Boundary limits of monotone Sobolev functions with variable exponent on uniform domains in a metric space

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Abstract

Our aim in this paper is to deal with boundary limits of monotone Sobolev functions with variable exponent on uniform domains in a metric space.

1 Introduction

A continuous function u on an open set D in the n-dimensional Euclidean space $\mathbf{R}^n$ is called monotone in the sense of Lebesgue (see [10]) if the equalities

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u$$

hold whenever G is a domain with compact closure $\overline{G} \subset D$. If u is a monotone function on D satisfying

$$\int_D |\nabla u(z)|^p \, dz < \infty \qquad \text{for some} \quad p > n-1,$$

then

$$|u(x) - u(y)| \le C(n,p) r^{1-n/p} \left(\int_{2B(x,r)} |\nabla u(z)|^p \, dz \right)^{1/p} \tag{1.1}$$

whenever $y \in B(x,r)$ with $2B(x,r) \subset D$, where $C(n,p)$ is a positive constant depending only on n and p (see [13, Chapter 8] and [18, Section 16]). Using this inequality (1.1), the first author and Mizuta proved Lindelöf theorems for monotone Sobolev functions on the half space of $\mathbf{R}^n$ in [5], as an extension of Mizuta [14, Theorem 2] and Manfredi-Villamor [11, 12]. This result was extended to a uniform domain by the first author [4]. Mizuta studied tangential boundary limits of monotone Sobolev functions with finite Dirichlet integral in the half space in [14]. Recently, Di Biase, the first author and the third author [1] gave Lindelöf theorems for monotone Sobolev functions in Orlicz spaces.

Variable exponent spaces have been studied in many articles over the past decade; for a survey see the recent book by Diening, Harjulehto, Hästö and Růžička [3]. Let $\mathbf{B}$

2000 Mathematics Subject Classification : Primary 31B25, 46E35

Key words and phrases : monotone Sobolev functions, nontangential limits, tangential limits, Lindelöf theorem, variable exponent, uniform domains

be the unit ball in $\mathbf{R}^n$. Lindelöf theorems for monotone Sobolev functions on variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbf{B})$ was investigated in [7].

For related results, see Koskela-Manfredi-Villamor [9], Villamor-Li [17], Mizuta [13] and the first author and Mizuta [6].

We denote by (X, d, μ) a metric measure spaces, where X is a set, d is a metric on X and μ is a Borel measure on X which is positive and finite in every balls. We write $d(x, y) = |x - y|$ for simplicity. A domain D in X with $\partial D \neq \emptyset$ is a uniform domain if there exist constants $A_1 \geq 1$ and $A_2 \geq 1$ such that each pair of points $x, y \in D$ can be joined by a rectifiable curve γ in D for which

$$\ell(\gamma) \leq A_1|x - y|, \tag{1.2}$$

$$\delta_D(z) \geq A_2 \min\{\ell(\gamma(x, z)), \ell(\gamma(y, z))\} \quad \text{for all } z \in \gamma, \tag{1.3}$$

where $\ell(\gamma)$, $\delta_D(z)$ and $\gamma(x, z)$ denote the length of γ, the distance from z to ∂D and the subarc of γ connecting x and z, respectively (see [16]). We denote by $B(x, r)$ the open ball centered at x with radius r and set $\lambda B(x, r) = B(x, \lambda r)$ for $\lambda > 0$.

In this paper, for $p > 1$, we are concerned with a positive continuous function $p(\cdot)$ on X satisfying the following conditions:

(p1) $p \leq p_- \equiv \inf_{x \in D} p(x) \leq p_+ \equiv \sup_{x \in D} p(x) < \infty$,

(p2) $|p(x) - p(y)| \leq \dfrac{C}{\log(e + 1/|x - y|)}$ for all $x, y \in \overline{D}$.

If $p(\cdot)$ satisfies (p2), we say that $p(\cdot)$ satisfies a log-Hölder condition.

In this paper, we are concerned with boundary limits of functions u on a uniform domain D for which there exist a constant $\alpha \in \mathbf{R}$ and a nonnegative function $g \in L^p_{loc}(D; \mu)$ such that

$$|u(x) - u(x')| \leq Cr \left(\fint_{\sigma B} g(z)^p d\mu(z) \right)^{1/p} \tag{1.4}$$

for every $x, x' \in B$ with $\sigma B \subset D$, where $\sigma > 1$, $B = B(y, r)$ and

$$\int_D g(z)^{p(z)} \delta_D(z)^\alpha d\mu(z) < 1. \tag{1.5}$$

Here we used the standard notation

$$\fint_E u(z) d\mu(z) = \frac{1}{\mu(E)} \int_E u(z) d\mu(z)$$

for a measurable set E with $0 < \mu(E) < \infty$. Let μ be a Borel measure on X satisfying the doubling condition:

$$\mu(2B) \leq c_0 \mu(B)$$

for every ball $B \subset X$. We further assume that

$$\frac{\mu(B')}{\mu(B)} \geq C \left(\frac{r'}{r} \right)^s \tag{1.6}$$

for all balls $B' = B(x', r')$ and $B = B(x, r)$ with $x', x \in \overline{D}$ and $B' \subset B$, where $s > 1$ (see e.g. [8]). Here note that if μ satisfies the doubling condition, then

$$\frac{\mu(B')}{\mu(B)} \geq c_0^{-2} \left(\frac{r'}{r}\right)^{\log_2 c_0}$$

for all balls $B' = B(x', r')$ and $B = B(x, r)$ with $x', x \in \overline{D}$ and $B' \subset B$ (see e.g. [2, Lemma 3.3]).

Let u be a function on D and let $\xi \in \partial D$. For $\beta \geq 1$ and $c > 0$, set

$$T_\beta(\xi; c) = \{x \in D : |x - \xi|^\beta \leq c\delta_D(x)\}.$$

We say u has a tangential limit of order β at ξ if the limit

$$\lim_{T_\beta(\xi;c) \ni x \to \xi} u(x)$$

exists for every $c > 0$. In particular, a tangential limit of order 1 is called nontangential limit.

Our first aim in this note is to establish the following theorem, as an extension of [14, Theorem 4]. See [1, Remark 3.1] for Orlicz spaces.

Theorem 1.1. *Let u be a function on a uniform domain D with $g \geq 0$ satisfying* (1.4) *and* (1.5) *and let $\beta \geq 1$. Suppose $p_+ < s + \alpha$ and set*

$$E_\beta = \left\{\xi \in \partial D : \limsup_{r \to 0} (r^{\beta(-p(\xi)+s+\alpha)-s} \mu(B(\xi, r)))^{-1} \int_{B(\xi,r)\cap D} g(z)^{p(z)} \delta_D(z)^\alpha \, d\mu(z) > 0\right\}.$$

If $\xi \in \partial D \setminus E_\beta$ and there exists a rectifiable curve γ in $T_\beta(\xi; c)$ tending to ξ along which u has a finite limit L, then u has a tangential limit L of order β at ξ.

Next we give the following result concerning the Lindelöf-type theorem, as an extension of [4], [5], [11] and [14] in the constant exponent case and the authors [7] in the variable exponent case. See [1, Theorem 1.1] for Orlicz spaces.

Theorem 1.2. *Let u be a function on a uniform domain D with $g \geq 0$ satisfying* (1.4) *and* (1.5). *Suppose $p_- > s + \alpha - 1$. If $\xi \in \partial D \setminus E_1$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.*

Theorems 1.1 and 1.2 are proved in the same way as Remark 3.1 and Theorem 1.1 in [1]. The key lemmas for our results are Lemmas 2.3 and 2.7 below.

Remark 1.3. Let $\beta \geq 1$. Let $h_\beta(r; x) = r^{\beta(-p(x)+s+\alpha)-s} \mu(B(x, r))$ for $x \in \partial D$ and $0 < r < \tilde{r}$, where $\tilde{r} > 0$. Assume that $h_\beta(\cdot; x)$ is non-decreasing on $(0, \tilde{r})$ for each $x \in \partial D$. For $E \subset X$ and $0 < r_0 < \tilde{r}$, let

$$H_{h_\beta}^{(r_0)}(E) = \inf\left\{\sum_j h_\beta(r_j; x_j); E \subset \bigcup_j B(x_j, r_j), 0 < r_j \leq r_0\right\}.$$

Since $H_{h_\beta}^{(r_0)}(E)$ increases as r_0 decreases, we define the generalized Hausdorff measure with respect to h_β by

$$H_{h_\beta}(E) = \lim_{r_0 \to +0} H_{h_\beta}^{(r_0)}(E).$$

Clearly, $H_{h_\beta}^{(r_0)}(E)$ and $H_{h_\beta}(E)$ are measures on X.

If g satisfies (1.5) and $p_- > s(1-1/\beta)+\alpha$, then $H_{h_\beta}(E_\beta) = 0$. In particular, if g satisfies (1.5) and $p_- > \alpha$, then $H_{h_1}(E_1) = 0$.

Corollary 1.4. *Let u be a monotone Sobolev function on a uniform domain D in $\mathbf{R}^n$ satisfying*

$$\int_D |\nabla u(z)|^{p(z)} \delta_D(z)^\alpha dz < \infty. \tag{1.7}$$

Suppose $n-1 < p_- \le p_+ < n+\alpha$. Set

$$E'_\beta = \left\{ \xi \in \partial D : \limsup_{r \to 0} r^{\beta(p(\xi)-n-\alpha)} \int_{B(\xi,r)\cap D} |\nabla u(z)|^{p(z)} \delta_D(z)^\alpha \ dz > 0 \right\}.$$

If $\xi \in \partial D \setminus E'_\beta$ and there exists a rectifiable curve γ in $T_\beta(\xi;c)$ tending to ξ along which u has a finite limit L, then u has a tangential limit L of order β at ξ.

Corollary 1.5. *Let u be a monotone Sobolev function on a uniform domain D in $\mathbf{R}^n$ satisfying* (1.7). *Suppose $p_- > \max\{n-1, n+\alpha-1\}$. Set*

$$E' = \left\{ \xi \in \partial D : \limsup_{r \to 0} r^{p(\xi)-\alpha-n} \int_{B(\xi,r)\cap D} |\nabla u(z)|^{p(z)} \delta_D(z)^\alpha \ dz > 0 \right\}.$$

If $\xi \in \partial D \setminus E'$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

2 Preliminary lemmas

Throughout this paper, let C denote various constants independent of the variables in question.

Let us begin with the following result borrowed from [7, Lemma 3].

Lemma 2.1. *Let $\{p_j\}$ be a sequence such that $p_* = \inf p_j > 1$ and $p^* = \sup p_j < \infty$. Then*

$$\sum |a_j b_j| \leqq 2 \left(\sum |a_j|^{p_j} \right)^{1/q} \left(\sum |b_j|^{p'_j} \right)^{1/q'},$$

where $1/p_j + 1/p'_j = 1$, $q = p_*$ if $\sum |a_j|^{p_j} \ge \sum |b_j|^{p'_j}$ and $q = p^*$ if $\sum |a_j|^{p_j} \le \sum |b_j|^{p'_j}$.

Lemma 2.2. (cf. [4, Lemma 1]) *Let D be a uniform domain. Then for each $\xi \in \partial D$ there exists a rectifiable curve γ_ξ in D ending at ξ such that*

$$\delta_D(z) \ge A_3 \ell(\gamma_\xi(\xi, z)) \tag{2.1}$$

for all $z \in \gamma_\xi$, where A_3 is a constant depending only on A_1 and A_2.

Fix $\xi \in \partial D$. For $x \in D$ such that x is close to ξ, set

$$r(x) = |\xi - x|.$$

Now, we give the following estimate of

$$F_u(x,y) = \min\{|u(x)-u(y)|^{p_-}, |u(x)-u(y)|^{p_+}\},$$

whenever x and y can be joined by a rectifiable curve γ in D such that

$$\delta_D(z) \geq A_0 \ell(\gamma(x,z)) \quad \text{and} \quad \sigma B(z) \subset B(\xi, c_0 r(x)) \tag{2.2}$$

for all $z \in \gamma$, where A_0 and c_0 are positive constants and $B(z) = B(z, \delta_D(z)/(2\sigma))$.

LEMMA 2.3. (cf. [1, Lemma 2.2]) *Let $\lambda \in \mathbf{R}$. Let u be a function on D with $g \geq 0$ satisfying* (1.4) *and* (1.5). *Suppose x and y can be joined by a rectifiable curve γ in D satisfying* (2.2) *and $r(x) < 1$.*

(1) *If $p_+ < s - \lambda$, then for each $x \in T_\beta(\xi; c)$*

$$\begin{aligned} F_u(x,y) &\leq Cr(x)^{\beta(p(\xi)-s+\lambda)+s}\mu(B(\xi,r(x)))^{-1}\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z) \\ &\quad + Cr(x)^{p_-}. \end{aligned}$$

(2) *If $p_- > s - \lambda$, then for each $x \in D$*

$$\begin{aligned} F_u(x,y) &\leq Cr(x)^{p(\xi)+\lambda}\mu(B(\xi,r(x)))^{-1}\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z) \\ &\quad + Cr(x)^{p_-}. \end{aligned}$$

Proof. We can take a finite chain of balls B_0, B_1, …, B_N such that

(i) $B_j = B(x_j)$, $x_j \in \gamma$, $x_0 = x$ and $y \in B_N$;

(ii) $\ell(\gamma(x_j, x_{j+1})) \geq \delta_D(x_j)/(2\sigma)$ and $\ell(\gamma(x, x_{j+1})) > \ell(\gamma(x, x_j))$;

(iii) $B_j \cap B_k \neq \emptyset$ if and only if $|j-k| \leq 1$.

See [6, Lemma 2.2]. By (ii) and (2.2), we have

$$\delta_D(x_j) \geq A_0\ell(\gamma(x,x_j)) \geq A_0\ell(\gamma(x,x_1)) \geq \frac{A_0}{2\sigma}\delta_D(x)$$

for $1 \leq j \leq N$ and

$$\delta_D(x_j) \leq |x_j - \xi| \leq c_0 r(x)$$

for $0 \leq j \leq N$, so that

(iv) $c_1\delta_D(x) \leq \delta_D(x_j) \leq c_0 r(x)$, where c_1 is a positive constant depending only on A_0 and σ.

Take a subsequence $\{x_{j_k}\}_{k=0}^{n}$ of $\{x_j\}_{j=0}^{N}$ such that $t < \delta_D(x_{j_k}) \le 2t$ for $t > 0$. Then we have by (ii)

$$\frac{1}{2\sigma}t \le \frac{1}{2\sigma}\delta_D(x_{j_k}) \le \ell(\gamma(x_{j_k}, x_{j_{k+1}})).$$

Since

$$\frac{1}{2\sigma}t(n-1) \le \ell(\gamma(x_{j_1}, x_{j_n})) \le \ell(\gamma(x, x_{j_n})) \le \frac{1}{A_0}\delta_D(x_{j_n}) \le \frac{2t}{A_0}$$

by (2.2), we have

(v) For each $t > 0$, the number of x_j such that $t < \delta_D(x_j) \le 2t$ is less than c_2, where c_2 is a positive constant depending only on A_0 and σ.

As in the proof of [6, Lemma 2.1], we see from (iii) that

(vi) $\sum_{j=0}^{N} \chi_{B_j}(z) \le c_3$, where χ_E denotes the characteristic function of E and c_3 is a positive constant depending only on the doubling constant of μ and σ.

Consider the function $p_*(x_j) = \inf_{z \in \sigma B_j} p(z)$. Since $p_*(x_j) \ge p$, we see that

$$|u(\zeta_1) - u(\zeta_2)| \le C\delta_D(x_j)\left(\frac{1}{\mu(\sigma B_j)}\int_{\sigma B_j} g(z)^{p_*(x_j)} d\mu(z)\right)^{1/p_*(x_j)}$$

for every $\zeta_1, \zeta_2 \in B_j$. Set $G_j = \{z \in \sigma B_j : g(z) \ge 1\}$. Then

$$\begin{aligned}\int_{\sigma B_j} g(z)^{p_*(x_j)} d\mu(z) &= \int_{G_j} g(z)^{p(z)} g(z)^{p_*(x_j)-p(z)} d\mu(z) + \int_{\sigma B_j \setminus G_j} g(z)^{p_*(x_j)} d\mu(z)\\ &\le \int_{\sigma B_j} g(z)^{p(z)} dz + \mu(\sigma B_j),\end{aligned}$$

so that we obtain by (1.5)

$$\begin{aligned}&|u(\zeta_1) - u(\zeta_2)|\\ \le\ & C\delta_D(x_j)\mu(\sigma B_j)^{-1/p_*(x_j)}\left(\int_{\sigma B_j} g(z)^{p(z)} d\mu(z)\right)^{1/p_*(x_j)} + C\delta_D(x_j)\\ \le\ & C\delta_D(x_j)^{1-\alpha/p_*(x_j)}\mu(\sigma B_j)^{-1/p_*(x_j)}\left(\int_{\sigma B_j} g(z)^{p(z)}\delta_D(z)^{\alpha} d\mu(z)\right)^{1/p_*(x_j)} + C\delta_D(x_j)\\ \le\ & C\delta_D(x_j)^{1-\alpha/p_*(x_j)}\mu(\sigma B_j)^{-1/p_*(x_j)}\left(\int_{\sigma B_j} g(z)^{p(z)}\delta_D(z)^{\alpha} d\mu(z)\right)^{1/p(x_j)} + C\delta_D(x_j)\end{aligned}$$

since $\delta_D(x_j)/2 \le \delta_D(z) \le 3\delta_D(x_j)/2$ for $z \in \sigma B_j$. Here note from (1.6) that

$$\begin{aligned}\mu(\sigma B_j)^{-1/p_*(x_j)} &= \mu(\sigma B_j)^{-1/p(x_j)}\mu(\sigma B_j)^{-(p(x_j)-p_*(x_j))/(p(x_j)p_*(x_j))}\\ &\le \mu(\sigma B_j)^{-1/p(x_j)}\left\{C\mu(B(\xi, c_0))\left(\frac{\delta_D(x_j)}{2c_0}\right)\right\}^{-s(p(x_j)-p_*(x_j))/(p(x_j)p_*(x_j))}\\ &\le C\mu(\sigma B_j)^{-1/p(x_j)}\delta_D(x_j)^{-C/\log(1/(e+\delta_D(x_j)))}\\ &\le C\mu(\sigma B_j)^{-1/p(x_j)}\end{aligned}$$

since $\delta_D(x_j) \le 2c_0$ by $\sigma B_j \subset B(\xi, c_0 r(x)) \subset B(\xi, c_0)$. Similarly, we have

$$C^{-1}\delta_D(x_j)^{1/p_*(x_j)} \le \delta_D(x_j)^{1/p(x_j)} \le C\delta_D(x_j)^{1/p_*(x_j)}.$$

Therefore, for $\lambda \in \mathbf{R}$, we find

$$\begin{aligned}
& |u(\zeta_1) - u(\zeta_2)| \\
\le\ & C\delta_D(x_j)^{1-\alpha/p(x_j)} \left(⨍_{\sigma B_j} g(z)^{p(z)} \delta_D(z)^{\alpha} d\mu(z) \right)^{1/p(x_j)} + C\delta_D(x_j) \\
\le\ & C\delta_D(x_j)^{1+\lambda/p(x_j)} \left(⨍_{\sigma B_j} g(z)^{p(z)} \delta_D(z)^{-\lambda} d\mu(z) \right)^{1/p(x_j)} + C\delta_D(x_j)
\end{aligned}$$

since $\delta_D(x_j)/2 \le \delta_D(z) \le 3\delta_D(x_j)/2$ for $z \in \sigma B_j$.

Set $p_j = p(x_j)$ and pick $z_j \in B_{j-1} \cap B_j$ for $1 \le j \le N$; set $z_0 = x$ and $z_{N+1} = y$. By the above inequality, we see that

$$\begin{aligned}
& |u(x) - u(y)| \\
\le\ & \sum_{j=0}^{N} |u(z_{j+1}) - u(z_j)| \\
\le\ & C \sum_{j=0}^{N} \delta_D(x_j)^{1+\lambda/p_j} \mu(\sigma B_j)^{-1/p_j} \left(\int_{\sigma B_j} g(z)^{p(z)} \delta_D(z)^{-\lambda} d\mu(z) \right)^{1/p_j} \\
& + C \sum_{j=0}^{N} \delta_D(x_j). \qquad (2.3)
\end{aligned}$$

Taking integers k_0 and k_1 such that $2^{-k_0-1} \le c_0 r(x) < 2^{-k_0}$ and $2^{-k_1-1} \le c_1\delta_D(x) < 2^{-k_1}$, we see from (iv) and (v) that

$$\begin{aligned}
\sum_{j=0}^{N} \delta_D(x_j) &\le \sum_{k=k_0}^{k_1} \left(\sum_{2^{-k-1} \le \delta_D(x_j) < 2^{-k}} \delta_D(x_j) \right) \\
&\le c_2 \sum_{k=k_0}^{k_1} 2^{-k} \le 2c_2 \int_{2^{-k_1-1}}^{2^{-k_0}} dt \le C \int_{c_1\delta_D(x)/2}^{2c_0 r(x)} dt \le C r(x).
\end{aligned}$$

Hence we have by Lemma 2.1

$$\begin{aligned}
& |u(x) - u(y)| \\
\le\ & C \left(\sum_{j=0}^{N} \delta_D(x_j)^{p_j'(1+\lambda/p_j)} \mu(\sigma B_j)^{-p_j'/p_j} \right)^{1/q'} \left(\sum_{j=0}^{N} \int_{\sigma B_j} g(z)^{p(z)} \delta_D(z)^{-\lambda} d\mu(z) \right)^{1/q} + C r(x) \\
\le\ & C \left(I^{q-1} \int_{\cup \sigma B_j} g(z)^{p(z)} \delta_D(z)^{-\lambda} d\mu(z) \right)^{1/q} + C r(x),
\end{aligned}$$

where q is a number in $\{\min p_j, \max p_j\}$ and

$$I = \sum_{j=0}^{N} \delta_D(x_j)^{p_j'(1+\lambda/p_j)} \mu(\sigma B_j)^{-p_j'/p_j}.$$

Since

$$\delta_D(x_j) \geq A_0 \ell(\gamma(x, x_j)) \geq A_0 |x - x_j|$$

by (2.2), we have

$$\begin{aligned} \left| \frac{p_j + \lambda}{p_j - 1} - \frac{p(x) + \lambda}{p(x) - 1} \right| &= \left| \frac{(\lambda + 1)(p(x) - p_j)}{(p(x) - 1)(p_j - 1)} \right| \\ &\leq C|p(x) - p_j| \leq \frac{C}{\log(1/|x - x_j|)} \leq \frac{C}{\log(1/\delta_D(x_j))} \end{aligned}$$

and

$$\left| \frac{p_j'}{p_j} - \frac{p(x)'}{p(x)} \right| = \left| \frac{p(x) - p_j}{(p(x) - 1)(p_j - 1)} \right| \leq \frac{C}{\log(1/|x - x_j|)} \leq \frac{C}{\log(1/\delta_D(x_j))}.$$

Therefore we obtain by (1.6)

$$\begin{aligned} I &\leq C \sum_{j=0}^{N} \delta_D(x_j)^{(p(x)+\lambda)/(p(x)-1)} \mu(\sigma B_j)^{-p'(x)/p(x)} \\ &\leq C \sum_{j=0}^{N} \delta_D(x_j)^{(p(x)+\lambda)/(p(x)-1)} \mu(B(\xi, r(x)))^{-p'(x)/p(x)} r(x)^{sp'(x)/p(x)} \delta_D(x_j)^{-sp'(x)/p(x)} \\ &= C \mu(B(\xi, r(x)))^{-p'(x)/p(x)} r(x)^{sp'(x)/p(x)} \sum_{j=0}^{N} \delta_D(x_j)^{(p(x)+\lambda-s)/(p(x)-1)} \\ &\leq C \left(\mu(B(\xi, r(x)))^{-1} r(x)^s \right)^{\frac{1}{p(x)-1}} \int_{c_1 \delta_D(x)/2}^{2c_0 r(x)} t^{\frac{p(x)-s+\lambda}{p(x)-1}} \frac{dt}{t}, \end{aligned}$$

where $1/p(x) + 1/p'(x) = 1$. First consider the case $p_+ < s - \lambda$ and $x \in T_\beta(\xi; c)$. Since $r(x)^\beta \leq c\delta_D(x)$ and $|x - x_j| \leq (1 + c_0)r(x)$, we see that

$$\begin{aligned} & \left| \frac{(p(x) - s + \lambda)(q - 1)}{p(x) - 1} - (p(\xi) - s + \lambda) \right| \\ = & \left| \frac{(p(x) - s + \lambda)(q - p(x))}{p(x) - 1} + (p(x) - p(\xi)) \right| \\ \leq & \; C|q - p(x)| + |p(x) - p(\xi)| \\ \leq & \; \frac{C}{\log(1/r(x))} \leq \frac{C}{\log(1/\delta_D(x))} \end{aligned}$$

and

$$\left| \frac{q - 1}{p(x) - 1} - 1 \right| \leq C|q - p(x)| \leq \frac{C}{\log(1/r(x))} \leq \frac{C}{\log(1/\delta_D(x))}.$$

Then we have

$$
\begin{aligned}
I^{q-1} &\le C\left(\mu(B(\xi,r(x)))^{-1}r(x)^s\right)^{\frac{q-1}{p(x)-1}}\delta_D(x)^{(p(x)-s+\lambda)(q-1)/(p(x)-1)} \\
&\le C\mu(B(\xi,r(x)))^{-1}r(x)^s\delta_D(x)^{p(\xi)-s+\lambda}
\end{aligned}
$$

since

$$
\left(\frac{\mu(B(\xi,r(x)))}{\mu(B(\xi,1))}\right)^{-C|q-p(x)|} \le Cr(x)^{-C|q-p(x)|} \le C
$$

by (1.6). Hence we obtain by (vi)

$$
\begin{aligned}
F_u(x,y) &\le |u(x)-u(y)|^q \\
&\le C\mu(B(\xi,r(x)))^{-1}r(x)^s\delta_D(x)^{p(\xi)-s+\lambda}\int_{\cup\sigma B_j} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z) \\
&\quad +Cr(x)^q \\
&\le C\mu(B(\xi,r(x)))^{-1}r(x)^{\beta(p(\xi)-s+\lambda)+s}\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z) \\
&\quad +Cr(x)^{p_-}.
\end{aligned}
$$

Next consider the case $p_- > s-\lambda$. Noting that

$$
\left|\frac{(p(x)-s+\lambda)(q-1)}{p(x)-1} - (p(\xi)-s+\lambda)\right| \le \frac{C}{\log(1/r(x))},
$$

we have

$$
\begin{aligned}
I^{q-1} &\le C\left((B(\xi,r(x)))^{-1}r(x)^s\right)^{\frac{q-1}{p(x)-1}} r(x)^{(p(x)-s+\lambda)(q-1)/(p(x)-1)} \\
&\le C(\mu(B(\xi,r(x)))^{-1}r(x)^{p(\xi)+\lambda}.
\end{aligned}
$$

Thus we can show the second part, in the same manner as the first part. □

REMARK 2.4. Let γ_1 be a rectifiable curve in D joining x and w satisfying (2.2), and let γ_2 be a rectifiable curve in D joining y and w satisfying (2.2). Suppose $r(x)=r(y)<1$.

(1) If $p_+ < s-\lambda$, then for each $x,y \in T_\beta(\xi;c)$

$$
\begin{aligned}
F_u(x,y) &\le Cr(x)^{\beta(p(\xi)-s+\lambda)+s}\mu(B(\xi,r(x)))^{-1}\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z) \\
&\quad +Cr(x)^{p_-}.
\end{aligned}
$$

(2) If $p_- > s-\lambda$, then for each $x,y \in D$

$$
\begin{aligned}
F_u(x,y) &\le Cr(x)^{p(\xi)+\lambda}\mu(B(\xi,r(x)))^{-1}\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z) \\
&\quad +Cr(x)^{p_-}.
\end{aligned}
$$

Remark 2.5. In Lemma 2.3, we can replace

$$\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z)$$

by

$$\int_{B(\xi,c_0r(x))\cap D} g(z)^{p(z)}\delta_D(z)^{\alpha}|r(x)-|z-\xi||^{-\lambda-\alpha}d\mu(z)$$

if $\alpha+\lambda>0$.

In fact, in the proof of Lemma 2.3, we can replace

$$\int_{\sigma B_j} g(z)^{p(z)}\delta_D(z)^{-\lambda}d\mu(z)$$

by

$$\int_{\sigma B_j} g(z)^{p(z)}\delta_D(z)^{\alpha}|r(x)-|z-\xi||^{-\lambda-\alpha}d\mu(z),$$

since

$$|r(x)-|z-\xi||\le|x-z|\le|x-x_j|+|x_j-z|\le\ell(\gamma(x,x_j))+\frac{\delta_D(x_j)}{2}\le\left(A_0+\frac{1}{2}\right)\delta_D(x_j)$$

and $\delta_D(x_j)\le 2\delta_D(z)$ for $z\in\sigma B_j$.

Remark 2.6. The number of balls B_0, B_1, ..., B_N in Lemma 2.3 is less than

$$\frac{c_2}{\log 2}\log\left(\frac{4cr(x)}{c_1\delta_D(x)}\right).$$

In fact,

$$\begin{aligned} N+1 &= \sum_{k=k_0}^{k_1}\#\{j:2^{-k-1}\le\delta_D(x_j)<2^{-k}\} \\ &\le \sum_{k=k_0}^{k_1}c_2=\frac{c_2}{\log 2}\int_{2^{-k_1-1}}^{2^{-k_0}}\frac{dt}{t}\le\frac{c_2}{\log 2}\int_{c_1\delta_D(x)/2}^{2cr(x)}\frac{dt}{t}=\frac{c_2}{\log 2}\log\left(\frac{4cr(x)}{c_1\delta_D(x)}\right), \end{aligned}$$

where we take k_0 and k_1 as in the proof of Lemma 2.3.

The following lemma can be proved using inequality (2.3) in the proof of Lemma 2.3.

Lemma 2.7. (cf. [1, Lemma 2.5]) *Let u be a function on a uniform domain D with $g\ge 0$ satisfying* (1.4) *and* (1.5). *If $\xi\in\partial D\setminus E_1$ and there exist a rectifiable curve γ_ξ in D ending at ξ satisfying* (2.1) *and a sequence $\{y_j\}$ such that $y_j\in\gamma_\xi$ and $2^{-j-1}\le|\xi-y_j|<2^{-j}$ and $u(y_j)$ has a finite limit L, then u has a nontangential limit L at ξ.*

Proof. Fix $\xi\in\partial D\setminus E_1$. Take $x_j\in T_1(\xi;c)$ with $2^{-j-1}\le|x_j-\xi|<2^{-j}$. Let γ be a rectifiable curve in D joining x_j and y_j satisfying (1.2) and (1.3). Take $y\in\gamma$ such that

$\ell(\gamma(x_j,y)) = \ell(\gamma(y_j,y))$, and set $\gamma_1 = \gamma(x_j,y)$ and $\gamma_2 = \gamma(y_j,y)$. Then γ_i satisfies (2.2) with $A_0 = A_2$ and $c_0 = 3(3A_1+1)/2$. In fact, we have by (1.3)

$$\delta_D(z) \geq A_2 \min\{\ell(\gamma(x_j,z)), \ell(\gamma(z,y_j))\} = A_2 \ell(\gamma_1(x_j,z))$$

for $z \in \gamma_1$. Take $w \in \sigma B(z)$ for $z \in \gamma_1$. Then note that

$$|w-\xi| \leq |w-z| + |z-\xi| \leq \frac{3}{2}|z-\xi| \leq \frac{3}{2}(r(x_j)+\ell(\gamma)) \leq \frac{3(3A_1+1)}{2} r(x_j)$$

since we have by (1.2)

$$\ell(\gamma) \leq A_1|x_j-y_j| \leq 3A_1 r(x_j).$$

Similarly, we have

$$\delta_D(z) \geq A_2 \ell(\gamma_2(y_j,z))$$

and $\sigma B(z) \subset B(\xi, c_o r(y_j))$ for $z \in \gamma_2$.

Then, for γ_i, we can take a finite chain of balls B^i_0, B^i_1, …, $B^i_{N_i}$ with $B^i_k = B(w^i_k)$ as in the proof of Lemma 2.3. By Remark 2.6, we note that N_i is less than a positive constant C_1, since

$$\frac{r(x_j)}{\delta_D(x_j)} \leq \frac{cr(x_j)}{|x_j-\xi|} = c$$

and

$$\frac{r(y_j)}{\delta_D(y_j)} \leq \frac{r(y_j)}{A_3|\xi-y_j|} = \frac{1}{A_3}$$

by (2.1). Further we note from the fact that $x_j \in T_1(\xi;c)$, (iv), (2.1) and (2.2) that

$$2^{-j-1} \leq |x_j-\xi| \leq c\delta_D(x_j) \leq \frac{c}{c_1}\delta_D(w^1_k) \leq \frac{c}{c_1}|w^1_k-\xi| \leq \frac{cc_0}{c_1}r(x_j) \leq \frac{cc_0}{c_1}2^{-j},$$

$$2^{-j-1} \leq |y_j-\xi| \leq \frac{1}{A_3}\delta_D(y_j) \leq \frac{1}{c_1A_3}\delta_D(w^2_k) \leq \frac{1}{c_1A_3}|w^2_k-\xi| \leq \frac{c_0}{c_1A_3}r(y_j) \leq \frac{c_0}{c_1A_3}2^{-j},$$

$$|w^1_k-\xi| \leq |w^1_k-x_j| + |x_j-\xi| \leq \frac{1}{A_0}\delta_D(w^1_k) + c\delta_D(x_j) \leq \left(\frac{1}{A_0}+\frac{c}{c_1}\right)\delta_D(w^1_k)$$

and

$$|w^2_k-\xi| \leq |w^2_k-y_j| + |y_j-\xi| \leq \frac{1}{A_0}\delta_D(w^2_k) + \frac{1}{A_3}\delta_D(y_j) \leq \left(\frac{1}{A_0}+\frac{1}{c_1A_3}\right)\delta_D(w^2_k),$$

where c_0, c_1 are positive constants appearing in the proof of Lemma 2.3. Therefore

$$C^{-1}2^{-j} \leq \delta_D(w^i_k) \leq C2^{-j}$$

and

$$C^{-1}|w^i_k-\xi| \leq \delta_D(w^i_k) \leq |w^i_k-\xi|.$$

Here we see from (1.6) that

$$\frac{\mu(\sigma B^1_k)}{\mu(B(\xi, c_0 r(x_j)))} \geq C\left(\frac{\delta_D(w^1_k)}{2c_0 r(x_j)}\right)^s \geq C$$

and

$$\frac{\mu(\sigma B_k^2)}{\mu(B(\xi, c_0 r(y_j)))} \geq C \left(\frac{\delta_D(w_k^2)}{2c_0 r(y_j)} \right)^s \geq C.$$

Hence, we obtain by (2.3) and (vi) in the proof of Lemma 2.3

$$\begin{aligned}
& |u(x_j) - u(y_j)| \\
\leq\ & |u(x_j) - u(y)| + |u(y_j) - u(y)| \\
\leq\ & C \sum_{i=1}^{2} \sum_{k=0}^{N_i} \delta_D(w_k^i)^{1-\alpha/p(w_k^i)} \left(\fint_{\sigma B_k^i} g(z)^{p(z)} \delta_D(z)^\alpha d\mu(z) \right)^{1/p(w_k^i)} + C \sum_{i=1}^{2} \sum_{k=0}^{N_i} \delta_D(w_k^i) \\
\leq\ & C \sum_{i=1}^{2} \sum_{k=0}^{N_i} \left(\delta_D(w_k^i)^{p(\xi)-\alpha} \mu(\sigma B_k^i)^{-1} \int_{\sigma B_k^i} g(z)^{p(z)} \delta_D(z)^\alpha d\mu(z) \right)^{1/p(w_k^i)} \\
& + C \sum_{i=1}^{2} \sum_{k=0}^{N_i} \delta_D(w_k^i) \\
\leq\ & C 2^{-j} + C \left(2^{-j(p(\xi)-\alpha)} \mu(B(\xi, 2^{-j}))^{-1} \int_{B(\xi, c_0 2^{-j})} g(z)^{p(z)} \delta_D(z)^\alpha d\mu(z) \right)^{1/p_+}
\end{aligned}$$

since we have by (p2)

$$\delta_D(w_k^i)^{-|p(\xi)-p(w_k^i)|} \leq \delta_D(w_k^i)^{-C/\log(e+1/|w_k^i-\xi|)} \leq C.$$

Since $\xi \in \partial \mathbf{B} \setminus E_1$ and $\lim_{j\to\infty} u(y_j) = L$, u has a nontangential limit L at ξ. □

3 Proof of Theorem 1.1

We may assume that for each $x \in T_\beta(\xi; c)$, there exists a point $y(x) \in \gamma$ such that $r(x) = r(y(x)) < 1$. As in the proof of Lemma 2.7, let γ_0 be a rectifiable curve in D joining x and $y(x)$ satisfying (1.2) and (1.3). Take $w \in \gamma_0$ such that $\ell(\gamma_0(x, w)) = \ell(\gamma_0(y(x), w))$, and set $\gamma_1 = \gamma_0(x, w)$ and $\gamma_2 = \gamma_0(y(x), w)$. Since $\xi \notin E_\beta$, we have by Lemma 2.3(1) with $\lambda = -\alpha$ and Remark 2.4

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} F_u(x, y(x)) = 0,$$

so that

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} |u(x) - u(y(x))| = 0.$$

Since $\lim_{x\to\xi} u(y(x)) = L$ by our assumption,

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} u(x) = L,$$

as required. □

4 Proof of Theorem 1.2

Take $\lambda \in \mathbf{R}$ such that $s + \alpha - p_- < \lambda + \alpha < 1$. Let γ_ξ be as in Lemma 2.2. For $r > 0$ sufficiently small, take $x(r) \in \gamma \cap \partial B(\xi, r)$ and $y(r) \in \gamma_\xi \cap \partial B(\xi, r)$. As in the proof of Lemma 2.7, let γ_0 be a rectifiable curve in D joining $x(r)$ and $y(r)$ satisfying (1.2) and (1.3). Take $w \in \gamma_0$ such that $\ell(\gamma_0(x(r), w)) = \ell(\gamma_0(y(r), w))$, and set $\gamma_1 = \gamma_0(x(r), w)$ and $\gamma_2 = \gamma_0(y(r), w)$. By Lemma 2.3(2), Remark 2.4 and Remark 2.5, we have

$$\begin{aligned} F_u(x(r), y(r)) \leq\ & C r^{p(\xi)+\lambda} \mu(B(\xi, r))^{-1} \int_{B(\xi, c_0 r) \cap D} g(z)^{p(z)} \delta_D(z)^\alpha |r - |z - \xi||^{-\lambda-\alpha} d\mu(z) \\ & + C r^{p_-}. \end{aligned}$$

Moreover, since $0 < \lambda + \alpha < 1$, we see that

$$\int_{2^{-j-1}}^{2^{-j}} |r - |z - \xi||^{-\lambda-\alpha}\, dr \leq C 2^{-j(1-\lambda-\alpha)}.$$

Hence it follows that

$$\begin{aligned} & \inf_{2^{-j-1} \leq r < 2^{-j}} F_u(x(r), y(r)) \\ \leq\ & C \int_{2^{-j-1}}^{2^{-j}} \left(r^{p(\xi)+\lambda} \mu(B(\xi, r))^{-1} \int_{B(\xi, c_0 r) \cap D} g(z)^{p(z)} \delta_D(z)^\alpha \, |r - |z - \xi||^{-\lambda-\alpha}\, d\mu(z) \right) \frac{dr}{r} \\ & + C (2^{-j})^{p_-} \\ \leq\ & C 2^{-j\{p(\xi)+\lambda-1\}} \mu(B(\xi, 2^{-j}))^{-1} \int_{B(\xi, c_0 2^{-j}) \cap D} g(z)^{p(z)} \delta_D(z)^\alpha \left(\int_{2^{-j-1}}^{2^{-j}} |r - |z - \xi||^{-\lambda-\alpha}\, dr \right) d\mu(z) \\ & + C (2^{-j})^{p_-} \\ \leq\ & C (2^{-j\{-p(\xi)+\alpha\}} \mu(B(\xi, 2^{-j}))^{-1} \int_{B(\xi, c_0 2^{-j}) \cap D} g(z)^{p(z)} \delta_D(z)^\alpha d\mu(z) + C (2^{-j})^{p_-}. \end{aligned}$$

Since $\xi \notin E_1$, we see that

$$\lim_{j \to \infty} \inf_{2^{-j-1} \leq r < 2^{-j}} F_u(x(r), y(r)) = 0.$$

Hence we find a sequence $\{r_j\}$ such that $2^{-j-1} \leq r_j < 2^{-j}$ and

$$\lim_{j \to \infty} F_u(x(r_j), y(r_j)) = 0.$$

Since u has a finite limit L at ξ along γ, we have

$$\lim_{j \to \infty} u(y(r_j)) = \lim_{j \to \infty} u(x(r_j)) = L.$$

Thus u has a nontangential limit L at ξ by Lemma 2.7. $\square$

Acknowledgements We would like to express our thanks to the referees for their kind comments and suggestions.

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