Hardy's inequality in Orlicz-Sobolev spaces of variable exponent

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Abstract

Our aim in this paper is to deal with a norm version of Hardy's inequality for Orlicz-Sobolev functions with $|\nabla u| \in L^{p(\cdot)} \log L^{p(\cdot)q(\cdot)}(\Omega)$ for an open set $\Omega \subset \mathbf{R}^n$. Here $p(\cdot)$ and $q(\cdot)$ are variable exponents satisfying log-Hölder and loglog-Hölder conditions, respectively. We are also concerned with the case when p attains the value 1 in some parts of the domain is included in the results.

1 Introduction and statement of results

In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see for example Kováčik-Rákosník [17], Musielak [24], Orlicz [25] and Růžička [26].

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. In this paper, following Cruz-Uribe and Fiorenza [1], we consider variable exponents $p(\cdot)$ and $q(\cdot)$ are continuous functions on $\mathbf{R}^n$ satisfying:

(p1) $1 \le p^- = \inf_{x\in\mathbf{R}^n} p(x) \le \sup_{x\in\mathbf{R}^n} p(x) = p^+ < \infty$;

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(p2) $|p(x)-p(y)| \le \dfrac{C_{\log}}{\log(e+1/|x-y|)}$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$;

(p3) $|p(x)-p(y)| \le \dfrac{C_{\log}}{\log(e+|x|)}$ whenever $|y| \ge |x|/2$;

(q1) $-\infty < q^- = \inf_{x\in\mathbf{R}^n} q(x) \le \sup_{x\in\mathbf{R}^n} q(x) = q^+ < \infty$;

(q2) $|q(x)-q(y)| \le \dfrac{C_{\log\log}}{\log(e+\log(e+1/|x-y|))}$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

Condition (p3) implies that $p_\infty = \lim_{|x|\to\infty} p(x)$ exists and

$$|p(x)-p_\infty| \le \frac{C_{\log}}{\log(e+|x|)} \qquad \text{for all } x \in \mathbf{R}^n. \tag{1.1}$$

Set

$$\Phi_{p(\cdot),q(\cdot)}(x,t) = \left(t(\log(c_0+t))^{q(x)}\right)^{p(x)};$$

here, we assume the existence of $c_0 > e$ such that

(Φ_1) $\Phi_{p(\cdot),q(\cdot)}(x,\cdot)$ is convex on $[0,\infty)$ for every $x \in \mathbf{R}^n$.

We note by a computation of the second derivative of $\Phi(x,\cdot)$ that if there is a positive constant C_0 such that

$$C_0(p(x)-1)+p(x)q(x) \ge 0, \tag{1.2}$$

then condition (Φ_1) holds. For example, if $p^- > 1$, then (1.2) is satisfied with $C_0 \ge -p^-q^-/(p^- -1)$; if $p^- = 1$ and $q^- \ge 0$, then (1.2) is also satisfied with every $C_0 \ge 0$. For later use it is convenient to see from (Φ_1) that

(Φ_2) $t^{-1}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is nondecreasing on $(0,\infty)$ for fixed $x \in \mathbf{R}^n$.

Let Ω be an open set in $\mathbf{R}^n$. Define the $\Phi_{p(\cdot),q(\cdot)}(\Omega)$ norm by

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi_{p(\cdot),q(\cdot)}\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

and denote by $\Phi_{p(\cdot),q(\cdot)}(\Omega)$ the space of all measurable functions f on Ω with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} < \infty$.

We define the variable exponent Sobolev–Orlicz space by

$$W^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega) = \{u \in \Phi_{p(\cdot),q(\cdot)}(\Omega) : |\nabla u| \in \Phi_{p(\cdot),q(\cdot)}(\Omega)\}.$$

The norm $\|u\|_{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega) = \|u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} + \||\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$ makes $W^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ a Banach space. Further we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ by $W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, which is extended to be 0 outside Ω. In case $q \equiv 0$, $\Phi_{p(\cdot),q(\cdot)}(\Omega)$ and $W^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ are denote by $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ for simplicity. For fundamental properties of these spaces, see, for example, Kováčik and Rákosník [17].

We denote by $B(x,r)$ the open ball centered at x of radius r. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

Recently Hästö [11, Theorem 3.2] proved the following:

THEOREM A. *Let $\Omega \neq \mathbf{R}^n$ be an open set. Suppose $1 < p^- \leq p^+ < \infty$. Assume that Ω satisfies the measure density condition, that is, there exists a constant $k > 0$ such that*

$$|B(z,r) \cap \Omega^c| \geq k|B(z,r)| \tag{1.3}$$

for every $z \in \partial\Omega$ and $r > 0$ (see [8]). *Then there exist positive constants C and b_0 such that the inequality*

$$\|\delta^{b-1}u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^{p(\cdot)}(\Omega)}$$

holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and all $0 \leq b < b_0$, where $\delta(x) = dist(x, \partial\Omega)$ and the constants C, b_0 depend on p^+, p^-, n, k.

This gives an Harjulehto, Hästö and Koskenoja [10, Theorem 3.3] in the case when Ω is bounded. In the constant exponent case, Theorem A is given by Hajłasz [7, Theorem 1]; for related results, see also Edmunds and Evans [4], Kufner and L. E. Persson [18], Wannebo [30], Kinnunen and Martio [15].

Our aim in this paper is to give Hardy's inequality for $W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, as an extension of Theorem A. For $0 \leq a \leq 1$, set

$$1/p_a^\sharp(x) = 1/p(x) - a/n$$

and

$$\Phi_{p_a^\sharp(\cdot),q(\cdot)}(x,t) = (t(\log(c_0+t))^{q(x)})^{p_a^\sharp(x)}.$$

Note that

$$\Phi_{p_a^\sharp(\cdot),q(\cdot)}(x,t) = \left(\Phi_{p(\cdot),q(\cdot)}(x,t)\right)^{p_a^\sharp(x)/p(x)}$$

and $\Phi_{p_a^\sharp(\cdot),q(\cdot)}$ also satisfies (Φ_1), since $p_a^\sharp(x)/p(x) \geq 1$.

THEOREM 1.1. *Let $\Omega \neq \mathbf{R}^n$ be an open set satisfying* (1.3)*. Suppose $1 < p^- \leq p^+ < \infty$, $0 < A \leq 1$ and $0 < A < n/p^+$. Then there exist $C > 0$ and $0 < b_0 < 1$ depending on A such that*

$$\|\delta^{a+b-1}u\|_{\Phi_{p_a^\sharp(\cdot),q(\cdot)}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$$

for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, $0 \leq a \leq A$ and $0 \leq b \leq b_0$.

This theorem extends Theorem A given by Hästö [11], whose crucial idea is a partition norm on $L^{p(\cdot)}(\mathbf{R}^n)$. We give a straightforward and simple proof of Theorem 1.1 in a quite different manner. In fact, we apply Poincaré's inequality and the boundedness of maximal functions, following the idea by Hedberg [13].

If $p^- > n$, then we do not need the measure density condition to derive Poincaré's inequality. In fact we will show the following result, which gives an extension of Harjulehto, Hästö and Koskenoja [10, Theorem 3.5].

Theorem 1.2. *Let $\Omega \neq \mathbf{R}^n$ be an open set. Suppose $n < p^- \leq p^+ < \infty$ and $0 < A < n/p^+$. Then there exist $C > 0$ and $0 < b_0 < 1$ depending on A such that*

$$\|\delta^{a+b-1}u\|_{\Phi_{p_a^\sharp(\cdot),q(\cdot)}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$$

for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, $0 \leq a \leq A$ and $0 \leq b \leq b_0$.

Finally we are concerned with the case $p^- \geq 1$:

Theorem 1.3. *Let $\Omega \neq \mathbf{R}^n$ be an open set satisfying* (1.3). $0 < A < \min\{1, n/p^+\}$ *and $\gamma > 1$. Let*

$$L(t) = (\log(c_0 + t + 1/t))^{-\gamma}.$$

Then there exists a constant $C > 0$ depending on A and γ such that

$$\int_\Omega \Phi_{p_a^\sharp(\cdot),q(\cdot)}(x, \delta(x)^{a-1}|u(x)|)L(\delta(x)^{a-1}|u(x)|)\, dx \quad \leq \quad C$$

for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ with $\||\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \leq 1$ and $0 \leq a \leq A$.

Theorem 1.3 is not always valid when $\gamma = 1$ (see Remark 4.3). The case $a = 1$, that is, Sobolev inequality for $W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, is given in [12, Theorem 1.1], which was an extension of the results by Harjulehto and Hästö [9, Proposition 4.2(1)] and Hästö [11, Theorem 3.4].

For further related results, see Samko [27, 28], Kokilashvili and Samko [16], Diening and Samko [3], Humberto and Samko[14] and Futamura, Mizuta and Shimomura [6].

2 Proof of Theorem 1.1

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

Denote by $W^{1,1}_{loc}(\mathbf{R}^n)$ the class of all functions u such that $\varphi u \in W^{1,1}(\mathbf{R}^n)$ for every $\varphi \in C_0^\infty(\mathbf{R}^n)$.

First let us begin with the following lemma (see the proof of Hajłasz [7, Proposition 1]).

Lemma 2.1. *Let $\Omega \neq \mathbf{R}^n$ be an open set satisfying* (1.3). *Then there exists a constant $C = C(n, k) > 0$ such that*

$$|u(x)| \leq C \int_{B(x,2\delta(x))} |x-y|^{1-n}|\nabla u(y)|\, dy$$

for almost every $x \in \Omega$, whenever $u \in W^{1,1}_{loc}(\mathbf{R}^n)$ and $u = 0$ outside Ω.

Proof. Let $u \in W^{1,1}_{loc}(\mathbf{R}^n)$ and $u = 0$ outside Ω. By (1.3), we obtain

$$\begin{aligned} |u(x)| &= \frac{1}{|B(x,2\delta(x)) \cap \Omega^c|} \int_{B(x,2\delta(x))\cap\Omega^c} |u(x) - u(y)| \, dy \\ &\le \frac{C}{|B(x,2\delta(x))|} \int_{B(x,2\delta(x))} |u(x) - u(y)| \, dy \\ &\le C \int_{B(x,2\delta(x))} |x-y|^{1-n} |\nabla u(y)| \, dy \end{aligned}$$

for almost every $x \in \Omega$. □

For a locally integrable function f on $\mathbf{R}^n$, we consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy.$$

We know the following result concerning the boundedness of maximal functions in $\Phi_{p(\cdot),q(\cdot)}(\Omega)$.

Lemma 2.2 ([5, Proposition 2.2]). *Suppose $p^- > 1$. Then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \le C \|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

Lemma 2.3. *Let $\Omega \ne \mathbf{R}^n$ be an open set satisfying (1.3). Suppose $1 < p^- \le p^+ < \infty$. Then there exist $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{b-1} u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C \|\delta^b |\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$$

for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ and $0 \le b \le b_0$.

Proof. Since $\Omega \ne \mathbf{R}^n$, without loss of generality, we may assume that the origin is on the boundary $\partial\Omega$, that is,

$$0 \in \partial\Omega. \tag{2.1}$$

We first treat $u \in C_0^\infty(\Omega)$. Applying Lemma 2.1 to $\delta^b u$, we have

$$\delta(x)^b |u(x)| \le C \int_{B(x,2\delta(x))} |x-y|^{1-n} \{ b\delta(y)^{b-1} |u(y)| + \delta(y)^b |\nabla u(y)| \} \, dy, \tag{2.2}$$

so that

$$\delta(x)^{b-1} |u(x)| \le CbM(\delta^{b-1}|u|)(x) + CM(\delta^b |\nabla u|)(x).$$

In view of Lemma 2.2, we find

$$\|\delta^{b-1} u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le Cb \|\delta^{b-1} u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} + C \|\delta^b |\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)},$$

which gives

$$(1 - Cb) \|\delta^{b-1} u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C \|\delta^b |\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}.$$

Now it suffices to take b_0 such that $1 - Cb_0 > 0$.

We next treat $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ with compact support. Then we can find a sequence $\varphi_j \in C_0^\infty(\Omega)$ such that $\varphi_j \to u$ in $W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$. Suppose $u = 0$ outside $B(0,R)$. Then we may assume that $\varphi_j = 0$ outside $B(0,2R)$. By the above discussions we have

$$\|\delta^{b-1}\varphi_j\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C\|\delta^b|\nabla\varphi_j|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}.$$

Since $\|\delta^b|\nabla(\varphi_j - u)|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$ tends to zero as $j \to \infty$, we obtain

$$\|\delta^{b-1}u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}.$$

Finally we treat a general $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$. For $N > 1$ we consider a continuous function h_N on $[0,\infty)$ and N^* such that $0 \le h_N \le 1$ on $[0,\infty)$, $h_N = 0$ on $[0,N]$, $h_N = 0$ on $[N^*,\infty]$, $h_N(t) \le t^{-1}$ for $t \in [N,N^*]$ and

$$\int_N^{N^*} h_N(t)dt = 1;$$

the existence of N^* is assured since $\displaystyle\int_N^\infty t^{-1}dt = \infty$. Set

$$H_N(x) = 1 - \int_0^{|x|} h_N(t)dt$$

for $x \in \mathbf{R}^n$. Then we know as above that

$$\begin{aligned}
\|\delta^{b-1}(H_N u)\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} &\le C\|\delta^b|\nabla(H_N u)|\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \\
&\le C\|\delta^b|\nabla H_N|u\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} + C\|\delta^b H_N|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \\
&\le C\||x|^b|\nabla H_N|u\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} + C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}.
\end{aligned}$$

Since $|\nabla H_N(x)| \le h_N(|x|) \le |x|^{-1}$ for $x \in \Omega$, $\||x|^b|\nabla H_N|u\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$ tends to 0 as $N \to \infty$, so that

$$\begin{aligned}
\|\delta^{b-1}u\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} &\le \liminf_{N\to\infty} \|\delta^{b-1}(H_N u)\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \\
&\le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)},
\end{aligned}$$

which completes the proof. □

Note that

$$\delta(x)^{a-1}\int_{B(x,2\delta(x))} |x-y|^{1-n}f(y)\,dy \le 2^{1-a}\int_{B(x,2\delta(x))} |x-y|^{a-n}f(y)\,dy \qquad (2.3)$$

for $x \in \Omega$, $0 \le a \le 1$ and nonnegative measurable function f on $\mathbf{R}^n$. To give its estimate, we prepare the following result.

LEMMA 2.4. *Let $0 < A < n/p^+$. Then there exists a constant $C > 0$ depending on A such that*

$$I \equiv \int_{B(x,2|x|)\setminus B(x,r)} |x-y|^{a-n} f(y)\, dy \le C r^{a-n/p(x)} (\log(c_0 + r^{-1}))^{-q(x)}$$

for all $x \in \mathbf{R}^n$, $r > 0$, $0 \le a \le A$ and $f \ge 0$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \le 1$.

This follows from [12, Lemmas 3.1 and 3.2]. Actually, since the estimate of the integral outside $B(x, 2|x|)$ given in the proof of [12, Lemma 3.1] is not needed here, we use [12, Lemma 3.1] if r is large, and both of [12, Lemmas 3.1 and 3.2] if r is small.

Next consider

$$J \equiv \delta(x)^{a-1} \int_{B(x,2\delta(x))} |x-y|^{1-n} f(y)\, dy$$

for $x \in \Omega$ and $f \ge 0$. When $a = 0$, we find

$$J \le CMf(x).$$

LEMMA 2.5. *Let $\Omega \ne \mathbf{R}^n$ be an open set. If $0 < A \le 1$ and $0 < A < n/p^+$, then there exists a constant $C > 0$ depending on A such that*

$$J \le C\{Mf(x)\}^{1-ap(x)/n} \{\log(c_0 + Mf(x))\}^{-ap(x)q(x)/n}$$

for all $x \in \Omega$, $0 \le a \le A$ and $f \ge 0$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le 1$.

Proof. We have only to consider the case $a > 0$. Let $f \ge 0$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le 1$.

First suppose $\{Mf(x)\}^{-p(x)/n}\{\log(c_0 + Mf(x))\}^{-p(x)q(x)/n} \le \delta(x)$; we set $f = 0$ outside Ω as before. For $0 < r < \delta(x)$, we have by (2.1) and Lemma 2.4

$$\begin{aligned} J &\le C\left\{\delta(x)^{a-1} r Mf(x) + \int_{B(x,2\delta(x))\setminus B(x,r)} |x-y|^{a-n} f(y)\, dy\right\} \\ &\le C\left\{r^a Mf(x) + r^{a-n/p(x)} (\log(c_0 + 1/r))^{-q(x)}\right\}. \end{aligned}$$

Now, letting $r = \{Mf(x)\}^{-p(x)/n}\{\log(c_0 + Mf(x))\}^{-p(x)q(x)/n}$ $(\le \delta(x))$, we have

$$J \le C\{Mf(x)\}^{1-ap(x)/n}\{\log(c_0 + Mf(x))\}^{-ap(x)q(x)/n}.$$

Next suppose $\{Mf(x)\}^{-p(x)/n}\{\log(c_0 + Mf(x))\}^{-p(x)q(x)/n} > \delta(x)$. Then we have

$$\begin{aligned} J &\le C\delta(x)^a Mf(x) \\ &\le C\{Mf(x)\}^{1-ap(x)/n}\{\log(c_0 + Mf(x))\}^{-ap(x)q(x)/n}, \end{aligned}$$

which proves the case $a > 0$. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ with $\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le 1$. We have by (2.2)

$$\delta(x)^{a+b-1}|u(x)| \le C\delta(x)^{a-1}\int_{B(x,2\delta(x))}|x-y|^{1-n}f(y)\,dy, \tag{2.4}$$

where $f(y) = b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|$. Here note from Lemma 2.3 that

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C.$$

Let $0 < A \le 1$, $0 < A < n/p^+$ and $0 \le a \le A$. Then we obtain by Lemma 2.5

$$\delta(x)^{a+b-1}|u(x)| \;\le\; CMf(x)^{p(x)/p_a^\sharp(x)}(\log(c_0 + Mf(x)))^{-ap(x)q(x)/n}.$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\|\delta^{a+b-1}u\|_{\Phi_{p_a^\sharp(\cdot),q(\cdot)}(\Omega)} &\le C\|Mf\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \\
&\le Cb\|\delta^{b-1}u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} + C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \\
&\le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)},
\end{aligned}$$

which proves the theorem. □

Remark 2.6. Let Ω be a bounded domain in $\mathbf{R}^n$ satisfying (1.3), $n \le p^- \le p^+ < \infty$ and $1 - 1/p^- > q^+$. Then there exist $C_1, C_2 > 0$ and $0 < b_0 < 1$ such that

$$\int_\Omega \exp(C_1(\delta(x)^{a+b-1}u(x))^{p(x)/(p(x)-p(x)q(x)-1)})\,dx \le C_2$$

for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ with $\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le 1$, $n/p^- \le a \le 1$ and $0 \le b \le b_0$.

In fact, (2.4) gives

$$\delta(x)^{a+b-1}|u(x)| \le C\int_{B(x,2\delta(x))}|x-y|^{a-n}f(y)\,dy,$$

where $f(y) = b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|$ and $n/p^- \le a \le 1$. As in the proof of Theorem 1.1 and [22, Lemma 4.4], we obtain

$$\delta(x)^{a+b-1}|u(x)| \le C(\log(c_0 + Mf(x)))^{(p(x)-p(x)q(x)-1)/p(x)},$$

so that

$$\exp\left(\left(C^{-1}\delta(x)^{a+b-1}|u(x)|\right)^{p(x)/(p(x)-p(x)q(x)-1)}\right) \le c_0 + Mf(x).$$

By integration we establish the required inequality.

3 Proof of Theorem 1.2

For a proof of Theorem 1.2, we prepare the following lemma instead of Lemma 2.3. We write $f \sim g$ if there exists a constant C so that $C^{-1}g \le f \le Cg$.

LEMMA 3.1. *Let $\Omega \neq \mathbf{R}^n$ be an open set. Suppose $n < p^- \le p^+ < \infty$. Then there exist $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{b-1}u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$$

for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ and $0 \le b \le b_0$.

Proof. Let $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$. Suppose $n < p^- \le p^+ < \infty$. Since $u \in W^{1,s}_{loc}(\mathbf{R}^n)$ with $n < s < p^-$, u is (Hölder) continuous in Ω and

$$|u(x)| \le C\left(\delta(x)^{s-n}\int_{B(x,2\delta(x))}|\nabla u(y)|^s\,dy\right)^{1/s}$$

for every $x \in \Omega$; for this, see also [7, Proposition 1], [15, (3.1)] and [23, Theorem 1]. This implies that

$$[\delta(x)^{-1}|u(x)|]^s \le C\delta(x)^{-n}\int_{B(x,2\delta(x))}|\nabla u(y)|^s\,dy. \tag{3.1}$$

Applying (3.1) to $\delta^b u$ as in Lemma 2.3, we have

$$[\delta(x)^{b-1}|u(x)|]^s \le C\delta(x)^{-n}\int_{B(x,2\delta(x))}[b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|]^s\,dy, \tag{3.2}$$

so that

$$[\delta(x)^{b-1}|u(x)|]^s \le C\left\{b^s M([\delta^{b-1}u]^s)(x) + M([\delta^b|\nabla u|]^s)(x)\right\}.$$

In view of Lemma 2.2, we find

$$\|[\delta^{b-1}u]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} \le C\left\{b^s\|[\delta^{b-1}u]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} + \|[\delta^b|\nabla u|]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)}\right\},$$

which gives

$$(1 - Cb^s)\|[\delta^{b-1}u]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} \le C\|[\delta^b|\nabla u|]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)}.$$

Since $\|f^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} \sim \|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$, we obtain

$$(1 - Cb^s)\|\delta^{b-1}u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}.$$

Now it suffices to take b_0 such that $1 - Cb_0^s > 0$.

□

Proof of Theorem 1.2. Let $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$. Suppose $n < s < p^- \le p^+ < \infty$ and

$$\|\delta^b |\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le 1.$$

For $0 \le a \le A < n/p^+$, we have by (3.2)

$$\begin{aligned} [\delta(x)^{a+b-1}|u(x)|]^s &\le C\delta(x)^{as-n}\int_{B(x,2\delta(x))} f(y)\,dy \\ &\le C\delta(x)^{as-As}\int_{B(x,2\delta(x))} |x-y|^{As-n} f(y)\,dy, \end{aligned}$$

where $f = [b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|]^s$. Here note from Lemma 3.1 that

$$\|f\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} \le C.$$

Hence, as in the proof of Lemma 2.5 with a, $p(x)$ and $q(x)$ replaced by as, $p(x)/s$ and $sq(x)$, respectively, we obtain

$$\delta(x)^{a+b-1}|u(x)| \le CMf(x)^{p(x)/(sp_a^\sharp(x))}(\log(c_0 + Mf(x)))^{-ap(x)q(x)/n}.$$

By Lemmas 2.2 and 3.1, we have

$$\begin{aligned} \|\delta^{a+b-1}u\|_{\Phi_{p_a^\sharp(\cdot),q(\cdot)}(\Omega)} &\le C\|Mf\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} \le C\|f\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} \\ &\le C\left\{b^s\|[\delta^{b-1}u]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)} + \|[\delta^b|\nabla u|]^s\|_{\Phi_{p(\cdot)/s,sq(\cdot)}(\Omega)}\right\} \\ &\le C\left\{b^s\|\delta^{b-1}u\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} + \|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}\right\} \\ &\le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}, \end{aligned}$$

which proves the theorem. □

4 Proof of Theorem 1.3

For a proof of Theorem 1.3, we prepare the following results.

Lemma 4.1 ([12, Lemmas 2.1 - 2.3]). *Let f be a nonnegative measurable function on Ω such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\{Mf(x)\}^{p(x)} \le C\{Mg(x)(\log(c_0 + Mg(x)))^{-p(x)q(x)} + C(1+|x|)^{-n}\},$$

and

$$\Phi_{p(\cdot),q(\cdot)}(x, Mf(x)) \le C\{Mg(x) + (1+|x|)^{-n}\},$$

where $g(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))$.

The next lemma is proved along the same lines as in Stein [29, Chapter 1]; see also [20, Lemma 2.5].

LEMMA 4.2. *Suppose* $1 < \gamma \le 2$. *Then there exists a constant* $C > 0$ *such that*

$$\int_\Omega Mg(x)(\log(c_0 + Mg(x) + Mg(x)^{-1}))^{-\gamma} dx \le C\|g\|_{L^1(\Omega)}$$

for all $g \in L^1(\Omega)$.

Proof. For $1 < \gamma \le 2$, we see that $t(\log(\gamma + t + 1/t))^{-\gamma}$ is increasing on $(0, \infty)$ and

$$t(\log(c_0 + t + 1/t))^{-\gamma} \le C(\gamma)t(\log(\gamma + t + 1/t))^{-\gamma}.$$

Hence

$$\begin{aligned} & \int_\Omega Mg(x)(\log(c_0 + Mg(x) + Mg(x)^{-1}))^{-\gamma} dx \\ \le\ & C(\gamma) \int_\Omega Mg(x)(\log(\gamma + Mg(x) + Mg(x)^{-1}))^{-\gamma} dx \\ =\ & C(\gamma) \int_0^\infty \lambda(t) d(t(\log(\gamma + t + t^{-1}))^{-\gamma}), \end{aligned}$$

where $\lambda(t) = |\{x \in \Omega : Mg(x) > t\}|$. Here we note from [29, Theorem 1, Chapter 1] that

$$\lambda(t) \le Ct^{-1} \int_{\{x\in\Omega:|g(x)|>t/2\}} |g(x)|\ dx$$

for $t > 0$. Now we obtain by Fubini's Theorem

$$\begin{aligned} & \int_\Omega Mg(x)(\log(c_0 + Mg(x) + Mg(x)^{-1}))^{-\gamma} dx \\ \le\ & C \int_\Omega |g(x)| \left\{ \int_0^{2|g(x)|} t^{-1} d(t(\log(\gamma + t + t^{-1}))^{-\gamma}) \right\} dx \\ \le\ & C \int_\Omega |g(x)|\ dx, \end{aligned}$$

as required. □

Proof of Theorem 1.3. We may assume that $1 < \gamma \le 2$. By Lemmas 2.1 and 2.5 we have

$$\delta(x)^{a-1}|u(x)| \le C\{M(|\nabla u|)(x)\}^{1-ap(x)/n}\{\log(c_0 + M(|\nabla u|)(x))\}^{-ap(x)q(x)/n}$$

for $x \in \Omega$ and $0 \le a \le A$ $(< \min\{1, n/p^+\})$. Hence Lemma 4.1 gives

$$\begin{aligned} & \Phi_{p_a^\sharp(\cdot),q(\cdot)}(x, \delta(x)^{a-1}|u(x)|)(\log(c_0 + \delta(x)^{a-1}|u(x)| + \delta(x)^{1-a}|u(x)|^{-1}))^{-\gamma} \\ \le\ & C\Phi_{p(\cdot),q(\cdot)}(x, M(|\nabla u|)(x))(\log(c_0 + M(|\nabla u|)(x) + M(|\nabla u|)(x)^{-1}))^{-\gamma} \\ \le\ & C\{Mg(x)(\log(c_0 + Mg(x) + Mg(x)^{-1}))^{-\gamma} + (1 + |x|)^{-n}(\log(c_0 + |x|))^{-\gamma}\}, \end{aligned}$$

where $g(y) = \Phi_{p(\cdot),q(\cdot)}(y, |\nabla u(y)|)$. Thus we obtain the required conclusion with the aid of Lemma 4.2.

□

Remark 4.3. Theorem 1.3 is not valid when $(p^- =)\ p^+ = 1$ and $\gamma = 1$.

To show this when $n = 1$, first consider the function

$$v(x) = (\log(\log(4/x)))^{-\beta}$$

for $\beta > 0$. Then $|\nabla v(x)| = \beta x^{-1}(\log(4/x))^{-1}(\log(\log(4/x)))^{-\beta-1}$ and

$$\int_0^1 |\nabla v(x)|\ dx < \infty.$$

On the other hand, if $0 < a < 1$ and $1/p_a^\sharp(x) = 1 - a$, then

$$\begin{aligned} & \int_0^{1/2} (x^{a-1}|v(x)|)^{p_a^\sharp(x)} (\log(c_0 + x^{a-1}|v(x)|))^{-1}\ dx \\ \geq\ & C \int_0^{1/2} x^{-1}(\log(\log(4/x)))^{-\beta/(1-a)}(\log(1/x))^{-1}\ dx = \infty \end{aligned}$$

when $-\beta/(1-a) + 1 \geq 0$, that is, $0 < \beta \leq 1 - a$. Now, letting $\varphi \in C^\infty((0,1))$ such that $\varphi(x) = 1$ for $x \in (0, 1/2)$ and $\lim_{x \to 1-0} \varphi(x) = 0$, one may consider $u(x) = \varphi(x)v(x)$ for $0 < x < 1$.

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