Musielak-Orlicz-Sobolev spaces on metric measure spaces

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Abstract

Our aim in this paper is to study Musielak-Orlicz-Sobolev spaces on metric measure spaces. We consider a Hajłasz type condition and a Newtonian type condition. We prove that Lipschitz continuous functions are dense as well as other basic properties. We study the relationship between these spaces, and discuss Lebesgue point theorem in these spaces. We also deal with the boundedness of the Hardy-Littlewood maximal operator on Musielak-Orlicz spaces. As an application of the boundedness of the Hardy-Littlewood maximal operator, we establish a generalization of Sobolev's inequality for Sobolev functions in Musielak-Orlicz-Hajłasz-Sobolev spaces.

1   Introduction

Sobolev spaces on metric measure spaces have been studied during the last two decades, see [6, 21, 23, 33, 52], etc.. The theory was generalized to Orlicz-Sobolev spaces on metric measure spaces in [4, 5, 54]. We refer [1, 2, 15, 55] for Sobolev spaces on $\mathbf{R}^N$, [9, 14] for variable exponent Sobolev spaces, [51] for Musielak-Orlicz spaces, [16] for the study of differential equations of divergence form in Musielak-Sobolev spaces and [17] for the study of uniform convexity of Musielak-Orlicz-Sobolev spaces and its applications to variational problems. In the last decade, variable exponent Sobolev spaces on metric measure spaces have been developed, see e.g. [19, 20, 31, 32, 49]. The purpose of this paper is to define Musielak-Orlicz-Sobolev spaces on metric measure spaces and prove the basic properties of such spaces.

There are two ways to define the first order Sobolev spaces on metric measure spaces. Hajłasz ([21]) showed that a p-integrable function u, $1 < p < \infty$, belongs to $W^{1,p}(\mathbf{R}^N)$ if and only if there exists a non-negative p-integrable function g such that

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)) \tag{1.1}$$

for almost every $x, y \in \mathbf{R}^N$. If we replace $|x - y|$ by the distance of the points x and y, (1.1) can be stated in metric measure spaces. Spaces defined by using (1.1) are called Hajłasz-Sobolev spaces. See also [23, 33]. The theory was generalized to Orlicz-Sobolev spaces by Aïssaoui ([4, 5]). For the Sobolev capacity on Hajłasz-Sobolev spaces, see [37]. By the classical Lebesgue differentiation theorem, almost every point is a Lebesgue point for a locally integrable function. For the Lebesgue point theorem in Hajłasz-Sobolev spaces, we refer the reader to [36].

Another way is to use weak upper gradients. A nonnegative Borel measurable function h is said to be an upper gradient of u if

$$|u(x) - u(y)| \leq \int_\gamma h \, ds \tag{1.2}$$

for every x, y and curve connecting x to y. Upper gradients were introduced by Heinonen and Koskela ([34]) as a tool to study quasiconformal maps. If (1.2) holds for a function u on every curve not belonging to an exceptional family of p-modulus zero in metric measure spaces, we call h a weak upper gradient of u. We call these spaces Newtonian spaces or Newton-Sobolev spaces. The study of Newton-Sobolev spaces was initiated by Shanmugalingam ([52]). See also [6]. The theory was generalized to Orlicz-Sobolev spaces by Tuominen ([54]).

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [9, 14]). See also [25, 26], etc.. Harjulehto, Hästö and Pere ([32]) studied basic properties of the variable exponent Hajłasz-Sobolev space and the variable exponent Newton-Sobolev space. For the Lebesgue point theorem in variable exponent spaces, see e.g. [24].

The Hardy-Littlewood maximal operator is a classical tool in harmonic analysis and studying Sobolev functions and partial differential equations and plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [7, 35, 41, 53], etc.). It is well known that the Hardy-Littlewood maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ (see [53]). See, e.g. [8] for Orlicz spaces, [10, 11] for variable exponent Lebesgue spaces $L^{p(\cdot)}$, [43, 48] for the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. These spaces are special cases of so-called Musielak-Orlicz spaces ([44, 51], etc.). We refer [16] for the study of differential equations of divergence form in Musielak-Sobolev spaces and [17] for the study of uniform convexity of Musielak-Orlicz-Sobolev spaces and its applications to variational problems. For general Musielak-Orlicz spaces, see [12]. In bounded doubling metric measure spaces, the boundedness of the Hardy-Littlewood maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was studied in [20, 31]. See also [3].

One of the important applications of the boundedness of the Hardy-Littlewood maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{L^{p^*}(\mathbf{R}^N)} \le C\|f\|_{L^p(\mathbf{R}^N)}$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$ (see, e.g. [2, Theorem 3.1.4]). This result was extended to Orlicz spaces in [8] and [50]. In the Euclidean setting, variable exponent versions were discussed in [13, 39, 40, 44, 48], etc.. For variable exponent versions on metric measure spaces, see, e.g. [20, 29].

In this paper, we define Musielak-Orlicz-Newton-Sobolev spaces as well as Musielak-Orlicz-Hajłasz-Sobolev spaces on metric measure spaces and prove the basic properties of such spaces.

This present paper is organized as follows. In Section 2, we define Musielak-Orlicz spaces on metric measure spaces.

In Section 3, we study basic properties of Musielak-Orlicz-Hajłasz-Sobolev spaces. We show that Lipschitz continuous functions are dense and study a related Sobolev type capacity. We prove that every point except in a small set is a Lebesgue point for Sobolev functions in Musielak-Orlicz-Hajłasz-Sobolev spaces.

In Section 4, we study basic properties of Musielak-Orlicz-Newton-Sobolev spaces. We show that Lipschitz continuous functions are dense if the measure is doubling and study a related Sobolev type capacity. We discuss Lebesgue point theorem in Musielak-Orlicz-Newton-Sobolev spaces.

In Section 5, we study the relationship between Musielak-Orlicz-Hajłasz-Sobolev spaces and Musielak-Orlicz-Newton-Sobolev spaces in a metric measure space (see Theorem 5.4).

In Section 6, we show that the Hardy-Littlewood maximal operator is bounded on Musielak-Orlicz spaces in our setting (see Theorem 6.3).

In Section 7, as an application of the boundedness of the Hardy-Littlewood maximal operator, we shall give a general version of Sobolev's inequality for Sobolev functions in Musielak-Orlicz-Hajłasz-Sobolev spaces (see Theorem 7.7). By treating such general setting, we can obtain new results (e.g., Corollaries 7.6 and 7.8).

In Section 8, we discuss Fuglede's theorem for Musielak-Orlicz-Sobolev spaces in the Euclidean setting.

2 Musielak-Orlicz spaces

Throughout this paper, let C denote various positive constants independent of the variables in question.

We denote by (X, d, μ) a metric measure space, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_\Omega = \sup\{d(x, y) : x, y \in \Omega\}$ for a set $\Omega \subset X$.

For a measurable function $Q(\cdot)$ satisfying

$$0 < Q^- := \inf_{x \in X} Q(x) \le \sup_{x \in X} Q(x) =: Q^+ < \infty,$$

we say that a measure μ is lower Ahlfors $Q(x)$-regular if there exists a constant $c_0 > 0$ such that

$$\mu(B(x, r)) \ge c_0 r^{Q(x)}$$

for all $x \in X$ and $0 < r < d_X$. Further, μ is Ahlfors $Q(x)$-regular if there exists a constant $c_1 > 0$ such that

$$c_1^{-1} r^{Q(x)} \le \mu(B(x, r)) \le c_1 r^{Q(x)}$$

for all $x \in X$ and $0 < r < d_X$. We say that the measure μ is a doubling measure, if there exists a constant $c_2 > 0$ such that $\mu(B(x, 2r)) \le c_2 \mu(B(x, r))$ for every $x \in X$ and $0 < r < d_X$. We say that X is a doubling space if μ is a doubling measure.

We consider a function

$$\Phi(x, t) = t\phi(x, t) : X \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions (Φ1) – (Φ4):

(Φ1) $\phi(\cdot, t)$ is measurable on X for each $t \ge 0$ and $\phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in X$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x, 1) \le A_1 \quad \text{for all } x \in X;$$

(Φ3) $\phi(x, \cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x, t) \le A_2 \phi(x, s) \quad \text{for all } x \in X \quad \text{whenever } 0 \le t < s;$$

(Φ4) there exists a constant $A_3 > 1$ such that

$$\phi(x, 2t) \le A_3 \phi(x, t) \quad \text{for all } x \in X \text{ and } t > 0.$$

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x \in X} \phi(x, t) \le \sup_{x \in X} \phi(x, t) < \infty$$

for each $t > 0$.

Let $\bar{\phi}(x, t) = \sup_{0 \le s \le t} \phi(x, s)$ and

$$\overline{\Phi}(x, t) = \int_0^t \bar{\phi}(x, r) \, dr$$

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for $x \in X$ and $t \geq 0$. Then $\overline{\Phi}(x, \cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \overline{\Phi}(x,t) \leq A_2\Phi(x,t) \tag{2.1}$$

for all $x \in X$ and $t \geq 0$.

By (Φ3), we see that

$$\Phi(x, at) \begin{cases} \leq A_2 a\Phi(x,t) & \text{if } 0 \leq a \leq 1 \\ \geq A_2^{-1} a\Phi(x,t) & \text{if } a \geq 1. \end{cases} \tag{2.2}$$

We shall also consider the following conditions:

(Φ5) for every $\gamma_1, \gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \geq 1$ such that

$$\phi(x,t) \leq B_{\gamma_1,\gamma_2}\phi(y,t)$$

whenever $d(x,y) \leq \gamma_1 t^{-1/\gamma_2}$ and $t \geq 1$;

(Φ6) there exist $x_0 \in X$, a function $g \in L^1(X)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in X$ and

$$B_\infty^{-1}\Phi(x,t) \leq \Phi(x',t) \leq B_\infty\Phi(x,t)$$

whenever $d(x', x_0) \geq d(x, x_0)$ and $g(x) \leq t \leq 1$.

EXAMPLE 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$, be measurable functions on X such that

(P1) $1 < p^- := \inf_{x \in X} p(x) \leq \sup_{x \in X} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x \in X} q_j(x) \leq \sup_{x \in X} q_j(x) =: q_j^+ < \infty$

for all $j = 1, \ldots, k$.

Set $L_c(t) = \log(c+t)$ for $c \geq e$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies (Φ1), (Φ2), (Φ3) and (Φ4).

$\Phi(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{L_e(1/d(x,y))}$$

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with a constant $C_p \geq 0$ and

(Q2) $q_j(\cdot)$ is j-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \leq \frac{C_{q_j}}{L_e^{(j)}(1/d(x,y))}$$

with constants $C_{q_j} \geq 0$, $j = 1, \ldots k$.

EXAMPLE 2.2. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on X satisfying (P1) and (Q1).
   Then,

$$\Phi(x,t) = (1+t)^{p_1(x)}(1+1/t)^{-p_2(x)}L_c(t)^{q_1(x)}L_c(1/t)^{-q_2(x)}$$

satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p_j^- > 1$, $j = 1,2$ or $q_j^- \geq 0$, $j = 1,2$. As a matter of fact, it satisfies (Φ3) if and only if $p_j(\cdot)$ and $q_j(\cdot)$ satisfy the following conditions:
   (1) $q_j(x) \geq 0$ at points x where $p_j(x) = 1$, $j = 1,2$;
   (2) $\sup_{x:p_j(x)>1}\{\min(q_j(x),0)\log(p_j(x)-1)\} < \infty$.
   Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if $p_1(\cdot)$ is log-Hölder continuous and $q_1(\cdot)$ is 2-log-Hölder continuous.

EXAMPLE 2.3. Let $\Phi(\cdot,\cdot)$ be as in Example 2.1 and fix $x_0 \in X$. Let κ and c be positive constants. If μ satisfies $\mu(B(x_0,r)) \leq cr^\kappa$ for all $r \geq 1$ and

(P3) $p(\cdot)$ is log-Hölder continuous at ∞, namely

$$|p(x) - p(x')| \leq \frac{C_{p.\infty}}{L_e(d(x,x_0))} \quad \text{for } d(x',x_0) \geq d(x,x_0)$$

   with a constant $C_{p,\infty} \geq 0$,

then $\Phi(\cdot,\cdot)$ satisfies (Φ6) with $g(x) = 1/(1+d(x,x_0))^{\kappa+1}$.

EXAMPLE 2.4. Let $\Phi(\cdot,\cdot)$ be as in Example 2.2 and fix $x_0 \in X$. Let κ and c be positive constants. If μ satisfies $\mu(B(x_0,r)) \leq cr^\kappa$ for all $r \geq 1$, $p_2(\cdot)$ satisfies (P3) and

(Q3) $q_2(\cdot)$ is 2-log-Hölder continuous at ∞, namely

$$|q_2(x) - q_2(x')| \leq \frac{C_{q_2,\infty}}{L_c^{(2)}(d(x,x_0))} \quad \text{for } d(x',x_0) \geq d(x,x_0)$$

   with a constant $C_{q_2,\infty} \geq 0$,

then $\Phi(\cdot,\cdot)$ satisfies (Φ6) with $g(x) = 1/(1+d(x,x_0))^{\kappa+1}$.

We say that u is a locally integrable function on X if u is an integrable function on all balls B in X. From now on, we assume that $\Phi(x,t)$ satisfies ($\Phi 1$), ($\Phi 2$), ($\Phi 3$) and ($\Phi 4$). Then the associated Musielak-Orlicz space

$$L^{\Phi}(X) = \left\{ f \in L^1_{loc}(X) \, ; \int_X \Phi\big(y, |f(y)|\big) \, d\mu(y) < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(X)} = \inf \left\{ \lambda > 0 \, ; \int_X \overline{\Phi}\big(y, |f(y)|/\lambda\big) \, d\mu(y) \le 1 \right\}$$

(cf. [51]).

For a measurable function f on X, we define the modular $\rho_{\Phi}(f)$ by

$$\rho_{\Phi}(f) = \int_X \overline{\Phi}(y, |f(y)|) \, d\mu(y).$$

LEMMA 2.5 ([46, Lemma 2.2] and [51, Theorem 8.14]). *Let* $\{f_i\}$ *be a sequence in* $L^{\Phi}(X)$. *Then* $\rho_{\Phi}(f_i)$ *converges to 0 if and only if* $\|f_i\|_{L^{\Phi}(X)}$ *converges to 0.*

3   Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,\Phi}(X)$

3.1   Basic properties

We say that a function $u \in L^{\Phi}(X)$ belongs to Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,\Phi}(X)$ if there exists a nonnegative function $g \in L^{\Phi}(X)$ such that

$$|u(x) - u(y)| \le d(x,y)(g(x) + g(y)) \tag{3.1}$$

for μ-almost every $x, y \in X$. Here, we call the function g a Hajłasz gradient of u. We define the norm

$$\|u\|_{M^{1,\Phi}(X)} = \|u\|_{L^{\Phi}(X)} + \inf \|g\|_{L^{\Phi}(X)},$$

where the infimum is taken over all Hajłasz gradients of u. For the case when $\Phi(x,t) = t^p$, the spaces $M^{1,p}(X)$ were first introduced by P. Hajłasz [21] as a generalization of the classical Sobolev spaces $W^{1,p}(\mathbf{R}^N)$ to the general setting of the quasi-metric measure spaces. For variable exponent spaces $M^{1,p(\cdot)}(X)$, see [32].

Since $L^{\Phi}(X)$ is a Banach space, standard arguments yield the following propositions (see [32]).

PROPOSITION 3.1 (cf. [32, Proposition 3.1]). *If* $L^{\Phi}(X)$ *is reflexive, then for every* $u \in M^{1,\Phi}(X)$, *there exist Hajłasz gradients of* u *which minimizes the norm. Moreover, if* $\|\cdot\|_{L^{\Phi}(X)}$ *is a uniformly convex norm, then there exists a unique Hajłasz gradient of* u *which minimizes the norm.*

REMARK 3.2. We say that $\Phi(x,t)$ is uniformly convex on X if for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$|a - b| \leq \varepsilon \max\{|a|, |b|\} \quad \text{or} \quad \overline{\Phi}\left(x, \frac{|a+b|}{2}\right) \leq (1 - \delta)\frac{\overline{\Phi}(x, |a|) + \overline{\Phi}(x, |b|)}{2}$$

for all $a, b \in \mathbf{R}$ and $x \in X$. By [14, Section 2.4], if $\Phi(x,t)$ is uniformly convex on X, then the norm $\|\cdot\|_{L^\Phi(X)}$ is a uniformly convex.

PROPOSITION 3.3 (cf. [32, Theorem 3.3]). $M^{1,\Phi}(X)$ is a Banach space.

PROPOSITION 3.4 (cf. [21, Theorem 5]). For every $u \in M^{1,\Phi}(X)$ and $\varepsilon > 0$, there exists a Lipschitz function $h \in M^{1,\Phi}(X)$ such that

(1) $\mu(\{x \in X : u(x) \neq h(x)\}) \leq \varepsilon$;

(2) $\|u - h\|_{M^{1,\Phi}(X)} \leq \varepsilon$.

Proof. For $u \in M^{1,\Phi}(X)$, we take $g \in L^\Phi(X)$ which is a Hajłasz gradient of u. Set

$$E_\lambda = \{x \in X : |u(x)| \leq \lambda \text{ and } g(x) \leq \lambda\}.$$

Note that u is Lipschitz continuous with a constant 2λ on E_λ. By McShane extension [47], we extend u to a Lipschitz function $\bar{u}$ on X, where

$$\bar{u}(x) = \inf_{y \in E_\lambda}\{u(y) + 2\lambda \operatorname{dist}(x, y)\}.$$

We modify this extension by truncating:

$$u_\lambda = (\operatorname{sign} \bar{u})\min\{|\bar{u}|, \lambda\}.$$

Then u_λ is Lipschitz continuous with a constant 2λ, $u = u_\lambda$ on E_λ and $|u_\lambda| \leq \lambda$. For every $\lambda > 1$, we see from (Φ2), (Φ3), (Φ4) and (2.2) that

$$\mu(\{x \in X : u(x) \neq u_\lambda(x)\}) \leq \mu(X \setminus E_\lambda)$$

$$\leq A_1 A_2 \int_{X \setminus E_\lambda} \Phi\left(x, \frac{|u(x)| + g(x)}{\lambda}\right) d\mu(x)$$

$$\leq A_1 A_2^2 \left\{\int_{X \setminus E_\lambda} \Phi\left(x, \frac{2|u(x)|}{\lambda}\right) d\mu(x) + \int_{X \setminus E_\lambda} \Phi\left(x, \frac{2g(x)}{\lambda}\right) d\mu(x)\right\}$$

$$\leq \frac{A_1 A_2^3}{\lambda} \left\{\int_{X \setminus E_\lambda} \Phi\left(x, 2|u(x)|\right) d\mu(x) + \int_{X \setminus E_\lambda} \Phi\left(x, 2g(x)\right) d\mu(x)\right\}$$

$$\leq \frac{2A_1 A_2^3 A_3}{\lambda} \left\{\int_{X \setminus E_\lambda} \Phi\left(x, |u(x)|\right) d\mu(x) + \int_{X \setminus E_\lambda} \Phi\left(x, g(x)\right) d\mu(x)\right\}.$$

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Hence we have $\mu(\{x \in X : u(x) \neq u_\lambda(x)\}) \to 0$ as $\lambda \to \infty$. Since $u_\lambda \leq \lambda \leq |u| + g$ in $X \setminus E_\lambda$, we find by ($\Phi 3$) and ($\Phi 4$) that

$$
\int_X \Phi(x, |u(x) - u_\lambda(x)|) \, d\mu(x)
$$

$$
= \int_{X \setminus E_\lambda} \Phi(x, |u(x) - u_\lambda(x)|) \, d\mu(x)
$$

$$
\leq A_2 \int_{X \setminus E_\lambda} \Phi(x, |u(x)| + |u_\lambda(x)|) \, d\mu(x)
$$

$$
\leq A_2^2 \int_{X \setminus E_\lambda} \{\Phi(x, 2|u(x)|) + \Phi(x, 2|u_\lambda(x)|)\} \, d\mu(x)
$$

$$
\leq 2A_2^2 A_3 \int_{X \setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, |u_\lambda(x)|)\} \, d\mu(x)
$$

$$
\leq 2A_2^3 A_3 \int_{X \setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, |u(x)| + g(x))\} \, d\mu(x)
$$

$$
\leq 4A_2^4 A_3^2 \int_{X \setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, |u(x)|) + \Phi(x, g(x))\} \, d\mu(x)
$$

$$
\leq 8A_2^4 A_3^2 \int_{X \setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, g(x))\} \, d\mu(x).
$$

Since $u, g \in L^\Phi(X)$ and $\mu(X \setminus E_\lambda) \to 0$ as $\lambda \to \infty$, $\rho_\Phi(u - u_\lambda)$ converges to 0 as $\lambda \to \infty$. Therefore, we see from Lemma 2.5 and (2.1) that $\|u - u_\lambda\|_{L^\Phi(X)}$ converges to 0 as $\lambda \to \infty$.

Next we consider the function $g_\lambda = (g + 3\lambda)\chi_{X \setminus E_\lambda}$, where χ_E denotes the characteristic function of E. Note that g_λ is a Hajłasz gradient of $u - u_\lambda$. We have by ($\Phi 3$) and ($\Phi 4$) that

$$
\int_X \Phi(x, g_\lambda(x)) \, d\mu(x) = \int_{X \setminus E_\lambda} \Phi(x, g(x) + 3\lambda) \, d\mu(x)
$$

$$
\leq 8A_2 A_3^3 \int_{X \setminus E_\lambda} \{\Phi(x, g(x)) + \Phi(x, \lambda)\} \, d\mu(x)
$$

$$
\leq 8A_2^2 A_3^3 \int_{X \setminus E_\lambda} \{\Phi(x, g(x)) + \Phi(x, |u(x)| + g(x))\} \, d\mu(x)
$$

$$
\leq 32A_2^3 A_3^4 \int_{X \setminus E_\lambda} \{\Phi(x, g(x)) + \Phi(x, |u(x)|)\} \, d\mu(x)
$$

and the above discussions implies that $\|g_\lambda\|_{L^\Phi(X)}$ converges to 0 as $\lambda \to \infty$. Thus the proposition is proved. $\qquad \square$

For a locally integrable function u on X and a ball $B(x, r) \subset X$, we define the mean integral:

$$
u_{B(x,r)} = \fint_{B(x,r)} u(y) \, d\mu(y) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) \, d\mu(y).
$$

We introduce a fractional sharp maximal operator. For every locally integrable function u on X, we define

$$u^\sharp(x) = \sup_{r>0} \frac{1}{r} \fint_{B(x,r)} |u(x) - u_{B(x,r)}| \, d\mu(x).$$

For a locally integrable function u on X, the Hardy-Littlewood maximal function Mu is defined by

$$Mu(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(y)| \, d\mu(y).$$

The following is a generalization of [22, Theorem 3.4], [23, Theorem 3.1] and [32, Theorem 5.2] (see also [18]).

For $a, b \in \mathbf{R}$, we write $a \sim b$ if $C^{-1}a \le b \le Ca$ for a constant $C > 0$.

THEOREM 3.5. *Let X be a doubling space. Suppose the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$. Then the following three statements are equivalent:*

(i) $u \in M^{1,\Phi}(X)$.

(ii) *$u \in L^\Phi(X)$ and there exists a nonnegative function $g \in L^\Phi(X)$ such that the Poincaré inequality*

$$\fint_{B(z,r)} |u(x) - u_{B(z,r)}| d\mu(x) \le Cr \fint_{B(z,r)} g(x) d\mu(x)$$

holds for every $z \in X$ and $r > 0$.

(iii) $u \in L^\Phi(X)$ and $u^\sharp \in L^\Phi(X)$.

Moreover, we obtain

$$\|u\|_{M^{1,\Phi}(X)} \quad \sim \quad \|u\|_{L^\Phi(X)} + \|u^\sharp\|_{L^\Phi(X)}$$

for all $u \in L^\Phi(X)$.

This theorem is proved as in [22, Theorem 3.4].

3.2 Sobolev capacity on Musielak-Orlicz-Hajłasz-Sobolev spaces

For $u \in M^{1,\Phi}(X)$, we define

$$\tilde{\rho}_\Phi(u) = \rho_\Phi(u) + \inf \rho_\Phi(g),$$

where the infimum is taken over all Hajłasz gradients of u. For $E \subset X$, we write

$$S_\Phi(E) = \{u \in M^{1,\Phi}(X) : u \ge 1 \text{ in an open set containing } E\}.$$

The Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces is defined by

$$C_\Phi(E) = \inf_{u \in S_\Phi(E)} \tilde{\rho}_\Phi(u).$$

In case $S_\Phi(E) = \emptyset$, we set $C_\Phi(E) = \infty$.

REMARK 3.6. We can redefine the Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces by

$$C_\Phi(E) = \inf_{u \in S'_\Phi(E)} \tilde{\rho}_\Phi(u)$$

since $M^{1,\Phi}(X)$ is lattice (see [37, Lemma 2.4]), where

$$S'_\Phi(E) = \{u \in S_\Phi(X) : 0 \le u \le 1\}.$$

The standard argument yields the following results (see [32, Theorem 3.11] and [37, Theorem 3.2, Remark 3.3 and Lemma 3.4].

PROPOSITION 3.7. *The set function $C_\Phi(\cdot)$ satisfies the following properties:*

(1) $C_\Phi(\cdot)$ *is an outer measure;*

(2) $C_\Phi(\emptyset) = 0$;

(3) $C_\Phi(E_1) \le C_\Phi(E_2)$ *for $E_1 \subset E_2 \subset X$;*

(4) $C_\Phi(E) = \inf_{E \subset U, U:open} C_\Phi(U)$ *for $E \subset X$ ($C_\Phi(\cdot)$ is an outer capacity);*

(5) *If $K_1 \supset K_2 \supset \cdots$ are compact sets on X, then*

$$\lim_{i \to \infty} C_\Phi(K_i) = C_\Phi\left(\bigcap_{i=1}^\infty K_i\right).$$

Further, as in the proof of [38, Theorem 4.1], we have the following result by [14, Theorem 2.2.8].

PROPOSITION 3.8. *If $L^\Phi(X)$ is reflexive and $E_1 \subset E_2 \subset \cdots$ are subsets of X, then*

$$\lim_{i \to \infty} C_\Phi(E_i) = C_\Phi\left(\bigcup_{i=1}^\infty E_i\right).$$

We say that a property holds C_Φ-q.e. in X, if it holds except of a set $F \subset X$ with $C_\Phi(F) = 0$.

THEOREM 3.9. *For each Cauchy sequence of functions in $M^{1,\Phi}(X) \cap C(X)$, there is a subsequence which converges pointwise C_Φ-q.e. in X. Moreover, the convergence is uniform outside a set of arbitrary small Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces.*

Proof. Let $\{u_i\}$ be a Cauchy sequence of functions in $M^{1,\Phi}(X) \cap C(X)$. Since, for all $0 < \varepsilon < 1$, $\|u\|_{M^{1,\Phi}(X)} < \varepsilon$ implies $\tilde{\rho}_\Phi(u) < \varepsilon$, we can take a subsequence of $\{u_i\}$, which we still denote by $\{u_i\}$, such that $\tilde{\rho}_\Phi(u_i - u_{i+1}) \le 2^{-i} A_2^{-1}(2A_3)^{-i-1}$ for each positive integers i. Consider the set

$$E_i = \{x \in X : |u_i(x) - u_{i+1}(x)| > 2^{-i}\}$$

and $F_j = \bigcup_{i=j}^\infty E_i$. Here note that $2^i|u_i - u_{i+1}| \in S_\Phi(E_i)$ by continuity of u_i. Since g_i is also a Hajłasz gradient of $|u_i - u_{i+1}|$ if g_i is a Hajłasz gradient of $u_i - u_{i+1}$, we have by (Φ4) and (2.1) that

$$C_\Phi(E_i) \le \tilde{\rho}_\Phi(2^i|u_i - u_{i+1}|) \le A_2(2A_3)^{i+1}\tilde{\rho}_\Phi(u_i - u_{i+1}) \le 2^{-i}.$$

Then it follows from Proposition 3.7 that

$$C_\Phi(F_j) \le \sum_{i=j}^\infty C_\Phi(E_i) \le 2^{-j+1}.$$

Hence, we obtain

$$C_\Phi\left(\bigcap_{j=1}^\infty F_j\right) \le \lim_{j \to \infty} C_\Phi(F_j) = 0$$

and $\{u_i\}$ converges in $X \setminus \bigcap_{j=1}^\infty F_j$. Moreover we find

$$|u_j(x) - u_k(x)| \le \sum_{i=j}^{k-1} |u_i(x) - u_{i+1}(x)| \le 2^{-j+1}$$

whenever $x \in X \setminus F_j$ for every $k > j$, which implies $\{u_i\}$ converges uniformly in $X \setminus F_j$. $\qquad\square$

We say that a function u is C_Φ-quasicontinuous on X if, for any $\varepsilon > 0$, there is a set E such that $C_\Phi(E) < \varepsilon$ and u is continuous on $X \setminus E$. By Proposition 3.4 and Theorem 3.9, we have the following result.

PROPOSITION 3.10. *For each $u \in M^{1,\Phi}(X)$, there is a C_Φ-quasicontinuous function $v \in M^{1,\Phi}(X)$ such that $u = v$ μ-a.e. in X.*

As in the proof of [37, Lemma 4.1], we have the following result.

LEMMA 3.11. $\mu(E) \le CC_\Phi(E)$ *for every $E \subset X$.*

In fact, note that for $u \in S_\Phi(E)$

$$\mu(E) \le A_1 A_2 \int_X \Phi(x, |u(x)|)\, d\mu(x) \le 2A_1 A_2 A_3 \rho_\Phi(u)$$

by (2.1), (Φ2) and (Φ3).

THEOREM 3.12. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi 5)$. *Then there exists a constant* $C > 0$ *such that*

$$C_\Phi(B(x_0, r)) \leq C\Phi(x_0, r^{-1})\mu(B(x_0, 2r))$$

for all $x_0 \in X$ *and* $0 < r \leq 1$.

Proof. Define

$$u(x) = \begin{cases} \dfrac{2r - d(x, x_0)}{r} & x \in B(x_0, 2r) \setminus B(x_0, r), \\ 1 & x \in B(x_0, r), \\ 0 & x \in X \setminus B(x_0, 2r) \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{1}{r} & x \in B(x_0, 2r), \\ 0 & x \in X \setminus B(x_0, 2r). \end{cases}$$

Then note from [37, Theorem 4.6] that g is a Hajłasz gradient of u and $u \in S_\Phi(B(x_0, r))$. Hence, we have by $(\Phi 2)$, $(\Phi 3)$, $(\Phi 5)$ and (2.1)

$$\begin{aligned} C_\Phi(B(x_0, r)) &\leq \int_{B(x_0, 2r)} \overline{\Phi}(x, u(x))\, d\mu(x) + \int_{B(x_0, 2r)} \overline{\Phi}(x, g(x))\, d\mu(x) \\ &\leq A_2 \int_{B(x_0, 2r)} \Phi(x, u(x))\, d\mu(x) + A_2 \int_{B(x_0, 2r)} \Phi(x, r^{-1})\, d\mu(x) \\ &\leq A_1 A_2^2 \mu(B(x_0, 2r)) + A_2 B_{2,1} \Phi(x_0, r^{-1})\mu(B(x_0, 2r)) \\ &\leq A_2(A_1^2 A_2^2 + B_{2,1})\Phi(x_0, r^{-1})\mu(B(x_0, 2r)), \end{aligned}$$

as required. $\qquad\qquad\square$

3.3 Lebesgue points in Musielak-Orlicz-Hajłasz-Sobolev spaces

Let X be a doubling space. We recall from [36, Section 3] the definition of the discrete maximal function. Fix $r > 0$ and let $B(x_i, r), i = 1, 2, \cdots$, be a family of balls covering X such that any point $x \in X$ belongs to at most θ balls $B(x_i, 6r)$. Here, θ can be chosen to depend only on the doubling constant c_2. Let $\{\varphi_i\}$ be a set of functions such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ in the complement of $B(x_i, 3r)$, $\varphi_i \geq c_3 > 0$ in $B(x_i, r)$, φ_i is Lipschitz with a constant c_3/r and $\sum_{i=1}^\infty \varphi_i = 1$ on X. We set

$$u_r(x) = \sum_{i=1}^\infty \frac{\varphi_i(x)}{\mu(B(x_i, 3r))} \int_{B(x_i, 3r)} |u(y)|\, d\mu(y).$$

Let $\{r_j\}$ be an enumeration of the positive rationals. For every radius r_j, we choose a covering $\{B(x_i, r_j)\}$ as above. We define the discrete maximal function related to covering $\{B(x_i, r_j)\}$ by

$$M^* u(x) = \sup_j u_{r_j}(x).$$

Note that the discrete maximal function related to covering $\{B(x_i, r_j)\}$ depends on the chosen coverings. However, by [36, Lemma 3.1], the inequalities

$$c_M^{-1} Mu(x) \leq M^* u(x) \leq c_M Mu(x) \qquad (3.2)$$

always hold for every $x \in X$ and every $u \in L^1_{loc}(X)$. Here the constant $c_M \geq 1$ depends only on the doubling constant.

LEMMA 3.13. *Let X be a doubling space. Suppose the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$. Then there exists a constant $C > 0$ such that*

$$C_\Phi(\{x \in X : Mu(x) > \lambda\}) \leq C\lambda^{-\log_2(2A_3)} \|u\|_{M^{1,\Phi}(X)}$$

for all $0 < \lambda < 1$ and $u \in M^{1,\Phi}(X)$ with $\|u\|_{M^{1,\Phi}(X)} \leq 1$.

Proof. Let $u \in M^{1,\Phi}(X)$ with $\|u\|_{M^{1,\Phi}(X)} \leq 1$ and let g be a Hajłasz gradient of u. By our assumption, there exists a constant $B_M > 0$ such that

$$\|Mv\|_{L^\Phi(X)} \leq B_M \|v\|_{L^\Phi(X)}$$

for all $v \in L^\Phi(X)$.

By (3.2), we have

$$\{x \in X : Mu(x) > \lambda\} \subset E_\lambda,$$

where $E_\lambda = \{x \in X : c_M M^* u(x) > \lambda\}$ is open since a supremum of continuous functions is lower semicontinuous.

Note from the proof of [36, Theorem 3.6] that $c_M M^* u / \lambda \in S_\Phi(E_\lambda)$ and cMg is a Hajłasz gradient of $M^* u$ for some constant $c \geq 1$. We have by (Φ3), (Φ4) and (2.2)

$$C_\Phi(E_\lambda)$$
$$\leq \int_X \overline{\Phi}(x, c_M M^* u(x)/\lambda) \, d\mu(x) + \int_X \overline{\Phi}(x, cc_M Mg(x)/\lambda) \, d\mu(x)$$
$$\leq A_2 \int_X \Phi(x, c_M M^* u(x)/\lambda) \, d\mu(x) + A_2 \int_X \Phi(x, cc_M Mg(x)/\lambda) \, d\mu(x)$$
$$\leq 2A_2^2 A_3 \left(\frac{cc_M}{\lambda}\right)^{\log_2(2A_3)} \left\{\int_X \Phi(x, M^* u(x)) \, d\mu(x) + \int_X \Phi(x, Mg(x)) \, d\mu(x)\right\}.$$

Since $\|Mu/B_M\|_{L^\Phi(X)} \leq \|u\|_{L^\Phi(X)} \leq 1$, we find by ($\Phi$3), ($\Phi$4), (2.2) and (3.2) that

$$\int_X \Phi(x, M^* u(x)) \, d\mu(x) \leq A_2 \int_X \Phi(x, c_M Mu(x)) \, d\mu(x)$$
$$\leq 2A_2^2 A_3 (c_M B_M)^{\log_2(2A_3)} \int_X \Phi(x, Mu(x)/B_M) \, d\mu(x)$$
$$\leq 4A_2^2 A_3^2 (c_M B_M)^{\log_2(2A_3)} \int_X \overline{\Phi}(x, Mu(x)/B_M) \, d\mu(x)$$
$$\leq 4A_2^2 A_3^2 (c_M B_M)^{\log_2(2A_3)} \|Mu/B_M\|_{L^\Phi(X)}$$
$$\leq 4A_2^2 A_3^2 (c_M B_M)^{\log_2(2A_3)} \|u\|_{L^\Phi(X)}.$$

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Similarly, we have

$$\int_X \Phi(x, Mg(x)) \, d\mu(x) \le 2A_2 A_3 (B_M)^{\log_2(2A_3)} \int_X \Phi(x, Mg(x)/B_M) \, d\mu(x)$$
$$\le 4A_2 A_3^2 (B_M)^{\log_2(2A_3)} \|g\|_{L^\Phi(X)}.$$

Thus we obtain the required result. $\qquad\square$

As in the proof of [36, Theorem 4.5], we can show the following result by Lemma 3.13.

THEOREM 3.14. *Let X be a doubling space and let $u \in M^{1,\Phi}(X)$. Suppose the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$. Then there exists a set $E \subset X$ of zero Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces such that*

$$\tilde{u}(x) = \lim_{r \to 0} u_{B(x,r)}$$

for every $x \in X \setminus E$, where $\tilde{u}$ is the C_Φ-quasicontinuous representative of u.

4 Musielak-Orlicz-Newton-Sobolev spaces $N^{1,\Phi}(X)$

4.1 Basic properties

A curve γ in X is a nonconstant continuous map $\gamma : I \to X$, where $I = [a, b]$ is a closed interval in $\mathbf{R}$. The image of γ is denoted by $|\gamma|$. Let Γ be a family of rectifiable curves in X. We denote by $F(\Gamma)$ the set of all admissible functions, that is, all Borel measurable functions $h : X \to [0, \infty]$ such that

$$\int_\gamma h \, ds \ge 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to path length. We define the Φ-modulus of Γ by

$$M_\Phi(\Gamma) = \inf_{h \in F(\Gamma)} \rho_\Phi(h).$$

If $F(\Gamma) = \emptyset$, then we set $M_\Phi(\Gamma) = \infty$.

LEMMA 4.1 (c.f. [30, Lemma 2.1]). *$M_\Phi(\cdot)$ is an outer measure.*

Proof. Since it is obvious that $M_\Phi(\emptyset) = 0$ and $\Gamma_1 \subset \Gamma_2$ implies $M_\Phi(\Gamma_1) \le M_\Phi(\Gamma_2)$, we show that $M_\Phi(\cdot)$ is a countably subadditive capacity. For $\varepsilon > 0$, we take $h_i \in F(\Gamma_i)$ such that

$$\int_X \overline{\Phi}(x, h_i(x)) \, d\mu(x) \le M_\Phi(\Gamma_i) + \varepsilon 2^{-i}.$$

We set $h = \sup_i h_i$. Noting that h satisfies $\int_\gamma h\,ds \geq 1$ for every $\gamma \in \bigcup_{i=1}^\infty \Gamma_i$, we have

$$M_\Phi\left(\bigcup_{i=1}^\infty \Gamma_i\right) \leq \rho_\Phi(h) \leq \sum_{i=1}^\infty \int_X \overline{\Phi}(x, h_i(x))\,d\mu(x) \leq \sum_{i=1}^\infty M_\Phi(\Gamma_i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we have the required result. $\qquad\square$

A family of curves Γ is said to be exceptional if $M_\Phi(\Gamma) = 0$. The following lemma is an extension of [32, Lemma 4.1]. The proof is the same as the proof of [30, Lemma 2.2].

LEMMA 4.2 (Fuglede's lemma). *Let $\{u_i\}$ be a sequence of nonnegative Borel functions in $L^\Phi(X)$ converging to zero in $L^\Phi(X)$. Then there exist a subsequence $\{u_{i_k}\}$ and an exceptional family Γ of rectifiable curves such that for every $\gamma \notin \Gamma$ we have*

$$\lim_{k \to \infty} \int_\gamma u_{i_k}\,ds = 0.$$

Let u be a real valued function on X. A nonnegative Borel measurable function h is said to be a Φ-weak upper gradient of u if there exists a family Γ of rectifiable curves with $M_\Phi(\Gamma) = 0$ and

$$|u(x) - u(y)| \leq \int_\gamma h\,ds$$

for every rectifiable curve $\gamma \notin \Gamma$ with endpoints x and y. Here note that basic properties of p-weak upper gradients can be extended to basic properties of Φ-weak upper gradients as in the proof of [6, Chapter 1].

We define the norm

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|h\|_{L^\Phi(X)},$$

where the infimum is taken over all Φ-weak upper gradients of u. We say that $u \in L^\Phi(X)$ belongs to Musielak-Orlicz-Newton-Sobolev space $N^{1,\Phi}(X)$ if $\|u\|_{N^{1,\Phi}(X)} < \infty$.

REMARK 4.3. Let u be a real valued function on X and let h be a Φ-weak upper gradient of u. Suppose Γ is a family of rectifiable curves γ satisfying that there exists a rectifiable subcurve γ' of γ, that is $|\gamma'| \subset |\gamma|$, such that

$$|u(x') - u(y')| \nleq \int_{\gamma'} h\,ds,$$

where x' and y' are endpoints of γ'. Then note that $M_\Phi(\Gamma) = 0$ (see [6, Lemma 1.40]).

LEMMA 4.4 (c.f. [36, Lemma 2.6] and [28, Lemma 3]). *Suppose $\{u_i\}$ are measurable functions. Let g_i be a Φ-weak upper gradient of u_i. If $u = \sup_i u_i$ is finite almost everywhere, then $g = \sup_i g_i$ is a Φ-weak upper gradient of u.*

For $u \in N^{1,\Phi}(X)$, we set

$$\hat{\rho}_\Phi(u) = \rho_\Phi(u) + \inf \rho_\Phi(h),$$

where the infimum is taken over all Φ-weak upper gradients of u. For $E \subset X$, we denote

$$s_\Phi(E) = \{u \in N^{1,\Phi}(X) : u \geq 1 \text{ on } E\}.$$

We define the capacity in Musielak-Orlicz-Newton-Sobolev spaces by

$$c_\Phi(E) = \inf_{u \in s_\Phi(E)} \hat{\rho}_\Phi(u).$$

In case $s_\Phi(E) = \emptyset$, we set $c_\Phi(E) = \infty$. For the definition of the Sobolev capacity, see [6, Section 6.2].

By Lemma 4.4, we have the following result.

PROPOSITION 4.5. *The set function $c_\Phi(\cdot)$ is an outer measure.*

Proof. Since it is obvious that $c_\Phi(\emptyset) = 0$ and $E_1 \subset E_2$ implies $c_\Phi(E_1) \leq c_\Phi(E_2)$, we only show that $c_\Phi(\cdot)$ is a countably subadditive capacity. Let E_i be subsets in X. We may assume that $\sum_{i=1}^{\infty} c_\Phi(E_i) < \infty$. For $\varepsilon > 0$, we take $u_i \in s_\Phi(E_i)$ such that

$$\int_X \overline{\Phi}(x, |u_i(x)|) \, d\mu(x) + \int_X \overline{\Phi}(x, h_i(x)) \, d\mu(x) \leq c_\Phi(E_i) + \varepsilon 2^{-i},$$

where h_i is a Φ-weak upper gradient of u_i. Set $u = \sup_i u_i$ and $h = \sup_i h_i$. Noting that $u \in L^\Phi(X)$ and $h \in L^\Phi(X)$, we find that h is a Φ-weak upper gradient of u by Lemma 4.4 and $u \in s_\Phi\left(\bigcup_{i=1}^{\infty} E_i\right)$. Hence, we have

$$
\begin{aligned}
c_\Phi\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \hat{\rho}_\Phi(u) \\
&\leq \sum_{i=1}^{\infty} \left\{ \int_X \overline{\Phi}(x, |u_i(x)|) \, d\mu(x) + \int_X \overline{\Phi}(x, h_i(x)) \, d\mu(x) \right\} \\
&\leq \sum_{i=1}^{\infty} c_\Phi(E_i) + \varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we have the required result. $\qquad\square$

We denote by Γ_E the family of all rectifiable curves whose image intersects the set E.

LEMMA 4.6. *Let $E \subset X$. If $c_\Phi(E) = 0$, then $M_\Phi(\Gamma_E) = 0$.*

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Proof. Let $E \subset X$ with $c_\Phi(E) = 0$. Then, for all positive integers i, we choose a functions $u_i \in N^{1,\Phi}(X)$ with Φ-weak upper gradients κ_i such that $u_i(x) \geq 1$ for every $x \in E$ and

$$\int_X \overline{\Phi}(x, |u_i(x)|) \, d\mu(x) + \int_X \overline{\Phi}(x, \kappa_i(x)) \, d\mu(x) \leq A_2^{-1}(2A_3)^{-i-1}.$$

Set $v_k = \sum_{i=1}^k |u_i|$. Then note that $h_k = \sum_{i=1}^k \kappa_i$ is a Φ-weak upper gradient of v_k. Since

$$\int_X \overline{\Phi}\left(x, \frac{|u_i(x)|}{2^{-i}}\right) d\mu(x) \leq A_2(2A_3)^i \int_X \Phi\left(x, |u_i(x)|\right) d\mu(x)$$

$$\leq A_2(2A_3)^{i+1} \int_X \overline{\Phi}\left(x, |u_i(x)|\right) d\mu(x) \leq 1$$

and

$$\int_X \overline{\Phi}\left(x, \frac{\kappa_i(x)}{2^{-i}}\right) d\mu(x) \leq 1$$

by (2.1) and (Φ4), we have

$$\|v_\ell - v_m\|_{L^\Phi(X)} \leq \sum_{i=m+1}^\ell \|u_i\|_{L^\Phi(X)} \leq 2^{-m}$$

and

$$\|h_\ell - h_m\|_{L^\Phi(X)} \leq \sum_{i=m+1}^\ell \|\kappa_i\|_{L^\Phi(X)} \leq 2^{-m}$$

for every $\ell > m$. Hence $\{v_k\}$ and $\{h_k\}$ are Cauchy sequences in $L^\Phi(X)$. Therefore, $\{h_k\}$ converges to a function h in $L^\Phi(X)$, which we may assume to be a Borel function. Setting $v(x) = \lim_{k \to \infty} v_k(x)$ for every $x \in X$, we find $v \in L^\Phi(X)$. Since $v_k(x) \geq k$ for $x \in E$, we have

$$E \subset E_\infty = \{x \in X : v(x) = \infty\}.$$

Hence it suffices to show that $M_\Phi(\Gamma_{E_\infty}) = 0$.

It follows from Lemma 4.2 that there exists a subsequence $\{h_{k_j}\}$ of $\{h_k\}$ such that there exists an exceptional family Γ_1 and

$$\lim_{j \to \infty} \int_\gamma |h_{k_j} - h| \, ds = 0$$

for all rectifiable curves $\gamma \notin \Gamma_1$. Set

$$\Gamma_2 = \left\{\gamma : \gamma \text{ is a rectifiable curve satisfying } \int_\gamma v \, ds = \infty\right\}$$

and

$$\Gamma_3 = \left\{ \gamma : \gamma \text{ is a rectifiable curve satisfying } \int_\gamma h \, ds = \infty \right\}.$$

We see from the convexity of $\overline{\Phi}$ that

$$M_\Phi(\Gamma_2) \leq \int_X \overline{\Phi}\left(x, \frac{v(x)}{i}\right) d\mu(x) \leq \frac{\|v\|_{L^\Phi(X)}}{i}$$

for all $i \geq \|v\|_{L^\Phi(X)}$. Hence $M_\Phi(\Gamma_2) = 0$. Similarly, $M_\Phi(\Gamma_3) = 0$. We denote by $\Gamma_{4,i}$ the exceptional family of rectifiable curves for u_i in Remark 4.3 and by Γ_4 the union of $\Gamma_{4,i}$. By Remark 4.3 and Lemma 4.1, we have $M_\Phi(\Gamma_4) = M_\Phi(\cup \Gamma_{4,i}) = 0$. Hence we find $M_\Phi(\Gamma_0) = 0$, where $\Gamma_0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

To complete the proof, we show that $\Gamma_{E_\infty} \subset \Gamma_0$. Suppose $\gamma \notin \Gamma_0$. Since $\gamma \notin \Gamma_2$, there is $y \in |\gamma|$ with $v(y) < \infty$. For any $x \in |\gamma|$, we find that

$$v_{k_j}(x) \leq v_{k_j}(y) + |v_{k_j}(x) - v_{k_j}(y)| \leq v_{k_j}(y) + \int_\gamma h_{k_j} \, ds$$

since $\gamma \notin \Gamma_4$. Letting $j \to \infty$, we have

$$v(x) = \lim_{j \to \infty} v_{k_j}(x) \leq v(y) + \int_\gamma h \, ds$$

since $\gamma \notin \Gamma_1$. Since $\gamma \notin \Gamma_3$ and $v(y) < \infty$, $v(x) < \infty$ for all $x \in |\gamma|$, which implies $\gamma \notin \Gamma_{E_\infty}$, as required. $\square$

The standard arguments and Lemma 4.6 yield the following proposition (see [32]).

PROPOSITION 4.7 (cf. [32, Theorem 4.4]). $N^{1,\Phi}(X)$ is a Banach space.

We say that X supports a $(1,1)$-Poincaré inequality if there exists a constant $C > 0$ such that for all open balls B in X,

$$\frac{1}{\mu(B)} \int_B |u(x) - u_B| \, d\mu(x) \leq C d_B \frac{1}{\mu(B)} \int_B h(x) \, d\mu(x)$$

holds whenever h is a Φ-weak upper gradient of u on B and u is integrable on B.

LEMMA 4.8. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. Assume that the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$. Then Lipschitz continuous functions are dense in $N^{1,\Phi}(X)$.*

Proof. Let $u \in N^{1,\Phi}(X)$ and let h be a Φ-weak upper gradient of u. By truncation, we may assume that u is a bounded function on X, say $|u| \leq u_0$ for $u_0 > 1$ (see [52, Lemma 4.3]). Set

$$E_\lambda = \{x \in X : Mh(x) > \lambda\}.$$

As in the proof of [32, Theorem 4.5], we can define

$$u_\lambda(x) = \lim_{r \to 0} u_{B(x,r)}$$

for all $x \in X \setminus E_\lambda$ and u_λ is $c\lambda$-Lipschitz in $X \setminus E_\lambda$ with some constant $c > 1$. We extend u_λ as a Lipschitz function to all of X by McShane extension [47], by setting

$$u_\lambda(x) = \inf_{y \in X \setminus E_\lambda} \{u_\lambda(y) + c\lambda d(x,y)\}.$$

We may assume that u_λ is still bounded by u_0 by truncation. Then we have by ($\Phi2$), ($\Phi3$) and ($\Phi4$) that

$$
\begin{aligned}
&\int_X \Phi(x, |u(x) - u_\lambda(x)|) \, d\mu(x) \\
&= \int_{E_\lambda} \Phi(x, |u(x) - u_\lambda(x)|) \, d\mu(x) \\
&\leq 2A_2^2 A_3 \left\{ \int_{E_\lambda} \Phi(x, |u(x)|) \, d\mu(x) + \int_{E_\lambda} \Phi(x, |u_\lambda(x)|) \, d\mu(x) \right\} \\
&\leq 4A_2^3 A_3 \int_{E_\lambda} \Phi(x, u_0) \, d\mu(x) \\
&\leq 8A_1 A_2^4 A_3^2 u_0^{\log_2(2A_3)} \mu(E_\lambda).
\end{aligned}
$$

Hence we see from the boundedness of the Hardy-Littlewood maximal operator on $L^\Phi(X)$, Lemma 2.5 and (2.1) that $u_\lambda \to u$ in $L^\Phi(X)$. Since E_λ is open and $u - u_\lambda$ is zero μ-a.e. in $X \setminus E_\lambda$, we may assume that the Φ-weak upper gradient of $u - u_\lambda$ is zero in $X \setminus E_\lambda$ (see [52, Lemma 4.3]). Since

$$\int_X \Phi(x, \lambda \chi_{E_\lambda}(x)) \, d\mu(x) \leq A_2 \int_X \Phi(x, Mh(x)) \, d\mu(x) < \infty$$

by the boundedness of the Hardy-Littlewood maximal operator on $L^\Phi(X)$, we find that the function $(c\lambda + h)\chi_{E_\lambda} \in L^\Phi(X)$ is a Φ-weak upper gradient of $u - u_\lambda$. Hence $u - u_\lambda \in N^{1,\Phi}(X)$ and therefore so is u_λ. We have

$$
\begin{aligned}
&\int_X \Phi(x, (c\lambda + h)\chi_{E_\lambda}(x)) \, d\mu(x) \\
&\leq 4A_2^3 A_3^2 c^{\log_2(2A_3)} \left\{ \int_{E_\lambda} \Phi(x, \lambda) \, d\mu(x) + \int_{E_\lambda} \Phi(x, h(x)) \, d\mu(x) \right\} \\
&\leq 4A_2^4 A_3^2 c^{\log_2(2A_3)} \left\{ \int_{E_\lambda} \Phi(x, Mh(x)) \, d\mu(x) + \int_{E_\lambda} \Phi(x, h(x)) \, d\mu(x) \right\}.
\end{aligned}
$$

Then the right hand side converges to zero as $\lambda \to \infty$. Hence $\{u_\lambda\}$ converges to u in $N^{1,\Phi}(X)$ by Lemma 2.5 and (2.1). $\qquad\square$

4.2 Lebesgue points in Musielak-Orlicz-Newton-Sobolev spaces

LEMMA 4.9. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. If the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$, then there exists a constant $C > 0$ such that*

$$c_\Phi(\{x \in X : Mu(x) > \lambda\}) \leq C\lambda^{-\log_2(2A_3)}\|u\|_{N^{1,\Phi}(X)}$$

for all $0 < \lambda < 1$ and $u \in N^{1,\Phi}(X)$ with $\|u\|_{N^{1,\Phi}(X)} \leq 1$.

Proof. Let $u \in N^{1,\Phi}(X)$ with $\|u\|_{N^{1,\Phi}(X)} \leq 1$ and $h \in L^\Phi(X)$ be a Φ-weak upper gradient of u. By (3.2), we have

$$\{x \in X : Mu(x) > \lambda\} \subset E_\lambda,$$

where $E_\lambda = \{x \in X : c_M M^*u(x) > \lambda\}$. Here, note from the boundedness of the Hardy-Littlewood maximal operator on $L^\Phi(X)$, Lemma 4.4 and [28, Lemma 5] that $M^*u \in L^\Phi(X)$ and $cMh \in L^\Phi(X)$ is a Φ-weak upper gradient of M^*u for some constant $c \geq 1$. Since $c_M M^*u/\lambda \in s_\Phi(E_\lambda)$, we have by ($\Phi$3), ($\Phi$4) and (2.2) that

$$
\begin{aligned}
&c_\Phi(E_\lambda)\\
&\leq \int_X \overline{\Phi}(x, c_M M^*u(x)/\lambda)\,d\mu(x) + \int_X \overline{\Phi}(x, cc_M Mh(x)/\lambda)\,d\mu(x)\\
&\leq 2A_2^2 A_3 \left(\frac{cc_M}{\lambda}\right)^{\log_2(2A_3)}\left\{\int_X \Phi(x, M^*u(x))\,d\mu(x) + \int_X \Phi(x, Mh(x))\,d\mu(x)\right\}.
\end{aligned}
$$

Thus, as in the proof of Lemma 3.13, we obtain the required result. $\square$

As in the proof of [28, Theorem 1], we can show the following result by Lemma 4.9.

THEOREM 4.10. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. If the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$ and $u \in N^{1,\Phi}(X)$, then there exists a set $E \subset X$ of zero Sobolev capacity in Musielak-Orlicz-Newton-Sobolev space such that*

$$u(x) = \lim_{r \to 0} u_{B(x,r)}$$

and

$$\lim_{r \to +0} \fint_{B(x,r)} |u(y) - u(x)|\,d\mu(y) = 0$$

for every $x \in X \setminus E$.

5 Equivalence of function spaces

Let $\mathbf{R}^N$ be the N-dimensional Euclidean space. In the case $X = \mathbf{R}^N$, let μ be the Lebesgue measure on $\mathbf{R}^N$ and let d be the Euclidean metric. We define the Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\mathbf{R}^N)$ by

$$W^{1,\Phi}(\mathbf{R}^N) = \{u \in L^{\Phi}(\mathbf{R}^N) : |\nabla u| \in L^{\Phi}(\mathbf{R}^N)\}.$$

The norm

$$\|u\|_{W^{1,\Phi}(\mathbf{R}^N)} = \|u\|_{L^{\Phi}(\mathbf{R}^N)} + \||\nabla u|\|_{L^{\Phi}(\mathbf{R}^N)}$$

makes $W^{1,\Phi}(\mathbf{R}^N)$ a Banach space.

We prove relations between Musielak-Orlicz-Hajłasz-Sobolev space and Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\mathbf{R}^N)$.

PROPOSITION 5.1. $M^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\Phi}(\mathbf{R}^N)$. *Moreover, if the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(\mathbf{R}^N)$, then $M^{1,\Phi}(\mathbf{R}^N) = W^{1,\Phi}(\mathbf{R}^N)$.*

Proof. First we show $M^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\Phi}(\mathbf{R}^N)$. Let $u \in M^{1,\Phi}(\mathbf{R}^N)$ and let $g \in L^{\Phi}(\mathbf{R}^N)$ be a Hajłasz gradient of u. Since $t \leq A_1 A_2 \Phi(x,t)$ for $t \geq 1$ by $(\Phi 2)$ and (2.2), we have $g \in L^1(B)$ for every ball B and hence ∇u exists and satisfies $|\nabla u(x)| \leq Cg(x)$ for a.e $x \in \mathbf{R}^N$ by [33, Remark 5.13]. Thus we have $M^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\Phi}(\mathbf{R}^N)$.

Next we prove the second claim. Let $u \in W^{1,\Phi}(\mathbf{R}^N)$. Then we have by [21, Section 2]

$$|u(x) - u(y)| \leq |x - y|(M|\nabla u|(x) + M|\nabla u|(y))$$

for a.e. $x, y \in \mathbf{R}^N$. By the boundedness of the Hardy-Littlewood maximal operator on $L^{\Phi}(\mathbf{R}^N)$, we find $M|\nabla u| \in L^{\Phi}(\mathbf{R}^N)$ is a Hajłasz gradients of u. Hence we obtain the required result. $\square$

THEOREM 5.2. $N^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\Phi}(\mathbf{R}^N)$. *Moreover, if $W^{1,\Phi}(\mathbf{R}^N)$ is reflexive and C^1-functions are dense in $W^{1,\Phi}(\mathbf{R}^N)$, then $N^{1,\Phi}(\mathbf{R}^N) = W^{1,\Phi}(\mathbf{R}^N)$.*

Proof. The proof of the first claim is exactly the same as the proof of [32, Theorem 5.3]. Hence we only show the second claim. Let $u \in W^{1,\Phi}(\mathbf{R}^N)$. Then we can take $\{u_i\} \subset W^{1,\Phi}(X) \cap C^1(X)$ such that u_i converges to u in $W^{1,\Phi}(X)$. By the proof of [30, Theorem 4.2], we see that the sum of absolute value of the distributional gradient of u_i is a Φ-weak upper gradient of u in $\mathbf{R}^N$. Hence we obtain the required results. $\square$

REMARK 5.3. By [45, Theorem 3.5], we know that C^1-functions are dense in $W^{1,\Phi}(\mathbf{R}^N)$ if $\Phi(x,t)$ satisfies $(\Phi 5)$ and $(\Phi 6)$.

THEOREM 5.4. *For $u \in M^{1,\Phi}(X)$, there exists a representative $\tilde{u}$ of u such that*

$$\|\tilde{u}\|_{N^{1,\Phi}(X)} \leq 4\|u\|_{M^{1,\Phi}(X)}$$

Further, if X is a doubling space that supports a $(1,1)$-Poincaré inequality and the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$, then $M^{1,\Phi}(X) \supset N^{1,\Phi}(X)$.

Proof. Let $u \in M^{1,\Phi}(X)$ and let $g \in L^{\Phi}(X)$ be a Hajłasz gradient of u. If u is continuous on X, we find that $4g$ is a Φ-weak upper gradient of u as in [52, Lemma 4.7]. Since continuous functions are dense in $M^{1,\Phi}(X)$ by Proposition 3.4, we can take $\{u_i\} \subset M^{1,\Phi}(X)$ such that u_i is continuous on X, u_i converges to u in $M^{1,\Phi}(X)$ and

$$\|u_n - u_m\|_{N^{1,\Phi}(X)} \leq 4\|u_n - u_m\|_{M^{1,\Phi}(X)}$$

for all positive integers n, m. Therefore, $\{u_i\} \subset N^{1,\Phi}(X)$ is a Cauchy sequence. Hence there exists a $\tilde{u} \in N^{1,\Phi}(X)$ such that

$$\|\tilde{u}\|_{N^{1,\Phi}(X)} \leq 4\|u\|_{M^{1,\Phi}(X)}$$

since $N^{1,\Phi}(X)$ is a Banach space by Proposition 4.7. Noting that $u(x) = \tilde{u}(x)$ for a.e. $x \in X$, we find that $\tilde{u}$ is a equivalence class of u in $M^{1,\Phi}(X)$.

By our assumption and Theorem 3.5, we obtain that $M^{1,\Phi}(X) \supset N^{1,\Phi}(X)$. $\quad\square$

6   Boundedness of the maximal operator on L^{Φ}

In this section, we show the boundedness of maximal operators on $L^{\Phi}(X)$. This proof is a minor change of [44] but, for reader's convenience, we give the proof.

For a nonnegative $f \in L^1_{loc}(X)$, let

$$I(f, x, r) = \frac{1}{\mu(B(x,r))} \int_{X \cap B(x,r)} f(y) \, d\mu(y)$$

and

$$J(f, x, r) = \frac{1}{\mu(B(x,r))} \int_{X \cap B(x,r)} \Phi\big(y, f(y)\big) \, d\mu(y).$$

LEMMA 6.1 (cf. [44, Lemma 3.1]). *Assume that μ is lower Ahlfors $Q(x)$-regular. Suppose $\Phi(x,t)$ satisfies ($\Phi5$). Then there exists a constant $C > 0$ such that*

$$\Phi\big(x, I(f; x, r)\big) \leq CJ(f; x, r)$$

for all $x \in X$, $r > 0$ and for all nonnegative $f \in L^1_{loc}(X)$ such that $f(y) \geq 1$ or $f(y) = 0$ for each $y \in X$ and $\|f\|_{L^{\Phi}(X)} \leq 1$.

Proof. Given f as in the statement of the lemma, $x \in X$ and $r > 0$, set $I = I(f; x, r)$ and $J = J(f; x, r)$. Note that $\|f\|_{L^{\Phi}(X)} \leq 1$ implies

$$J \leq 2A_3\mu(B(x,r))^{-1} \leq 2A_3c_0^{-1}r^{-Q(x)}$$

for $0 < r < d_X$ by (2.1) and lower Ahlfors $Q(x)$-regularity of μ.

By ($\Phi2$) and (2.2), $\Phi\big(y, f(y)\big) \geq (A_1A_2)^{-1}f(y)$, since $f(y) \geq 1$ or $f(y) = 0$. Hence $I \leq A_1A_2J$. Thus, if $J \leq 1$, then

$$\Phi(x, I) \leq (A_1A_2J)A_2\phi(x, A_1A_2) \leq CJ.$$

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Next, suppose $J > 1$. Since $\Phi(x, t) \to \infty$ as $t \to \infty$, there exists $K \geq 1$ such that

$$\Phi(x, K) = \Phi(x, 1)J.$$

Then $K \leq A_2 J$ by (2.2). With this K, we have

$$\int_{X \cap B(x,r)} f(y)\, d\mu(y) \leq K\mu(B(x,r)) + A_2 \int_{X \cap B(x,r)} f(y)\frac{\phi(y, f(y))}{\phi(y, K)}\, d\mu(y).$$

Since

$$1 \leq K \leq A_2 J \leq 2A_2 A_3 c_0^{-1} r^{-Q(x)} \leq Cr^{-Q^+},$$

by ($\Phi 5$) there is $\beta > 0$, independent of f, x, r, such that

$$\phi(x, K) \leq \beta\phi(y, K) \quad \text{for all } y \in B(x, r).$$

Thus, we have by ($\Phi 2$)

$$\int_{X \cap B(x,r)} f(y)\, d\mu(y) \leq K\mu(B(x,r)) + \frac{A_2\beta}{\phi(x, K)} \int_{X \cap B(x,r)} f(y)\phi(y, f(y))\, d\mu(y)$$

$$= K\mu(B(x,r)) + A_2\beta\mu(B(x,r))\frac{J}{\phi(x, K)}$$

$$= K\mu(B(x,r))\left(1 + \frac{A_2\beta}{\phi(x, 1)}\right) \leq K\mu(B(x,r))\left(1 + A_1 A_2\beta\right).$$

Therefore

$$I \leq (1 + A_1 A_2\beta)K.$$

By ($\Phi 2$), ($\Phi 3$) and ($\Phi 4$), we obtain

$$\Phi(x, I) \leq C\Phi(x, K) \leq CJ$$

with constants $C > 0$ independent of f, x, r, as required. $\qquad\square$

LEMMA 6.2 (cf. [44, Lemma 3.2]). *Suppose $\Phi(x, t)$ satisfies ($\Phi 6$). Then there exists a constant $C > 0$ such that*

$$\Phi\big(x, I(f; x, r)\big) \leq C\left\{J(f; x, r) + \Phi(x, g(x))\right\}$$

for all $x \in X$, $r > 0$ and for all nonnegative $f \in L^1_{loc}(X)$ such that $g(y) \leq f(y) \leq 1$ or $f(y) = 0$ for each $y \in X$, where g is the function appearing in ($\Phi 6$).

Proof. Given f as in the statement of the lemma, $x \in X$ and $r > 0$, let $I = I(f; x, r)$ and $J = J(f; x, r)$.

By Jensen's inequality, we have

$$\overline{\Phi}(x, I) \leq \frac{1}{\mu(B(x, r))} \int_{X \cap B(x,r)} \overline{\Phi}\big(x, f(y)\big)\, d\mu(y).$$

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In view of (2.1),

$$\Phi(x, I) \le 2A_2 A_3 \frac{1}{\mu(B(x,r))} \int_{X \cap B(x,r)} \Phi\big(x, f(y)\big)\, d\mu(y).$$

If $d(x, x_0) \ge d(y, x_0)$, then $\Phi\big(x, f(y)\big) \le B_\infty \Phi\big(y, f(y)\big)$ by ($\Phi6$), where x_0 is the point appearing in ($\Phi6$).

Let $d(x, x_0) < d(y, x_0)$. If $g(x) < f(y)$, then $\Phi\big(x, f(y)\big) \le B_\infty \Phi\big(y, f(y)\big)$ by ($\Phi6$) again. If $g(x) \ge f(y)$, then $\Phi\big(x, f(y)\big) \le A_2 \Phi\big(x, g(x)\big)$ by ($\Phi3$). Hence,

$$\Phi\big(x, f(y)\big) \le C \left\{ \Phi\big(y, f(y)\big) + \Phi\big(x, g(x)\big) \right\}$$

in any case. Therefore, we obtain the required inequality. $\qquad\square$

THEOREM 6.3 (cf. [44, Theorem 4.1]). *Assume that X is a doubling space and μ is lower Ahlfors $Q(x)$-regular. Suppose $\Phi(x,t)$ satisfies ($\Phi5$), ($\Phi6$) and further assume:*

($\Phi3^*$) $t \mapsto t^{-\varepsilon_0} \phi(x, t)$ *is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon_0 > 0$.*

Then the Hardy-Littlewood maximal operator M is bounded from $L^\Phi(X)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^\Phi(X)} \le C \|f\|_{L^\Phi(X)}$$

for all $f \in L^\Phi(X)$.

We use the following result which is a special case of the theorem when $\Phi(x, t) = t^{p_0}$ ($p_0 > 1$) (see [33, Theorem 2.2]).

LEMMA 6.4. *Let $p_0 > 1$. Suppose X is a doubling space. Then there exists a constant $\tilde{c} > 0$ depending only on p_0 and c_2 for which the following holds: If f is a measurable function such that*

$$\int_X |f(y)|^{p_0}\, d\mu(y) \le 1,$$

then

$$\int_X [Mf(x)]^{p_0}\, d\mu(x) \le \tilde{c}.$$

Proof of Theorem 6.3. Set $p_0 = 1 + \varepsilon_0$ for $\varepsilon_0 > 0$ in condition ($\Phi3^*$) and consider the function

$$\Phi_0(x, t) = \Phi(x, t)^{1/p_0}.$$

Then $\Phi_0(x, t)$ also satisfies all the conditions (Φj), $j = 1, 2, \ldots, 6$. In fact, it trivially satisfies (Φj) for $j = 1, 2, 4, 5, 6$ with the same g for ($\Phi6$). Since

$$\Phi_0(x, t) = t\phi_0(x, t) \quad \text{with} \quad \phi_0(x, t) = \big[t^{-\varepsilon_0} \phi(x, t)\big]^{1/p_0},$$

condition ($\Phi 3^*$) implies that $\Phi_0(x, t)$ satisfies ($\Phi 3$).

Let $f \geq 0$ and $\|f\|_{L^\Phi(X)} \leq 1$. Let $f_1 = f\chi_{\{x:f(x)\geq 1\}}$, $f_2 = f\chi_{\{x:g(x)\leq f(x)<1\}}$ with g in ($\Phi 6$) and $f_3 = f - f_1 - f_2$.

Since $\Phi(x, t) \geq 1/(A_1 A_2)$ for $t \geq 1$ by ($\Phi 2$) and (2.2),

$$\Phi_0(x, t) \leq (A_1 A_2)^{1-1/p_0} \Phi(x, t)$$

if $t \geq 1$. Hence there is a constant $\lambda > 0$ such that $\|f_1\|_{L^{\Phi_0}(X)} \leq \lambda$ whenever $\|f\|_{L^\Phi(X)} \leq 1$. Applying Lemma 6.1 to Φ_0 and f_1/λ, we have

$$\Phi_0\big(x, Mf_1(x)\big) \leq CM\Phi_0\big(\cdot, f_1(\cdot)\big)(x).$$

Hence

$$\Phi\big(x, Mf_1(x)\big) \leq C\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} \tag{6.1}$$

for all $x \in X$ with a constant $C > 0$ independent of f.

Next, applying Lemma 6.2 to Φ_0 and f_2, we have

$$\Phi_0(x, Mf_2(x)) \leq C\left[M\Phi_0\big(\cdot, f_2(\cdot)\big)(x) + \Phi_0\big(x, g(x)\big)\right].$$

Noting that $\Phi_0(x, g(x)) \leq Cg(x)$ by (2.2) and ($\Phi 2$), we have

$$\Phi\big(x, Mf_2(x)\big) \leq C\left\{\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} + g(x)^{p_0}\right\} \tag{6.2}$$

for all $x \in X$ with a constant $C > 0$ independent of f.

Since $0 \leq f_3 \leq g \leq 1$, $0 \leq Mf_3 \leq Mg \leq 1$. Hence we have

$$\Phi\big(x, Mf_3(x)\big) \leq A_2\Phi_0(x, Mg(x))^{p_0} \leq C[Mg(x)]^{p_0} \tag{6.3}$$

for all $x \in X$ with a constant $C > 0$ independent of f.

Combining (6.1), (6.2) and (6.3), and noting that $g(x) \leq Mg(x)$ for a.e. $x \in X$, we obtain

$$\Phi\big(x, Mf(x)\big) \leq C\left\{\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} + [Mg(x)]^{p_0}\right\} \tag{6.4}$$

for a.e. $x \in X$ with a constant $C > 0$ independent of f.

In view of (2.1),

$$\int_X \Phi_0(y, f(y))^{p_0} d\mu(y) = \int_X \Phi(y, f(y)) d\mu(y) \leq 2A_3$$

for all $x \in X$. Hence, applying Lemma 6.4 to $(2A_3)^{-1/p_0}\Phi_0(y, f(y))$, we have

$$\int_X \Big[M\Phi_0\big(\cdot, f(\cdot)\big)(y)\Big]^{p_0} d\mu(y) \leq C$$

with a constant $C > 0$ independent of f.

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By Lemma 6.4, we obtain

$$\int_X [Mg(y)]^{p_0}\, d\mu(y) \le C$$

by $g \in L^{p_0}(X)$.

Thus, by (6.4), we finally obtain

$$\int_X \Phi\big(y, Mf(y)\big)\, d\mu(y) \le C.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

COROLLARY 6.5. *Suppose μ is Ahlfors $Q(x)$-regular. Let $\Phi(x,t)$ be as in Examples 2.1 and 2.4. Then the Hardy-Littlewood maximal operator M is bounded from $L^\Phi(X)$ into itself.*

In fact, $\Phi(x,t)$ satisfies $(\Phi3^*)$ with $\varepsilon_0 = (p^- - 1)/2$.

As in the proof of Theorem 6.3, we can show the following lemma.

LEMMA 6.6. *Assume that X is a bounded doubling space. Suppose $\Phi(x,t)$ satisfies $(\Phi3^*)$ and $(\Phi5)$. Then the Hardy-Littlewood maximal operator M is bounded from $L^\Phi(X)$ into itself.*

COROLLARY 6.7. *Assume that X is a bounded doubling space. Let $\Phi(x,t)$ be as in Example 2.1. Then the Hardy-Littlewood maximal operator M is bounded from $L^\Phi(X)$ into itself.*

By Proposition 5.1 and Theorem 6.3, we have the following result.

PROPOSITION 6.8. *Suppose $\Phi(x,t)$ satisfies $(\Phi3^*)$, $(\Phi5)$ and $(\Phi6)$. Then $M^{1,\Phi}(\mathbf{R}^N) = W^{1,\Phi}(\mathbf{R}^N)$.*

7 Sobolev's inequality

In this section, we show the Sobolev-type inequality on Musielak-Orlicz-Hajłasz-Sobolev spaces . For this purpose, we first prove Sobolev's inequality for Riesz-type operator in Musielak-Orlicz spaces.

LEMMA 7.1 (cf. [44, Lemma 5.1]). *Let $H(x,t)$ be a positive function on $X \times (0,\infty)$ satisfying the following conditions:*

(H1)　$H(x,\cdot)$ *is continuous on $(0,\infty)$ for each $x \in X$;*

(H2)　*there exists a constant $K_1 \ge 1$ such that*

$$K_1^{-1} \le H(x,1) \le K_1 \quad \text{for all } x \in X;$$

(H3) $t \mapsto t^{-\varepsilon'} H(x, t)$ is uniformly almost increasing for $\varepsilon' > 0$; namely there exists a constant $K_2 \geq 1$ such that

$$t^{-\varepsilon'} H(x, t) \leq K_2 s^{-\varepsilon'} H(x, s) \quad \text{for all } x \in X \quad \text{whenever } 0 < t < s;$$

Set
$$H^{-1}(x, s) = \sup\{t > 0 \,;\, H(x, t) < s\}$$

for $x \in X$ and $s > 0$. Then:

(1) $H^{-1}(x, \cdot)$ is nondecreasing.

(2)
$$H^{-1}(x, \lambda s) \leq (K_2 \lambda)^{1/\varepsilon'} H^{-1}(x, s)$$

for all $x \in X$, $s > 0$ and $\lambda \geq 1$.

(3)
$$H(x, H^{-1}(x, t)) = t$$

for all $x \in X$ and $t > 0$.

(4)
$$K_2^{-1/\varepsilon'} t \leq H^{-1}(x, H(x, t)) \leq K_2^{2/\varepsilon'} t$$

for all $x \in X$ and $t > 0$.

(5)
$$\min\left\{1, \left(\frac{s}{K_1 K_2}\right)^{1/\varepsilon'}\right\} \leq H^{-1}(x, s) \leq \max\{1, (K_1 K_2 s)^{1/\varepsilon'}\}$$

for all $x \in X$ and $s > 0$.

REMARK 7.2. $H(x, t) = \Phi(x, t)$ satisfies (H1), (H2) and (H3) with $K_1 = A_1$, $K_2 = A_2$ and $\varepsilon' = 1$.

LEMMA 7.3. *Assume that X is a bounded space. Suppose μ is lower Ahlfors $Q(x)$-regular and $\Phi(x, t)$ satisfies ($\Phi 5$). Then there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x, r))} \int_{X \cap B(x, r)} f(y) \, d\mu(y) \leq C \Phi^{-1}(x, r^{-Q(x)})$$

for all $x \in X, 0 < r < d_X$ and $f \geq 0$ satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^\Phi(X)} \leq 1$. Then we have $\int_X \Phi(y, f(y)) \, d\mu(y) \leq 2A_3$ by (2.1). By Lemma 6.1, ($\Phi 2$), ($\Phi 3$) and ($\Phi 4$), we obtain

$$\Phi\left(x, \frac{1}{\mu(B(x, r))} \int_{X \cap B(x, r)} f(y) \, d\mu(y)\right) \leq C(1 + \mu(B(x, r))^{-1})$$

$$\leq C(1 + r^{-Q(x)}) \leq C_1 r^{-Q(x)}$$

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for some constant $C_1 > 1$ and for all $x \in X$ and $0 < r < d_X$. Hence, we find by Lemma 7.1 with $H = \Phi$

$$\frac{1}{\mu(B(x,r))} \int_{X \cap B(x,r)} f(y) \, d\mu(y) \leq A_2 \Phi^{-1}\left(x, C_1 r^{-Q(x)}\right) \leq C_1 A_2^2 \Phi^{-1}\left(x, r^{-Q(x)}\right),$$

as required. $\qquad\qquad\square$

For an open set $\Omega \subset X, f \in L^1_{loc}(X)$ and $\alpha > 0$, we define the Riesz-type operator $J_\alpha^\Omega f$ of order α by

$$J_\alpha^\Omega f(x) = \sum_{2^i \leq 2d_\Omega} \frac{2^{i\alpha}}{\mu(B(x,2^i))} \int_{\Omega \cap B(x,2^i)} |f(y)| \, d\mu(y).$$

If μ is a doubling measure, then $I_\alpha^\Omega f(x) \leq C J_\alpha^\Omega f(x)$ for a.e $x \in X$, where

$$I_\alpha^\Omega f(x) = \int_\Omega \frac{d(x,y)^\alpha |f(y)|}{\mu(B(x,r))} \, d\mu(y)$$

is the usual Riesz potential of order α (see e.g. [23]).

LEMMA 7.4. *Suppose X is a bounded space and μ is lower Ahlfors $Q(x)$-regular. Assume that $\Phi(x,t)$ satisfies (Φ5) and*

(Φμ) *there exist constants $\gamma > 0$ and $A_4 \geq 1$ such that*

$$s^{\gamma+\alpha} \Phi^{-1}(x, s^{-Q(x)}) \leq A_4 t^{\gamma+\alpha} \Phi^{-1}(x, t^{-Q(x)})$$

   for all $x \in X$ whenever $0 \leq t < s$.

Then there exists a constant $C > 0$ such that

$$\sum_{\delta < 2^i \leq 2d_X} \frac{2^{i\alpha}}{\mu(B(x,2^i))} \int_{X \cap B(x,2^i)} f(y) \, d\mu(y) \leq C\delta^\alpha \Phi^{-1}(x, \delta^{-Q(x)})$$

for all $x \in X, 0 < \delta < d_X$ and $f \geq 0$ satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^\Phi(X)} \leq 1$. By Lemmas 7.1 and 7.3 and (Φμ), we have

$$\sum_{\delta < 2^i \leq 2d_X} \frac{2^{i\alpha}}{\mu(B(x,2^i))} \int_{X \cap B(x,2^i)} f(y) \, d\mu(y) \leq C \sum_{\delta < 2^i \leq 2d_X} 2^{i\alpha} \Phi^{-1}(x, 2^{-iQ(x)})$$

$$\leq C \int_\delta^\infty t^\alpha \Phi^{-1}(x, t^{-Q(x)}) \frac{dt}{t}$$

$$\leq C\delta^\alpha \Phi^{-1}(x, \delta^{-Q(x)}),$$

as required. $\qquad\qquad\square$

Note that $(\Phi\mu)$ implies

$$\lim_{t\to\infty} t\Phi(x,t)^{-\alpha/Q(x)} = \infty \qquad \text{uniformly in } x \in X. \tag{7.1}$$

We consider a function $\Psi_\alpha(x,t) : X \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ1) $\Psi_\alpha(\cdot, t)$ is measurable on X for each $t \geq 0$ and $\Psi_\alpha(x, \cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;

(Ψ2) there is a constant $A_5 \geq 1$ such that

$$\Psi_\alpha(x, at) \leq A_5 a \Psi_\alpha(x, t)$$

for all $x \in X$, $t > 0$ and $0 \leq a \leq 1$;

($\Psi\Phi\mu$) there exists a constant $A_6 \geq 1$ such that

$$\Psi_\alpha\left(x,\, t\Phi(x,t)^{-\alpha/Q(x)}\right) \leq A_6 \Phi(x,t)$$

for all $x \in X$ and $t > 0$.

Note: (Ψ2) implies that $\Psi_\alpha(x, \cdot)$ is uniformly almost increasing on $[0,\infty)$; (Ψ2), (7.1) and ($\Psi\Phi\mu$) imply that $\Psi_\alpha(\cdot, t)$ is bounded on X for each $t > 0$.

THEOREM 7.5. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Suppose $\Phi(x,t)$ satisfies $(\Phi3^*)$, $(\Phi5)$ and $(\Phi\mu)$; $\Psi_\alpha(x,t)$ satisfies $(\Psi1)$, $(\Psi2)$ and $(\Psi\Phi\mu)$. Then there exist constants $C_1, C_2 > 0$, such that*

$$\int_X \Psi_\alpha(x, J_\alpha^X f(x)/C_1)\, d\mu(x) \leq C_2$$

for all $f \geq 0$ satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

Proof. Let f be a nonnegative measurable function on X satisfying $\|f\|_{L^\Phi(X)} \leq 1$. Write

$$J_\alpha^X f(x) = \sum_{2^i \leq \delta} \frac{2^{i\alpha}}{\mu(B(x, 2^i))} \int_{X \cap B(x, 2^i)} f(y)\, d\mu(y)$$

$$+ \sum_{\delta < 2^i < 2d_X} \frac{2^{i\alpha}}{\mu(B(x, 2^i))} \int_{X \cap B(x, 2^i)} f(y)\, d\mu(y)$$

$$=: J_1 + J_2.$$

We have by Lemma 7.4

$$J_2 \leq C\delta^\alpha \Phi^{-1}(x, \delta^{-Q(x)}).$$

Since $J_1 \leq C\delta^\alpha Mf(x)$, we find that

$$J_\alpha^X f(x) \leq C\left\{\delta^\alpha Mf(x) + \delta^\alpha \Phi^{-1}(x, \delta^{-Q(x)})\right\}.$$

Here, let $\delta = \min\{d_X, \Phi(x, Mf(x))^{-1/Q(x)}\}$.

If $d_X \le \Phi(x, Mf(x))^{-1/Q(x)}$, then note from Lemma 7.1 that

$$Mf(x) \le A_2 \Phi^{-1}(x, d_X^{-Q(x)}) \le A_2 \max\{1, A_1 A_2 d_X^{-Q(x)}\} \le C.$$

Therefore $J_\alpha^X f(x) \le C$.

Next, if $d_X > \Phi(x, Mf(x))^{-1/Q(x)}$, then we have

$$\Phi^{-1}(x, \delta^{-Q(x)}) = \Phi^{-1}(x, \Phi(x, Mf(x))) \le A_2^2 Mf(x)$$

in view of Lemma 7.1. Hence we see that

$$J_\alpha^X f(x) \le C_1 \max\left\{Mf(x)\Phi(x, Mf(x))^{-\alpha/Q(x)}, 1\right\}$$

for some constant $C_1 > 0$. By (Ψ2) and ($\Psi\Phi\mu$), we find

$$\Psi_\alpha(x, J_\alpha^X f(x)/C_1) \le A_5 \left\{\Psi_\alpha(x, Mf(x)\Phi(x, Mf(x))^{-\alpha/Q(x)}) + \Psi_\alpha(x, 1)\right\}$$
$$\le C \left\{\Phi(x, Mf(x)) + 1\right\}.$$

Hence, by Lemma 6.6

$$\int_X \Psi_\alpha(x, J_\alpha^X f(x)/C_1) \, d\mu(x) \le C \left\{\int_X \Phi(x, Mf(x)) \, d\mu(x) + \mu(X)\right\} \le C_2$$

for some constant $C_2 > 0$, as required. $\qquad\square$

COROLLARY 7.6. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Let $\Phi(x, t)$ be as in Example 2.1 and set*

$$\Psi_\alpha(x, t) = \left(t \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)/p(x)}\right)^{p^\sharp(x)}$$

for all $x \in X$ and $t > 0$, where $1/p^\sharp(x) = 1/p(x) - \alpha/Q(x)$. Suppose

$$\operatorname{ess\,sup}_{x \in X}(\alpha p(x) - Q(x)) < 0. \tag{7.2}$$

Then there exists a constant $C > 0$ such that

$$\int_X \Psi_\alpha(x, J_\alpha^X f(x)) \, d\mu(x) \le C$$

for all $f \ge 0$ satisfying $\|f\|_{L^\Phi(X)} \le 1$.

Proof. First note that

$$\Phi^{-1}(x, t) \sim t^{1/p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{-q_j(x)/p(x)}$$

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for all $x \in X$ and $t > 0$. Therefore, by (7.2), there exists a constant $\gamma > 0$ such that

$$t^{\gamma+\alpha}\Phi^{-1}(x, t^{-Q(x)}) \sim t^{\gamma+\alpha-Q(x)/p(x)} \prod_{j=1}^{k}(L_c^{(j)}(t^{-1}))^{-q_j(x)/p(x)}$$

is uniformly almost decreasing on t. Hence $\Phi(x, t)$ satisfies ($\Phi\mu$). Similarly, since $t^{-1}\Psi_\alpha(x, t)$ is uniformly almost increasing on t, we see that $\Psi_\alpha(x, t)$ satisfies ($\Psi2$).

Finally, since

$$\Psi_\alpha\left(x, t\Phi(x, t)^{-\alpha/Q(x)}\right) = \Psi_\alpha\left(x, t^{p(x)/p^\sharp(x)} \prod_{j=1}^{k}(L_c^{(j)}(t))^{-\alpha q_j(x)/Q(x)}\right) \leq C\Phi(x, t)$$

for all $x \in X$ and $t > 0$, we see that $\Psi_\alpha(x, t)$ satisfies ($\Psi\Phi\mu$). Hence we obtain the required result by Theorem 7.5. $\qquad\square$

THEOREM 7.7. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Suppose $\Phi(x, t)$ satisfies ($\Phi3^*$), ($\Phi5$) and ($\Phi\mu$); $\Psi_1(x, t)$ satisfies ($\Psi1$), ($\Psi2$) and ($\Psi\Phi\mu$). Then for each ball $B \subset X$, there exist constants $C_1, C_2 > 0$ such that*

$$\int_B \Psi_1(x, |u(x) - u_B|/C_1)\, d\mu(x) \leq C_2$$

for all u satisfying $\|u\|_{M^{1,\Phi}(X)} \leq 1$.

Proof. Let $u \in M^{1,\Phi}(X)$ and let $g \in L^\Phi(X)$ be a Hajłasz gradient of u. Integrating both sides in (3.1) over y and x, we obtain the Poincaré inequality

$$\fint_B |u(x) - u_B|\, d\mu(x) \leq C d_B \fint_B g(x)\, d\mu(x)$$

for every ball $B \subset X$. Here if μ is a doubling measure, then we have by [23, Theorem 5.2]

$$|u(x) - u_B| \leq C J_1^X g(x)$$

for μ-a.e. $x \in B$. Hence we obtain the Sobolev-type inequality on Musielak-Orlicz-Hajłasz-Sobolev spaces by Theorem 7.5. $\qquad\square$

COROLLARY 7.8. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Let $\Phi(x, t)$ and $\Psi_1(x, t)$ be as in Corollary 7.6. Suppose*

$$\operatorname{ess\,sup}_{x \in X}(p(x) - Q(x)) < 0.$$

Then for each ball $B \subset X$, there exists a constant $C > 0$ such that

$$\int_B \Psi_1(x, |u(x) - u_B|)\, d\mu(x) \leq C$$

for all u satisfying $\|u\|_{M^{1,\Phi}(X)} \leq 1$.

8 Appendix

8.1 Musielak-Orlicz-Sobolev capacity in $\mathbf{R}^N$

For $u \in W^{1,\Phi}(\mathbf{R}^N)$, we define

$$\breve{\rho}_\Phi(u) = \rho_\Phi(u) + \rho_\Phi(\nabla u).$$

For $E \subset \mathbf{R}^N$, we denote

$$T_\Phi(E) = \{u \in W^{1,\Phi}(\mathbf{R}^N) : u \geq 1 \text{ in an open set containing } E\}.$$

The Musielak-Orlicz-Sobolev Cap_Φ-capacity is defined by

$$Cap_\Phi(E) = \inf_{u \in T_\Phi(E)} \breve{\rho}_\Phi(u).$$

In case $T_\Phi(E) = \emptyset$, we set $Cap_\Phi(E) = \infty$.

REMARK 8.1. Let $u, v \in W^{1,\Phi}(\mathbf{R}^N)$. Since

$$\int_{B(x,1)} |u(x)| \, dx + \int_{B(x,1)} |\nabla u(x)| \, dx$$
$$\leq 2|B(x,1)| + A_1 A_2 \left\{ \int_{B(x,1)} \Phi(x, |u(x)|) \, dx + \int_{B(x,1)} \Phi(x, |\nabla u(x)|) \, dx \right\}$$
$$\leq 2|B(x,1)| + 2A_1 A_2 A_3 \breve{\rho}_\Phi(u)$$

for all $x \in \mathbf{R}^N$ by (2.1), ($\Phi 2$) and ($\Phi 3$), we find $u \in W^{1,1}_{loc}(\mathbf{R}^N)$, where $|E|$ denotes its Lebesgue measure for a set $E \subset \mathbf{R}^N$. As in the proof of [27, Theorem 2.2], we have $\min\{u, v\}, \max\{u, v\} \in W^{1,\Phi}(\mathbf{R}^N)$,

$$\nabla \min\{u, v\}(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \leq v\} \\ \nabla v(x) & \text{for a.e. } x \in \{u \geq v\} \end{cases}$$

and

$$\nabla \max\{u, v\}(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \geq v\} \\ \nabla v(x) & \text{for a.e. } x \in \{u \leq v\}. \end{cases}$$

LEMMA 8.2. *Let $\{u_j\}$ and $\{v_j\}$ be sequences in $W^{1,\Phi}(\mathbf{R}^N)$. Assume that $\{\breve{\rho}_\Phi(u_j)\}$ is bounded. If $\{\breve{\rho}_\Phi(u_j - v_j)\}$ converges to zero, then $\{\breve{\rho}_\Phi(u_j) - \breve{\rho}_\Phi(v_j)\}$ converges to zero.*

Proof. We have by ($\Phi 3$) and ($\Phi 4$) that

$$\Phi(x, |v_j(x)|) \leq A_2 \Phi(x, |u_j(x) - v_j(x)| + |u_j(x)|)$$
$$\leq 2A_2^2 A_3 \{\Phi(x, |u_j(x) - v_j(x)|) + \Phi(x, |u_j(x)|)\}$$

for all $x \in \mathbf{R}^N$. Hence $\{\breve{\rho}_\Phi(v_j)\}$ is also bounded. By [42, Lemma 1.5], for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$|\overline{\Phi}(x, t_1) - \overline{\Phi}(x, t_2)| \leq \varepsilon \left\{ \overline{\Phi}(x, t_1) + \overline{\Phi}(x, t_2) \right\} + C(\varepsilon)\overline{\Phi}(x, |t_1 - t_2|)$$

for all $x \in \mathbf{R}^N$ and $t_1, t_2 \geq 0$. Therefore we have

$$\begin{aligned}
|\breve{\rho}_\Phi(u_j) - \breve{\rho}_\Phi(v_j)| &\leq \varepsilon \left\{ \breve{\rho}_\Phi(u_j) + \breve{\rho}_\Phi(v_j) \right\} + C(\varepsilon)\breve{\rho}_\Phi(u_j - v_j) \\
&\leq 2M\varepsilon + C(\varepsilon)\breve{\rho}_\Phi(u_j - v_j)
\end{aligned}$$

since $\breve{\rho}_\Phi(u_j) \leq M$ and $\breve{\rho}_\Phi(v_j) \leq M$ for some constant $M > 0$. Hence we find

$$\lim_{j \to \infty} |\breve{\rho}_\Phi(u_j) - \breve{\rho}_\Phi(v_j)| \leq 2M\varepsilon,$$

as required. $\qquad\square$

The standard argument and Lemma 8.2 yield the following results (see [27, Theorems 3.1 and 3.2]).

PROPOSITION 8.3. *The set function $Cap_\Phi(\cdot)$ satisfies the following conditions:*

(1) $Cap_\Phi(\emptyset) = 0$;

(2) *if $E_1 \subset E_2 \subset \mathbf{R}^N$, then $Cap_\Phi(E_1) \leq Cap_\Phi(E_2)$;*

(3) $Cap_\Phi(\cdot)$ *is an outer capacity;*

(4) *for $E_1, E_2 \subset \mathbf{R}^N$,*

$$Cap_\Phi(E_1 \cup E_2) + Cap_\Phi(E_1 \cap E_2) \leq Cap_\Phi(E_1) + Cap_\Phi(E_2);$$

(5) *if $K_1 \supset K_2 \supset \cdots$ are compact sets on $\mathbf{R}^N$, then*

$$\lim_{i \to \infty} Cap_\Phi(K_i) = Cap_\Phi\left(\bigcap_{i=1}^{\infty} K_i \right);$$

(6) *if $W^{1,\Phi}(\mathbf{R}^N)$ is reflexive and $E_1 \subset E_2 \subset \cdots$ are subsets of $\mathbf{R}^N$, then*

$$\lim_{i \to \infty} Cap_\Phi(E_i) = Cap_\Phi\left(\bigcup_{i=1}^{\infty} E_i \right);$$

(7) *if $W^{1,\Phi}(\mathbf{R}^N)$ is reflexive and $E_i \subset \mathbf{R}^N$ for $i = 1, 2, \cdots$, then*

$$Cap_\Phi\left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} Cap_\Phi(E_i).$$

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We say that a property holds Cap_Φ-q.e. in $\mathbf{R}^N$, if it holds except of a set $F \subset \mathbf{R}^N$ with $Cap_\Phi(F) = 0$. As in the proof of Theorem 3.9, we have the following result.

THEOREM 8.4 (c.f. [27, Lemma 5.1]). *Suppose $W^{1,\Phi}(\mathbf{R}^N)$ is reflexive. Then, for each Cauchy sequence of functions in $W^{1,\Phi}(\mathbf{R}^N) \cap C(\mathbf{R}^N)$, there is a subsequence which converges pointwise Cap_Φ-q.e. in $\mathbf{R}^N$. Moreover, the convergence is uniform outside a set of arbitrary small Musielak-Orlicz-Sobolev Cap_Φ-capacity.*

We say that a function $u : \mathbf{R}^N \to \mathbf{R}$ is Cap_Φ-quasicontinuous if for every $\varepsilon > 0$, there exists a open set E with $Cap_\Phi(E) < \varepsilon$ such that u restricted to $\mathbf{R}^N \setminus E$ is continuous.

COROLLARY 8.5 (c.f. [27, Theorem 5.2]). *Suppose $W^{1,\Phi}(\mathbf{R}^N)$ is reflexive and C^1-functions are dense in $W^{1,\Phi}(\mathbf{R}^N)$. Then $u \in W^{1,\Phi}(\mathbf{R}^N)$ has a Cap_Φ-quasicontinuous representative of u.*

8.2 Fuglede's theorem in $\mathbf{R}^N$

LEMMA 8.6 (c.f. [30, Lemma 3.1]). *Suppose C^1-functions are dense in $W^{1,\Phi}(\mathbf{R}^N)$. Let $E \subset \mathbf{R}^N$. If $Cap_\Phi(E) = 0$, then $M_\Phi(\Gamma_E) = 0$.*

Proof. Let $E \subset X$ with $Cap_\Phi(E) = 0$. Then, for all positive integer i, we choose functions $u_i \in W^{1,\Phi}(\mathbf{R}^N) \cap C^1(\mathbf{R}^N)$ such that $u_i(x) \geq 1$ for every $x \in E$ and $\breve{\rho}_\Phi(u_i) \leq A_2^{-1}(2A_3)^{-i-1}$. Set $v_k = \sum_{i=1}^k |u_i|$. Since

$$\breve{\rho}_\Phi\left(\frac{u_i}{2^{-i}}\right) \leq A_2(2A_3)^{i+1}\breve{\rho}_\Phi(u_i) \leq 1$$

by (2.1) and (Φ4), we have $\|u_i\|_{W^{1,\Phi}(\mathbf{R}^N)} \leq 2^{-i}$. Therefore

$$\|v_\ell - v_m\|_{W^{1,\Phi}(\mathbf{R}^N)} \leq \sum_{i=m+1}^\ell \|u_i\|_{W^{1,\Phi}(\mathbf{R}^N)} \leq 2^{-m}$$

for every $\ell > m$. Hence $\{v_k\}$ is a Cauchy sequence in $W^{1,\Phi}(\mathbf{R}^N)$. Setting $v(x) = \lim_{k\to\infty} v_k(x)$ for every $x \in X$, we see that $v \in W^{1,\Phi}(\mathbf{R}^N)$ is a Borel function. Thus, as in the proof of Lemma 4.6, we have the required result. $\square$

We say that $u : \mathbf{R}^N \to \mathbf{R}$ is absolutely continuous on lines, $u \in ACL(\mathbf{R}^N)$, if u is absolutely continuous on almost every line segment in $\mathbf{R}^N$ parallel to the coordinate axes. Note that an ACL function has classical derivatives almost everywhere. An ACL function is said to belong to $ACL^\Phi(\mathbf{R}^N)$ if $|\nabla u| \in L^\Phi(\mathbf{R}^N)$. Since $W^{1,\Phi}(\mathbf{R}^N) \hookrightarrow W^{1,1}(\mathbf{R}^N)$ locally, we obtain the following result.

LEMMA 8.7.
$$ACL^\Phi(\mathbf{R}^N) \cap L^\Phi(\mathbf{R}^N) = W^{1,\Phi}(\mathbf{R}^N).$$

Let $u : \mathbf{R}^N \to \mathbf{R}$ and Γ be the family on rectifiable curves $\gamma : [0, \ell(\gamma)] \to \mathbf{R}^N$ such that $u \circ \gamma$ is not absolutely continuous on $[0, \ell(\gamma)]$. We say that u is absolutely continuous on curves, $u \in ACC_\Phi(\mathbf{R}^N)$, if $M_\Phi(\Gamma) = 0$. It is clear that $ACC_\Phi(\mathbf{R}^N) \subset ACL(\mathbf{R}^N)$. An ACC_Φ function is said to belong to $ACC^\Phi(\mathbf{R}^N)$ if $|\nabla u| \in L^\Phi(\mathbf{R}^N)$.

The proof of the following theorem is the same as the proof of [30, Theorem 4.2].

THEOREM 8.8 (c.f. [30, Theorem 4.2]). *Suppose $W^{1,\Phi}(\mathbf{R}^N)$ is reflexive and C^1-functions are dense in $W^{1,\Phi}(\mathbf{R}^N)$. Then*

$$ACC^\Phi(\mathbf{R}^N) \cap L^\Phi(\mathbf{R}^N) = W^{1,\Phi}(\mathbf{R}^N).$$

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