Exponential integrability for logarithmic potentials of functions in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}$ over non-doubling measure spaces

Sachihiro Kanemori, Takao Ohno and Tetsu Shimomura

Abstract

In this paper, we are concerned with exponential integrability for logarithmic potentials of functions in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}$ over non-doubling measure spaces. Here q satisfies the loglog-Hölder condition.

1 Introduction

The properties of the logarithmic potentials were studied by some authors (see e.g. [7], [8], [9], [10] and [12]). Our aim in this paper is to establish exponential integrability for logarithmic potentials of functions in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}$ over non-doubling measure spaces, as an extension of [11, Theorem 8.1] in the Euclidean setting.

We denote by (X, d, μ) a metric measure spaces, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_X = \sup\{d(x, y) : x, y \in X\}$. We assume that $0 < d_X < \infty$,

$$\mu(\{x\}) = 0$$

for $x \in X$ and $\mu(B(x, r)) > 0$ for $x \in X$ and $r > 0$ for simplicity. In the present paper, we do not postulate on μ the "so called" doubling condition. Recall that a Radon measure μ is said to be doubling if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in \mathrm{supp}(\mu)(= X)$ and $r > 0$. Otherwise μ

2000 Mathematics Subject Classification : Primary 46E35; Secondary 46E30.

Key words and phrases : exponential integrability, logarithmic potential, variable exponent, metric measure space, non-doubling measure

is said to be non-doubling. Assume that there exist positive constants K_0 and s such that, for all balls $B(x,r)$ with center $x \in X$ and of radius $0 < r < d_X$,

$$\mu(B(x,r)) \le K_0 r^s \tag{1.1}$$

(see e.g. [1], [5] and [6]).

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. For a survey, see [3] and [4].

In this paper, following Cruz-Uribe and Fiorenza [2], we consider a variable exponent $q(\cdot) : X \to [0,1)$ such that

$$|q(x) - q(y)| \le \frac{C_q}{\log(e + \log(e + 1/d(x,y)))} \qquad \text{for all } x, y \in X \tag{1.2}$$

with a constant $C_q \ge 0$.

Define the norm by

$$\|f\|_{L(\log L)^{q(\cdot)}(X)} = \inf\left\{ \lambda > 0 : \int_X \left|\frac{f(x)}{\lambda}\right| \left(\log\left(e + \left|\frac{f(x)}{\lambda}\right|\right)\right)^{q(x)} d\mu(x) \le 1 \right\}$$

and denote by $L(\log L)^{q(\cdot)}(X)$ the space of all measurable functions f on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} < \infty$.

We define the logarithmic potential for a locally integrable function f on X by

$$Lf(x) = \int_X \left(\log^+(1/d(x,y))\right) f(y)\, d\mu(y),$$

where $\log^+ r = \max\{0, \log r\}$. Here it is natural to assume that

$$\int_X (\log(e + d(x_0,y)))|f(y)|\, d\mu(y) < \infty \tag{1.3}$$

for some $x_0 \in X$ since this implies

$$\left| \int_X \log(1/d(x,y)) f(y)\, d\mu(y) \right| < \infty$$

for μ-a.e. in X (see [7, Lemma 1] and [9, Theorem 6.1, Chapter 2]).

In [11], we studied exponential integrability for logarithmic potentials of functions in $L(\log L)^{q(\cdot)}(\mathbf{R}^N)$ in the Euclidean setting. Our main aim in the present paper is to establish exponential integrability for Lf in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}(X)$ over non-doubling measure spaces, as an extension of [11, Theorem 8.1].

Theorem 1.1. *There exist constants $c_1, c_2 > 0$ such that*

$$\int_X \exp\left((c_1 Lf(x))^{1/(1-q(x))}\right) d\mu(x) \le c_2$$

for all nonnegative measurable functions f on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$.

Corollary 1.2. *There exists a constant $c_3 > 0$ such that*

$$\int_X \left\{\exp\left((c_3 Lf(x))^{1/(1-q(x))}\right) - 1\right\} d\mu(x) \le 1$$

for all nonnegative measurable functions f on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$.

Our strategy is to give an estimate of Lf by use of a logarithmic type potential

$$\int_X \mu(B(x,4r))^{-1}(\log(e+1/r))^{-\beta} f(y)(\log(e+f(y)))^{q(y)}\, d\mu(y)$$

with $\beta > 1$, which plays a role of maximal functions.

The sharpness of the exponent will be discussed in Section 4.

In the final section, we show the continuity for logarithmic potentials of functions in $L^{p(\cdot)}(\log L)^{r(\cdot)}(X)$ over non-doubling measure spaces, as an extension of [11, Theorem 8.4] and [9, Theorem 9.1, Section 5.9] (see Section 5 for the definition of $L^{p(\cdot)}(\log L)^{r(\cdot)}(X)$). For related results, see [12].

2 Preliminary lemmas

Throughout this paper, let C denote various positive constants independent of the variables in question.

To prove Theorem 1.1, we estimate Lf by the logarithmic potential

$$J = \int_X \rho_{-\beta}(d(x,y))g(y)\, d\mu(y),$$

where $\rho_{-\beta}(r) = \mu(B(x,4r))^{-1}(\log(e+1/r))^{-\beta}$ with $\beta > 1$ and $g(y) = f(y)(\log(e+f(y)))^{q(y)}$.

Lemma 2.1. *Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. Then there is a constant $C > 0$ such that*

$$F \equiv \int_{B(x,\delta)} \rho_{-\beta}(d(x,y))f(y)\, d\mu(y) \le CJ\left\{(\log(e+J))^{-q(x)} + (\log(e+1/\delta))^{-q(x)}\right\}$$

for all $x \in X$ and $0 < \delta < d_X$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. We have for $k > 0$

$$F \le k\int_{B(x,d_X)} \rho_{-\beta}(d(x,y))\, d\mu(y) + \int_{B(x,\delta)} \rho_{-\beta}(d(x,y))f(y)\left(\frac{\log(e+f(y))}{\log(e+k)}\right)^{q(y)} d\mu(y).$$

Since $\beta > 1$, we have

$$
\begin{aligned}
& \int_{B(x,d_X)} \rho_{-\beta}(d(x,y))\, d\mu(y) \\
= & \sum_{j=1}^{\infty} \int_{X\cap(B(x,2^{-j+1}d_X)\setminus B(x,2^{-j}d_X))} \mu(B(x,4d(x,y)))^{-1}(\log(e+1/d(x,y)))^{-\beta}\, d\mu(y) \\
\le & \sum_{j=1}^{\infty} \int_{X\cap(B(x,2^{-j+1}d_X)\setminus B(x,2^{-j}d_X))} \mu(B(x,2^{-j+2}d_X))^{-1}(\log(e+1/(2^{-j+1}d_X)))^{-\beta}\, d\mu(y) \\
\le & \sum_{j=1}^{\infty}(\log(e+1/(2^{-j+1}d_X)))^{-\beta} \\
\le & C.
\end{aligned}
$$

If $J \le \delta^{-1}$, then we set $k = J(\log(e+J))^{-q(x)}$. Since $\delta \le J^{-1}$, we see from (1.2) that

$$(\log(e+k))^{-q(y)} \le C(\log(e+J))^{-q(x)}$$

for $y \in B(x,\delta)$. Consequently it follows that

$$F \le CJ(\log(e+J))^{-q(x)}.$$

If $J > \delta^{-1}$, then we set $k = \delta^{-1}(\log(e+1/\delta))^{-q(x)}$ and obtain

$$
\begin{aligned}
F &\le C\left\{\delta^{-1}(\log(e+1/\delta))^{-q(x)} + J(\log(e+1/\delta))^{-q(x)}\right\} \\
&\le CJ(\log(e+1/\delta))^{-q(x)}.
\end{aligned}
$$

Now the result follows. □

LEMMA 2.2. *Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. Then there is a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\, d\mu(y) \le C(\log(e+1/\delta))^{-q(x)+1}$$

for all $x \in X$ and $0 < \delta < d_X$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. Let $0 < \gamma < s$, where s is a constant appearing in (1.1). For $y \in X \setminus B(x,\delta)$ and $0 < \delta < d_X$, set $N(x,y) = d(x,y)^{-\gamma}$. Let j_0 be the smallest integer such that

$2^{j_0}\delta \ge d_X$. We have by (1.1)

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))N(x,y)d\mu(y) \\
&= \sum_{j=1}^{j_0} \int_{X\cap(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta))} \log^+(1/d(x,y))N(x,y)\,d\mu(y) \\
&\le \sum_{j=1}^{j_0} \int_{X\cap(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta))} \log^+(1/(2^{j-1}\delta))(2^{j-1}\delta)^{-\gamma} d\mu(y) \\
&\le C\sum_{j=1}^{j_0} \log^+(1/(2^{j-1}\delta))(2^{j-1}\delta)^{s-\gamma} \\
&\le C
\end{aligned}$$

since $\gamma < s$. Hence, we see from (1.2) that

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\,d\mu(y) \\
&\le \int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))N(x,y)\,d\mu(y) \\
&\quad + \int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\left(\frac{\log(e+f(y))}{\log(e+N(x,y))}\right)^{q(y)} d\mu(y) \\
&\le C\left\{1 + \int_{X\setminus B(x,\delta)} (\log(e+1/d(x,y)))^{-q(y)+1} g(y)\,d\mu(y)\right\} \\
&\le C\left\{1 + (\log(e+1/\delta))^{-q(x)+1} \int_{X\setminus B(x,\delta)} g(y)\,d\mu(y)\right\} \\
&\le C(\log(e+1/\delta))^{-q(x)+1},
\end{aligned}$$

where $g(y) = f(y)(\log(e+f(y)))^{q(y)}$. Thus this lemma is proved. □

3 Proof of Theorem 1.1.

Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. For $x \in X$ and $0 < \delta < d_X$, write

$$\begin{aligned}
Lf(x) &= \int_{B(x,\delta)} \log^+(1/d(x,y))f(y)\,d\mu(y) + \int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\,d\mu(y) \\
&= I_1 + I_2.
\end{aligned}$$

For $\beta > 1$, we infer from Lemma 2.1 and (1.1) that

$$\begin{aligned}
I_1 &\le C\delta^s(\log(e+1/\delta))^{1+\beta} \int_{B(x,\delta)} \rho_{-\beta}(d(x,y))f(y)\,d\mu(y) \\
&\le C\delta^s(\log(e+1/\delta))^{1+\beta} J\left\{(\log(e+1/\delta))^{-q(x)} + (\log(e+J))^{-q(x)}\right\}.
\end{aligned}$$

Hence, in view of Lemma 2.2, we find

$$
\begin{aligned}
Lf(x) \ \le \ & C\Big[\delta^s(\log(e+1/\delta))^{1+\beta}J\left\{(\log(e+1/\delta))^{-q(x)}+(\log(e+J))^{-q(x)}\right\} \\
& \quad + (\log(e+1/\delta))^{-q(x)+1}\Big].
\end{aligned}
$$

Now, considering $\delta=\min\{d_X, J^{-1/s}(\log(e+J))^{-\beta/s}\}$, we find

$$
Lf(x)\le C(\log(e+J))^{-q(x)+1}.
$$

Hence

$$
\exp\left((c_1 Lf(x))^{1/(1-q(x))}\right)\le e+J.
$$

By using Fubini's theorem, we obtain

$$
\begin{aligned}
& \int_X \exp\left((c_1 Lf(x))^{1/(1-q(x))}\right)\,d\mu(x) \\
\le\ & \int_X (e+J)\,d\mu(x) \\
\le\ & \int_X g(y)\left(\int_X \frac{(\log(e+1/d(x,y)))^{-\beta}}{\mu(B(x,4d(x,y)))}d\mu(x)\right)\,d\mu(y)+C \\
\le\ & \int_X g(y)\left(\sum_{j=1}^{\infty}\int_{X\cap(B(y,2^{-j+1}d_X)\setminus B(y,2^{-j}d_X))}\frac{(\log(e+1/d(x,y)))^{-\beta}}{\mu(B(y,2d(x,y)))}\,d\mu(x)\right)\,d\mu(y)+C \\
\le\ & \int_X g(y)\left(\sum_{j=1}^{\infty}\int_{X\cap(B(y,2^{-j+1}d_X)\setminus B(y,2^{-j}d_X))}\frac{(\log(e+1/(2^{-j+1}d_X)))^{-\beta}}{\mu(B(y,2^{-j+1}d_X))}\,d\mu(x)\right)\,d\mu(y)+C \\
\le\ & \int_X g(y)\left(\sum_{j=1}^{\infty}(\log(e+1/(2^{-j+1}d_X)))^{-\beta}\right)\,d\mu(y)+C \\
\le\ & c_2,
\end{aligned}
$$

since $\beta>1$. This completes the proof of the theorem. □

4 Sharpness

Let $X=B(0,1)\subset \mathbf{R}^N$ and $q(\cdot)=q$. For $\delta>0$, consider the function

$$
u(x)=\int_{B(0,1)}\log^{+}(1/|x-y|)f(y)\,dy
$$

with

$$
f(y)=|y|^{-N}(\log(e/|y|))^{\delta-2}\qquad \text{for } y\in B(0,1).
$$

Then f satisfies

$$
\int_{B(0,1)} f(y)\,(\log(e+f(y)))^{q}\,dy<\infty \tag{4.1}
$$

if and only if $\delta - 1 + q < 0$. We see that

$$u(x) \geq C \int_{\{y \in B(0,1/2):|y|>|x|\}} \log^+(1/|y|) f(y)\, dy \geq C(\log(e/|x|))^{\delta}$$

for $|x| < 1/2$. Hence, if $\beta\delta > 1$, then

$$\int_{B(0,1)} \exp\left(u(x)^{\beta}\right) dx = \infty. \tag{4.2}$$

If $\beta > 1/(1-q)$, then we can choose δ such that

$$1/\beta < \delta < 1 - q.$$

In this case, both (4.1) and (4.2) hold. This implies that the exponent $1/(1-q)$ in Theorem 1.1 is sharp.

5 Continuity

In this section, we consider variable exponents $p(\cdot) : X \to [1, \infty)$ and $r(\cdot) : X \to (-\infty, \infty)$ such that

$$-\infty < \inf_{x \in X} r(x) \leq \sup_{x \in X} r(x) < \infty. \tag{5.1}$$

Define the norm by

$$\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} = \inf \left\{ \lambda > 0 : \int_X \left| \frac{f(x)}{\lambda} \right|^{p(x)} \left(\log \left(e + \left| \frac{f(x)}{\lambda} \right| \right) \right)^{r(x)} d\mu(x) \leq 1 \right\}$$

and denote by $L^{p(\cdot)}(\log L)^{r(\cdot)}(X)$ the space of all measurable functions f on X with $\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} < \infty$.

THEOREM 5.1 (cf. [9, Theorem 9.1, Section 5.9]). *Let $p(\cdot)$ and $r(\cdot)$ be two variable exponents on X satisfying* (5.1) *such that*

$$p(x) > 1 \qquad \text{or} \qquad r(x) \geq 1$$

for all $x \in X$. If f is a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} < \infty$, then Lf is continuous on X.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} < \infty$. Then note that

$$\int_X f(y)(\log(e + f(y)))\, d\mu(y) < \infty.$$

Hence, it follows from [7, Theorem 1] that Lf is continuous on X by (1.3). □

References

[1] A. Björn and J. Björn, Nonlinear potential theory on metric spaces. EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zurich, 2011.

[2] D. Cruz-Uribe and A. Fiorenza, $L \log L$ results for the maximal operator in variable L^p spaces, Trans. Amer. Math. Soc. **361** (2009), 2631–2647.

[3] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue spaces. Foundations and harmonic analysis. Applied and Numerical Harmonic Analysis. Birkhauser/Springer, Heidelberg, 2013.

[4] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Math. **2017**, Springer-Verlag, Berlin, 2011.

[5] J. Garciá-Cuerva and A. E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, Studia Math. **162** (2004), 245–261.

[6] P. Hajłasz and P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. **145**, 2000.

[7] M. G. Hajibayov, Continuity of logarithmic potentials, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **25** (2006), 41–46.

[8] N. S. Landkof, Foundations of modern potential theory, Springer-Verlag, Berlin Heidelberg New York, 1972.

[9] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[10] Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions. Hiroshima Math. J. **23** (1993), no. 1, 79–153.

[11] Y. Mizuta, T. Ohno and T. Shimomura, Sobolev embeddings for Riesz potential spaces of variable exponents near 1 and Sobolev's exponent, Bull. Sci. Math. **134** (2010), 12–36.

[12] T. Ohno, Continuity properties for logarithmic potentials of functions in Morrey spaces of variable exponent, Hiroshima Math. J. **38** (2008), 363–383.

Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
and
Faculty of Education and Welfare Science
Oita University
Dannoharu Oita-city 870-1192, Japan
E-mail : t-ohno@oita-u.ac.jp
and
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail : tshimo@hiroshima-u.ac.jp