\leq \quad C\int_{G\setminus B(z,2r)} |z-y|^{-n}(\log(e+1/|z-y|))^{-1}g(y)\ dy \end{aligned}$$

for $x \in B(z,r)$. Hence, as in the proof of Lemma 2.7, we see that

$$\begin{aligned} \int_{B(z,r)} L_2(x)\ dx \quad &\leq \quad Cr^n\int_{G\setminus B(z,2r)} |z-y|^{-n}(\log(e+1/|z-y|))^{-1}g(y)\ dy \\ &\leq \quad Cr^n\int_{2r}^{2d_G} t^{-\nu(z)}(\log(e+1/t))^{-\beta(z)-1}\frac{dt}{t} \\ &\leq \quad Cr^{n-\nu(z)}(\log(e+1/r))^{-\beta(z)-1}. \end{aligned}$$

Thus this lemma is proved. □

Proof of Theorem 3.1. We may assume that $f \geq 0$. For $\varepsilon = \min\{\alpha_-/2, q_0/2\}$ and $x \in \mathbf{R}^n$, set $L = L_\varepsilon f(x)$.

For $\delta > 0$, write

$$\begin{aligned} I_{\alpha(x)} f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\, dy + \int_{G \setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\, dy \\ &= I_1(\delta) + I_2(\delta). \end{aligned}$$

In view of Lemma 3.4, we find

$$\begin{aligned} I_1(\delta) &\leq \int_{B(x,\delta)} |x-y|^{\alpha(x)-n-\varepsilon}\, dy \\ &\quad + \int_{\{y \in B(x,\delta): |x-y|^{-\varepsilon} < f(y)\}} |x-y|^{\alpha(x)-n} \left(\frac{(\log(e+f(y)))}{\log(e+|x-y|^{-\varepsilon})} \right)^{\varepsilon} f(y)\, dy \\ &\leq C\{\delta^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ &\quad + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)} (\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+1} L\} \end{aligned}$$

with $L = L_\varepsilon f(x)$. Moreover, Lemma 2.8 yields

$$I_2(\delta) \leq C\delta^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)},$$

so that

$$\begin{aligned} I_{\alpha(x)} f(x) &\leq C\{\delta^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/\delta))^{-(q(x)+\beta(x))/p(x)} \\ &\quad + \delta^{\alpha(x)+(p(x)-1)\nu(x)/p(x)} (\log(e+1/\delta))^{\beta(x)-(q(x)+\beta(x))/p(x)+1} L\}. \end{aligned}$$

Now, letting $\delta = \min\{d_G, L^{-1/\nu(x)} (\log(e+L))^{-(\beta(x)+1)/\nu(x)}\}$, we obtain

$$I_{\alpha(x)} f(x) \leq C\left\{1 + L^{1/p^*(x)} (\log(e+L))^{-\alpha(x)\beta(x)/\nu(x) - q(x)/p(x) + 1/p^*(x)}\right\}.$$

In view of Lemma 3.5, we find

$$\begin{aligned} & ⨍_{B(z,r)} \Psi(x, I_{\alpha(x)} f(x)) (\log(e + I_{\alpha(x)} f(x)))^{-1}\, dx \\ \leq\ & C ⨍_{B(z,r)} (1+L)\, dx \leq C r^{-\nu(z)} (\log(e+1/r))^{-\beta(z)}, \end{aligned}$$

which completes the proof of Theorem 3.1. □

EXAMPLE 3.6. Let

$$\omega(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\log(1/|t|) & \text{when } 0 < |t| < r_0, \\ 1/\log(1/r_0) & \text{when } |t| \geq r_0 \end{cases}$$

and

$$\eta(t) = \begin{cases} 0 & \text{when } t = 0, \\ 1/\log\log(1/|t|) & \text{when } 0 < |t| < r_0, \\ 1/\log\log(1/r_0) & \text{when } |t| \geq r_0 \end{cases}$$

for $0 < r_0 < 1/4$. Consider

$$p(x) = p(x_1, x_2) = 1 + a\omega(x_2),$$

and

$$q(x) = q(x_1, x_2) = q + b\eta(x_2),$$

where $a > 0$, $q > 0$ and $b > 0$. Let $\gamma \in \mathbf{R}$. If

$$f(y) = |y_2|^{-1}(\log(e + 1/|y_2|))^{-\gamma},$$

then note that

$$⨍_{B(z,r)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy \le Cr^{-1}(\log(e + 1/r))^{-\beta}$$

for all $z \in \mathbf{B} = B(0, 1)$ and $r > 0$, when $\beta = \gamma - 1 - q > 0$. Further, for $0 < \alpha < 1$, we have

$$I_\alpha f(x) \ge C|x_2|^{\alpha-1}(\log(e + 1/|x_2|))^{-\gamma+1}$$

for $x \in B(0, 1)$. Take γ such that $\gamma < \delta + 1 + q$ for $\delta > 0$. Then we see that

$$\begin{aligned} & ⨍_{B(0,r)} I_\alpha f(x)^{p^*(x)}(\log(e + I_\alpha f(x)))^{(q(x)/p(x)+\alpha\beta)p^*(x)-1+\delta} dx \\ \ge\ & C ⨍_{B(0,r)} |x_2|^{-1}(\log(e + 1/|x_2|))^{-\beta-1+\delta} dx = \infty \end{aligned}$$

for all $0 < r < 1$ and $\delta > 0$. This implies that Theorem 3.1 is best possible as to the exponents appearing in the Morrey condition.

4 Sobolev's inequality in the case $p_- > 1$

In this section, we are concerned with the case $p_- > 1$. In this case, (1.1) holds for $K \ge -q_-/(p_- - 1)$.

We first show the boundedness of the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_B ⨍_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x.

THEOREM 4.1. *Suppose $p_- > 1$ and $\nu_- > 0$ Then there exists a constant $C > 0$ such that*

$$⨍_{B(z,r)} Mf(x)^{p(x)}(\log(e + Mf(x)))^{q(x)} dx \le Cr^{-\nu(z)}(\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$, $0 < r < d_G$ and f with $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

REMARK 4.2. For the constant case, we refer the reader to [25].

To prove Theorem 4.1, we prepare several lemmas. Let us begin with the following result, which is a consequence of [20, Theorem 1].

Lemma 4.3 ([20, Theorem 1]). *Suppose $p_0 > 1$ and $\nu_- > 0$. Let f be a measurable function on G satisfying*

$$\fint_{B(x,r)} |f(y)|^{p_0}\, dy \le r^{-\nu(x)}(\log(e+1/r))^{-\beta(x)} \tag{4.1}$$

for all $x \in G$ and $0 < r < d_G$. Then there exists a constant $C > 0$ such that

$$\fint_{B(z,r)} Mf(x)^{p_0}\, dx \le C r^{-\nu(x)}(\log(e+1/r))^{-\beta(x)}$$

for all $z \in G$ and $0 < r < d_G$, where the constant C is independent of f satisfying (4.1).

Lemma 4.4. *Suppose $\nu_- > 0$. Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$ such that*

$$f(x) \ge 1 \quad \text{or} \quad f(x) = 0 \quad \text{for each } x \in G. \tag{4.2}$$

Then there exists a constant $C > 0$ such that

$$Mf(x)^{p(x)}(\log(e+Mf(x)))^{q(x)} \le CMg(x)$$

for all $x \in G$, where $g(y) = f(y)^{p(y)}(\log(e+f(y)))^{q(y)}$. In the above, the constant C is independent of f.

Proof. Let

$$H = H_{x,r} = \fint_{B(x,r)} g(y)\, dy.$$

We shall show

$$\fint_{B(x,r)} f(y)\, dy \le CH^{1/p(x)}(\log(e+H))^{-q(x)/p(x)} \tag{4.3}$$

for all $x \in G$ and $0 < r < d_G$. Then

$$Mf(x) \le CMg(x)^{1/p(x)}(\log(e+Mg(x)))^{-q(x)/p(x)}.$$

This implies the desired conclusion.

To show (4.3), first consider the case when $H \ge 1$. Set

$$k = H^{1/p(x)}(\log(e+H))^{-q(x)/p(x)}.$$

Then we have

$$\begin{aligned} \fint_{B(x,r)} f(y)\, dy &\le k + C\fint_{B(x,r)} f(y)\left(\frac{f(y)}{k}\right)^{p(y)-1}\left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} dy \\ &= k + C\fint_{B(x,r)} g(y)k^{-p(y)+1}(\log(e+k))^{-q(y)} dy. \end{aligned}$$

Since

$$H \le r^{-\nu(x)}(\log(e+1/r))^{-\beta(x)}$$

for all $x \in G$ and $0 < r < d_G$, we obtain for $y \in B(x,r)$, as in the proof of Lemma 2.5,

$$k^{-p(y)} \le Ck^{-p(x)} = CH^{-1}(\log(e+H))^{q(x)}$$

and

$$(\log(e+k))^{-q(y)} \le C(\log(e+k))^{-q(x)} \le C(\log(e+H))^{-q(x)}.$$

Consequently (4.3) follows.

In the case $H \le 1$, we find

$$H \le CH^{1/p(x)}(\log(e+H))^{-q(x)/p(x)}.$$

Since $f(y) \ge 1$ or $f(y) = 0$ for each $y \in G$, we have

$$g(y) = f(y) \cdot f(y)^{p(y)-1}(\log(e+f(y)))^{q(y)} \ge Cf(y)$$

for some $C > 0$ and hence

$$\fint_{B(x,r)} f(y)\,dy \le CH.$$

This shows (4.3). □

Proof of Theorem 4.1. We may assume that $f \ge 0$. Write

$$f = f\chi_{\{y:f(y)\ge 1\}} + f\chi_{\{y:f(y)<1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E. Take p_0 such that $1 < p_0 < p_-$. Since

$$\begin{aligned}&\fint_{B(x,r)} f_1(y)^{p(y)/p_0}(\log(e+f_1(y)))^{q(y)/p_0}dy\\ &\le C\fint_{B(x,r)} f_1(y)^{p(y)}(\log(e+f_1(y)))^{q(y)}\,dy \le Cr^{-\nu(x)}(\log(e+1/r))^{-\beta(x)}\end{aligned}$$

for all $x \in G$ and $0 < r < d_G$, applying Lemma 4.4 with $p(x)$ and $q(x)$ replaced by $p(x)/p_0$ and $q(x)/p_0$, respectively, we obtain

$$Mf_1(x)^{p(x)/p_0}(\log(e+Mf_1(x)))^{q(x)/p_0} \le CMg_1(x),$$

where $g_1(y) = f_1(y)^{p(y)/p_0}(\log(e+f_1(y)))^{q(y)/p_0}$. Note that g_1 satisfies (4.1). Since $Mf_2 \le 1$, it follows that

$$Mf(x)^{p(x)}(\log(e+Mf(x)))^{q(x)} \le C(1+Mg_1(x)^{p_0}).$$

Hence, by Lemma 4.3, we see that

$$\begin{aligned}\fint_{B(z,r)} Mf(x)^{p(x)}(\log(e+Mf(x)))^{q(x)}\,dx &\le C\fint_{B(z,r)}(1+Mg_1(x)^{p_0})\,dx\\ &\le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)}\end{aligned}$$

for all $z \in G$ and $0 < r < d_G$, as required. □

Now we give a Morrey version of Sobolev's inequality for Riesz potentials. Let p^* and Ψ be as in (2.2) and (2.3), respectively.

THEOREM 4.5. *Suppose that $p_- > 1$ and (2.1) holds. Then there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(x, |I_{\alpha(x)} f(x)|) dx \le C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$, $0 < r < d_G$ and f satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

This theorem gives the following norm version, which is simpler than Corollaries 2.4 and 3.2 .

COROLLARY 4.6 (cf. [16, Theorem 4.3]). *Suppose that $p_- > 1$ and (2.1) holds. Then there exists a constant $C > 0$ such that*

$$\|I_{\alpha(\cdot)} f\|_{L^{\Psi,\nu,\beta}(G)} \le C \|f\|_{L^{\Phi,\nu,\beta}(G)}.$$

REMARK 4.7. If $p(x) = p > 1$, $q(x) = q \in \mathbf{R}$, $\nu(x) = n$ and $\beta(x) = 0$, then $p^* = np/(n - \alpha p)$ and the operator I_α is bounded from $L^p(\log L)^q(G)$ to $L^{p^*}(\log L)^{p^* q/p}(G)$, which is shown by O'Neil [26, Theorem 4.7].

For further related results, we refer the reader to the papers [3, 16, 17].

REMARK 4.8. Theorem 4.5 is best possible as to the exponents appearing in the Morrey condition.

Proof of Theorem 4.5. We may assume that $f \ge 0$, as before. By Lemma 2.8, we find

$$\begin{aligned} I_{\alpha(x)} f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) dy + \int_{G \setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y) dy \\ &\le C \left\{ \delta^{\alpha(x)} M f(x) + \delta^{\alpha(x) - \nu(x)/p(x)} (\log(e + 1/\delta))^{-(q(x)+\beta(x))/p(x)} \right\}. \end{aligned}$$

Considering

$$\delta = \min \left\{ d_G, Mf(x)^{-p(x)/\nu(x)} (\log(e + Mf(x)))^{-(q(x)+\beta(x))/\nu(x)} \right\},$$

we have

$$\begin{aligned} I_{\alpha(x)} f(x) &\le C \left\{ 1 + Mf(x)^{1 - \alpha(x) p(x)/\nu(x)} (\log(e + Mf(x)))^{-\alpha(x)(q(x)+\beta(x))/\nu(x)} \right\} \\ &= C \left\{ 1 + Mf(x)^{p(x)/p^*(x)} (\log(e + Mf(x)))^{-\alpha(x)(q(x)+\beta(x))/\nu(x)} \right\}. \end{aligned}$$

Then we find

$$\Psi(x, I_{\alpha(x)} f(x)) \le C \{ 1 + Mf(x)^{p(x)} (\log(e + Mf(x)))^{q(x)} \}$$

for all $x \in G$. It follows from Theorem 4.1 that

$$\fint_{B(z,r)} \Psi(x, I_{\alpha(x)} f(x)) \, dx \le C r^{-\nu(z)} (\log(e + 1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$, as required. □

5 Trudinger's inequality

This section is concerned with Morrey version of Trudinger's type exponential integrability for Riesz potentials, in case

$$\operatorname*{ess\,inf}_{x\in\mathbf{R}^n}(\alpha(x)-\nu(x)/p(x))\geq 0, \tag{5.1}$$

which is equivalent to

$$\operatorname*{ess\,sup}_{x\in\mathbf{R}^n}(1/p(x)-\alpha(x)/\nu(x))\leq 0.$$

Set

$$\Gamma(x,r)=c_0\int_1^r(\log(e+t))^{-(q(x)+\beta(x))/p(x)}\frac{dt}{t}$$

for $x\in\mathbf{R}^n$ and $r\geq 2$, where we choose c_0 such that $\inf_{x\in\mathbf{R}^n}\Gamma(x,2)=2$. For convenience, set $\Gamma(x,r)=(\Gamma(x,2)/2)r$ when $r<2$. Note that there exists a constant $C>0$ such that

$$C^{-1}\leq\frac{\Gamma(x,r^2)}{\Gamma(x,r)}\leq C\quad\text{for}\quad x\in\mathbf{R}^n\text{ and }r\geq 2,$$

since $-(q(x)+\beta(x))/p(x)$ is bounded. Let

$$s_x=\sup_{r\geq 2}\Gamma(x,r)=c_0\int_1^\infty(\log(1+t))^{-(q(x)+\beta(x))/p(x)}\frac{dt}{t}.$$

Then $2<s_x\leq\infty$ and $\Gamma(x,\cdot)$ is bijective from $[0,\infty)$ to $[0,s_x)$. We denote by $\Gamma^{-1}(x,\cdot)$ the inverse function of $\Gamma(x,\cdot)$. If $s_x<\infty$, we set $\Gamma^{-1}(x,r)=\infty$ for $r\geq s_x$.

THEOREM 5.1. *Suppose $\nu_->0$ and (5.1) holds. Let ε be a measurable function on $\mathbf{R}^n$ such that*

$$\operatorname*{ess\,inf}_{x\in\mathbf{R}^n}\,(\nu(x)/p(x)-\varepsilon(x))>0\ \textit{and}\ 0<\varepsilon_-\leq\varepsilon_+<\alpha_-. \tag{5.2}$$

Then there exist constants $c_1,c_2>0$ such that

$$⨍_{B(z,r)}\Gamma^{-1}\left(x,\frac{|I_{\alpha(x)}f(x)|}{c_1}\right)dx\leq c_2\,r^{\varepsilon(z)-\nu(z)/p(z)}$$

for all $z\in G$, $0<r<d_G$ and f satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)}\leq 1$. In the above $|I_{\alpha(x)}f(x)|/c_1<s_x$ for a.e. $x\in B(z,r)$.

REMARK 5.2. Let $\alpha,p,q,\nu,\beta,\varepsilon$ be all constants and $0<\varepsilon<\alpha$.

(1) If $q+\beta<p$, then, for $r\geq 2$,

$$C^{-1}\Gamma(r)\leq(\log(e+r))^{1-(q+\beta)/p}\leq C\Gamma(r)$$

and

$$\Gamma^{-1}(C^{-1}r)\leq\exp(r^{p/(p-q-\beta)})\leq\Gamma^{-1}(Cr).$$

(2) If $q+\beta=p$, then, for $r\geq 2$,

$$C^{-1}\Gamma(r)\leq \log(\log(e+r))\leq C\Gamma(r)$$

and

$$\Gamma^{-1}(C^{-1}r)\leq \exp\exp(r)\leq \Gamma^{-1}(Cr).$$

COROLLARY 5.3. *Under the assumptions in Theorem 5.1, there exist constants* $c_1,c_2>0$ *such that*

(1) *in case* $\operatorname{ess\,sup}_{x\in\mathbf{R}^n}\,(q(x)+\beta(x))/p(x)<1$,

$$⨍_{B(z,r)} \exp\left(\frac{|I_\alpha f(x)|^{p(x)/(p(x)-q(x)-\beta)}}{c_1}\right)dx\leq c_2 r^{\varepsilon(z)-\nu/p(z)};$$

(2) *in case* $\operatorname{ess\,inf}_{x\in\mathbf{R}^n}\,(q(x)+\beta(x))/p(x)\geq 1$,

$$⨍_{B(z,r)} \exp\left(\exp\left(\frac{|I_\alpha f(x)|}{c_1}\right)\right)dx\leq c_2 r^{\varepsilon(z)-\nu/p(z)}$$

for all $z\in G$, $0<r<d_G$ *and* f *satisfying* $\|f\|_{L^{\Phi,\nu,\beta}(G)}\leq 1$.

REMARK 5.4. When p,q,β,α,ν are all constants such that $p=1$, $q=0$, $\beta<1$ and $\alpha=\nu$, this is due to Corollaries 4.6 and 4.8 in [12]. In particular, the case $p=1$, $q=\beta=0$, $\alpha=\nu=1$ and $r=d_G$ coincides with the result by Trudinger [30]. A weaker result is shown by Mizuta and Shimomura [15, Theorem 4.4].

To prove the theorem, we use the following lemmas. The first lemma can be proved with minor changes of the proof of Lemma 2.8.

LEMMA 5.5. *Suppose that* $\nu_->0$ *and* (5.1) *holds. Then there exists a constant* $C>0$ *such that*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-n}f(y)\,dy\leq C\Gamma(x,1/\delta)$$

for all $x\in G$, $0<\delta<d_G$ *and* $f\geq 0$ *satisfying* $\|f\|_{L^{\Phi,\nu,\beta}(G)}\leq 1$.

LEMMA 5.6. *Let* ε *be a measurable function on* G *satisfying* (5.2). *Setting* $\rho(z,r)=r^{\varepsilon(z)}(\log(e+1/r))^{(q(z)+\beta(z))/p(z)}$, *define*

$$I_{\rho(z)}f(x)=\int_G \frac{\rho(z,|x-y|)}{|x-y|^n}f(y)\,dy.$$

Then there exists a constant $C>0$ *such that*

$$⨍_{B(z,r)} I_{\rho(z)}f(x)dx\leq Cr^{\varepsilon(z)-\nu(z)/p(z)}$$

for all $z\in G$, $0<r<d_G$ *and* $f\geq 0$ *satisfying* $\|f\|_{L^{\Phi,\nu,\beta}(G)}\leq 1$.

Proof. Write

$$
\begin{aligned}
I_{\rho(z)}f(x) &= \int_{B(z,2r)} \frac{\rho(z,|x-y|)}{|x-y|^n} f(y)\,dy + \int_{G\setminus B(z,2r)} \frac{\rho(z,|x-y|)}{|x-y|^n} f(y)\,dy \\
&= I_1(x) + I_2(x).
\end{aligned}
$$

By Fubini's theorem and Lemma 2.6, we have

$$
\begin{aligned}
\int_{B(z,r)} I_1(x)\,dx &= \int_{B(z,2r)} \left(\int_{B(z,r)} \frac{\rho(z,|x-y|)}{|x-y|^n}\,dx \right) f(y)\,dy \\
&\le \int_{B(z,2r)} \left(\int_{B(y,3r)} \frac{\rho(z,|x-y|)}{|x-y|^n}\,dx \right) f(y)\,dy \\
&= n\sigma_n \int_{B(z,2r)} \left(\int_0^{3r} \frac{\rho(z,t)}{t}\,dt \right) f(y)\,dy \\
&\le C\rho(z,3r) \int_{B(z,2r)} f(y)dy \\
&\le C\rho(z,3r)(2r)^{n-\nu(z)/p(z)}(\log(e+1/(2r)))^{-(q(z)+\beta(z))/p(z)} \\
&\le Cr^{n+\varepsilon(z)-\nu(z)/p(z)}.
\end{aligned}
$$

For I_2, note that

$$
I_2(x) \le C \int_{G\setminus B(z,2r)} \frac{\rho(z,|z-y|)}{|z-y|^n} f(y)\,dy \quad \text{for} \quad x \in B(z,r),
$$

since there exists a constant $C > 0$ such that

$$
C^{-1} \le \frac{\rho(z,r)}{\rho(z,s)} \le C \quad \text{for} \quad z \in G,\ \frac{1}{2} \le \frac{r}{s} \le 2.
$$

Hence we have by Lemmas 2.6 and 2.7

$$
\begin{aligned}
I_2(x) &\le C \int_{2r}^{2d_G} \frac{\rho(z,t)}{t^n} t^{n-\nu(z)/p(z)}(\log(e+1/t))^{-(q(z)+\beta(z))/p(z)} \frac{dt}{t} \\
&\le C \int_{2r}^{2d_G} t^{\varepsilon(z)-\nu(z)/p(z)} \frac{dt}{t} \\
&\le Cr^{\varepsilon(z)-\nu(z)/p(z)}.
\end{aligned}
$$

Thus this lemma is proved. □

Proof of Theorem 5.1. We have only to treat nonnegative f with $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$. By Lemma 5.5 we find

$$
\begin{aligned}
I_{\alpha(x)}f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\,dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-n} f(y)\,dy \\
&= \int_{B(x,\delta)} |x-y|^{\alpha(x)-\varepsilon(z)}(\log(e+1/|x-y|))^{-(q(z)+\beta(z))/p(z)} \frac{\rho(z,|x-y|)}{|x-y|^n} f(y)\,dy \\
&\quad +C\Gamma(x,1/\delta) \\
&\le C\left\{ \delta^{\alpha(x)-\varepsilon(z)}(\log(e+1/\delta))^{-(q(z)+\beta(z))/p(z)} I_{\rho(z)}f(x) + \Gamma(x,1/\delta) \right\}
\end{aligned}
$$

for $\delta > 0$. Considering

$$\delta = \min\left\{ d_G, \left(\frac{\Gamma(x, I_{\rho(z)}f(x))(\log(e + I_{\rho(z)}f(x)))^{(q(z)+\beta(z))/p(z)}}{I_{\rho(z)}f(x)} \right)^{1/(\alpha(x)-\varepsilon(z))} \right\},$$

we have the inequality

$$I_{\alpha(x)}f(x) \le c_1 \max\left\{1, \Gamma(x, I_{\rho(z)}f(x))\right\},$$

for some constant $c_1 > 0$. Since $1 \le \Gamma(x,1) = \Gamma(x,2)/2$, $\Gamma^{-1}(x,1) \le 1$. Then

$$\fint_{B(z,r)} \Gamma^{-1}\left(x, \frac{I_{\alpha(x)}f(x)}{c_1}\right) dx \le \fint_{B(z,r)} \left\{1 + I_{\rho(z)}f(x)\right\} dx$$

for all $z \in G$ and $0 < r < d_G$. Hence Lemma 5.6 gives the conclusion. □

6 Continuity

In this section we are concerned with continuity for Riesz potentials when (5.1) and the following condition hold:

$$\mathcal{H}(x,r) \equiv \int_0^r t^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t} < \infty.$$

In this case $\mathcal{H}(x,r) \to 0$ as $r \to 0$ and $\mathcal{H}(x,r) \le \mathcal{H}(x,2r) \le C\mathcal{H}(x,r)$ for some constant $C > 0$ independent of $x \in \mathbf{R}^n$ and $0 < r < \infty$.

THEOREM 6.1. *Let $0 < \theta \le 1$ and $\gamma(x) = \alpha(x) - \nu(x)/p(x)$. Suppose that $\alpha \in \mathrm{Lip}_\theta(G)$, $\nu_- > 0$ and $0 \le \gamma_- \le \gamma_+ < \theta$. If f is a measurable function on G satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$, then $I_{\alpha(x)}f$ is continuous on G. Moreover, there exists a constant $C > 0$ such that*

$$|I_{\alpha(x)}f(x) - I_{\alpha(z)}f(z)| \le C\{\mathcal{H}(x,|x-z|) + \mathcal{H}(z,|x-z|)\}$$

for all $x, z \in G$, where the constant C is independent of f satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$. That is, the operator $I_{\alpha(\cdot)}$ is bounded from $L^{\Phi,\nu,\beta}(G)$ to $\Lambda_{\mathcal{H}}(G)$.

COROLLARY 6.2. *Let $0 < \theta \le 1$ and $\gamma(x) = \alpha(x) - n/p(x)$. Suppose $\alpha \in \mathrm{Lip}_\theta(G)$ and $0 < \gamma_- \le \gamma_+ < \theta$. Then the operator $I_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(G)$ to $\mathrm{Lip}_{\gamma(\cdot)}(G)$.*

REMARK 6.3. The case when α, p are constants and $n = 1$ is the result of Hardy-Littlewood [6, Theorem 12].

COROLLARY 6.4. *Let α, ν and β be constants. Suppose*

$$0 \le \alpha - \nu/p_- \le \alpha - \nu/p_+ < 1, \quad \beta > \operatorname*{ess\,sup}_{x \in \mathbf{R}^n}(p(x) - q(x)).$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned}&|I_\alpha f(x) - I_\alpha f(z)| \\ &\qquad \le C\big\{|x-z|^{\alpha-\nu/p(x)}(\log(e+1/|x-z|))^{-(q(x)+\beta)/p(x)+1} \\ &\qquad\qquad + |x-z|^{\alpha-\nu/p(z)}(\log(e+1/|x-z|))^{-(q(z)+\beta)/p(z)+1}\big\},\end{aligned}$$

for all $x, z \in G$ and for all f satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

REMARK 6.5. If $p(x) = 1$ and $q(x) = 0$, the corollary above is a special case of [21, Theorem 3.3]. If $p(x) = 1$, $q(x) = 0$, $\alpha = \nu$ and $\beta > 1$, the corollary above is [11, Theorem 1.1 (3)], where α, ν and β are constants. See also [21, 29].

To prove the theorems, we need the following lemmas.

LEMMA 6.6. *Let $0 < \theta \le 1$. Suppose $\alpha \in \mathrm{Lip}_\theta(G)$. Then there exists a constant $C > 0$ such that*

$$||x-y|^{\alpha(x)-n} - |z-y|^{\alpha(z)-n}| \le C(|x-z||x-y|^{\alpha(x)-n-1} + |x-z|^\theta |x-y|^{\alpha(x)-n-\theta}),$$

for all $x, y, z \in G$ satisfying $|x-y| \ge 2|x-z|$.

Proof. Let $r = |x-y|$ and $s = |z-y|$. Then $1/2 \le r/s \le 2$ and

$$\begin{aligned}|r^{\alpha(x)-n} - s^{\alpha(z)-n}| &\le |r^{\alpha(x)-n} - s^{\alpha(x)-n}| + |s^{\alpha(x)-n} - s^{\alpha(z)-n}| \\ &= |r-s||\alpha(x)-n|\,\tilde r^{\,\alpha(x)-n-1} + |\alpha(x)-\alpha(z)||\log s|\, s^{\tilde\alpha-n} \\ &\le C(|x-z|r^{\alpha(x)-n-1} + |x-z|^\theta s^{\alpha(x)-n-\theta} s^{\tilde\alpha-\alpha(x)}) \\ &\le C(|x-z|r^{\alpha(x)-n-1} + |x-z|^\theta r^{\alpha(x)-n-\theta}),\end{aligned}$$

where $\tilde r = (1-t)r + ts$ and $\tilde\alpha = (1-u)\alpha(x) + u\alpha(z)$ for some $0 < t, u < 1$. □

The following two lemmas can be proved in the same manner as Lemma 2.8.

LEMMA 6.7. *Suppose $\nu_- > 0$. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,r)\setminus B(x,s)} |x-y|^{\alpha(x)-n} f(y)dy \le C\int_s^r t^{\alpha(x)-\nu(x)/p(x)}(\log(e+1/t))^{-(q(x)+\beta(x))/p(x)}\frac{dt}{t}$$

for all $x \in G$, $0 < 2s < r < \infty$ and for all $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

LEMMA 6.8. *Let $\theta > 0$. Suppose $\nu_- > 0$ and*

$$\operatorname*{ess\,sup}_{x\in G}(\alpha(x) - \nu(x)/p(x)) < \theta.$$

Then there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^n\setminus B(x,r)} |x-y|^{\alpha(x)-n-\theta} f(y)\,dy \le Cr^{\alpha(x)-\nu(x)/p(x)-\theta}(\log(e+1/r))^{-(q(x)+\beta(x))/p(x)},$$

for all $x \in G$, $r > 0$ and for all $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\nu,\beta}(G)} \le 1$.

Proof of Theorem 6.1. We may assume that $f \geq 0$. Write

$$\begin{aligned} I_{\alpha(x)}f(x) - I_{\alpha(z)}f(z) \\ = \int_{B(x,2|x-z|)} |x-y|^{\alpha(x)-n} f(y)\,dy - \int_{B(x,2|x-z|)} |z-y|^{\alpha(z)-n} f(y)\,dy \\ + \int_{G\setminus B(x,2|x-z|)} (|x-y|^{\alpha(x)-n} - |z-y|^{\alpha(z)-n}) f(y)\,dy \end{aligned}$$

for $x, z \in G$. Using Lemma 6.7, we have

$$\begin{aligned} \int_{B(x,2|x-z|)} |x-y|^{\alpha(x)-n} f(y)\,dy &\leq C \int_0^{2|x-z|} t^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/t))^{-(q(x)+\beta(x))/p(x)} \frac{dt}{t} \\ &\leq C\mathcal{H}(x, |x-z|), \end{aligned}$$

and

$$\begin{aligned} \int_{B(x,2|x-z|)} |x-y|^{\alpha(z)-n} f(y)\,dy &\leq \int_{B(z,3|x-z|)} |x-y|^{\alpha(z)-n} f(y)\,dy \\ &\leq C\mathcal{H}(z, |x-z|). \end{aligned}$$

On the other hand, by Lemmas 6.6 and 6.8, we have

$$\begin{aligned} &\int_{G\setminus B(x,2|x-z|)} ||x-y|^{\alpha(x)-n} - |z-y|^{\alpha(z)-n}| f(y)\,dy \\ &\leq C\Big\{ |x-z| \int_{G\setminus B(x,2|x-z|)} |x-y|^{\alpha(x)-n-1} f(y)\,dy \\ &\qquad + |x-z|^{\theta} \int_{G\setminus B(x,2|x-z|)} |x-y|^{\alpha(x)-n-\theta} f(y)\,dy \Big\} \\ &\leq C|x-z|^{\alpha(x)-\nu(x)/p(x)} (\log(e+1/|x-z|))^{-(q(x)+\beta(x))/p(x)} \\ &\leq C\mathcal{H}(x, |x-z|). \end{aligned}$$

Then we have the conclusion. □

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