Integrability of maximal functions for generalized Lebesgue spaces with variable exponent

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Abstract

Our aim in this paper is to deal with integrability of maximal functions for generalized Lebesgue spaces with variable exponent. Our exponent approaches 1 on some part of the domain, and hence the integrability depends on the shape of that part and the speed of the exponent approaching 1.

1 Introduction

Let G be a bounded open set in the n-dimensional Euclidean space $\mathbf{R}^n$. Following Orlicz [8] and Kováčik and Rákosník [6], we consider a positive continuous function $p(\cdot)$ on G and the space of all measurable functions f on G satisfying

$$\int_G \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy < \infty$$

for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_G \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \leq 1 \right\}.$$

In recent years, the generalized Lebesgue spaces $L^{p(\cdot)}$ have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see Růžička [9].

We denote by $B(x, r)$ the open ball centered at x of radius r. For a locally integrable function f on a bounded open set $G \subset \mathbf{R}^n$, we consider the maximal function on G defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{G \cap B(x,r)} |f(y)| dy.$$

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In this paper we study the boundedness of Hardy-Littlewood maximal operator in variable exponent Lebesgue spaces. The Hardy-Littlewood maximal operator is effectively used in many fields of real analysis.

In classical (constant exponent) Lebesgue spaces, we know the following basic facts about the maximal operator (see the book by Stein [11, Chapter 1]): if $q > 1$, then

$$\|Mf\|_q \leq C\|f\|_q$$

for all $f \in L^q(G)$. Since this is not true when $q = 1$, we consider the space $L \log L(G)$ of measurable functions f on G whose norm

$$\|f\|_{L \log L} = \inf \left\{ \lambda > 0 : \int_G \left| \frac{f(y)}{\lambda} \right| \log \left(2 + \left| \frac{f(y)}{\lambda} \right| \right) dy \leq 1 \right\}$$

is finite. It is known from Stein [10] and [11, Sections I.1 and I.5] that the maximal operator has the following weaker integrability for $q = 1$, that is,

$$\|Mf\|_1 \leq C\|f\|_{L \log L}.$$

Conversely, if $\|Mf\|_1 < \infty$, then we deduce that $f \in L \log L(G)$.

In connection with these classical results, a natural question arises about conditions on $p(\cdot)$ implying the inequality

$$\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Diening [3] has shown that this remains true for variable exponents $p(\cdot)$ satisfying so called a log-Hölder condition, which is stated in the following:

Theorem A. *Suppose $p(\cdot)$ is a continuous function on G satisfying the following conditions:*

(p1) $1 < \operatorname{ess\,inf}_G \; p(x) \leq \operatorname{ess\,sup}_G \; p(x) < \infty$;

(p2) $|p(x) - p(y)| \leq \dfrac{c}{-\log|x-y|}$ *whenever* $x \in G, y \in G$ *and* $|x - y| < 1/2$.

Then M is bounded on $L^{p(\cdot)}(G)$, i.e.

$$\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)} \qquad \textit{whenever } f \in L^{p(\cdot)}(G).$$

The conclusion of Theorem A implies that

$$\|Mf\|_1 \leq C\|f\|_{p(\cdot)} \qquad \text{whenever } f \in L^{p(\cdot)}(G). \tag{1.1}$$

If $p(\cdot)$ approaches 1 on a compact set K in G, that is, $\operatorname{ess\,inf}_G \; p(x) = 1$, then this is not always true (see Cruz-Uribe, Fiorenza and Neugebauer [2, Theorem 1.7]). In this case, Hästö [5] has obtained an interesting class of variable exponents for which (1.1) is still valid. We are also interested in the shape of K, which is characterized by use of Minkowski content. When $p(\cdot)$ and K are given, our aim in this paper is to show that the maximal operator maps $L^{p(\cdot)}$ into an Orlicz space; as a consequence it is shown that the maximal operator maps $L^{p(\cdot)}$ into L^1.

As in Hästö [5], letting $\log_{(1)} t = \log t$ and $\log_{(m+1)} t = \log(\log_{(m)} t)$ for $m = 1, 2, ...$, we consider a function $p(\cdot)$ satisfying a log-Hölder condition such that $p(0) = 1$,

$$p(r) = 1 + \frac{a}{\log(1/r)} + \frac{b \log_{(2)}(1/r)}{\log(1/r)} + \frac{c \log_{(3)}(1/r)}{\log(1/r)} \tag{1.2}$$

for $0 < r \le r_0$ and $p(r) = p(r_0)$ for $r > r_0$, where the numbers a, b, c and r_0 are chosen so that $p(r)$ is nondecreasing on $[0, \infty)$. For a compact set K in $\mathbf{R}^n$, we define

$$K(r) = \{x \in \mathbf{R}^n : \delta_K(x) < r\},$$

where $\delta_K(x)$ denotes the distance of x from K. For $\alpha \ge 0$, we say that the Minkowski $(n - \alpha)$-content of K is finite if

$$|K(r)| \le Cr^\alpha \qquad \text{for small } r > 0,$$

where $|E|$ denotes the Lebesgue measure of a set E. Note here that if K is a singleton, then its Minkowski 0-content is finite, and if K is a spherical surface, then its Minkowski $(n - 1)$-content is finite. As another examples of K, we may consider fractal type sets like Cantor sets or Koch curves. Now we define a variable exponent $p(\cdot)$ by

$$p(x) = p(\delta_K(x)) \tag{1.3}$$

for $x \in \mathbf{R}^n$; set $p(x) = 1$ on K.

We are ready to state the result by Hästö [5].

Theorem B (cf. Hästö [5]). *Let K be a compact subset of a bounded open set G and $p(\cdot)$ be given as above. If $b > 1$, $c = 0$ and the Minkowski $(n - 1)$-content of K is finite, then* (1.1) *is satisfied for all $f \in L^{p(\cdot)}(G)$.*

Futamura and the first author [4] have proved that the conclusion is still valid when $b = 1$ and $c = 0$. They have also proved that the conclusion is not true when $b = 1$ and $c < 0$. Our aim in this paper is to establish the following result.

Theorem C. *Let $p(r)$ be of the form* (1.2). *For a compact set $K \subset G$ whose Minkowski $(n - \alpha)$-content is finite, set $p(x) = p(\delta_K(x))$ for $x \in G$, as in* (1.3).

(i) *If $b > 0$, then*

$$\int_G |f(x)|(\log(1 + |f(x)|))^{b\alpha}(\log(1 + (\log(1 + |f(x)|))))^{c\alpha}\, dx \le C,$$

or equivalently,

$$\int_G Mf(x)(\log(1 + Mf(x)))^{b\alpha-1}(\log(1 + (\log(1 + Mf(x)))))^{c\alpha}\, dx \le C$$

for all functions f on G with $\|f\|_{p(\cdot)} \le 1$.

(ii) *If* $b = 0$ *and* $c > 0$, *then*

$$\int_G |f(x)|(\log(1 + (\log(1 + |f(x)|))))^{c\alpha}\, dx \le C,$$

or equivalently,

$$\int_G Mf(x)(\log(1 + Mf(x)))^{-1}(\log(1 + (\log(1 + Mf(x)))))^{c\alpha-1}\, dx \le C$$

for all functions f *on* G *with* $\|f\|_{p(\cdot)} \le 1$.

Part (i) with $b = 1/\alpha$ and $c = 0$ gives an extension of Theorem B by Hästö [5] as well as Futamura and the first author [4].

2 Maximal functions

Throughout this paper, let C denote various constants independent of the variables in question. Further let G denote a bounded open set in $\mathbf{R}^n$.

For a positive continuous nonincreasing function k on $(0, \infty)$, assume that there exist $\varepsilon_0 > 0$ and $r_0 > 0$ such that $k(0) = \infty$ and

(k) $(\log(1/r))^{-\varepsilon_0} k(r)$ is nondecreasing on $(0, r_0)$.

Here we may assume that

$$k(r_0) \ge e^{\varepsilon_0}. \tag{2.1}$$

By (k) we see that

$$C^{-1}k(r) \le k(r^2) \le Ck(r) \qquad \text{whenever } 0 < r < r_0, \tag{2.2}$$

which implies the doubling condition on k. Our typical example of k is of the form

$$k(r) = a(\log_{(1)}(1/r))^b(\log_{(2)}(1/r))^c$$

for $r \in (0, r_0)$, where the numbers a, b, c and r_0 are chosen so that $k(r)$ is nonincreasing and positive on $(0, r_0]$.

LEMMA 2.1. *There exists* $0 < r^* < r_0$ *such that* $\log k(r)/\log(1/r)$ *is nondecreasing on* $(0, r^*)$.

Proof. Let $0 < r_1 < r_2 < r_0$. By (k), we have

$$\begin{aligned}
\frac{\log k(r_1)}{\log(1/r_1)} &\le \varepsilon_0 \frac{\log_{(2)}(1/r_1) - \log_{(2)}(1/r_2)}{\log(1/r_1)} + \frac{\log k(r_2)}{\log(1/r_1)} \\
&= \varepsilon_0 \frac{\log_{(2)}(1/r_1) - \log_{(2)}(1/r_2)}{\log(1/r_1)} + \frac{\log k(r_2)}{\log(1/r_2)}\left(1 + \frac{\log(r_1/r_2)}{\log(1/r_1)}\right) \\
&= \frac{\log k(r_2)}{\log(1/r_2)} + \frac{1}{\log(1/r_1)}\left\{\varepsilon_0 \log\left(\frac{\log(1/r_1)}{\log(1/r_2)}\right) + \frac{\log(r_1/r_2)}{\log(1/r_2)} \log k(r_2)\right\}.
\end{aligned}$$

Since $\log(1+t) < t$ for $t > 0$,

$$\log\left(\frac{\log(1/r_1)}{\log(1/r_2)}\right) \leq \frac{\log(r_2/r_1)}{\log(1/r_2)},$$

so that

$$\frac{\log k(r_1)}{\log(1/r_1)} - \frac{\log k(r_2)}{\log(1/r_2)} \quad \leq \quad \frac{1}{\log(1/r_1)}\left(\frac{\log(r_2/r_1)}{\log(1/r_2)}\right)(\varepsilon_0 - \log k(r_2)) < 0$$

by (2.1). □

Consider a positive continuous function $p(\cdot)$ such that $p(0) = 1$,

$$p(r) = p_{k,\alpha}(r) = 1 + \frac{\log k(r)}{\alpha \log(1/r)} \tag{2.3}$$

for $0 < r \leq r_0$ and $p(r) = p(r_0)$ when $r > r_0$. Here we choose r_0 so small that $p(r)$ is nondecreasing on $[0, \infty)$. Then it is worth to see that $p(\cdot)$ is continuous on the interval $[0, \infty)$, which is stated as follows:

Lemma 2.2. *$p(\cdot)$ is continuous at $r = 0$ (from the right).*

Lemma 2.3. (cf. [4, Lemma 2.1]). *Let K be a compact set in G whose Minkowski $(n-\alpha)$-content is finite. Then*

$$\int_G \delta_K(x)^{-\alpha}(\log(1+\delta_K(x)^{-1}))^{-\beta}\, dx < \infty$$

for every $\beta > 1$.

Proof. For each integer j, set $K_j = \{x \in G : 2^{-j} \leq \delta_K(x) < 2^{-j+1}\}$. Then we have for $\beta > 1$

$$\begin{aligned}
\int_G \delta_K(x)^{-\alpha}(\log(1+\delta_K(x)^{-1}))^{-\beta}\, dx &= \sum_{j=-j_0}^{\infty} \int_{K_j} \delta_K(x)^{-\alpha}(\log(1+\delta_K(x)^{-1}))^{-\beta}\, dx \\
&\leq \sum_{j=-j_0}^{\infty} (2^{-j})^{-\alpha}(\log(1+2^{j-1}))^{-\beta}|K(2^{-j+1})| \\
&\leq \sum_{j=-j_0}^{\infty} Cj^{-\beta} < \infty,
\end{aligned}$$

as required. □

Let K be a compact set in G. For $p(x) = p(\delta_K(x))$ with $p(r) = p_{k,\alpha}(r)$, consider the $L^{p(\cdot)}(G)$ norm defined by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leq 1\right\}.$$

We denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$.

Lemma 2.4. (cf. [4, Lemma 2.3]). *Suppose the Minkowski $(n-\alpha)$-content of K is finite. If f is a measurable function on G with $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G |f(x)|k(|f(x)|^{-1})\, dx \le C.$$

Proof. Consider the set

$$G' = \{x \in K(r_0) : |f(x)| < \delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta}\}$$

for $\beta > 1+\varepsilon_0$; here we set $\delta(x) = \delta_K(x)$ for simplicity. If $x \in G'$, then we have by (k)

$$\begin{aligned} |f(x)|k(|f(x)|^{-1}) &\le \delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta} k(\delta(x)^{\alpha}(\log(1/\delta(x)))^{\beta}) \\ &\le C\delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta+\varepsilon_0}. \end{aligned}$$

Hence it follows from Lemma 2.3 that

$$\int_{G'} |f(x)|k(|f(x)|^{-1})\, dx \le C$$

since $\beta > 1+\varepsilon_0$.

If $x \notin G'$ and $\delta(x) < r_0$, then $|f(x)| \ge \delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta}$, so that

$$\delta(x) \ge C|f(x)|^{-1/\alpha}(\log|f(x)|)^{-\beta/\alpha}.$$

Hence, in view of Lemma 2.1 and (2.2), we see that

$$\begin{aligned} \frac{\log k(\delta(x))}{\alpha \log(1/\delta(x))} \log|f(x)| &\ge \frac{\log k(C|f(x)|^{-1/\alpha}(\log|f(x)|)^{-\beta/\alpha})}{\alpha \log(C|f(x)|^{1/\alpha}(\log|f(x)|)^{\beta/\alpha})} \log|f(x)| \\ &\ge \frac{\log(Ck(|f(x)|^{-1}))}{\log|f(x)| + \beta\log(C\log|f(x)|)} \log|f(x)| \\ &= \log(Ck(|f(x)|^{-1}))\left(1 - \frac{\beta\log(C\log|f(x)|)}{\log|f(x)| + \beta\log(C\log|f(x)|)}\right) \\ &\ge \log k(|f(x)|^{-1}) - C, \end{aligned}$$

which yields

$$\begin{aligned} |f(x)|^{p(x)-1} &= \exp\left(\frac{\log k(\delta(x))}{\alpha\log(1/\delta(x))}\log|f(x)|\right) \ge \exp(\log k(|f(x)|^{-1}) - C) \\ &= Ck(|f(x)|^{-1}). \end{aligned}$$

Thus it follows that

$$\int_{K(r_0)\setminus G'} |f(x)|k(|f(x)|^{-1})\, dx \le C\int_G |f(x)|^{p(x)}\, dx \le C.$$

Finally, since $p(x) \geq p_1 > 1$ when $\delta(x) \geq r_0$, we find

$$\int_{G\setminus K(r_0)} |f(x)|k(|f(x)|^{-1})\ dx \leq C \int_G |f(x)|^{p(x)}\ dx + C \leq C.$$

The required assertion is now proved. □

Let Φ_k be a nonnegative and nondecreasing function on $[0, \infty)$ such that

$$\Phi_k(2t) \leq C\Phi_k(t)$$

and

$$C^{-1}k(t^{-1}) \leq \int_1^t \frac{\Phi_k(s)}{s^2} ds \leq Ck(t^{-1})$$

for all $t > 2$. Then note that

$$C^{-1} \int_1^t \frac{\Phi_k(s)}{s^2} ds \leq \int_1^t s^{-1} d\Phi_k(s) \leq C \int_1^t \frac{\Phi_k(s)}{s^2} ds \tag{2.4}$$

for all $t > 2$, since $t^{-1}\Phi_k(t) \leq C \int_1^t \Phi_k(s)s^{-2}ds$ by the doubling property of Φ_k.

The next lemma is an extension of Stein [11, Chapter 1], whose proof will be done along the same lines as in Stein [11, Chapter 1]; for another proof, see Cianchi [1].

LEMMA 2.5. *For a locally integrable function f on G, the following are equivalent:*

(i) $\displaystyle\int_G |f(x)|k(|f(x)|^{-1})\ dx \leq C;$

(ii) $\displaystyle\int_G \Phi_k(Mf(x))\ dx \leq C.$

Proof. Note that

$$\int_G \Phi_k(Mf(x))\ dx \ \ = \ \ \int_0^\infty \lambda(t)d\Phi_k(t),$$

where $\lambda(t) = |\{x \in G : Mf(x) > t\}|$.

First suppose (i) holds. We note from [11, Theorem 1, Chapter 1] that

$$\lambda(t) \leq Ct^{-1} \int_{\{x\in G:|f(x)|>t/2\}} |f(x)|\ dx$$

for $t > 1$. Hence we obtain by Fubini's Theorem and (2.2)

$$\begin{aligned}\int_G \Phi_k(Mf(x))\,dx &\le C\int_{\{x\in G:|f(x)|>1/2\}} |f(x)| \left\{\int_1^{2|f(x)|} t^{-1} d\Phi_k(t)\right\} dx + C \\ &\le C\int_G |f(x)|k(|f(x)|^{-1})\,dx + C,\end{aligned}$$

which implies (ii).

Conversely, suppose (ii) holds. In view of [11, 5.2 (b) of Section 5, Chapter 1], we find $\gamma > 0$ such that

$$\lambda(t) \ge Ct^{-1}\int_{\{x\in G:|f(x)|>\gamma t\}} |f(x)|\,dx$$

for $t > 1$. Hence the implication (ii) $\Rightarrow$ (i) follows by the same argument as before. □

3 Proof of Theorem C

By Lemmas 2.4 and 2.5, we have the following result.

THEOREM 3.1. *Let Φ_k be as in Lemma 2.5. Suppose the Minkowski $(n-\alpha)$-content of K is finite. If $p(x) = p_{k,\alpha}(x)$ and $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G |f(x)|k(|f(x)|^{-1})\,dx \le C,$$

or equivalently,

$$\int_G \Phi_k(Mf(x))\,dx \le C.$$

REMARK 3.2. *Let $k(1/t) = A(\log t)^b(\log_{(2)} t)^c$ for large t, where $A > 0$.*

(i) If $b > 0$, then we can take

$$\Phi_k(t) = t(\log t)^{b-1}(\log(\log t))^c$$

for large t.

(ii) If $b = 0$ and $c > 0$, then we can take

$$\Phi_k(t) = t(\log t)^{-1}(\log(\log t))^{c-1}$$

for large t.

Thus Theorem 3.1 gives Theorem C in the Introduction.

4 Further remarks

In this section, we give some remarks on Theorem C.

REMARK 4.1. *If $k(t) = \log(1 + 1/t)$, then we can take $\Phi_k(t) = t$, so that Theorem 3.1 implies that the maximal operator M maps $L^{p(\cdot)}(G)$ into $L^1(G)$. One of the referees kindly informed us that the integrability also follows from a result by Musielak [7, §8]; more precisely, if we find a constant $C > 0$ and $g \in L^1(G)$ such that*

$$t \log(1 + t) \le Ct^{p(x)} + g(x) \qquad \text{for all } t \ge 0, \tag{4.1}$$

then $Mf \in L^1(G)$ for all $f \in L^{p(\cdot)}(G)$. Further he suggested us to take t such that $t \log(1 + t) \ge 2Ct^{p(x)}$ and obtain the integrability of the maximal operator. In fact, letting

$$t(x) = \left(\frac{\varepsilon(p(x) - 1)}{-\log(\varepsilon(p(x) - 1))} \right)^{-1/(p(x)-1)}$$

for $\varepsilon > 0$, we see that

$$t \log(1 + t) \le Ct^{p(x)} \qquad \text{for all } t \ge t(x),$$

because

$$(p(x) - 1) \log t - \log(\log t) \ge (p(x) - 1) \log t(x) - \log(\log t(x)) \ge \log C$$

when $\varepsilon(p(x) - 1)$ is small. Now, since (4.1) holds for $g(x) = t(x) \log(1 + t(x))$, the problem is to discuss when g is integrable. If

$$p(r) = 1 + \frac{b \log_{(2)}(1/r)}{\log(1/r)}$$

with $b > 1/\varepsilon$ and $p(x) = p(\delta_K(x))$, then

$$g(x) \le C\delta_K(x)^{-1/b}(\log \delta_K(x)^{-1})^{-2}$$

whenever $\delta_K(x)$ is small enough, say $\delta_K(x) \le r_0$, and our Lemma 2.3 would be applicable.

However, it seems that our discussions are simple and straightforward.

REMARK 4.2. *Let K be a compact subset of $\mathbf{R}^n$ and $0 < \alpha < n$.*

(i) If there is a finite Borel measure μ on K such that

$$\mu(B(x, r)) \ge Cr^{n-\alpha} \qquad \text{whenever } 0 < r < r_0 \text{ and } x \in K,$$

then

$$\limsup_{r \to 0} |K(r)|/r^{\alpha} < \infty.$$

(ii) If there is a finite Borel measure μ on K such that $\mu(K) > 0$ and

$$\mu(B(x,r)) \le Cr^{n-\alpha} \qquad \text{whenever } 0 < r < r_0 \text{ and } x \in K,$$

then

$$\liminf_{r\to 0} \ |K(r)|/r^\alpha > 0.$$

To prove these facts, noting that $K(r) = \bigcup_{x\in K} B(x,r)$, by a covering lemma (see [11, Lemma 1.6, Chapter 1], we can find a disjoint family $\{B(x_j,r)\}$ such that

$$\bigcup_j B(x_j, 5r) \supseteq K(r).$$

By our assumption,

$$|K(r)| \le C\sum_j (5r)^n \le C\sum_j r^\alpha \mu(B(x_j,r)) \le Cr^\alpha \mu\Big(\bigcup_j B(x_j,r)\Big) \le Cr^\alpha \mu(K),$$

which implies

$$\limsup_{r\to 0} \ |K(r)|/r^\alpha \le C\mu(K) < \infty.$$

Thus (i) follows.

Since $K(r) \supset \bigcup_j B(x_j,r)$, we find

$$|K(r)| \ge C\sum_j r^n \ge C\sum_j r^\alpha \mu(B(x_j,5r)) \ge Cr^\alpha \sum_j \mu(B(x_j,5r)) \ge Cr^\alpha \mu(K),$$

which gives

$$\liminf_{r\to 0} \ |K(r)|/r^\alpha \ge C\mu(K) > 0.$$

Thus (ii) is proved.

Remark 4.3. *For $\alpha > 0$, let K be a compact subset of G such that*

$$C^{-1}r^\alpha \le |K(r)| \le Cr^\alpha \qquad \text{for } 0 < r < r_0.$$

Set $\delta(x) = \delta_K(x)$ for simplicity. Then Theorem C is seen to be sharp in the following sense: if $b > 0$, $c > 0$,

$$p(x) = 1 + \frac{b\log_{(2)}(1/\delta(x))}{\log(1/\delta(x))} - \frac{c\log_{(3)}(1/\delta(x))}{\log(1/\delta(x))}$$

when $\delta(x) \le r_0$ and $\inf_{\{x:\delta(x)>r_0\}} p(x) > 1$, then we can find $f \in L^{p(\cdot)}(G)$ satisfying

$$\int_G |f(x)|(\log(1+|f(x)|))^{b\alpha}\, dx = \infty,$$

which implies that

$$\int_G Mf(x)(\log(1+Mf(x)))^{b\alpha-1}\ dx = \infty.$$

For this purpose, we consider the function

$$f(x) = \delta(x)^{-\alpha}(\log 1/\delta(x))^{-b\alpha-1}(\log_{(2)}(1/\delta(x)))^{-1}\ \Big(\ \le \delta(x)^{-\alpha}\ \Big)$$

for $x \in G$ with $\delta(x) \le r_0$; set $f(x) = 0$ when $\delta(x) > r_0$. Then

$$\int_G f(x)(\log(1+f(x)))^{b\alpha}\ dx \ \ \ge\ \ C\int_0^{r_0} t^{-1}(\log(1/t))^{-1}(\log_{(2)}(1/t))^{-1}dt = \infty.$$

Further, we have for $t = \delta(x) \le r_0$

$$\begin{aligned} f(x)^{p(x)-1} &\le \exp\left(\alpha\log(1/t)\frac{b\log_{(2)}(1/t) - c\log_{(3)}(1/t)}{\log(1/t)}\right) \\ &= (\log(1/t))^{b\alpha}(\log_{(2)}(1/t))^{-c\alpha}, \end{aligned}$$

so that

$$\int_G f(x)^{p(x)}\ dx \ \ \le\ \ C\int_0^{r_0} t^{-1}(\log(1/t))^{-1}(\log_{(2)}(1/t))^{-c\alpha-1}dt < \infty.$$

REMARK 4.4. *Let $p(r)$ be a continuous function on $[0,\infty)$ such that $p(0) = p_0 > 1$,*

$$p(r) = p_0 + \frac{a}{\log(1/r)} + \frac{b\log_{(2)}(1/r)}{\log(1/r)} + \frac{c\log_{(3)}(1/r)}{\log(1/r)},$$

for $0 < r \le r_0 < 1/4$ and $p(r) = p(r_0)$ for $r > r_0$; here the numbers a, $b \ge 0$, c and $r_0 > 0$ are chosen so that $p(r)$ is nondecreasing on $[0,\infty)$. For a compact set $K \subset G$ whose Minkowski $(n-\alpha)$-content is finite, define $p(x) = p(\delta_K(x))$ for $x \in G$. Then, letting

$$k(1/r) = e^{a\alpha}(\log r)^{b\alpha}(\log_{(2)} r)^{c\alpha}$$

for large $r > 0$, we see as in Lemma 2.4 that

$$\int_G |f(x)|^{p_0}k(|f(x)|^{-1})^{1/p_0}dx \le C$$

for all functions f on G with $\|f\|_{p(\cdot)} \le 1$. Hence, in the same way as Theorem 3.1, we can prove that

$$\int_G |f(x)|^{p_0}(\log(1+|f(x)|))^{b\alpha/p_0}(\log(1+(\log(1+|f(x)|))))^{c\alpha/p_0}\ dx \le C, \tag{4.2}$$

or equivalently,

$$\int_G Mf(x)^{p_0}(\log(1+Mf(x)))^{b\alpha/p_0}(\log(1+(\log(1+Mf(x)))))^{c\alpha/p_0}\ dx \le C$$

for all functions f on G with $\|f\|_{p(\cdot)} \le 1$.

Remark 4.5. *We here show that (4.2) is good in as far as the exponents are concerned. For this purpose, let $K = \partial B(0,1)$, whose Minkowski $(n-1)$-content is finite. Consider*

$$p(r) = p_0 + \frac{a}{\log(1/r)} + \frac{b\log_{(2)}(1/r)}{\log(1/r)}$$

for $0 < r \le r_0 < 1/4$ and $p(r) = p(r_0)$ for $r > r_0$. Define

$$p(x) = p(\delta_K(x)) = p(|1-|x||).$$

Further define

$$f(x) = \begin{cases} \delta_K(x)^{-1/p_0}(\log(1/\delta_K(x)))^{-\beta} & \text{when } \delta_K(x) \le r_0, \\ 0 & \text{when } \delta_K(x) > r_0. \end{cases}$$

Then the following are equivalent:

(i) $\displaystyle\int_{B(0,2)} |f(x)|^{p(x)}\, dx < \infty$;

(ii) $\displaystyle\int_{B(0,2)} |f(x)|^{p_0}(\log(1+|f(x)|))^{b/p_0}\, dx < \infty$;

(iii) $-\beta p_0 + b/p_0 < -1$.

We next give another example. Let K be a set of one point, say, $K = \{0\}$, whose Minkowski 0-content is finite. Consider $p(x) = p(\delta_K(x)) = p(|x|)$ and

$$f(x) = \begin{cases} |x|^{-n/p_0}(\log(1/|x|))^{-\beta} & \text{when } |x| \le r_0, \\ 0 & \text{when } |x| > r_0. \end{cases}$$

Then the following are equivalent:

(i) $\displaystyle\int_{B(0,1)} |f(x)|^{p(x)}\, dx < \infty$;

(ii) $\displaystyle\int_{B(0,1)} |f(x)|^{p_0}(\log(1+|f(x)|))^{bn/p_0}\, dx < \infty$;

(iii) $-\beta p_0 + bn/p_0 < -1$.

We may further consider many types of fractal sets as examples of compact sets K. But we do not go into details.

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