SOBOLEV'S THEOREM FOR RIESZ POTENTIALS OF FUNCTIONS IN GRAND MORREY SPACES OF VARIABLE EXPONENT

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Abstract. In this paper we first study the boundedness of the maximal operator in grand Morrey spaces of variable exponent. As an application of the boundedness of maximal operator, we give Sobolev's inequality for Riesz potentials of functions in grand Morrey spaces of variable exponent, as an extension of Meskhi [19]. Further we are concerned with Trudinger's type exponential integrability and the continuity for Riesz potentials.

1. Introduction

Let $\mathbf{R}^N$ denote the N-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r and denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbf{R}^N$. In our discussions, the boundedness of the Hardy-Littlewood maximal operator is a crucial tool as in Hedberg [14]. It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ (see [27]).

In 1938, Morrey [21] considered the integral growth condition on derivatives over balls, in order to study the existence and regularity for partial differential equations. A family of functions with the integral growth condition is then called a Morrey space after his name. A systematical study for Morrey spaces was done by Peetre [23] in 1969. Chiarenza-Frasca [4] generalized the boundedness of the maximal operator by replacing Lebesgue spaces by Morrey spaces $L^{p,\nu}(\mathbf{R}^N)$, where Morrey space $L^{p,\nu}(\mathbf{R}^N)$ is a family of $f \in L^1_{loc}(\mathbf{R}^N)$ satisfying the Morrey condition

$$\sup_{x \in \mathbf{R}^N, r>0} r^{\nu} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p} < \infty$$

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for $\nu > 0$ (see also Nakai [22] and Mizuta-Shimomura [20]).

In [5], Diening showed that the maximal operator was bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^N)$ if the variable exponent $p(\cdot)$, which is a constant outside a ball, satisfies the locally log-Hölder condition and $\inf p(x) > 1$ (see condition (P2) in Section 2). In the mean time, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. These spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics; see [24]. In the case of bounded open sets, Almeida-Hasanov-Samko [2], Guliyev-Hasanov-Samko [12, 13] and Mizuta-Shimomura [20] studied the boundedness of the maximal operator for the variable exponent Morrey spaces.

Grand Lebesgue spaces were introduced in [15] for the sake of study of the integrability of the Jacobian. Grand Lebesgue spaces have been considered in various fields: in the theory of partial differential equations (see e.g. [16, 17, 25, 26]) and in the study of maximal operators (see e.g. [8]). In particular, in the theory of partial differential equations, it turns out that they are the right spaces in which N-harmonic equations $\mathrm{div}(|\nabla u|^{N-2}\nabla u) = \mu$ have to be considered (see [9, 11]). Further they have been studied in their own (see e.g. [3, 10]). Fiorenza-Gupta-Jain [7] studied the boundedness of the maximal operator in the grand Lebesgue spaces $L^{p)}([0,1])$ (see also [18]). Meskhi [19] generalized the boundedness of the maximal operator by replacing grand Lebesgue spaces by grand Morrey spaces $L^{p),\nu,\theta}(G)$, where G is bounded open set in $\mathbf{R}^N$ and grand Morrey space $L^{p),\nu,\theta}(G)$ is a family of $f \in L^1_{loc}(G)$ satisfying the grand Morrey condition

$$\sup_{x\in G, 0<r<d_G, 0<\varepsilon<p-1} \varepsilon^{\theta} r^{\nu} \left(\frac{1}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|^{p-\varepsilon}\, dy \right)^{1/(p-\varepsilon)} < \infty$$

for $\nu > 0$ and $\theta > 0$.

Our first aim in this paper is to establish the boundedness of the maximal operator in grand Morrey spaces of variable exponent, as an extension of Meskhi [19].

For $0 < \alpha < N$ and a locally integrable function f on G, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_G |x-y|^{\alpha-N} f(y)\, dy.$$

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|U_\alpha f\|_{L^{p^*}(\mathbf{R}^N)} \le C\|f\|_{L^p(\mathbf{R}^N)}$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where $1/p^* = 1/p - \alpha/N$. Sobolev's inequality has been studied in many articles and settings. If $f \in L^{p,\nu}(\mathbf{R}^N)$, then it is shown (see Adams [1] and Peetre [23]) that $U_\alpha f$ satisfies Sobolev's inequality whenever $\nu > \alpha$ and $1 < p < \infty$. Diening [6] dealt with Sobolev's embeddings for Riesz potentials with functions in $L^{p(\cdot)}(\mathbf{R}^N)$. In the case of bounded open sets, Almeida-Hasanov-Samko [2] and Mizuta-Shimomura [20] have established embedding results for Riesz potentials of functions in the variable exponent Morrey spaces. The version for the generalized variable exponent Morrey space $L^{p(\cdot),\omega}(G)$ was discussed by Guliyev-Hasanov-Samko [12, 13]. Further, Meskhi [19] studied Sobolev's embeddings for Riesz potentials of functions in the grand Morrey spaces.

Our second aim in this paper, as an application of the boundedness of maximal operator, is to establish Sobolev type inequalities for Riesz potentials of functions in grand Morrey spaces of variable exponent, as an extension of Meskhi [19]. Further, in Sections 5 and 6, we are concerned with Trudinger's type exponential integrability and the continuity for $U_\alpha f(x)$.

2. Preliminaries

Let G be a bounded open set in $\mathbf{R}^N$ whose diameter is denoted by d_G. Consider a function $p(\cdot)$ on G such that

(P1) $1 < p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$; and

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{c_p}{\log(e + 1/|x-y|)} \quad \text{for } x, y \in G$$

with a constant $c_p \ge 0$; $p(\cdot)$ is referred to as a variable exponent.

For a locally integrable function f on G, set

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\}.$$

In what follows, set $f = 0$ outside G. We denote by $L^{p(\cdot)}(G)$ the class of locally integrable functions f on G satisfying $\|f\|_{L^{p(\cdot)}(G)} < \infty$.

For $0 < \varepsilon < p^- - 1$, set $p_\varepsilon(x) = p(x) - \varepsilon$. For $\nu > 0$ and $\theta > 0$, we denote by $L^{p(\cdot)-0,\nu,\theta}(G)$ the class of locally integrable functions f on G satisfying

$$\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} = \sup_{x\in G, 0<r<d_G, 0<\varepsilon<p^- -1} \varepsilon^\theta r^\nu \left(\frac{1}{|B(x,r)|}\right)^{1/p_\varepsilon(x)} \times \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} < \infty.$$

The space $L^{p(\cdot)-0,\nu,\theta}(G)$ is referred to as a grand Morrey space of variable exponent.

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

LEMMA 2.1. *Let* $0 < \varepsilon_0 < p^- - 1$. *Then*

$$\sup_{x\in G, 0<r<d_G, 0<\varepsilon\le\varepsilon_0} \varepsilon^\theta r^\nu \left(\frac{1}{|B(x,r)|}\right)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \sim \|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)}$$

for all $f \in L^1_{loc}(G)$.

Proof. We may assume that $f \ge 0$ and

$$\sup_{x\in G, 0<r<d_G, 0<\varepsilon\le\varepsilon_0} \varepsilon^\theta r^\nu \left(\frac{1}{|B(x,r)|}\right)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \le 1.$$

Then note from (P1) and (P2) that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p_{\varepsilon_0}(y)}\, dy \le Cr^{-\nu p_{\varepsilon_0}(x)}$$

for all $x \in G$ and $0 < r < d_G$, since $r^{p(y)} \le Cr^{p(x)}$ whenever $|x-y| < r$ by (P2). If $x \in G$, $0 < r < d_G$ and $\varepsilon_0 < \varepsilon < p^- - 1$, then we obtain

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p_\varepsilon(y)}\, dy &\le \frac{1}{|B(x,r)|}\int_{B(x,r)} r^{-\nu p_\varepsilon(y)}\, dy \\ &+ \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p_\varepsilon(y)} \left(\frac{f(y)}{r^{-\nu}}\right)^{p_{\varepsilon_0}(y)-p_\varepsilon(y)} dy \\ &\le Cr^{-\nu p_\varepsilon(x)} + r^{\nu(\varepsilon_0-\varepsilon)}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p_{\varepsilon_0}(y)}dy \le Cr^{-\nu p_\varepsilon(x)},\end{aligned}$$

which proves the lemma. □

3. BOUNDEDNESS OF MAXIMAL FUNCTIONS

We present the boundedness of maximal functions in grand Morrey spaces of variable exponent, as an extension of Meskhi [19]. We recall the notion of maximal functions of locally integrable functions f on G, which are in fact defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\, dy.$$

LEMMA 3.1. *Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)\, dy \le C r^{-\nu}$$

for all $x \in G, 0 < r < d_G$ and $0 < \varepsilon < p^- - 1$, where $g(y) = \varepsilon^\theta f(y)$.

Here, taking $\varepsilon = (p^- - 1)(\log(e + d_G/r))^{-1}$, we find

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy \le C r^{-\nu} (\log(2d_G/r))^\theta. \tag{1}$$

Proof of Lemma 3.1. Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$. Then note from (P2) that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)^{p_\varepsilon(y)}\, dy \le C r^{-\nu p_\varepsilon(x)}$$

for all $x \in G, 0 < r < d_G$ and $0 < \varepsilon < p^- - 1$. Hence, we find from (P2)

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)\, dy &\le r^{-\nu} + \frac{1}{|B(x,r)|} \int_{B(x,r)} g(y) \left(\frac{g(y)}{r^{-\nu}}\right)^{p_\varepsilon(y)-1} dy \\ &\le r^{-\nu} + C r^{\nu(p_\varepsilon(x)-1)} \frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)^{p_\varepsilon(y)}\, dy \le C r^{-\nu}, \end{aligned}$$

as required. □

We denote by χ_E the characteristic function of E.

LEMMA 3.2. *Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$. Set $g_j(y) = \varepsilon^\theta f(y) \chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)}$ for $0 < \varepsilon < p^- - 1$ and $j \ge 1$. Then there exists a constant $C > 0$ such that*

$$M g_j(z) \le C r^{-\nu} 2^{-\nu j}$$

for all $z \in B(x,r)$ and $0 < \varepsilon < p^- - 1$.

Proof. Let $z \in B(x,r)$. Noting that $g_j(y) = 0$ for $y \in B(z,(2^j - 1)r)$, we have by Lemma 3.1 and (P2)

$$M g_j(z) = \sup_{t > (2^j-1)r} \frac{1}{|B(z,t)|} \int_{B(z,t)} g_j(y)\, dy \le C \sup_{t > (2^j-1)r} t^{-\nu} \le C 2^{-\nu j} r^{-\nu},$$

as required. □

LEMMA 3.3 ([5, Theorem 3.5]). *Suppose that $p_0(\cdot)$ is a function on G such that*

$$1 < p_0^- := \inf_{x\in G} p_0(x) \le \sup_{x\in G} p_0(x) =: p_0^+ < \infty; \text{and}$$

$$|p_0(x) - p_0(y)| \le \frac{c_{p_0}}{\log(2d_G/|x-y|)}$$

for all $x, y \in G$, where $c_{p_0} \ge 0$ is a constant. Then there exists a constant $c_0 > 0$ depending only on p_0^-, p_0^+, c_{p_0} and $|G|$ such that

$$\|Mf\|_{L^{p_0(\cdot)}(G)} \le c_0 \|f\|_{L^{p_0(\cdot)}(G)}$$

for all $f \in L^{p_0(\cdot)}(G)$.

THEOREM 3.4 (cf. [19, Theorem 3.1]). *The maximal operator : $f \to Mf$ is bounded from $L^{p(\cdot)-0,\nu,\theta}(G)$ to $L^{p(\cdot)-0,\nu,\theta}(G)$, that is,*

$$\|Mf\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le C\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \qquad \text{for all } f \in L^{p(\cdot)-0,\nu,\theta}(G).$$

Proof. Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$. Let $x \in G, 0 < r < d_G$ and $0 < \varepsilon < (p^- - 1)/2$ be fixed. Set $g(y) = \varepsilon^\theta f(y)$.

For each positive integer j, set $g_j = g\chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)}$ and $g_0 = g\chi_{B(x,2r)}$. Here, we find by (P2) and Lemma 3.2

$$\|Mg_j\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \le C2^{-\nu j} r^{-\nu} |B(x,r)|^{1/p_\varepsilon(x)}$$

for $j \ge 1$. By Lemma 3.3, we have

$$\|Mg_0\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \le C\|g\|_{L^{p_\varepsilon(\cdot)}(B(x,2r))},$$

where the constant C does not depend on ε with $0 < \varepsilon < (p^- - 1)/2$. Hence

$$\begin{aligned}\|Mg\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} &\le \|Mg_0\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} + \sum_{j=1}^\infty \|Mg_j\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\ &\le C\left\{\|g\|_{L^{p_\varepsilon(\cdot)}(B(x,2r))} + |B(x,r)|^{1/p_\varepsilon(x)} r^{-\nu} \sum_{j=1}^\infty 2^{-\nu j}\right\} \\ &\le C\left\{|B(x,2r)|^{1/p_\varepsilon(x)}(2r)^{-\nu} + |B(x,r)|^{1/p_\varepsilon(x)} r^{-\nu}\right\},\end{aligned}$$

so that

$$\sup_{x\in G, 0<r<d_G, 0<\varepsilon<(p^- -1)/2} r^\nu \left(\frac{1}{|B(x,r)|}\right)^{1/p_\varepsilon(x)} \|Mg\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \le C$$

for all $x \in G, 0 < r < d_G$ and $0 < \varepsilon < (p^- - 1)/2$. Hence, we obtain the required results by Lemma 2.1. □

4. Sobolev's inequality

Now we show the Sobolev type inequality for Riesz potentials in grand Morrey spaces of variable exponent, as an extension of Meskhi [19].

Theorem 4.1 (cf. [19, Theorems 5.3 and 5.4]). *Suppose $\alpha < \nu \le N$ and $1/p^*(x) = (\nu-\alpha)/(\nu p(x))$. Then there exists a constant $C > 0$ such that*

$$\|U_\alpha f\|_{L^{p^*(\cdot)-0,\nu-\alpha,\theta}(G)} \le C\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)}.$$

Proof. Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$. Let $x \in G, 0 < r < d_G$ and $0 < \varepsilon < \min\{p^- - 1, ((p^*)^- - 1)(\nu-\alpha)/\nu\}$ be fixed. For $z \in B(x,r)$ and $\delta > 0$, we write

$$\begin{aligned} U_\alpha f(z) &= \int_{B(z,\delta)} |z-y|^{\alpha-N} f(y)\,dy + \int_{G\setminus B(z,\delta)} |z-y|^{\alpha-N} f(y)\,dy \\ &= U_1(z) + U_2(z). \end{aligned}$$

First note that $U_1(z) \le C\delta^\alpha Mf(z)$. To estimate U_2, set $g(y) = \varepsilon^\theta f(y)$. Then we have by Lemma 3.1

$$\varepsilon^\theta U_2(z) = \sum_{j=1}^\infty \int_{B(z,2^j\delta)\setminus B(z,2^{j-1}\delta)} |z-y|^{\alpha-N} g(y)\,dy$$

$$\le C\sum_{j=1}^\infty (2^j\delta)^\alpha \frac{1}{|B(x,2^j\delta)|}\int_{B(z,2^j\delta)} g(y)\,dy \le C\sum_{j=1}^\infty (2^j\delta)^{\alpha-\nu} \le C\delta^{\alpha-\nu}.$$

Hence

$$U_\alpha g(z) \le C\left\{\delta^\alpha Mg(z) + \delta^{\alpha-\nu}\right\}.$$

Here, letting $\delta = \{Mg(z)\}^{-1/\nu}$, we establish

$$U_\alpha g(z) \le CMg(z)^{1-\alpha/\nu}.$$

Now Theorem 3.4 gives

$$\int_{B(x,r)} \{\varepsilon^\theta U_\alpha f(z)\}^{(p_\varepsilon)^*(z)}\,dz \le C\int_{B(x,r)} \{Mg(z)\}^{p_\varepsilon(z)}\,dz \le Cr^{N-\nu p_\varepsilon(x)},$$

where $1/(p_\varepsilon)^*(x) = (\nu-\alpha)/(\nu(p_\varepsilon(x))$ by definition. Since $(p_\varepsilon)^*(z) = p^*(z) - \nu\varepsilon/(\nu-\alpha) = (p^*)_{\nu\varepsilon/(\nu-\alpha)}(z)$, we find

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \{\varepsilon^\theta U_\alpha f(z)\}^{(p^*)_{\nu\varepsilon/(\nu-\alpha)}(z)}\,dz \le Cr^{-(\nu-\alpha)(p^*(z))_{\nu\varepsilon/(\nu-\alpha)}(x)},$$

so that we obtain the required results by Lemma 2.1. □

5. Exponential integrability

Lemma 5.1. *Let f be a nonnegative function on G satisfying* (1) *with $\nu = \alpha$ for all $x \in G$ and $0 < r < d_G$. For $0 < \eta < \alpha$, there exists a constant $C > 0$ such that*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} I_\eta f(x)\,dx \le C r^{\eta-\alpha}(\log(2d_G/r))^\theta$$

for $z \in G$ and $0 < r < d_G$.

Proof. For $x \in B(z,r)$, write

$$\begin{aligned} I_\eta f(x) &= \int_{B(z,2r)} |x-y|^{\eta-N} f(y)\,dy + \int_{G\setminus B(z,2r)} |x-y|^{\eta-N} f(y)\,dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

By Fubini's theorem and (1), we have

$$\begin{aligned} \int_{B(z,r)} I_1(x)\,dx &= \int_{B(z,2r)} \left(\int_{B(z,r)} |x-y|^{\eta-N}\,dx \right) f(y)\,dy \\ &\le \int_{B(z,2r)} \left(\int_{B(y,3r)} |x-y|^{\eta-N}\,dx \right) f(y)\,dy \\ &\le C r^\eta \int_{B(z,2r)} f(y)\,dy \le C r^{\eta-\alpha}|B(z,r)|(\log(2d_G/r))^\theta. \end{aligned}$$

For I_2, note that

$$I_2(x) \le C \int_{G\setminus B(z,2r)} |z-y|^{\eta-N} f(y)\,dy$$

for $x \in B(z,r)$. Hence we have

$$\begin{aligned} I_2(x) &\le C \int_{2r}^{4d_G} t^{\eta-N} \left(\int_{B(z,t)} f(y)\,dy \right) \frac{dt}{t} \\ &\le C \int_{2r}^{4d_G} t^{\eta-\alpha}(\log(2d_G/t))^\theta \frac{dt}{t} \le C r^{\eta-\alpha}(\log(2d_G/r))^\theta. \end{aligned}$$

Thus this lemma is proved. □

Theorem 5.2. *For $0 < \eta < \alpha$ there exist constants $c_1, c_2 > 0$ such that*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left(c_1 U_\alpha f(x)^{1/(\theta+1)} \right) dx \le c_2 r^{\eta-\alpha}$$

for all $z \in G$ and $0 < r < d_G$, whenever f is a nonnegative measurable function on G satisfying $\|f\|_{L^{p(\cdot)-0,\alpha,\theta}(G)} \le 1$.

Proof. Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p(\cdot)-0,\alpha,\theta}(G)} \le 1$. Then we have by (1)

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \le Cr^{-\alpha}(\log(2d_G/r))^{\theta}$$

for all $x \in G$ and $0 < r < d_G$.

For $x \in B(z,r), 0 < \delta < d_G$ and $0 < \eta < \alpha$, write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-N} f(y)\,dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\,dy \\ &= U_1(x) + U_2(x). \end{aligned}$$

Then we have $U_1(x) \le \delta^{\alpha-\eta} I_\eta f(x)$, and by (1)

$$\begin{aligned} U_2(x) &\le C\int_\delta^{2d_G} t^{\alpha-N}\left(\int_{B(x,t)} f(y)\,dy\right)\frac{dt}{t} \\ &\le C\int_\delta^{2d_G} (\log(2d_G/t))^{\theta}\,\frac{dt}{t} \le C(\log(2d_G/\delta))^{\theta+1}. \end{aligned}$$

Hence it follows that

$$U_\alpha f(x) \le C\left\{\delta^{\alpha-\eta} I_\eta f(x) + (\log(2d_G/\delta))^{\theta+1}\right\}.$$

Here, letting $\delta = \min\{d_G, \{I_\eta f(x)\}^{-1/(\alpha-\eta)}(\log(e + I_\eta f(x)))^{(\theta+1)/(\alpha-\eta)}\}$, we have the inequality

$$U_\alpha f(x) \le C(\log(e + I_\eta f(x)))^{\theta+1}.$$

Then, in view of Lemma 5.1, there exist constants $c_1, c_3 > 0$ such that

$$\begin{aligned} &\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left(c_1 U_\alpha f(x)^{1/(\theta+1)}\right) dx \\ \le\ & \frac{1}{|B(z,r)|}\int_{B(z,r)} \{e + I_\eta f(x)\}\,dx \le c_3 r^{\eta-\alpha}(\log(2d_G/r))^{\theta} \end{aligned}$$

for all $z \in G$ and $0 < r < d_G$. Since $c_3 r^{\eta-\alpha}(\log(2d_G/r))^{\theta} \le c_2(\eta')r^{\eta'-\alpha}$ for all $0 < r < d_G$ when $0 < \eta' < \eta$, the proof of the present theorem is completed. □

6. Continuity

THEOREM 6.1. *If $\alpha - 1 < \nu < \alpha$, then there exists a constant $C > 0$ such that*

$$|U_\alpha f(x) - U_\alpha f(z)| \le C|x-z|^{\alpha-\nu}(\log(2d_G/|x-z|))^{\theta}$$

for all $x, z \in G$*, whenever* f *is a nonnegative measurable function on* G *satisfying* $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$.

Proof. Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p(\cdot)-0,\nu,\theta}(G)} \le 1$. For $x, z \in G$ write $\rho = |x - z|$ and

$$|U_\alpha f(x) - U_\alpha f(z)| \le \int_{B(x,2\rho)} |x-y|^{\alpha-N} f(y)\,dy + \int_{B(x,2\rho)} |z-y|^{\alpha-N} f(y)\,dy$$
$$+ \int_{G\setminus B(x,2\rho)} \left| |x-y|^{\alpha-N} - |z-y|^{\alpha-N} \right| f(y)\,dy = U_1 + U_2 + U_3.$$

Using (1), we have

$$U_1 \le C\rho^{\alpha-\nu}(\log(2d_G/\rho))^\theta$$

and

$$U_2 \le \int_{B(z,3\rho)} |z-y|^{\alpha-N} f(y)\,dy \le C\rho^{\alpha-\nu}(\log(2d_G/\rho))^\theta.$$

Moreover, by (1), we have

$$U_3 \le C\rho \int_{G\setminus B(x,2\rho)} |x-y|^{\alpha-1-N} f(y)\,dy \le C\rho^{\alpha-\nu}(\log(2d_G/\rho))^\theta.$$

Thus we have the conclusion. □

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