# INTEGRABILITY OF MAXIMAL FUNCTIONS FOR GENERALIZED LEBESGUE SPACES $L^{p(\cdot)}(\log L)^{q(\cdot)}$

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Abstract. We study the integrability of maximal functions for generalized Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, where the exponent $p(\cdot)$ approaches 1 on some part of the domain. Our integrability results depend on the shape of that part and the speed of the exponent approaching 1.

## 1. Introduction

A crucial tool in the development of the function space theory is the boundedness of Hardy-Littlewood maximal operator. For a locally integrable function $f$ on $\mathbf{R}^n$, the maximal function is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $|B(x,r)|$ denotes the volume of the open ball $B(x,r)$ centered at $x$ of radius $r > 0$.

In the classical Lebesgue $L^p$ spaces, we know (see the book by Stein [14, Chapter 1]) that if $p > 1$, then

$$\|Mf\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)}$$

for all $f \in L^p(\mathbf{R}^n)$. Unfortunately, this is not true when $p = 1$ even if we are restricted to a bounded domain $G$ in $\mathbf{R}^n$. Thus we need to consider the space $L\log L(G)$ of measurable functions $f$ on $G$ whose norm

$$\|f\|_{L\log L(G)} = \inf\left\{\lambda > 0 : \int_G \frac{|f(y)|}{\lambda} \log\left(e + \frac{|f(y)|}{\lambda}\right) dy \le 1\right\}$$

is finite. It is known from Stein [13] and [14, Sections I.1 and I.5] that $Mf \in L^1(G)$ if and only if $f \in L\log L(G)$.

Following Orlicz [11], Kováčik-Rákosník [6] and Musielak [10], we consider a positive continuous function $p(\cdot)$ on $G$ whose value is not less than 1, and the space of all measurable functions $f$ on $G$ satisfying

$$\int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty$$

for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}.$$

In recent years, the generalized Lebesgue spaces $L^{p(\cdot)}$ have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see Růžička [12].

In this paper we study the integrability of the maximal operator in the Lebesgue space of variable exponent approaching 1 in some part of $G$. Recently, Hästö [5] has obtained an interesting class of variable exponents approaching 1 in some part of $G$ such that $Mf \in L^1(G)$ for all

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$f \in L^{p(\cdot)}(G)$. As in Hästö [5], letting $\log_{(1)} t = \log t$ and $\log_{(m+1)} t = \log(\log_{(m)} t)$ for $m = 1, 2, ...$, we consider a function $\omega$ on the interval $[0, \infty)$ such that $\omega(0) = 0$ and

$$\omega(r) = \frac{a_1}{\log(1/r)} + \frac{a_2 \log_{(2)}(1/r)}{\log(1/r)} + \frac{a_3 \log_{(3)}(1/r)}{\log(1/r)}$$

for $0 < r \le r_1$, where the numbers $a_1, a_2, a_3$ and $0 < r_1 < 1/e$ are chosen so that $\omega(r)$ is nondecreasing on $[0, r_1]$; set $\omega(r) = \omega(r_1)$ for $r > r_1$.

THEOREM A (cf. Hästö [5]). *Let $x_0 \in G$ and consider a variable exponent $p(\cdot)$ such that*

$$p(x) = 1 + \omega(|x - x_0|) \tag{1.1}$$

*for $x \in G$. If $a_2 > 1/n$, then the maximal operator $\mathcal{M} : f \to Mf$ is bounded from $L^{p(\cdot)}(G)$ into $L^1(G)$, that is,*

$$\|Mf\|_{L^1(G)} \le C\|f\|_{L^{p(\cdot)}(G)}.$$

Futamura and Mizuta [4] have proved that the conclusion is still valid when $a_2 = 1/n$ and $a_3 \ge 0$.

In this paper, following Cruz-Uribe, Fiorenza and Neugebauer [2], we further consider a function $q(\cdot)$ such that $q(0) = q_0 \in \mathbf{R}$ and

$$q(x) = q_0 + \eta(|x - x_0|) \tag{1.2}$$

for $x \in G$, where $\eta(r) = \dfrac{b_1}{\log_{(2)}(1/r)} + \dfrac{b_2 \log_{(3)}(1/r)}{\log_{(2)}(1/r)}$ for $0 < r \le r_2$, and $\eta(r) = \eta(r_2)$ for $r > r_2$. Here the numbers $b_1$, $b_2$ and $0 < r_2 < 1/e$ are chosen so that $\eta(r)$ is nondecreasing on $(0, \infty)$.

Now we set

$$\Phi(x, t) = t^{p(x)}(\log(c_0 + t))^{q(x)},$$

where $c_0$ is chosen that $\Phi(\cdot, t)$ is a convex function of $t$. Define the norm

$$\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \Phi(x, |f(x)|/\lambda)\, dx \le 1\right\}$$

for a measurable function $f$ on $G$. We denote by $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ the family of all measurable functions $f$ on $G$ such that $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} < \infty$.

Our aim in this paper is to establish the following result, as an extension of the recent result by Cruz-Uribe, Fiorenza and Neugebauer [2].

THEOREM B. *Let $p(\cdot)$ and $q(\cdot)$ be of the form* (1.1) *and* (1.2).

(i) *If $na_2 + q_0 > 0$, then*

$$\int_G Mf(x)(\log(1 + Mf(x)))^{na_2+q_0-1}(\log(1 + (\log(1 + Mf(x)))))^{na_3+b_2}\, dx \le C;$$

(ii) *if $na_2 + q_0 = 0$ and $na_3 + b_2 > 0$, then*

$$\int_G Mf(x)(\log(1 + Mf(x)))^{-1}(\log(1 + (\log(1 + Mf(x)))))^{na_3+b_2-1}\, dx \le C$$

*for all measurable functions $f$ on $G$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$, where $C$ denotes a positive constant independent of $f$.*

This also gives extensions of Theorem A by Hästö [5], Futamura-Mizuta [4] and Mizuta-Ohno-Shimomura [9].

## 2. Variable exponents

Throughout this paper, let $C$ denote various positive constants independent of the variables in question. Further let $G$ denote a bounded open set in $\mathbf{R}^n$.

We say that a positive nondecreasing function $\varphi$ on the interval $[0,\infty)$ satisfies $(\varphi)$ if there exist $\varepsilon_1 > 0$ and $0 < r_1 < 1$ such that

$(\varphi)$ $(\log(1/r))^{-\varepsilon_1}\varphi(1/r)$ is nondecreasing on $(0, r_1)$.

Similarly, a positive nondecreasing function $\psi$ on the interval $[0,\infty)$ is said to satisfy $(\psi)$ if there exist $\varepsilon_2 > 0$ and $0 < r_2 < 1/e$ such that

$(\psi)$ $(\log_{(2)}(1/r))^{-\varepsilon_2}\psi(1/r)$ is nondecreasing on $(0, r_2)$.

Consider positive nondecreasing functions $\varphi$ and $\psi$ satisfying $(\varphi)$ and $(\psi)$, respectively. Set

$$\varepsilon_0 = \max\{\varepsilon_1, \varepsilon_2\}.$$

For the sake of convenience, we assume that

$(\varphi')$ $\varphi(t) \ge e^{\varepsilon_0}$ for all $t > 0$,
$(\psi')$ $\psi(t) \ge e^{\varepsilon_0}$ for all $t > 0$.

First we give the following results, which can be derived by conditions $(\varphi)$ and $(\varphi')$.

**Lemma 1.** ([8, Lemma 3.1, Section 5.3], [9, Lemmas 2.1 and 2.2]).

(i) *$\varphi(r)$ is of log-type, that is, there exists $C > 0$ such that*

$$C^{-1}\varphi(r) \le \varphi(r^2) \le C\varphi(r) \qquad \textit{whenever } r > 0. \tag{2.1}$$

(ii) *For $\gamma > 0$, there exists $C > 0$ such that*

$$t^{-\gamma}\varphi(t) \le Cs^{-\gamma}\varphi(s) \qquad \textit{whenever } t \ge s > 0.$$

(iii) *There exists $0 < \tilde{r}_1 < r_1$ such that $\omega(r) = \log\varphi(1/r)/\log(1/r)$ is nondecreasing on $[0, \tilde{r}_1]$.*

Further, we see from $(\psi)$ and $(\psi')$ that $\psi$ satisfies (i), (ii) and

(iv) there exists $0 < \tilde{r}_2 < r_2$ such that $\eta(r) = \log\psi(1/r)/\log(\log(1/r))$ is nondecreasing on $[0, \tilde{r}_2]$.

Condition (2.1) implies the doubling condition on $\varphi$, that is, there exists a constant $C > 1$ such that

$$\varphi(r) \le \varphi(2r) \le C\varphi(r) \qquad \text{whenever } r > 0.$$

Our typical example of $\varphi$ is of the form

$$\varphi(r) = a_1(\log r)^{a_2}(\log_{(2)} r)^{a_3}$$

for large $r$, where $a_1 > 0$, $a_2 \ge 0$ and $a_3 \in \mathbf{R}$; similarly, that of $\psi$ is of the form

$$\psi(r) = b_1(\log_{(2)} r)^{b_2}$$

for large $r$, where $b_1 > 0$ and $b_2 \ge 0$.

For simplicity, set

$$r_0 = \min\{\tilde{r}_1, \tilde{r}_2\}.$$

Consider

$$\omega(r) = \frac{\log\varphi(1/r)}{\log(1/r)}$$

for $0 < r \le r_0$; set $\omega(r) = \omega(r_0)$ for $r > r_0$. We further consider

$$\eta(r) = \frac{\log\psi(1/r)}{\log_{(2)}(1/r)}$$

for $0 < r \le r_0$; set $\eta(r) = \eta(r_0)$ for $r > r_0$.

For a compact set $K$ in $\mathbf{R}^n$, we define

$$K(r) = \{x \in \mathbf{R}^n : \delta_K(x) \le r\}$$

for $r \geq 0$, where $\delta_K(x)$ denotes the distance of $x$ from $K$. For $0 < \alpha \leq n$, we say that the $(n-\alpha)$-dimensional upper Minkowski content of $K$ is finite if

$$|K(r)| \leq Cr^{\alpha} \qquad \text{for small } r > 0;$$

see the book by Mattila [7]. Note here that if $K$ is a singleton, then its 0-dimensional upper Minkowski content is finite, and if $K$ is a spherical surface, then its $(n-1)$-dimensional upper Minkowski content is finite. As examples of $K$, we may consider fractal type sets like Cantor sets or Koch curves.

Now we define variable exponents $p(\cdot)$ and $q(\cdot)$ by

$$p(x) = 1 + \omega(\delta_K(x))$$

and

$$q(x) = q_0 + \eta(\delta_K(x))$$

for $x \in \mathbf{R}^n$ and $q_0 \in \mathbf{R}$.

We here remark the following easy result.

**Lemma 2.** (cf. [9, Lemma 2.2]). *$p(\cdot)$ and $q(\cdot)$ are continuous on $\mathbf{R}^n$.*

## 3. Integrability results

Let us begin with the following result.

**Lemma 3.** ([9, Lemma 2.3]). *Let $K$ be a compact set in $G$ whose $(n-\alpha)$-dimensional upper Minkowski content is finite. Then*

$$\int_G \delta_K(x)^{-\alpha}(\log(1+\delta_K(x)^{-1}))^{-\beta}\, dx < \infty$$

*for every $\beta > 1$.*

**Lemma 4.** (cf. [4, Lemma 2.3], [9, Lemma 2.4]). *Suppose the $(n-\alpha)$-dimensional upper Minkowski content of $K$ is finite. If $f$ is a measurable function on $G$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \leq 1$, then*

$$\int_G |f(x)|(\log(e+|f(y)|))^{q_0}\varphi(|f(x)|)^{\alpha}\psi(|f(x)|)\, dx \leq C.$$

*Proof.* Consider the set

$$G' = \{x \in K(r_0) \cap G : |f(x)| < \delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta}\},$$

where the constant $\beta$ is determined later and $\delta(x) = \delta_K(x)$ for simplicity. If $x \in G'$, then we have by $(\varphi)$ and $(\psi)$

$$\begin{aligned}
&|f(x)|(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\alpha}\psi(|f(x)|) \\
&\leq C\delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta}(\log(1/\delta(x)))^{q_0}\varphi(1/\delta(x))^{\alpha}\psi(1/\delta(x)) \\
&\leq C\delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta+q_0+\varepsilon_3},
\end{aligned}$$

where $\varepsilon_3 > \varepsilon_1\alpha$. If we take $\beta$ so large that $\beta > 1 + q_0 + \varepsilon_3$, then it follows from Lemma 3 that

$$\int_{G'} |f(x)|(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\alpha}\psi(|f(x)|)dx \leq C.$$

If $x \notin G'$ and $\delta(x) < r_0$, then $|f(x)| \geq \delta(x)^{-\alpha}(\log(1/\delta(x)))^{-\beta}$, so that

$$\delta(x) \geq C|f(x)|^{-1/\alpha}(\log|f(x)|)^{-\beta/\alpha}.$$

Hence, in view of Lemma 1, we see that

$$\begin{aligned}\frac{\log\varphi(1/\delta(x))}{\log(1/\delta(x))}\log|f(x)| &\geq \frac{\log\varphi(C|f(x)|^{1/\alpha}(\log|f(x)|)^{\beta/\alpha})}{\log(C|f(x)|^{1/\alpha}(\log|f(x)|)^{\beta/\alpha})}\log|f(x)| \\ &\geq \frac{\alpha\log(C\varphi(|f(x)|))}{\log|f(x)|+C\log(C\log|f(x)|)}\log|f(x)| \\ &= \alpha\log(C\varphi(|f(x)|))\left(1-\frac{C\log(C\log|f(x)|)}{\log|f(x)|+C\log(C\log|f(x)|)}\right) \\ &\geq \alpha\log\varphi(|f(x)|)-C,\end{aligned}$$

which yields

$$\begin{aligned}|f(x)|^{p(x)-1} &= \exp\left(\frac{\log\varphi(1/\delta(x))}{\log(1/\delta(x))}\log|f(x)|\right)\geq\exp\left(\alpha\log\varphi(|f(x)|)-C\right) \\ &= C\varphi(|f(x)|)^{\alpha}.\end{aligned}$$

Similarly, we have

$$\frac{\log\psi(1/\delta(x))}{\log_{(2)}(1/\delta(x))}\log_{(2)}|f(x)|\geq\log\psi(|f(x)|)-C,$$

which yields

$$(\log|f(x)|)^{q(x)-q_0}\geq C\psi(|f(x)|).$$

Thus it follows that

$$\begin{aligned}&\int_{K(r_0)\setminus G'}|f(x)|(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\alpha}\psi(|f(x)|)dx \\ &\leq C\int_G|f(x)|^{p(x)}(\log(c_0+|f(x)|))^{q(x)}\,dx\leq C.\end{aligned}$$

Finally, since $p(x)\geq p_1>1$ when $\delta(x)\geq r_0$, we find

$$\begin{aligned}&\int_{G\setminus K(r_0)}|f(x)|(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\alpha}\psi(|f(x)|)dx \\ &\leq C\int_G|f(x)|^{p(x)}(\log(c_0+|f(x)|))^{q(x)}\,dx+C\leq C.\end{aligned}$$

The required assertion is now proved. $\square$

For simplicity, set

$$\lambda(t)=t(\log(e+t))^{q_0}\varphi(t)^{\alpha}\psi(t)$$

and consider a continuous function $\Lambda$ such that

$$C^{-1}\frac{\lambda(t)}{t}\leq\int_1^t\frac{\Lambda(s)}{s^2}ds\leq C\frac{\lambda(t)}{t}\qquad\text{whenever } t>2.$$

The next lemma is an extension of Stein [14, Chapter 1], whose proof will be done along the same lines as in Stein [14, Chapter 1] (see [9]); for another proof, see Cianchi [1].

**Lemma 5.** ([9, Lemma 2.5]). *For a locally integrable function $f$ on $G$, the following are equivalent:*

(i) $\displaystyle\int_G\lambda(|f(x)|)\,dx\leq A_1$;

(ii) $\displaystyle\int_G\Lambda(Mf(x))\,dx\leq A_2$,

*where $A_1,A_2>1$ are constants (independent of $f$) such that $C^{-1}A_1\leq A_2\leq CA_1$.*

## 4. Proof of Theorem B

By Lemmas 4 and 5, we have the following result.

**Theorem 1.** *Let $\lambda$ and $\Lambda$ be as before Lemma 5. Suppose the $(n-\alpha)$-dimensional upper Minkowski content of $K$ is finite. If $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$, then*

$$\int_G \lambda(|f(x)|)\, dx \le C,$$

*or equivalently,*

$$\int_G \Lambda(Mf(x))\, dx \le C.$$

*Remark* 1. Let $\varphi(r) = A(\log r)^{a_2}(\log_{(2)} r)^{a_3}$ for large $r$, where $A, a_2$ and $a_3$ are chosen so that $\varphi$ is nondecreasing. Moreover, let $\psi(r) = B(\log_{(2)} r)^{b_2}$ for large $r$, where $B > 0$ and $b_2 \ge 0$. In this case, we can take

$$\lambda(r) = r(\log r)^{a_2\alpha+q_0}(\log_{(2)} r)^{a_3\alpha+b_2}$$

for large $r$.

(i) If $a_2\alpha + q_0 > 0$, then we can take

$$\Lambda(r) = r(\log r)^{a_2\alpha+q_0-1}(\log_{(2)} r)^{a_3\alpha+b_2}$$

for large $r$.

(ii) If $a_2\alpha + q_0 = 0$ and $a_3\alpha + b_2 > 0$, then we can take

$$\Lambda(r) = r(\log r)^{-1}(\log_{(2)} r)^{a_3\alpha+b_2-1}$$

for large $r$.

Thus Theorem 1 gives Theorem B in the Introduction and the following corollary.

**Corollary 1.** *For $\alpha > 0$, let $K$ be a compact subset of $G$ whose $(n-\alpha)$-dimensional upper Minkowski content is finite. Consider the exponents*

$$p(x) = 1 + \frac{a_1}{\log(1/\delta_K(x))} + \frac{a_2 \log_{(2)}(1/\delta_K(x))}{\log(1/\delta_K(x))}$$

*and*

$$q(x) = q_0 + \frac{b_1}{\log_{(2)}(1/\delta_K(x))} + \frac{b_2 \log_{(3)}(1/\delta_K(x))}{\log_{(2)}(1/\delta_K(x))}$$

*when $\delta(x) \le r_0$. If $a_2, b_2 > 0$ and $a_1, b_1, q_0 \in \mathbf{R}$ are chosen so that $a_2\alpha + q_0 > 0$, $\omega(r) = a_1/\log(1/r) + a_2(\log_{(2)}(1/r))/\log(1/r)$ and $\eta(r) = b_1/\log_{(2)}(1/r) + b_2(\log_{(3)}(1/r))/\log_{(2)}(1/r)$ are nondecreasing on $(0, r_0]$, then*

$$\int_G |f(x)|(\log(1+|f(x)|))^{a_2\alpha+q_0}(\log_{(2)}(1+|f(x)|))^{b_2}\, dx < \infty, \tag{4.1}$$

*or equivalently,*

$$\int_G Mf(x)(\log(1+Mf(x)))^{a_2\alpha+q_0-1}(\log_{(2)}(1+Mf(x)))^{b_2}\, dx < \infty$$

*for all $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$.*

## 5. Further remarks

In this section, we discuss the sharpness of Theorem 1 (Corollary 1).

If $K$ is a compact subset of $G$, then we write

$$\delta(x) = \delta_K(x)$$

for simplicity.

*Remark* 2. We show that the exponents in (4.1) are sharp under the condition that

$$C^{-1} r^{\alpha} \le |K(r)| \le C r^{\alpha} \qquad \text{for } 0 < r < r_0.$$

In fact, if we replace $q(x)$ by

$$\tilde{q}(x) = q_0 + \frac{b_1}{\log_{(2)}(1/\delta(x))} + \frac{b_2 \log_{(3)}(1/\delta(x))}{\log_{(2)}(1/\delta(x))} - \frac{b_3 \log_{(4)}(1/\delta(x))}{\log_{(2)}(1/\delta(x))}$$

with $b_3 > 0$, then we can find $f \in L^{p(\cdot)}(\log L)^{\tilde{q}(\cdot)}(G)$ such that

$$\int_G |f(x)|(\log(1+|f(x)|))^{a_2\alpha+q_0}(\log_{(2)}(1+|f(x)|))^{b_2}\, dx = \infty.$$

To show this, we consider the function

$$f(x) = \delta(x)^{-\alpha}(\log(1/\delta(x)))^{-a_2\alpha-q_0-1}(\log_{(2)}(1/\delta(x)))^{-b_2-1}(\log_{(3)}(1/\delta(x)))^{-1}$$

for $x \in G$ with $\delta(x) \le r_0$; set $f(x) = 0$ when $\delta(x) > r_0$. Then

$$\begin{aligned} &\int_G f(x)(\log(1+f(x)))^{a_2\alpha+q_0}(\log_{(2)}(1+f(x)))^{b_2}\, dx \\ \ge\ & C\int_0^{r_0} t^{-1}(\log(1/t))^{-1}(\log_{(2)}(1/t))^{-1}(\log_{(3)}(1/t))^{-1}dt = \infty. \end{aligned}$$

Further, since $f(x) \le \delta(x)^{-\alpha}$, we have for $t = \delta(x) \le r_0$

$$f(x)^{p(x)-1} \le \exp\left(\alpha \log(1/t)\frac{a_1 + a_2\log_{(2)}(1/t)}{\log(1/t)}\right) = C(\log(1/t))^{a_2\alpha}$$

and

$$\begin{aligned} (\log f(x))^{\tilde{q}(x)-q_0} &\le C\exp\left(\log_{(2)}(1/t)\frac{b_1 + b_2\log_{(3)}(1/t) - b_3\log_{(4)}(1/t)}{\log_{(2)}(1/t)}\right) \\ &= C(\log_{(2)}(1/t))^{b_2}(\log_{(3)}(1/t))^{-b_3}, \end{aligned}$$

so that

$$\begin{aligned} &\int_G f(x)^{p(x)}(\log(c_0+f(x)))^{\tilde{q}(x)}\, dx \\ &\le C\int_0^{r_0} t^{-1}(\log(1/t))^{-1}(\log_{(2)}(1/t))^{-1}(\log_{(3)}(1/t))^{-b_3-1}dt < \infty. \end{aligned}$$

*Remark* 3. Let

$$p(x) = 1 + \frac{a_1}{\log(1/|x|)} + \frac{a_2\log_{(2)}(1/|x|)}{\log(1/|x|)} + \frac{a_3\log_{(3)}(1/|x|)}{\log(1/|x|)}$$

and

$$q(x) = q_0 + \frac{b_1}{\log_{(2)}(1/|x|)} + \frac{b_2\log_{(3)}(1/|x|)}{\log_{(2)}(1/|x|)}$$

for $x \in B_0 = B(0, r_0)$ with $0 < r_0 < 1/e$, where $a_2, a_3, b_2 > 0$, $a_1, b_1, q_0 \in \mathbf{R}$, $a_2 n + q_0 > 0$, $\omega(r) = \{a_1 + a_2\log_{(2)}(1/r) + a_3\log_{(3)}(1/r)\}/\log(1/r)$ and $\eta(r) = \{b_1 + b_2\log_{(3)}(1/r)\}/\log_{(2)}(1/r)$ are nondecreasing on $(0, r_0]$. Then we can find $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(B_0)$ such that $Mf$ fails to belong to $L^{p(\cdot)}(\log L)^{q(\cdot)}(B_0)$.

For this purpose, consider

$$f(y) = |y|^{-n}(\log(1/|y|))^{-\beta}$$

for $y \in B_0$ and $\beta$ such that $a_2n+q_0+1 < \beta < a_2n+q_0+2$ or $\beta = a_2n+q_0+1$ and $a_3n+b_2 < -1$. Then we see that

$$|f(y)|^{p(y)-1} \le C(\log(1/|y|))^{a_2n}(\log_{(2)}(1/|y|))^{a_3n}$$

and

$$(\log|f(y)|)^{q(y)-q_0} \le C(\log(1/|y|))^{b_2}$$

for $y \in B_0$, so that

$$\begin{aligned} &\int_{B_0} |f(y)|^{p(y)}(\log(c_0+|f(y)|))^{q(y)}\,dy \\ \le\ & C\int_0^{1/e} t^{-1}(\log(1/t))^{-\beta+a_2n+q_0}(\log_{(2)}(1/t))^{a_3n+b_2}dt < \infty \end{aligned}$$

by our assumption on $\beta$.

On the other hand,

$$Mf(x) \ge \frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)} |y|^{-n}(\log(1/|y|))^{-\beta}dy \ge C|x|^{-n}(\log(1/|x|))^{-\beta+1}$$

since $\beta > 1$. Hence

$$\begin{aligned} &\int_{B_0} Mf(x)^{p(x)}(\log(c_0+Mf(x)))^{q(x)}\,dx \\ \ge\ & C\int_0^{1/e} t^{-1}(\log(1/t))^{-\beta+a_2n+q_0+1}(\log_{(2)}(1/t))^{a_3n+b_2}dt = \infty, \end{aligned}$$

which implies that $Mf \notin L^{p(\cdot)}(\log L)^{q(\cdot)}(B_0)$.

*Remark* 4. Let

$$p(x) = p_0 + \frac{a_1}{\log(1/|x|)}$$

and

$$q(x) = q_0 + \frac{b_1}{\log_{(2)}(1/|x|)}$$

for $x \in B_0 = B(0,r_0)$ with $0 < r_0 < 1/e$, where $a_1, b_1, q_0 \in \mathbf{R}$, $p_0 > 1$ and $\inf_{x\in B_0} p(x) > 1$. Then, in view of Diening [3], we see that $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(B_0)$ if and only if $Mf \in L^{p(\cdot)}(\log L)^{q(\cdot)}(B_0)$.

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